Dominic Amann, BSc

Boundary element methods for Helmholtz transmission problems

MASTERARBEIT

zur Erlangung des akademischen Grades

Diplom-Ingenieur

Masterstudium Technomathematik

eingereicht an der

Technischen Universität Graz

Betreuer:
Prof. Dr. O. Steinbach

Institut für Numerische Mathematik

Graz, November 2014

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Abstract

In this thesis we compare several boundary integral formulations for three dimensional Helmholtz transmission problems. When studying boundary integral equations for the Helmholtz equation, eigenvalues of the Laplace operator, also called spurious modes, may cause difficulties. If the considered wave number corresponds to a spurious mode, certain boundary integral operators lose their injectivity, which may lead to non-uniquely solvable boundary integral equations. Whether a wave number belongs to an eigenvalue of the Laplace operator is dependent on the domain and can in general not be determined easily.

At first we consider the model problem of a bounded Lipschitz domain and constant wave numbers. In a second step we expand our considerations by allowing the wave numbers to be piecewise constant. For both cases we discuss three direct boundary integral formulations that overcome the problem mentioned above, meaning we can establish unique solvability independent of the wave numbers. The presented formulations are the single trace formulation, the Steklov–Poincaré operator formulation and the local multi trace formulation. For the discretization of these formulations we apply a Galerkin scheme which leads to systems of linear equations. To efficiently solve these systems with iterative solvers, we examine the formulations's compatibility with operator preconditioning and how such preconditioners can be constructed. Finally numerical examples are presented to confirm our findings and compare the three formulations.

Zusammenfassung

Diese Arbeit vergleicht verschiedene Randintegralformulierungen für das dreidimensionale Helmholtz Transmissionsproblem. Untersucht man die Helmholtz-Gleichung mit Hilfe von Randintegralgleichungen, so können Eigenwerte des Laplace Operators, so genannte Spurious Modes, Probleme verursachen. Angenommen, die betrachtete Wellenzahl korrespondiert zu einem Spurious Mode, so verlieren bestimmte Randintegraloperatoren ihre Injektivität. Dies kann in weiterer Folge zu nicht eindeutig lösbaren Systemen von Randintegralgleichungen führen. Ob eine Wellenzahl zu einem Eigenwert des Laplace Operators gehört, hängt vom betrachteten Gebiet ab und kann im Allgemeinen nicht ohne Aufwand bestimmt werden.

In dieser Arbeit wird als Modellproblem zunächst ein beschränktes Lipschitz Gebiet mit konstanten Wellenzahlen betrachtet. In einem zweiten Schritt werden die Betrachtungen auf stückweise konstante Wellenzahlen erweitert. Für beide Fälle werden drei direkte Randintegralformulierungen vorgestellt, welche stabil in Hinsicht auf Spurious Modes sind. Dies bedeutet, dass ihre eindeutige Lösbarkeit unabhängig von der Wahl der Wellenzahlen gewährleistet ist. Die vorgestellten Formulierungen sind die Single Trace Formulierung, die Steklov–Poincaré Operator Formulierung und die Multi Trace Formulierung. Diese Formulierungen werden mit Hilfe einer Galerkin-Methode diskretisiert, was zu linearen Gleichungssystemen führt. Um für diese Systeme iter-

ative Lösungsverfahren effizient anwenden zu können, untersuchen wir, ob sich die betrachteten Formulierungen für die Vorkonditionierung mit geeignet gewählten Operatoren verwenden lassen und wie solche Vorkonditioner aufgestellt werden können. Abschließend werden numerische Beispiele betrachtet, um die theoretischen Aussagen zu bekräftigen und die drei Formulierungen zu vergleichen.

Vorwort

Ich möchte mich bei meinem Betreuer Herrn Professor Steinbach für die engagierte Unterstützung beim Verfassen dieser Arbeit bedanken. Sein Feedback war eine hilfreiche Ressource, ohne die diese Diplomarbeit nicht möglich gewesen wäre. Ein besonderer Dank geht an Arno Kimeswenger, der mir den Einstieg in Bem++ maßgeblich erleichtert hat und bei Fragen helfend zur Seite stand. Diesbezüglich geht auch ein Dank an das Team von Bem++, das ihre Software für jeden zugänglich zur Verfügung stellt. Abschließend möchte ich dem gesamten Institut für numerische Mathematik für die offene und angenehme Atmosphäre danken, die mein Studium geprägt hat.

Contents

Introduction

In this thesis we discuss how to compute approximate solutions for the three dimensional Helmholtz transmission problem. This problem is a coupling of at least two Helmholtz equations, each of whom is defined as a partial differential equation

$$-\Delta u(x) - \kappa^2 u(x) = 0$$

where Δ is the Laplace differential operator and u the unknown function. The wave number κ is a given material constant, which we assume to be real and positive. This equation can be used to model linear acoustics when assuming time harmonic behaviour of the acoustic waves [8]. In this case, the unknown complex function u is either the velocity potential or the pressure and κ is the wave length depending on the medium.

We now formulate the Helmholtz transmission problem, which describes the scattering of an incoming wave by a penetrable scatterer $\Omega \subset \mathbb{R}^3$. The solution consists of two functions u_{int} and u_{ext} that solve the local partial differential equations

$$\begin{aligned} -\Delta u_{\text{int}}(x) - \kappa_{\text{int}}^2 u_{\text{int}}(x) &= 0 \qquad \text{for } x \in \Omega, \\ -\Delta u_{\text{ext}}(x) - \kappa_{\text{ext}}^2 u_{\text{ext}}(x) &= 0 \qquad \text{for } x \in \Omega^c := \mathbb{R}^3 \setminus \overline{\Omega}. \end{aligned}$$

The two functions are connected via the boundary $\Gamma := \partial\Omega$ where they fulfil transmission conditions

$$\begin{aligned} u_{\text{int}}(x) - u_{\text{ext}}(x) &= f(x) \qquad \text{for } x \in \Gamma, \\ \frac{\partial}{\partial n_x} u_{\text{int}}(x) - \frac{\partial}{\partial n_x} u_{\text{ext}}(x) &= g(x) \qquad \text{for } x \in \Gamma, \end{aligned}$$

with the unit normal vector n_x and given transmission data (f, g). Finally the solution in the exterior u_{ext} has to satisfy the Sommerfeld radiation condition to ensure uniqueness. This is the simplest form of the Helmholtz transmission problem, a more general model allows for a piecewise constant interior wave number κ_{int}.

Since in most cases it is not possible to derive an analytical solution, we have to resort to approximate solutions. These can be computed by numerical methods such as the finite element method [2, 19] or the boundary element method [15, 19]. In this thesis we consider the boundary element method which is based on the property that any solution of the Helmholtz equation u is fully described by its trace u and its normal derivative $\frac{\partial u}{\partial n_x}$ on the boundary. These traces can be determined as the solution of boundary integral equations formulated in the appropriate function spaces on the boundary. In a last step these spaces are replaced by finite dimensional discrete

function spaces which leads to a system of linear equations and thus an approximate solution.

There are several advantages of the boundary element method over the finite element method for this model problem. First of all the domain we have to consider is reduced from $\mathbb{R}^3$ to the surface of Ω and secondly, the boundary element method can handle unbounded domains and radiation conditions naturally. Unfortunately, boundary integral equations may suffer from spurious modes which are connected to eigenvalues of the interior Laplace operator. These modes cause the loss of injectivity for certain boundary integral operators, which can result in a non unique solution even though the model problem stated above is uniquely solvable.

Thus we are interested in boundary integral equations that are uniquely solvable for all wave numbers. The approaches presented in this thesis are two Steklov–Poincaré operator formulations [20, 22], the single trace formulation [21] and the local multi trace formulation [5, 12]. Some other formulations not discussed here are the global multi trace formulation [4] and the boundary element tearing and interconnecting method [22].

All these formulations result in systems of linear equations that can be solved using iterative solvers. One obstacle is that increasing the dimension of the finite dimensional functions spaces, which leads to more accurate approximate solutions, also increases the number of required iteration steps and hence computation time. The solution to this problem is preconditioning, where we modify the systems with the aim that the number of iterations is independent of the dimension of the discrete spaces. One such preconditioning strategy, which is based on operators of opposite order [11, 19], is presented and discussed for three of the formulations.

This thesis is structured as follows. The first chapter introduces the abstract setting for coercive operators as well as the appropriate function spaces. Chapter 2 presents all standard boundary integral operators and the Steklov–Poincaré operator, which is a Dirichlet to Neumann mapping. The third chapter explains how we can translate a continuous operator equation into a discrete setting and introduces the operator preconditioning strategy. In Chapter 4 we return to the model problem and discuss the four presented formulations with regard to their unique solvability, discretization and preconditioning. Chapter 5 considers the more general case, where the domain is not homogeneous but has a piecewise constant wave number κ_{int}. The last chapter confirms the theoretical results and compares the formulations with numerical examples.

1 Fundamentals

This chapter gives a short introduction to the fundamental concepts and notations we require later on to study the Helmholtz transmission problem.

The first two sections present two different abstract settings for operator equations of the form $Au = f$ as well as sufficient properties to ensure their unique solvability. One is the classical approach of the Fredholm alternative [13, 19], the other a combination of the Fredholm alternative and Lion's lemma [5, 12].

The remainder of this chapter introduces the so–called Sobolev spaces [3, 13, 15, 19], which are spaces of functions whose derivatives are square integrable up to a given order. Starting from a domain $\Omega \subset \mathbb{R}^3$, the definition is extended to cover boundaries, open subsets of boundaries and skeleton boundaries. All considerations are restricted to the case when $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain.

1.1 Variational formulations for coercive operators

We consider a bounded linear operator $A : X \to X'$ where X is a complex Hilbert space. An operator is called bounded if there exists a constant $c_2^A > 0$ such that

$$\|Av\|_{X'} \leq c_2^A \|v\|_X$$

holds true for all $v \in X$. For a given $f \in X'$ the goal is to find a solution $u \in X$ such that the operator equation

$$Au = f$$

is satisfied. This model complies with the Steklov–Poincaré operator formulation and the single trace formulation we discuss later on. The first step is to rewrite the operator equation into an equivalent variational formulation to find $u \in X$ such that

$$\langle Au, v \rangle = \langle f, v \rangle \tag{1.1}$$

holds for all $v \in X$ where $\langle \cdot, \cdot \rangle$ is the duality pairing on $X' \times X$. For a proof of the equivalence of these problems see [19]. Since A is linear and bounded it induces a sesquilinear form $a(\cdot, \cdot) : X \times X \to \mathbb{C}$ by

$$a(u, v) := \langle Au, v \rangle .$$

On the other hand, let $a(\cdot, \cdot)$ be a given sesquilinear form, then there exists a linear bounded operator $A : X \to X'$ such that this equation is satisfied. This allows us to switch between these two formulations without restrictions. Note that this also implies that every property of an operator has a counterpart in the sesquilinear form.

The properties we assumed so far for A are not sufficient to ensure unique solvability of (1.1). One possible set of such properties is given in the Fredholm alternative.

Theorem 1.1 (Fredholm alternative)**.** *[13, Thm 5.3.1] Let $K : X \to X$ be a compact operator, then exactly one of the following alternatives is true:*

- *The homogeneous equation $(I - K)u = 0$ has a non trivial solution $u \in X$.*
- *For any $g \in X$ there exists an element $u \in X$ such that $(I - K)u = g$ and $\|u\|_X \leq c\|g\|_{X'}$ with a constant $c > 0$ independent of g.*

A linear operator is said to be compact if the image of the unit ball is relatively compact in the image space [3]. Although we want to use this result, we would like to get rid of the auxiliary operator K and instead formulate conditions on the operator A itself.

Definition 1. Let X be a Hilbert space. An operator $A : X \to X'$ is called coercive if there exists a compact operator $T_A : X \to X'$ and a constant $c_1^A > 0$ such that the Gårding inequality

$$\Re \langle (A + T_A)v, v \rangle \geq c_1^A \|v\|_X^2 \tag{1.2}$$

holds for all $v \in X$.

Definition 2. Let $A : X \to X'$ be a linear operator. If it follows from

$$Au = 0$$

that $u = 0$, we call A injective.

Assuming that A fulfils these properties we can use the Fredholm alternative to show unique solvability of the continuous variational formulation as stated in the following lemma.

Lemma 1.2. *[19, Theorem 3.15] Let $A : X \to X'$ be a bounded, injective and coercive linear operator, i.e. (1.2) holds true. Then the operator equation $Au = f$ has for any $f \in X'$ a unique solution $u \in X$ that satisfies*

$$\|u\|_X \leq c\|f\|_{X'}$$

with a constant $c > 0$ independent of f.

1.2 Variational formulations and Lion's lemma

In this section we consider a bounded and linear operator $A : X \to \Pi'$ where Π is a subset of the complex Hilbert space X. For a given element $f \in X'$, note that $\Pi \subset X$ implies $X' \subset \Pi'$, we have to find $u \in X$ such that

$$Au = f.$$

This model problem corresponds to the multi trace formulation. Since $Au \in \Pi'$, the equation can only be satisfied in the sense of Π' and the variational formulation reads

$$a(u,\varphi) := \langle Au, \varphi \rangle = \langle f, \varphi \rangle \quad \forall \varphi \in \Pi. \tag{1.3}$$

The necessary properties to ensure the existence of a unique solution are quite similar to those we have encountered in the previous section. The result is given in the following theorem which is a derivation of Lion's projection lemma [12, Lemma 9] and the Fredholm alternative.

Theorem 1.3. *[12, Lemma 10] Let X be a Hilbert space and Π a dense subspace of X. Let $a(\cdot,\cdot) : X \times \Pi \to \mathbb{C}$ be a sesquilinear form and $T : X \to X'$ be a compact and continuous operator. Further we assume:*

- *For every $\varphi \in \Pi$, the linear form $u \to a(u,\varphi)$ is continuous in X.*
- *The sesquilinear form $a(\cdot,\cdot)$ is coercive on $\Pi \times \Pi$ with T as the compact operator. That means there exists $c > 0$ such that*

$$\Re\{a(\varphi,\varphi) + \langle T\varphi, \varphi \rangle\} \geq c\|\varphi\|_X^2 \qquad \forall \varphi \in \Pi.$$

- *The sesquilinear form $a(\cdot,\cdot)$ is injective, which means $a(u,\varphi) = 0$ for all $\varphi \in \Pi$ implies $u = 0$.*

Let these assumptions be fulfilled, then for any $f \in X'$ there exists an element $u \in X$ as the unique solution of the variational problem (1.3) satisfying the stability estimate

$$\|u\|_X \leq \tilde{c}\|f\|_{X'}$$

where $\tilde{c} > 0$ is independent of f.

1.3 Sobolev spaces

In this section we present the Hilbert spaces we use to formulate the variational formulation of the Helmholtz transmission problem, the so–called Sobolev spaces. Although we are mainly interested in functions defined on the boundary, we also introduce spaces in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$. We then explain how to connect these spaces, defined in the domain and on the boundary, by using trace operators. In this thesis we only give a short introduction, for details see [3, 15].

The basis for all further spaces is the space of square integrable functions

$$L_2(\Omega) := \left\{ u : \Omega \to \mathbb{C} | u \text{ is Lebesgue measurable}, \int_\Omega |u|^2 \, dx < \infty \right\}$$

with the inner product $\langle u, v \rangle_{L_2(\Omega)} := \int_\Omega u\bar{v}dx$ and the norm $\|u\|^2_{L_2(\Omega)} := \langle u, u \rangle_{L_2(\Omega)}$. For an integer $s > 0$ we then define the space of functions whose weak derivatives up to the order s are square integrable

$$H^s(\Omega) := \{ u \in L_2(\Omega) | \partial^\alpha u \in L_2(\Omega) \; \forall \alpha : |\alpha| \leq s \}.$$

As before we can equip this space with an inner product

$$\langle u, v\rangle_{H^s(\Omega)} := \sum_{|\alpha|\leq s} \int_\Omega \partial^\alpha u(x)\overline{\partial^\alpha v(x)}dx$$

that induces a norm $\|u\|^2_{H^s(\Omega)} := \langle u, u\rangle_{H^s(\Omega)}$. For the more general case when $s \in \mathbb{R}_+$ there exists a $\lambda \in (0, 1)$ and a $\lfloor s\rfloor \in \mathbb{N}$ such that $s = \lfloor s\rfloor + \lambda$. With this, the $H^s(\Omega)$ norm is defined as

$$\|v\|^2_{H^s(\Omega)} := \sum_{|\alpha|<\lfloor s\rfloor} \|\partial^\alpha v\|^2_{L^2(\Omega)} + \sum_{|\alpha|<\lfloor s\rfloor} \int_\Omega\int_\Omega \frac{|\partial^\alpha v(x) - \partial^\alpha v(y)|^2}{|x-y|^{3+2\lambda}}dxdy.$$

The space $H^s(\Omega)$ is then the closure of $\{\varphi \in C^\infty(\Omega) \,|\, \|\varphi\|_{H^s(\Omega)} < \infty\}$ with respect to the $H^s(\Omega)$ norm.

Considering the unbounded complement of Ω, $\Omega^c := \mathbb{R}^3 \setminus \overline{\Omega}$, we further define the space where the square integrability is only required locally as

$$H^s_{\text{loc}}(\Omega^c) := \{u | u\varphi \in H^s(\Omega^c)\ \forall \varphi \in C^\infty_0(\mathbb{R}^3)\}.$$

Since we want to solve the partial differential equation by using functions defined on the boundary $\Gamma := \partial\Omega$, we are interested in corresponding Sobolev spaces. These are obtained in a similar fashion as before, with the main difference that we have to use spaces with a non integer order s. Let $s \in (0, 1)$, then we define the $L_2(\Gamma)$ respectively the Sobolev–Slobodeckii norm

$$\|u\|_{L_2(\Gamma)} := \left[\int_\Gamma u(x)\overline{u(x)}ds_x\right]^{1/2},$$
$$\|u\|_{H^s(\Gamma)} := \left[\|u\|^2_{L_2(\Gamma)} + \int_\Gamma\int_\Gamma \frac{|u(x) - u(y)|^2}{|x-y|^{2+2s}}ds_x ds_y\right]^{1/2}.$$

With these norms we get corresponding spaces

$$L_2(\Gamma) := \{v : \Gamma \to \mathbb{C} | v \text{ is Lebesgue measurable, } \|v\|_{L_2(\Gamma)} < \infty\},$$
$$H^s(\Gamma) := \{v \in L_2(\Gamma) \mid \|v\|_{H^s(\Gamma)} < \infty\} \qquad \text{for } s \in (0, 1).$$

The definition of the $L_2(\Gamma)$ norm already indicates how an $L_2(\Gamma)$ inner product looks like, that is

$$\langle u, v\rangle_{L_2(\Gamma)} := \int_\Gamma u(x)\overline{v(x)}ds_x \quad \text{for } u, v \in L_2(\Gamma).$$

So far, we have only considered $s > 0$, this inner product now allows us to define Sobolev spaces of negative order as dual spaces. That means for $s \in (0, 1)$ we set $H^{-s}(\Gamma) := [H^s(\Gamma)]'$ with the norm

$$\|t\|_{H^{-s}(\Gamma)} := \sup_{0\neq v\in H^s(\Gamma)} \frac{\langle t, v\rangle_\Gamma}{\|v\|_{H^s(\Gamma)}}.$$

The appearing duality pairing is the extension of the $L_2(\Gamma)$ inner product, that is

$$\langle u, v\rangle_\Gamma := \int_\Gamma u(x)\overline{v(x)}ds_x$$

for $u \in H^s(\Gamma)$ and $v \in H^{-s}(\Gamma)$. Both the L_2 inner product and the duality pairing are sesquilinear forms that are antilinear in the second argument.

Until now we only considered $s \in (0, 1)$ which is all that is possible in the case of a Lipschitz domain. Sobolev spaces on the boundary of higher orders can only be defined if the the boundary satisfies further regularity assumptions. More specifically, if Ω is a C^ℓ domain then we can extend the definitions above to $H^s(\Gamma)$ provided $s < \ell$, for details see [15].

The next step is to connect these spaces on the boundary with the ones in the domain. Let $u \in C^1(\overline{\Omega})$, then we can simply evaluate u as well as the normal derivative of u on the boundary. By normal derivative we mean

$$\frac{\partial}{\partial n}u(x) := \nabla u(x) \cdot n(x) \qquad \text{for } x \in \Gamma$$

with the outward unit normal vector n. We now want to find more general formulations of these two evaluations, of particular interest is their application to functions in $H^1(\Omega)$. The trace theorem [14, Theorem 3.38] states that the Dirichlet trace operator

$$\gamma_0^{int} : H^1(\Omega) \to H^{1/2}(\Gamma),$$

is a bounded operator. For the normal derivative, or Neumann trace operator, we have to further assume that $\Delta u \in L_2(\Omega)$. Since we are interested in functions that solve the homogeneous Helmholtz equation, this is satisfied and we can use [15, Thm 2.7.7], which gives us that

$$\gamma_1^{int} : H^1(\Delta, \Omega) \to H^{-1/2}(\Gamma)$$

is a bounded operator with

$$H^1(\Delta, \Omega) := \{u \in H^1(\Omega) \,|\, \Delta u \in L_2(\Omega)\}.$$

Note that this definition is only correct if we consider a bounded domain Ω, the case of an unbounded domain is considered later. As desired, these operators are extensions of the evaluations discussed before, that means for $u \in C^1(\overline{\Omega})$ and $x \in \Gamma$ it holds that

$$\gamma_0^{int}u(x) = u(x) \quad \text{and} \quad \gamma_1^{int}u(x) = n(x) \cdot \nabla u(x).$$

The same can be done for functions that are defined in Ω^c, which leads to the exterior trace operators

$$\gamma_0^{ext} : H^1_{\text{loc}}(\Omega^c) \to H^{1/2}(\Gamma),$$
$$\gamma_1^{ext} : H^1(\Delta, \Omega^c) \to H^{-1/2}(\Gamma),$$

with

$$H^1(\Delta, \Omega^c) := \{u \in H^1_{\text{loc}}(\Omega^c) \,|\, \Delta u \in L_2^{\text{comp}}(\Omega^c)\}.$$

The space $H^1_{\text{loc}}(\Omega)$ was already introduced, the yet undefined space is the space of L_2 functions with a compact support

$$L_2^{\text{comp}}(\Omega^c) := \bigcup_{K \subset \mathbb{R}^3, K \text{ compact}} \{u \in L_2^{\text{loc}}(\Omega^c) \,|\, \text{supp}(u) \subset K\}.$$

If Ω is bounded then it holds that $H^1_{\text{loc}}(\Omega) = H^1(\Omega)$ and $L_2^{\text{comp}}(\Omega) = L_2(\Omega)$. So we see that the former definition of $H^1(\Delta, \Omega)$ is a special case of $H^1(\Delta, \Omega^c)$.

Note that the normal vector is the same as above, so we consider the inward unit normal vector with respect to Ω^c. That means for a function $u \in C^1(\mathbb{R}^3)$ we get

$$\gamma_1^{int} u(x) = \gamma_1^{ext} u(x) \quad \text{for } x \in \Gamma.$$

1.4 Spaces on parts of the boundary

In this section we present Sobolev spaces on open subsets of the boundary as well as piecewise defined spaces, for further details see [15, 19]. These spaces are used for the more general model problem in Chapter 5, in particular the local multi trace formulation in Section 5.5. Let Γ be a closed surface fragmented into open subsets Γ_i such that it can be written as

$$\Gamma = \bigcup_{i=1}^{J} \overline{\Gamma}_i \quad \text{and } \Gamma_i \cap \Gamma_j = \emptyset \text{ for } i \neq j.$$

First, we define the Sobolev space on each of these parts for $s \in (0, 1)$ as

$$H^s(\Gamma_i) := \{v = w_{|\Gamma_i} | w \in H^s(\Gamma)\}$$

with the corresponding norm

$$\|v\|_{H^s(\Gamma_i)} := \inf\{\|w\|_{H^s(\Gamma)} | w \in H^s(\Gamma); w_{|\Gamma_i} = v\}.$$

The index s is assumed to satisfy $|s| < 1$ in the entire section. A well known subspace of $H^s(\Gamma_i)$ is given if we restrict it to functions with compact support,

$$\widetilde{H}^s(\Gamma_i) := \{v = w_{|\Gamma_i} | w \in H^s(\Gamma), \text{ supp}(w) \subset \overline{\Gamma}_i\} \quad \text{for } s \in (0, 1).$$

Spaces of negative order are defined as the dual spaces with respect to the localized L_2 inner product. Let $s > 0$, then we set

$$H^{-s}(\Gamma_i) := [\widetilde{H}^s(\Gamma_i)]',$$
$$\widetilde{H}^{-s}(\Gamma_i) := [H^s(\Gamma_i)]',$$

with the norms

$$\|w\|_{H^{-s}(\Gamma_i)} := \sup_{0\neq v\in\widetilde{H}^s(\Gamma_i)} \frac{\langle w, v\rangle_{\Gamma_i}}{\|v\|_{H^s(\Gamma_i)}},$$
$$\|w\|_{\widetilde{H}^{-s}(\Gamma_i)} := \sup_{0\neq v\in H^s(\Gamma_i)} \frac{\langle w, v\rangle_{\Gamma_i}}{\|v\|_{H^s(\Gamma_i)}}.$$

From these definition we conclude the following inclusions for $s > 0$:

$$\widetilde{H}^s(\Gamma_i) \subset H^s(\Gamma_i) \subset L_2(\Gamma_i) \subset \widetilde{H}^{-s}(\Gamma_i) \subset H^{-s}(\Gamma_i).$$

So far, we looked at the functions on each subset of Γ individually, the next step is to combine these functions. In doing so we get a function which is defined on the complete boundary Γ and has a certain regularity on each segment Γ_i. Let $s > 0$, then these are the so–called piecewise defined spaces

$$H^s_{\mathrm{pw}}(\Gamma) := \{v \in L^2(\Gamma) | v_{|\Gamma_i} \in H^s(\Gamma_i) \text{ for } i = 1, \ldots, J\},$$
$$\widetilde{H}^s_{pw}(\Gamma) := \prod_{i=1}^{J} \widetilde{H}^s(\Gamma_i),$$

with the corresponding norm

$$\|v\|_{H^s_{\mathrm{pw}}(\Gamma)} := \left[\sum_{i=1}^{J} \left\|v_{|\Gamma_i}\right\|^2_{H^s(\Gamma_i)}\right]^{1/2}.$$

The dual spaces for $s > 0$ are given by

$$\widetilde{H}^{-s}_{pw}(\Gamma) := \prod_{i=1}^{J} \widetilde{H}^{-s}(\Gamma_i) = \left[H^s_{pw}(\Gamma)\right]',$$
$$H^{-s}_{pw}(\Gamma) := \prod_{i=1}^{J} H^{-s}(\Gamma_i) = \left[\widetilde{H}^s_{pw}(\Gamma)\right]'.$$

Similar to above, here we have the inclusions for $s > 0$

$$\widetilde{H}^s_{pw}(\Gamma) \subset H^s(\Gamma) \subset H^s_{pw}(\Gamma) \subset L_2(\Gamma) \subset \widetilde{H}^{-s}_{pw}(\Gamma) \subset H^{-s}(\Gamma) \subset H^{-s}_{pw}(\Gamma).$$

Of particular interest for our further considerations is the pair $\widetilde{H}^{1/2}_{pw}(\Gamma) \times \widetilde{H}^{-1/2}_{pw}(\Gamma)$. The following lemma tells us that these form a dense subset of the classical trace spaces $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$.

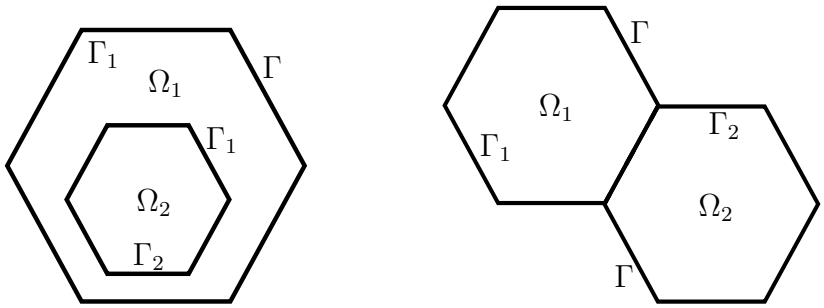
Lemma 1.4. *[5, Lemma 2.15] Let Γ be the boundary of a Lipschitz domain, then there hold the dense embeddings*

$$\widetilde{H}^{1/2}_{pw}(\Gamma) \overset{d}{\hookrightarrow} H^{1/2}(\Gamma),$$
$$\widetilde{H}^{-1/2}_{pw}(\Gamma) \overset{d}{\hookrightarrow} H^{-1/2}(\Gamma).$$

1.5 Spaces on a skeleton boundary

Up to this point we only considered the case of a single domain Ω an its closed boundary Γ. In this section we introduce spaces on the boundary of several domains that may share parts of their boundary. These spaces are used in the single trace formulation as well as the Steklov–Poincaré operator formulation for composite structures in Chapter 5. To keep notations simple we restrict ourself to two domains, the extension to an arbitrary number of domains is then straight forward.

Let Ω be a Lipschitz domain which is divided into two disjoint Lipschitz subdomains Ω_i with boundaries $\Gamma_i := \partial\Omega_i$ for $i \in \{1, 2\}$. Figure 1.1 shows the two scenarios that may appear. The first one, see Figure 1.1a, is that one domain is enclosed by the other, which means it either holds $\Gamma_1 \subset \Gamma_2$ or $\Gamma_2 \subset \Gamma_1$. This case leads to the standard Sobolev spaces and hence we neglect it from here on out. The more interesting scenario is that both domains share part of their boundary with Ω, i.e. $\partial\Omega \cap \Gamma_i \neq \emptyset$ for $i \in \{1, 2\}$. This case is pictured in Figure 1.1b.



(a) $\Gamma_2 \cap \Gamma = \emptyset$ (b) $\Gamma_i \cap \Gamma \neq \emptyset$ for $i \in \{1, 2\}$

Figure 1.1: A structure Ω composed of two subdomains Ω_1 and Ω_2.

First we define the skeleton boundary as the union of all individual boundaries $\Gamma_S := \Gamma_1 \cup \Gamma_2$ and the exterior domain $\Omega_0 := \mathbb{R}^3 \setminus (\overline{\Omega}_1 \cup \overline{\Omega}_2)$ with $\Gamma_0 := \partial\Omega_0$. For any function $u \in H^1(\Delta, \mathbb{R}^3)$ we apply the trace operators for all three domains and get

$$\gamma_0^{int} u_{|\Omega_i} \in H^{1/2}(\Gamma_i), i \in \{1, 2\} \qquad \gamma_0^{ext} u_{|\Omega_0} \in H^{1/2}(\Gamma_0),$$
$$\gamma_1^{int} u_{|\Omega_i} \in H^{-1/2}(\Gamma_i), i \in \{1, 2\} \qquad \gamma_1^{ext} u_{|\Omega_0} \in H^{-1/2}(\Gamma_0).$$

Taking the Dirichlet trace on the whole skeleton we have found a function whose restriction to any boundary Γ_i is in $H^{1/2}(\Gamma_i)$. This motivates the definition of Sobolev spaces on the skeleton by starting from the product space of local trace spaces $H^s(\Gamma_i)$ and enforcing continuity across shared boundaries. We denote the product space for $s \in (-1, 1)$ as

$$H_T^s(\Gamma_S) := \prod_{i=0}^{2} H^s(\Gamma_i).$$

For any $u \in H_T^s(\Gamma_S)$ let its elements be $u_i \in H^s(\Gamma_i)$ for $i \in \{0, 1, 2\}$, that means $u = (u_0, u_1, u_2)^\top$. Then the skeleton Dirichlet trace space $H^{1/2}(\Gamma_S)$ is defined as

$$H^{1/2}(\Gamma_S) := \{u \in H_T^{1/2}(\Gamma_S) | u_i = u_j \text{ in } H^{1/2}(\Gamma_{ij}) \text{ for } i \neq j, i, j = 0, 1, 2\}.$$

Let $u = (u_0, u_1, u_2) \in H^{1/2}(\Gamma_S)$, then we define the restriction operator $\cdot_{|\Gamma_i}$ as

$$u_{|\Gamma_i} = u_i \quad \text{in } H^{1/2}(\Gamma_i), i \in \{0,1,2\}.$$

The norm of $H^{1/2}(\Gamma_S)$ is given by

$$\|u\|_{H^{1/2}(\Gamma_S)} := \left(\sum_{i=0}^{2} \left\|u_{|\Gamma_i}\right\|^2_{H^{1/2}(\Gamma_i)}\right)^{1/2}.$$

When considering the Neumann datum, we have to take into account that the normal derivative has an orientation depending on the domain it comes from. In particular, for almost all $x \in \Gamma_{12}$ we get $\gamma_1^{int} u_{|\Omega_1}(x) = -\gamma_1^{int} u_{|\Omega_2}(x)$ while $\gamma_1^{int} u_{|\Omega_1}(x) = \gamma_1^{ext} u_{|\Omega_0}$ holds true for almost all $x \in \Gamma_{10}$. One way to write this continuity condition is by means of local duality pairings and exploiting that traces in $H^{1/2}(\Gamma_S)$ are unique on each interface. Let the components of $t \in H_T^{-1/2}(\Gamma_S)$ be t_i, then we can write $t = (t_0, t_1, t_2)^\top$ and define

$$H^{-1/2}(\Gamma_S) := \left\{ t \in H_T^{-1/2}(\Gamma_S) \middle| \left\langle t_0, u_{|\Gamma_0} \right\rangle_{\Gamma_0} - \sum_{i=1}^{2} \left\langle t_i, u_{|\Gamma_i} \right\rangle_{\Gamma_i} = 0, \quad \forall u \in H^{1/2}(\Gamma_S) \right\},$$

$$\|t\|_{H^{-1/2}(\Gamma_S)} := \sup_{0 \neq u \in H^{1/2}(\Gamma_S)} \frac{\langle u, t \rangle_{\Gamma_S}}{\|u\|_{H^{1/2}(\Gamma_S)}}.$$

Instead of a simple restriction we define a mapping $\mathbb{L}_i : H^{-1/2}(\Gamma_S) \to H^{-1/2}(\Gamma_i)$ as

$$\mathbb{L}_i t = t_i \quad i \in \{0,1,2\}.$$

This discrepancy to the notation used earlier, where we just restricted the Dirichlet skeleton trace, is to emphasize that the Neumann skeleton trace is not unique on interfaces but may change the sign.

Naturally we are interested in a duality pairing between these two spaces, unfortunately we cannot express it for arbitrary functions $(u,t) \in H^{1/2}(\Gamma_S) \times H^{-1/2}(\Gamma_S)$. To illustrate the occurring problem, let $u_0 \in H^{1/2}(\Gamma_0)$ and define the adjoint operator to $\mathbb{L}_0$ by $\mathbb{L}_0^* : H^{1/2}(\Gamma_0) \to H^{1/2}(\Gamma_S)$. Then for $t \in H^{-1/2}(\Gamma_S)$ we see that

$$\langle u_0, \mathbb{L}_0 t \rangle_{\Gamma_0} = \langle \mathbb{L}_0^* u_0, t \rangle_{\Gamma_S}.$$

Hence, we can describe the duality pairing in the case that one of the functions is a composition of a function living on one boundary and an adjoint restriction operator. Let $(\cdot_{|\Gamma_i})^* : H^{-1/2}(\Gamma_S) \to H^{-1/2}(\Gamma_i)$ be the adjoint operator of the restriction for the Dirichlet trace, then for $i \in \{0,1,2\}$ we set

$$\begin{aligned} \langle \mathbb{L}_i^* u_i, t \rangle_{\Gamma_S} &:= \langle u_i, \mathbb{L}_i t \rangle_{\Gamma_i} \quad \forall u_i \in H^{1/2}(\Gamma_i), t \in H^{-1/2}(\Gamma_S), \\ \left\langle (\cdot_{|\Gamma_i})^* t_i, u \right\rangle_{\Gamma_S} &:= \left\langle t_i, u_{|\Gamma_i} \right\rangle_{\Gamma_i} \quad \forall t_i \in H^{-1/2}(\Gamma_i), u \in H^{1/2}(\Gamma_S). \end{aligned}$$

It is not obvious if and how this definition can be extended to arbitrary functions in $H^{1/2}(\Gamma_S) \times H^{-1/2}(\Gamma_S)$. The naive approach would be to add the local duality pairings up, but since the normal derivative has a change in sign on Γ_{12}, the terms on Γ_{12} would cancel on another out. Fortunately, for the formulations we are going to consider it is sufficient to be able to compute the special cases mentioned above.

The next lemma allows us to estimate the norm of the Neumann trace on one boundary by the norm on the remaining boundaries which is required for the Steklov–Poincaré operator formulation.

Lemma 1.5. *Let $t \in H^{-1/2}(\Gamma_S)$ and $i \in \{0,1,2\}$, then it holds*

$$\|\mathbb{L}_i t\|_{H^{-1/2}(\Gamma_i)} \leq \sum_{j\neq i, j=0}^{2} \|\mathbb{L}_j t\|_{H^{-1/2}(\Gamma_j)}.$$

Proof. Let $i \in \{0,1,2\}$ be arbitrary but fixed, then the local norm is defined as

$$\|\mathbb{L}_i t\|_{H^{-1/2}(\Gamma_i)} = \sup_{0\neq u_i \in H^{1/2}(\Gamma_i)} \frac{\langle u_i, \mathbb{L}_i t\rangle_{\Gamma_i}}{\|u_i\|_{H^{1/2}(\Gamma_i)}} = \sup_{0\neq \tilde{u}_i \in \widetilde{H}^{1/2}_{pw}(\Gamma_i)} \frac{\langle \tilde{u}_i, \mathbb{L}_i t\rangle_{\Gamma_i}}{\|\tilde{u}_i\|_{H^{1/2}(\Gamma_i)}}.$$

For the second equality we used Lemma 1.4 which tells us that $\widetilde{H}^{1/2}_{pw}(\Gamma_i)$ is dense in $H^{1/2}(\Gamma_i)$. Let $\tilde{u}_i \in \widetilde{H}^{1/2}_{pw}(\Gamma_i)$ be arbitrary but fixed, then we can define a skeleton Dirichlet trace function

$$\tilde{u} := (\tilde{u}_0, \tilde{u}_1, \tilde{u}_2)^\top \in H^{1/2}(\Gamma_S), \qquad \text{for } j \neq i : \tilde{u}_j := \begin{cases} \tilde{u}_{i|\Gamma_{ij}} & \text{on } \Gamma_{ij} \\ 0 & \text{elsewhere} \end{cases}.$$

Furthermore we get $\tilde{u}_j \in \widetilde{H}^{1/2}_{pw}(\Gamma_i)$ for all $j \in \{0,1,2\}$. Exploiting the definition of $H^{-1/2}(\Gamma_S)$ we can rewrite the dividend of the norm expression as

$$\begin{aligned} \left|\langle \tilde{u}_i, \mathbb{L}_i t\rangle_{\Gamma_i}\right| &= \left|\left\langle \tilde{u}_{|\Gamma_i}, \mathbb{L}_i t\right\rangle_{\Gamma_i}\right| \\ &\leq \sum_{j\neq i, j=0}^{2} \left|\left\langle \tilde{u}_{|\Gamma_j}, \mathbb{L}_j t\right\rangle_{\Gamma_j}\right| \\ &\leq \sum_{j\neq i, j=0}^{2} \|\tilde{u}_j\|_{H^{1/2}(\Gamma_i)} \|\mathbb{L}_j t\|_{H^{-1/2}(\Gamma_i)}. \end{aligned}$$

The last step is to estimate the norm $\tilde{u}_j$, for this recall that it holds $\tilde{u}_j = \chi_{ij}\tilde{u}_{i|\Gamma_{ij}}$ with the characteristic function χ_{ij} which is one on Γ_{ij} and zero everywhere else. Then it easy to see that for $j \neq i$

$$\|\tilde{u}_j\|_{H^{1/2}(\Gamma_j)} = \left\|\tilde{u}_{i|\Gamma_{ij}}\right\|_{H^{1/2}(\Gamma_{ij})} \leq \|\tilde{u}_i\|_{H^{1/2}(\Gamma_i)}.$$

Finally we use these results to estimate the $H^{-1/2}(\Gamma_i)$ norm by

$$\|\mathbb{L}_i t\|_{H^{-1/2}(\Gamma_i)} = \sup_{0\neq \tilde{u}_i \in \widetilde{H}^{1/2}_{pw}(\Gamma_i)} \frac{\langle \tilde{u}_i, \mathbb{L}_i t\rangle_{\Gamma_i}}{\|\tilde{u}_i\|_{H^{1/2}(\Gamma_i)}}. \leq \sup_{0\neq \tilde{u}_i \in \widetilde{H}^{1/2}_{pw}(\Gamma_i)} \frac{\|\tilde{u}_i\|_{H^{1/2}(\Gamma_i)}}{\|\tilde{u}_i\|_{H^{1/2}(\Gamma_i)}} \sum_{j\neq i, j=0}^{2} \|\mathbb{L}_j t\|_{H^{-1/2}(\Gamma_i)},$$

therefore showing that we can estimate the local $H^{-1/2}$ norm by the sum of the other $H^{-1/2}$ norms. □

2 Boundary integral equations

In this chapter we present the concept of boundary integral equations for the Helmholtz equation. The main idea is that a solution of the Helmholtz equation for a given domain Ω is fully described by its Dirichlet and Neumann traces on the boundary. Therefore it is sufficient to find these traces and restrict all our considerations to the boundary.

To find these so–called Cauchy traces we establish boundary integral operators and show that they fulfil certain relations, provided the Cauchy traces come from a solution of the Helmholtz problem. This system of relations is called the Calderón projection and is the starting point of the derivation of systems of boundary integral equations in Chapters 4 and 5.

Sections 2.2 and 2.3 discuss properties of the boundary integral operators presented in Section 2.1, namely injectivity and coercivity. In the last section we present and study the Steklov–Poincaré operator, which is a composition of previously introduced operators. For a more comprehensive discussion of the topic we refer to [13, 14, 15, 19]

2.1 Boundary integral operators

Let $\kappa \in \mathbb{R}_+$ with $\mathbb{R}_+ := \{x \in \mathbb{R} \,|\, x > 0\}$ be a given wave number, $\Omega \subset \mathbb{R}^3$ a bounded Lipschitz domain and $\Gamma := \partial\Omega$ its boundary. We start with the three dimensional fundamental solution of the Helmholtz equation, that is

$$U_\kappa^*(x,y) := \frac{1}{4\pi} \frac{e^{i\kappa|x-y|}}{|x-y|}.$$

With this, we can define two boundary integral potentials that solve the Helmholtz equation in $\mathbb{R}^3 \setminus \Gamma$. These are the single layer potential

$$(\widetilde{V}_\kappa w)(x) := \int_\Gamma U_\kappa^*(x,y) w(y) ds_y, \qquad x \in \mathbb{R}^3 \setminus \Gamma,$$

defined for $w \in H^{-1/2}(\Gamma)$ and the double layer potential

$$(W_\kappa v)(x) := \int_\Gamma v(y) \frac{\partial}{\partial n_y} U_\kappa^*(x,y) ds_y, \qquad x \in \mathbb{R}^3 \setminus \Gamma,$$

defined for $v \in H^{1/2}(\Gamma)$. It holds that $\widetilde{V}_\kappa : H^{-1/2}(\Gamma) \to H^1(\Omega) \cap H^1_{loc}(\Omega^c)$ and $W_\kappa : H^{1/2}(\Gamma) \to H^1(\Omega) \cap H^1_{loc}(\Omega^c)$ are continuous and linear operators [19]. Moreover, for

any $w \in H^{-1/2}(\Gamma)$, $v \in H^{1/2}(\Gamma)$ and $x \in \mathbb{R}^3 \setminus \Gamma$ these functions solve the Helmholtz equation, that is

$$\begin{aligned} -\Delta(\widetilde{V}_\kappa w)(x) - \kappa^2(\widetilde{V}_\kappa w)(x) &= 0, \\ -\Delta(W_\kappa v)(x) - \kappa^2(W_\kappa v)(x) &= 0. \end{aligned}$$

The single layer potential and the double layer potential further satisfy the Sommerfeld radiation condition. As mentioned earlier, we are interested in keeping our considerations restricted to the boundary. Hence we take the Dirichlet and Neumann traces of these two potentials which results in the well known relations

$$\begin{aligned} \gamma_0^{int}(\widetilde{V}_\kappa w)(x) = \gamma_0^{ext}(\widetilde{V}_\kappa w)(x) &= (V_\kappa w)(x), \\ \gamma_1^{int}(\widetilde{V}_\kappa w)(x) &= \frac{1}{2}w(x) + (K'_\kappa w)(x), \\ \gamma_1^{ext}(\widetilde{V}_\kappa w)(x) &= -\frac{1}{2}w(x) + (K'_\kappa w)(x), \\ \gamma_0^{int}(W_\kappa v)(x) &= -\frac{1}{2}v(x) + (K_\kappa v)(x), \\ \gamma_0^{ext}(W_\kappa v)(x) &= \frac{1}{2}v(x) + (K_\kappa v)(x), \\ \gamma_1^{int}(W_\kappa v)(x) = \gamma_1^{ext}(W_\kappa v)(x) &= -(D_\kappa v)(x) \end{aligned}$$

for almost all $x \in \Gamma$. In these equations we have introduced new operators, namely

$$\begin{aligned} \text{the single layer boundary integral operator } V_\kappa &: H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), \\ \text{the double layer boundary integral operator } K_\kappa &: H^{1/2}(\Gamma) \to H^{1/2}(\Gamma), \\ \text{the adjoint double layer boundary integral operator } K'_\kappa &: H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma) \\ \text{and the hypersingular boundary integral operator } D_k &: H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma). \end{aligned}$$

All these operators are continuous and linear, therefore bounded [19]. Representations of these operators for $v \in H^{1/2}(\Gamma)$ and $w \in H^{-1/2}(\Gamma)$ are given by

$$\begin{aligned} (V_\kappa w)(x) &:= \int_\Gamma U_\kappa^*(x,y)w(y)ds_y, \\ (K_\kappa v)(x) &:= \int_\Gamma v(y)\frac{\partial}{\partial n_y}U_\kappa^*(x,y)ds_y, \\ (K'_\kappa w)(x) &:= \int_\Gamma w(y)\frac{\partial}{\partial n_x}U_\kappa^*(x,y)ds_y, \\ (D_\kappa v)(x) &:= -\frac{\partial}{\partial n_x}\int_\Gamma v(y)\frac{\partial}{\partial n_y}U_\kappa^*(x,y)ds_y. \end{aligned}$$

For notes on how these integrals are to be understood and can be evaluated, see [15, 19, 22]. Even though the name suggests otherwise, the adjoint double layer boundary integral operator is not the $L_2(\Gamma)$–adjoint of the double layer boundary integral operator. How these two operators are connected is stated in the following lemma.

Lemma 2.1. *[20, Section 2] For all $\kappa \in \mathbb{R}$, $v \in H^{1/2}(\Gamma)$ and $\tau \in H^{-1/2}(\Gamma)$ it holds*

$$\langle K_\kappa v, \tau \rangle_\Gamma = \langle v, K'_{-\kappa} \tau \rangle_\Gamma.$$

A similar result holds true for the single layer boundary integral operator.

Lemma 2.2. *[20, Section 2] For all $\kappa \in \mathbb{R}$ and $(t, \tau) \in [H^{-1/2}(\Gamma)]^2$ it holds*

$$\langle V_\kappa \tau, t \rangle_\Gamma = \langle \tau, V_{-\kappa} t \rangle_\Gamma.$$

Let $u \in H^1(\Omega)$ be a weak solution of the interior Helmholtz problem

$$-\Delta u(x) - \kappa^2 u(x) = 0 \quad \text{for } x \in \Omega,$$

then it can be expressed by its Dirichlet and Neumann traces with the representation formula [19, Chapter 5]

$$u(x) = (\widetilde{V}_\kappa \gamma_1^{int} u)(x) - (W_\kappa \gamma_0^{int} u)(x) \quad \text{for } x \in \Omega. \tag{2.1}$$

Thus it is sufficient to know the Cauchy traces of the solution u. All in this thesis considered formulations are so–called direct approaches, that means the goal is to compute the Cauchy traces. An alternative are indirect formulations that consider a potential approach, for example $u = (Vw)$ with an unknown density $w \in H^{-1/2}(\Gamma)$. Applying the trace operators to the representation formula results in the interior Calderón projection which can be written in matrix form as

$$\begin{pmatrix} \gamma_0^{int} u \\ \gamma_1^{int} u \end{pmatrix} = \begin{pmatrix} \frac{1}{2} I - K_\kappa & V_\kappa \\ D_\kappa & \frac{1}{2} I + K'_\kappa \end{pmatrix} \begin{pmatrix} \gamma_0^{int} u \\ \gamma_1^{int} u \end{pmatrix}. \tag{2.2}$$

Since this is a projection, that is $\mathcal{C}^2 = \mathcal{C}$, we can derive the following relations.

Corollary 2.3. *[19, Corollary 6.19] For all wave numbers $\kappa \in \mathbb{R}$ it holds*

$$\begin{aligned} V_\kappa D_\kappa &= \left(\frac{1}{2} I + K_\kappa\right)\left(\frac{1}{2} I - K_\kappa\right), \\ D_\kappa V_\kappa &= \left(\frac{1}{2} I + K'_\kappa\right)\left(\frac{1}{2} I - K'_\kappa\right), \\ V_\kappa K'_\kappa &= K_\kappa V_\kappa, \\ K'_\kappa D_\kappa &= D_\kappa K_\kappa. \end{aligned}$$

If we consider the exterior Helmholtz problem

$$-\Delta u(x) - \kappa^2 u(x) = 0 \quad \text{for } x \in \Omega^c$$

instead, we have to further demand that the solution $u \in H^1_{loc}(\Omega^c)$ satisfies the Sommerfeld radiation condition

$$\lim_{r \to \infty} \int_{|x|=r} \left| \frac{\partial}{\partial n_x} u_(x) - i\kappa u(x) \right|^2 ds_x = 0 \tag{2.3}$$

to ensure uniqueness for any given boundary conditions. A solution for this problem can be fully described by its traces using the representation formula [19, Chapter 5]

$$u(x) = -(\widetilde{V}_\kappa \gamma_1^{ext} u)(x) + (W_\kappa \gamma_0^{ext} u)(x) \quad \text{for } x \in \Omega^c. \tag{2.4}$$

Applying the Dirichlet and Neumann trace operators to this expression yields the exterior Calderón projection

$$\begin{pmatrix} \gamma_0^{ext} u \\ \gamma_1^{ext} u \end{pmatrix} = \begin{pmatrix} \frac{1}{2} I + K_\kappa & -V_\kappa \\ -D_\kappa & \frac{1}{2} I - K'_\kappa \end{pmatrix} \begin{pmatrix} \gamma_0^{ext} u \\ \gamma_1^{ext} u \end{pmatrix}. \tag{2.5}$$

All above conditions hold true for $\kappa = 0$, which gives us the special case of the Laplace instead of the Helmholtz equation. We denote the Laplace boundary integral operators by dismissing the sub–index, that means $V := V_0$, $D := D_0$ and so forth. One notable difference is that, in case of the Laplace problem, the exterior solution has to satisfy a different radiation condition.

2.2 Injectivity of boundary integral operators

The presented operators may or may not be injective, depending on the wave number κ and the geometry Ω. For example, let $u \in H^1(\Omega)$ be a solution of the interior homogeneous Dirichlet problem

$$-\Delta u = \kappa^2 u \quad \text{in } \Omega, \quad \gamma_0^{int} u = 0 \quad \text{on } \Gamma. \tag{2.6}$$

Then the first line of the interior Calderón projection (2.2) reads

$$0 = -V_\kappa \gamma_1^{int} u.$$

That means if the problem (2.6) has a non trivial solution, then V_κ is not injective. We get a similar result for $(\frac{1}{2} I - K'_\kappa)$ if we consider the second equation instead of the first one. As the next lemma shows, the other direction is true as well, if V_κ is not injective, then there exists a non trivial solution of the homogeneous Dirichlet problem.

Lemma 2.4. *[20, Section 2] Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain with the boundary Γ. For $t \in H^{-1/2}(\Gamma)$ and $\kappa \in \mathbb{R}$ the following statements are equivalent:*

1. *The pair $(\kappa, t) \in \mathbb{R} \times H^{-1/2}(\Gamma)$ is a solution for the single layer boundary integral operator eigenvalue problem $V_\kappa t = 0$.*
2. *The pair $(\kappa, t) \in \mathbb{R} \times H^{-1/2}(\Gamma)$ is a solution for the adjoint boundary integral operator eigenvalue problem $(\frac{1}{2} I - K'_\kappa)$.*
3. *The interior Dirichlet eigenvalue problem*

$$-\Delta u_\lambda(x) = \lambda u_\lambda(x) \quad \text{for } x \in \Omega, \qquad \gamma_0^{int} u_\lambda(x) = 0 \quad \text{for } x \in \Gamma$$

has a solution $u_\lambda \in H^1(\Omega)$ with $\gamma_1^{int} u_\lambda = t$ and $\lambda := \kappa^2$.

A similar statement holds true when we consider the interior Neumann problem instead.

Lemma 2.5. *[20, Section 2] Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain with the boundary Γ. For $u \in H^{1/2}(\Gamma)$ and $\kappa \in \mathbb{R}$ the following statements are equivalent:*

1. *The pair $(\kappa, u) \in \mathbb{R} \times H^{1/2}(\Gamma)$ is a solution of the hypersingular boundary integral operator eigenvalue problem $D_\kappa u = 0$.*
2. *The pair $(\kappa, u) \in \mathbb{R} \times H^{1/2}(\Gamma)$ is a solution of the double layer boundary integral operator eigenvalue problem $(\frac{1}{2}I + K_\kappa)u = 0$.*
3. *The interior Neumann eigenvalue problem*

$$-\Delta u_\mu(x) = \mu u_\mu(x) \quad \text{for } x \in \Omega, \qquad \gamma_1^{int} u_\mu(x) = 0 \quad \text{for } x \in \Gamma$$

has a non trivial solution $u_\mu \in H^1(\Omega)$ with $\gamma_0^{int} u_\mu = u$ and $\mu := \kappa^2$.

This absence of injectivity is a problem if we aim to make use of Lemma 1.2 or Theorem 1.3 which both demand injective operators. The idea is to combine different operators in such a way that the resulting system is injective.

2.3 Coercivity of boundary integral operators

Recall the definition of coercivity (1.2), that is for an operator $A : X \to X'$ there exists a compact operator $T_A : X \to X'$ such that the Gårding inequality

$$\Re\langle (A + T_A)v, v\rangle \geq c\|v\|_X^2$$

is satisfied for a $c > 0$ and all $v \in X$. The first step to show this property is the next lemma that gives us such compact operators for all four boundary integral operators.

Lemma 2.6. *[15, Lemma 3.9.8] The operators*

$$\begin{aligned} V_\kappa - V &: H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), \\ D_\kappa - \widetilde{D} &: H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma), \\ K_\kappa - K &: H^{1/2}(\Gamma) \to H^{1/2}(\Gamma), \\ K'_\kappa - K' &: H^{1/2}(\Gamma) \to H^{1/2}(\Gamma), \end{aligned}$$

are compact. The operator $\widetilde{D}$ is the stabilised hypersingular operator defined as

$$\langle \widetilde{D}u, v\rangle_\Gamma := \langle Du, v\rangle_\Gamma + \langle u, V^{-1}1\rangle_\Gamma \overline{\langle v, V^{-1}1\rangle_\Gamma}$$

for all $(u, v) \in H^{1/2}(\Gamma)$, see [20].

Together with the well known ellipticity properties of the Laplace boundary integral operators V and $\widetilde{D}$, see [19], this gives us the desired coercivity for the single layer boundary integral operator and the hypersingular boundary integral operator.

Lemma 2.7. *The operators $V_\kappa : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and $D_\kappa : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ are coercive for $\kappa \in \mathbb{R}$ with the compact operators $T_V = V - V_\kappa$ and $T_D = \widetilde{D} - D_\kappa$.*

2.4 The Steklov–Poincaré operator

In this section we discuss a Dirichlet to Neumann mapping, the Steklov–Poincaré operator. To consider interior and exterior problems at the same time, we define κ_1 as the wave number in Ω and κ_0 for the exterior domain. Suppose κ_1 is not a Dirichlet eigenvalue of Ω, then V_{κ_1} is invertible and we can solve the first line of the interior Calderón operator (2.2) for

$$\gamma_1^{int} u = V_{\kappa_1}^{-1} \left(\frac{1}{2} + K_{\kappa_1} \right) \gamma_0^{int} u.$$

Combined with the second line of the Calderón operator this yields

$$\gamma_1^{int} u = \left[D_{\kappa_1} + \left(\frac{1}{2} + K'_{\kappa_1} \right) V_{\kappa_1}^{-1} \left(\frac{1}{2} I + K_{\kappa_1} \right) \right] \gamma_0^{int} u.$$

We have at least two representations of an operator that, for given Dirichlet datum, returns the Neumann datum of the corresponding solution for the interior Helmholtz equation. We define

$$\begin{aligned} S_{\kappa_1}^{int} :=& D_{\kappa_1} + \left(\frac{1}{2} I + K'_{\kappa_1} \right) V_{\kappa_1}^{-1} \left(\frac{1}{2} I + K_{\kappa_1} \right) \\ =& V_{\kappa_1}^{-1} \left(\frac{1}{2} I + K_{\kappa_1} \right) \end{aligned}$$

as the interior Steklov–Poincaré operator. Now we want to study certain properties of this operator, namely injectivity and coercivity. Showing coercivity would be simple if K_{κ_1} is the adjoint of K'_{κ_1}, which is unfortunately not the case for $\kappa_1 \neq 0$ as we have seen in Lemma 2.1. One key element in construction of the needed compact operator is given in the next lemma.

Lemma 2.8. *The operator* $(K'_\kappa - K'_{-\kappa}) : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ *is compact.*

Proof. From Lemma 2.6 we know that the operators

$$T_{K'_\kappa} := K' - K'_\kappa \quad \text{and} \quad T_{K'_{-\kappa}} := K' - K'_{-\kappa}$$

are compact. Therefore, their difference

$$T_{K'_{-\kappa}} - T_{K'_\kappa} = K'_\kappa - K'_{-\kappa}$$

is compact as well. □

Lemma 2.9. *[7, Lemma 3] Let* V_{κ_1} *be invertible, that is* κ_1^2 *is not a Dirichlet eigenvalue, then* $S_{\kappa_1}^{int}$ *is coercive.*

Proof. Let $v \in H^{1/2}(\Gamma)$ be arbitrary but fixed. We consider the symmetric representation of the Steklov–Poincaré operator. From Lemma 2.6 we know that D_{κ_1} is coercive with the compact operator T_D and can write, with a yet undefined operator T,

$$\begin{aligned}\Re\langle[S^{int}_{\kappa_1} + T_D + T]v, v\rangle_\Gamma &= \Re\langle\left[D_{\kappa_1} + T_D + \left(\frac{1}{2}I + K'_{\kappa_1}\right) V^{-1}_{\kappa_1}\left(\frac{1}{2}I + K_{\kappa_1}\right) + T\right] v, v\rangle_\Gamma \\ &\geq c_1^D \|v\|^2_{H^{1/2}(\Gamma)} + \Re\langle\left[\left(\frac{1}{2}I + K'_{\kappa_1}\right) V^{-1}_{\kappa_1}\left(\frac{1}{2}I + K_{\kappa_1}\right) + T\right] v, v\rangle_\Gamma.\end{aligned}$$

For the inequality we used the coercivity of D_{κ_1}. The operator that we use for the remainder is motivated by Lemma 2.1. We set

$$T := -\left(K'_{\kappa_1} - K'_{-\kappa_1}\right) V^{-1}_{\kappa_1}\left(\frac{1}{2}I + K_{\kappa_1}\right) + \tilde{T},$$

the operator $\tilde{T}$ will be defined later. With that, the remainder reads

$$\begin{aligned}&\Re\langle\left[\left(\frac{1}{2}I + K'_{-\kappa_1}\right) V^{-1}_{\kappa_1}\left(\frac{1}{2}I + K_{\kappa_1}\right) + \tilde{T}\right] v, v\rangle_\Gamma \\ &\qquad = \Re\langle V^{-1}_{\kappa_1}\left(\frac{1}{2}I + K_{\kappa_1}\right) v, \left(\frac{1}{2}I + K_{\kappa_1}\right) v\rangle_\Gamma + \Re\langle\tilde{T}v, v\rangle_\Gamma \\ &\qquad = \Re\langle V^{-1}_{\kappa_1}\left(\frac{1}{2}I + K_{\kappa_1}\right) v, V_{\kappa_1} V^{-1}_{\kappa_1}\left(\frac{1}{2}I + K_{\kappa_1}\right) v\rangle_\Gamma + \Re\langle v, \tilde{T}^* v\rangle_\Gamma.\end{aligned}$$

We already know that V_{κ_1} is coercive with the compact operator T_V, motivating the definition

$$\tilde{T} := \left(\frac{1}{2}I + K'_{-\kappa_1}\right) V^{-1}_{-\kappa_1} T^*_V V^{-1}_{\kappa_1}\left(\frac{1}{2}I + K_{\kappa_1}\right)$$

where T^*_V is the adjoint of T_V. Here we used Lemma 2.2 to find the adjoint of V_{κ_1}, that is $V_{-\kappa_1}$. With this choice it follows

$$\Re\langle\left[\left(\frac{1}{2}I + K'_{-\kappa_1}\right) V^{-1}_{\kappa_1}\left(\frac{1}{2}I + K_{\kappa_1}\right) + \tilde{T}\right] v, v\rangle_\Gamma \geq 0$$

and furthermore

$$\Re\langle[S^{int}_{\kappa_1} + T_S]v, v\rangle_\Gamma \geq c_1^D \|v\|^2_{H^{1/2}(\Gamma)}$$

with

$$T_S := T_D - \left(K'_{\kappa_1} - K'_{-\kappa_1}\right) V^{-1}_{\kappa_1}\left(\frac{1}{2}I + K_{\kappa_1}\right) + \left(\frac{1}{2}I + K'_{-\kappa_1}\right) V^{-1}_{-\kappa_1} T^*_V V^{-1}_{\kappa_1}\left(\frac{1}{2}I + K_{\kappa_1}\right).$$

This operator is compact since the space of compact operators form a two–sided operator ideal in the space of bounded operators, see [3]. This means if K is a compact operator and B a bounded one, then KB and BK are compact operators. For the third element we need Schauders theorem which tells us that the adjoint of a compact operator is again a compact operator. In summation we established that T_S is a compact operator and thus $S^{int}_{\kappa_1}$ is coercive. □

Lemma 2.10. *Assume κ_1^2 is neither a Dirichlet nor a Neumann eigenvalue, then $S_{\kappa_1}^{int}$ is injective.*

Proof. We have to show that, if $u \in H^{1/2}(\Gamma)$ solves $S_{\kappa_1}^{int} u = 0$, it follows $u = 0$. Since κ_1^2 is not a Dirichlet eigenvalue, we can apply the operator $S_{\kappa_1}^{int}$ and there exists at least one solution $u \in H^{1/2}(\Gamma)$ of the problem $S_{\kappa_1}^{int} u = 0$, the zero function. Let now u be such a solution. From the definition of $S_{\kappa_1}^{int}$ we know that $t = S_{\kappa_1}^{int} u$ is the Neumann datum of the interior Dirichlet problem to find $v \in H^1(\Omega)$ such that

$$-\Delta v - \kappa_1^2 v = 0 \quad \text{in } \Omega, \quad \gamma_0^{int} v = u \quad \text{on } \Gamma.$$

This means that we can find u as the Dirichlet trace of the solution for the interior Neumann problem, find $\hat{v} \in H^1(\Omega)$ that solves

$$-\Delta \hat{v} - \kappa_1^2 \hat{v} = 0 \quad \text{in } \Omega, \quad \gamma_1^{int} \hat{v} = t = S_{\kappa_1}^{int} u = 0 \quad \text{on } \Gamma.$$

Since κ_1^2 is not a Neumann eigenvalue, we know that this problem only has the trivial solution $v = 0$ in Ω. From this it follows $u = \gamma_0^{int} v = 0$. □

Next we consider the exterior domain Ω^c and again we assume for the moment that κ_0^2 is not a Dirichlet eigenvalue. Then we can solve the first line of the exterior Calderón operator (2.5) for $\gamma_1^{ext} u$ and use this result in the second line which gives us the symmetric formulation of the exterior Steklov–Poincaré operator

$$\gamma_1^{ext} u = -S_{\kappa_0}^{ext} \gamma_0^{ext} u := -\left[D_{\kappa_0} + \left(\frac{1}{2} I - K'_{\kappa_0} \right) V_{\kappa_0}^{-1} \left(\frac{1}{2} I - K_{\kappa_0} \right) \right] \gamma_0^{ext} u. \tag{2.7}$$

Naturally there is a non symmetrical representation of the exterior Steklov–Poincaré operator that we get by solving the first equation of the Calderón projection. However, this form is only well defined if κ_0^2 is not a Dirichlet eigenvalue whereas we will show that the symmetric form is well defined independent of the wave number. Hence we restrict our considerations to the representation (2.7) for the remainder of this thesis.

Lemma 2.11. *[22, Section 5.4] The symmetric formulation of the exterior Steklov–Poincaré operator (2.7) is well defined for all wave numbers $\kappa_0 \in \mathbb{R}$.*

Proof. Assume that κ_0^2 is not a Dirichlet eigenvalue, then V_{κ_0} is invertible and the exterior Steklov–Poincaré operator is well defined as a combination of well defined operators.

The second possible case, κ_0^2 is a Dirichlet eigenvalue, needs a bit more consideration. The operator D_{κ_0} is well defined for all wave numbers, so we only have to consider the remaining part. Let $u \in H^{1/2}(\Gamma)$ be arbitrary but fixed, then we set $g := \left(\frac{1}{2} I - K_{\kappa_0} \right) u \in H^{1/2}(\Gamma)$. From [9, Corollary 3.3] it follows $g \in \mathrm{Imag}(V_{\kappa_0})$ which allows us to apply $V_{\kappa_0}^{-1}$, which is not uniquely defined since we can add any element $\tilde{t} \in \ker(V_{\kappa_0})$. Let $\tilde{t}$ be such an element, that is $V_{\kappa_0} \tilde{t} = 0$, then Lemma 2.4 gives us

$$\left(\frac{1}{2} I - K'_{\kappa_0} \right) \tilde{t} = 0.$$

Let t be the unique element of the factor space $H^{-1/2}(\Gamma)/\ker(V_{\kappa_0})$ that solves $V_{\kappa_0}t = g$, then it follows for $\alpha \in \mathbb{C}$ and $\tilde{t} \in \ker(V_{\kappa_0})$

$$\begin{aligned}\left(\frac{1}{2}I - K'_{\kappa_0}\right) V_{\kappa_0}^{-1} g &= \left(\frac{1}{2}I - K'_{\kappa_0}\right)(t + \alpha\tilde{t}) \\ &= \left(\frac{1}{2}I - K'_{\kappa_0}\right) t.\end{aligned}$$

From Lemma 2.4 we know that $(\frac{1}{2}I - K'_{\kappa_0})$ and V_{κ_0} have the same kernel and thus the exterior Steklov–Poincaré operator is well defined. □

Lemma 2.12. *The exterior Steklov–Poincaré operator $S^{ext}_{\kappa_0}$ is coercive for all wave numbers $\kappa_0 \in \mathbb{R}$.*

Proof. This proof follows the proof of Lemma 2.9 with slight variations since we consider the exterior case instead of the interior one. First we have to note that in Lemma 2.4 we only consider the square of κ, therefore $(\frac{1}{2}I - K'_{\kappa})$ and $(\frac{1}{2}I - K'_{-\kappa})$ have the same kernel. Therefore, as in the proof of Lemma 2.11, the operators

$$\begin{aligned}\left(\frac{1}{2}I - K'_{-\kappa_0}\right) V_{\kappa_0}^{-1} \left(\frac{1}{2}I - K_{\kappa_0}\right) &: H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma), \\ \left(K'_{\kappa_0} - K'_{-\kappa_0}\right) V_{\kappa_0}^{-1} \left(\frac{1}{2}I - K_{\kappa_0}\right) &: H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)\end{aligned}$$

are well defined, even though V_{κ_1} might not be invertible on the whole of $H^{1/2}(\Gamma)$. With these operators we construct, as in the proof for the interior case, the compact operator

$$T_S := T_D + \left(K'_{\kappa_0} - K'_{-\kappa_0}\right) V_{\kappa_0}^{-1} \left(\frac{1}{2}I - K_{\kappa_0}\right) + \left(\frac{1}{2}I - K'_{-\kappa_0}\right) V_{-\kappa_0}^{-1} T_V^* V_{\kappa_0}^{-1} \left(\frac{1}{2}I - K_{\kappa_0}\right)$$

and get for all $v \in H^{1/2}(\Gamma)$

$$\Re\langle (S^{ext}_{\kappa_0} + T_S)v, v\rangle_\Gamma \geq c_1^D \|v\|^2_{H^{1/2}(\Gamma)}.$$

□

Lemma 2.13. *The exterior Steklov–Poincaré operator $S^{ext}_{\kappa_0}$ is injective for all wave numbers $\kappa_0 \in \mathbb{R}$.*

Proof. The proof is the same as for Lemma 2.10 with the simplification that the exterior Dirichlet and Neumann problems possess unique solutions for all wave numbers κ_0. This is due to the Sommerfeld radiation condition (2.3) that we enforce in the case of unbounded domains. □

3 Boundary element method

In the previous chapters we presented several boundary integral operators which will be used to derive boundary integral equations. In this chapter we discuss how these results translate if we consider finite dimensional test and trial spaces.

The first section of this chapter is a continuation of Section 1.1 for a finite dimensional subspace X_h of X with a Galerkin–Bubnov scheme [19]. In Section 3.2 we discuss discrete spaces that approximate the Sobolev spaces introduced in Section 1.3. Recall that the Steklov–Poincaré operator is not given in a closed form but as the composition of boundary integral operators, involving an inverse. Therefore its discretization needs consideration, this is discussed in Section 3.3. Section 3.4 deals with the preconditiong [11, 16] of the systems we derived earlier.

3.1 Galerkin scheme for coercive operators

In this section we continue with the abstract setting presented in Section 1.1. We consider a bounded linear operator $A : X \to X'$ and a given right hand side $f \in X'$. We then want to find an element $u \in X$ as the solution of $Au = f$ or the equivalent variational problem

$$\langle Au, v\rangle = \langle f, v\rangle \qquad \forall v \in X.$$

Assuming that A fulfils all requirements stated in Lemma 1.2, the unique solvability is given by the Fredholm alternative. The next step is to discretize the problem, this means we want to find a different problem with only finite dimensions whose solution approximates the solution of the continuous problem. For this we consider a subspace $X_h \subset X$ with a basis $\{\varphi_\ell\}_{\ell=1}^N$, that is

$$\operatorname{span}\{\varphi_\ell\}_{\ell=1}^N = X_h \quad \text{and} \quad \dim X_h = N.$$

The sub–index h is a reference to the mesh size, a characteristic quantity of the discrete spaces presented in the next section. Since X_h is a subset of X we call it a conforming test and trial space, in this thesis we only consider such discrete spaces.

The discrete variational problem is now to find $u_h \in X_h$ such that

$$\langle Au_h, v_h\rangle = \langle f, v_h\rangle \qquad \forall v_h \in X_h. \tag{3.1}$$

The considered space X_h is finite dimensional, hence we can express any element $v_h \in X_h$, by means of the basis, as

$$v_h = \sum_{\ell=1}^{N} v_\ell \varphi_\ell.$$

With this we have found an isomorphism between X_h and $\mathbb{C}^N$ which enables us to identify any function in X_h with the vector of its basis coefficients and vice versa. Using this isomorphism in the discrete variational formulation for u_h and testing with φ_k yields

$$\sum_{\ell=1}^{N} u_\ell \langle A\varphi_\ell, \varphi_k\rangle = \langle f, \varphi_k\rangle \quad \text{for } k = 1, \dots, N.$$

Since the set $\{\varphi_\ell\}_{\ell=1}^N$ is a basis of X_h this problem is equivalent to solving the discrete variational formulation (3.1). Rewriting in a matrix vector form yields

$$A_h \underline{u} = \underline{f}$$

with $A \in \mathbb{C}^{N\times N}, \underline{f} \in \mathbb{C}^N$ and $\underline{u} \in \mathbb{C}^N$ defined by

$$\begin{aligned} A_h[k,\ell] &:= \langle A\varphi_\ell, \varphi_k\rangle, & \ell, k &= 1, \dots, N,\\ \underline{f}[k] &:= \langle f, \varphi_k\rangle, & k &= 1, \dots, N,\\ \underline{u}[\ell] &:= u_\ell, & \ell &= 1, \dots, N. \end{aligned}$$

This finally is a form we can solve using direct or iterative solvers and Cea's lemma tells us how good this approximate solution is.

Lemma 3.1 (Cea's lemma)**.** *[19, Theorem 8.10] Let $X_h \subset X$ be finite dimensional and let the Babuska–Brezzi–Ladyshenskaya (BBL) condition*

$$\sup_{v_h\in X_h, v_h\neq 0} \frac{|a(u_h, v_h)|}{\|v_h\|_X} \geq c\|u_h\|_X \tag{3.2}$$

be satisfied for $c > 0$ and all $u_h \in X_h$. This inequality is also known as the discrete inf–sup condition. If A is bounded, linear and coercive then the discrete problem (3.1) has a unique solution that satisfies the stability estimate

$$\|u_h\|_X \leq \frac{1}{c}\|f\|_{X'}$$

as well as the quasi–optimal error estimate

$$\|u - u_h\|_X \leq \left(1 + \frac{c_2^A}{c}\right) \inf_{v_h\in X_h} \|u - v_h\|_X.$$

The constant c_2^A is the boundedness constant of A.

In order to apply this lemma, we need to show the stability condition (3.2) as well as an estimate for the infimum that appears in the quasi–optimal error estimate. Both these properties are connected with how well X_h approximates X. In order to describe this, we switch from considering a single subspace X_h to a family of subspaces.

Definition 3. Let $(X_{h_n})_{n\in\mathbb{N}}$ be a sequence of conforming spaces $X_{h_n} \subset X$. Then the sequence is said to approximate X if

$$\lim_{n\to\infty} \inf_{v_h\in X_{h_n}} \|v - v_h\|_X = 0 \qquad \forall v \in X.$$

Assuming that we can construct such a sequence of subspaces, we easily see from Cea's Lemma that u_h converges to u. The only property left is to show that the BBL condition (3.2) is satisfied. This follows if X_h approximates X well enough, as is shown in the next theorem.

Theorem 3.2. *[9, 19] Let $(X_{h_n})_{n\in\mathbb{N}}$ be an approximating sequence of subspaces. If A is coercive and injective, then there exists a $n_0 \in \mathbb{N}$ such that the discrete inf–sup condition (3.2) is satisfied for all $n > n_0$.*

Cea's lemma assumes that the right hand side $\langle f, \varphi_k\rangle$ is evaluated exactly. When using the boundary element method, this usually is not the case. It is possible that the right hand side itself contains an operator, for example $f = Bg$ with a bounded linear operator $B : Y \to X'$ and a given element $g \in Y$ Then the variational formulation is

$$\langle Au, v\rangle = \langle Bg, v\rangle \qquad \forall v \in X$$

Since our implementation only covers the application of operators to basis functions $\psi_k \in Y_h \subset Y$ we cannot compute f. One possibility to overcome this problem is to apply an interpolation or projection operator to g first which results in a discrete function $g_h \in Y_h$. With this we get the perturbed system

$$\langle A\tilde{u}_h, \varphi_k\rangle = \langle Bg_h, \varphi_k\rangle \qquad \forall \varphi_k \in X_h$$

with the solution $\tilde{u}_h$ that is not a solution of the original discrete problem (3.1). However, the Strang lemma [15, Theorem 4.2.11] shows that the asymptotic order of convergence is not affected, provided the approximation of the right hand side g_h is good enough.

3.2 Discrete trial and test spaces

This section introduces finite dimensional subspaces for Sobolev spaces which allow us to apply the theory discussed in Section 3.1. In particular we present discrete spaces that serve as approximations of $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$. It is sufficient to consider real valued basis functions since the imaginary part of a function can be expressed using complex coefficient vectors.

Let Γ be the boundary of a Lipschitz domain that admits a decomposition into N disjoint triangles τ_ℓ which can be written as

$$\Gamma = \Gamma_h = \bigcup_{\ell=1}^{N} \overline{\tau}_\ell.$$

This equation introduces the mesh Γ_h of the boundary Γ. In general geometries do not allow for such a triangulation, in those cases we have to consider an approximation $\Gamma_h \approx \Gamma$ which introduces an additional consistency error. We have seen similar disturbances before when we considered approximations of the right hand side. Once again,

the Strang lemma, see [15, Section 4.2.4] or [2, Section 3.1], tells us that the order of convergence is not affected, provided the approximation Γ_h of Γ is good enough.

To discuss properties of this discretization we need quantifiable measurements of the elements. Therefore we define for each τ_ℓ

$$\begin{aligned} \text{the volume } \Delta_\ell &:= \int_{\tau_\ell} ds_x, \\ \text{the local mesh size } h_\ell &:= \sqrt{\Delta_\ell} \\ \text{and the diameter } d_\ell &:= \sup_{x,y\in\tau_\ell} |x-y|. \end{aligned}$$

The maximum over all local mesh sizes is a common measure for how fine the mesh is and is therefore simply called mesh size

$$h := \max_{\ell=1,\dots,N} h_\ell.$$

To distinguish meshes with regards to how fine they are, one often considers either the mesh size h or the number of elements N. These two characteristics behave in the three dimensional case as

$$h \sim \sqrt{\frac{1}{N}}.$$

At last, we define the minimum mesh size

$$h_{\min} := \min_{\ell=1,\dots,N} h_\ell.$$

Now let us consider a sequence of meshes with an increasing number of elements N which leads to smaller elements τ_ℓ and therefore to a decrease of h and $h_{\min}$. These meshes are called globally quasi–uniform if there exists a constant $c_G \geq 1$ such that

$$\frac{h}{h_{\min}} \leq c_G \tag{3.3}$$

holds for all meshes Γ_N. Note that since c_G is a constant, it is in particular independent of N. A second mesh property we consider is shape regularity, i.e. that there exists a constant $c_B > 0$ independent of N that satisfies

$$d_\ell \leq c_B h_\ell \quad \text{for } \ell = 1, \dots, N. \tag{3.4}$$

On such a shape regular mesh we introduce the space of piecewise constant functions

$$S_h^0(\Gamma) := \operatorname{span}\{\varphi_\ell^0\}_{\ell=1}^N$$

with

$$\varphi_\ell^0(x) := \begin{cases} 1 & x \in \tau_\ell, \\ 0 & \text{else.} \end{cases}$$

This space is used as a conforming discrete space for $H^{-1/2}(\Gamma)$ with the following approximation properties.

Lemma 3.3. *[19, Theorem 10.4] Let Γ be a Lipschitz domain that admits a shape regular discretization, i.e. (3.4) is satisfied. Let $\sigma \in [-1, 0]$ and $u \in H^s(\Gamma)$ with $s \in [\sigma, 1]$. Then there holds for a constant $c > 0$*

$$\inf_{v_h \in S_h^0(\Gamma)} \|u - v_h\|_{H^\sigma(\Gamma)} \leq c h^{s-\sigma} |u|_{H^s(\Gamma)}.$$

The second discrete space we consider is the space of piecewise linear, globally continuous functions $S_h^1(\Gamma)$. Each element in this space is determined by its values in the nodes of the mesh. Let $\{x_k\}_{k=1}^M$ be the nodes of the mesh Γ_N, then we set

$$S_h^1(\Gamma) := \operatorname{span}\{\varphi_k^1\}_{k=1}^M$$

with

$$\varphi_k^1(x) := \begin{cases} 1 & x = x_k, \\ 0 & x = x_\ell, \ell \neq k, \\ \text{linear} & \text{else.} \end{cases}$$

This space is a subspace of $H^{1/2}(\Gamma)$ with the following approximation properties.

Lemma 3.4. *[19, Theorem 10.9] [15, Theorem 4.3.22] Let Γ be the polyhedral surface of a Lipschitz domain that admits a shape regular discretization, i.e. (3.4) is satisfied. Let $\sigma \in [0, 1]$ and $u \in H^s(\Gamma)$ with $s \in [\sigma, 2]$, then there holds for a constant $c > 0$*

$$\inf_{v_h \in S_h^1(\Gamma)} \|u - v_h\|_{H^\sigma(\Gamma)} \leq c h^{s-\sigma} |u|_{H^s(\Gamma)}.$$

These two lemmata show that, as h goes to zero, the infimum goes to zero as well if shape regularity is provided. Therefore we can construct a sequence of approximating discrete spaces by reducing the mesh size for each new level. One common strategy is to subdivide the mesh, for triangles this means to insert new vertices at the midpoints of edges and hence divide each triangle into four smaller ones.

3.3 Discrete Steklov–Poincaré operator

Recall that for a given operator $A : X \to X'$ we can define its discrete counterpart by its application to all basis functions. This discrete operator is then fully described by the matrix

$$A_h[\ell, k] := \langle A\varphi_k, \varphi_\ell \rangle \qquad k, \ell = 1, \ldots, N,$$

where $\{\varphi_\ell\}_{\ell=1}^N$ is a basis of X. However, if we consider the exterior Steklov–Poincaré operator (2.7) in its symmetric form

$$S_\kappa^{ext} = D_\kappa + \left(\frac{1}{2} I - K_\kappa'\right) V_\kappa^{-1} \left(\frac{1}{2} I - K_\kappa\right),$$

then such a computation is not possible due to the inverse operator V_κ^{-1} which we cannot apply directly. Instead, let $v \in H^{1/2}(\Gamma)$ be arbitrary but fixed. Then we define

$$w := V_\kappa^{-1} \left(\frac{1}{2} I - K_\kappa\right) v \in H^{-1/2}(\Gamma),$$

which by definition is a solution of

$$w \in H^{-1/2}(\Gamma): \quad \langle V_\kappa w, \tau\rangle_\Gamma = \left\langle \left(\frac{1}{2}I - K_\kappa\right) v, \tau \right\rangle_\Gamma \qquad \forall \tau \in H^{-1/2}(\Gamma).$$

This motivates the introduction of a discrete function $w_h \in S_h^0(\Gamma) \subset H^{-1/2}(\Gamma)$ defined as the solution of

$$w_h \in S_h^0(\Gamma): \quad \langle V_\kappa w_h, \tau_h\rangle_\Gamma = \left\langle \left(\frac{1}{2}I - K_\kappa\right) v, \tau_h \right\rangle_\Gamma \qquad \forall \tau_h \in S_h^0(\Gamma).$$

With this we set

$$\widetilde{S}_\kappa^{ext} v := D_\kappa v + \left(\frac{1}{2}I - K'_\kappa\right) w_h$$

as an approximation of the Steklov–Poincaré operator. Its matrix is given by

$$S_h[\ell, k] := \left\langle \widetilde{S}_\kappa^{ext} \varphi_k^1, \varphi_\ell^1 \right\rangle \qquad \forall k, \ell = 1, \ldots, M$$

with the basis $\{\varphi_k^1\}_{k=1}^M$ of $S_h^1(\Gamma)$. The question is now if this disturbed discrete problem still has a unique solution and if that solution approximates the solution of the continuous problem. The Strang lemma [15, Theorem 4.2.11] tells us, that both these properties are satisfied for the considered case. Let $u \in H^{1/2}(\Gamma)$ be the unique solution of

$$\langle S_\kappa^{ext} u, v\rangle_\Gamma = \langle f, v\rangle_\Gamma \quad \forall v \in H^{1/2}(\Gamma),$$

then the disturbed discrete variational problem

$$u_h \in S_h^1(\Gamma): \quad \langle \widetilde{S}_\kappa^{ext} u_h, v_h\rangle_\Gamma = \langle f, v_h\rangle_\Gamma \quad \forall v_h \in S_h^1(\Gamma)$$

has an unique solution that satisfies the error estimate

$$\|u - u_h\|_{H^{1/2}(\Gamma)} \leq c \inf_{v_h \in S_h^1} (\|u - v_h\|_{H^{1/2}(\Gamma)} + \delta_h \|v_h\|_{H^{1/2}(\Gamma)}).$$

with a continuity constant $\delta_h > 0$. At first glance, the additional error term might look bad, but for the Steklov–Poincaré operator and the choice $S_h^1(\Gamma)$ we get the same asymptotic error behaviour as we would for the undisturbed system. The same can be done for the interior Steklov–Poincaré operator if we assume that the operator V_κ is invertible.

3.4 Preconditioning

In the previous section we derived a system of linear equations which remains to be solved. We do not solve the system directly but use an iterative solver, in our case GMRES [18, Section 5.2]. The downside of such a solver is that the number of iterations steps required for a fixed accuracy depends on the condition number $\kappa_2(A_h)$ of the system matrix A_h. For boundary integral equations of the first kind, which

our formulations are, this condition number increases as the mesh size h decreases [15]. This grow of the condition number that manifests in an increase of required iteration steps can be observed for numerical examples in Section 6.3 and Section 6.4. To counteract this behaviour we use a preconditioning strategy. This means we solve a modified system with a condition number that is independent of the mesh size and has the same solution as the original system [11, 19].

As a model we consider the operator equation

$$Au = f$$

for a bounded and coercive operator $A : X \to X'$ and $f \in X'$. The corresponding discrete system is

$$A_h \underline{u} = \underline{f}$$

with a conforming test and trial space $X_h \subset X$. If all assumptions of Lemma 3.1 are satisfied, both of these problems have a unique solution.

The idea is to find a regular matrix $C_{A,h} \in \mathbb{C}^{N \times N}$ and solve the modified problem

$$C_{A,h}^{-1} A_h \underline{u} = C_{A,h}^{-1} \underline{f}.$$

Since $C_{A,h}$ is regular this system has the same solution as the original one.

In order to be a good preconditioner the matrix $C_{A,h}$ should comply with two criteria. The first criteria is that the new system matrix $C_{A,h}^{-1} A_h$ has a condition number that is bounded independent of the mesh size h. To ensure this, both matrices need to be similar in some sense. The second criteria is that we have to be able to compute and apply the inverse of $C_{A,h}$ efficiently.

The two trivial choices of $C_{A,h}$ that illustrate the trade–off between these two properties are A itself and the identity I. If we choose $C_{A,h} = A_h$ the modified system is the identity with the condition number 1, but to apply $C_{A,h}^{-1}$ we have to compute the inverse of A_h which was the original problem. For $C_{A,h} = I$ we know $C_{A,h}^{-1} = I$ which is trivial to apply but the condition number of the modified system is the same as for the original system.

For the use in boundary element methods there exist several approaches to construct feasible matrices $C_{A,h}^{-1}$. Such preconditioning techniques include, amongst others, the multigrid method [15, Section 6.5] and the artificial multilevel boundary element preconditioners [16]. In this thesis we use and present the concept of preconditioning using boundary integral operators of opposite order [11, 19].

Let $B : X' \to X$ be a bounded operator and consider the composition $BA : X \to X$. This means that B corresponds to $C_{A,h}^{-1}$ from the previous setting. In order to estimate the norm of the resulting system A and B have to satisfy stability conditions

$$\sup_{0 \neq v_h \in X_h} \frac{|\langle Au_h, v_h \rangle|}{\|v_h\|_X} \geq c_1^A \|u_h\|_X \qquad \forall u_h \in X_h \subset X, \tag{3.5}$$

$$\sup_{0 \neq w_h \in X_h'} \frac{|\langle Bw_h, z_h \rangle|}{\|w_h\|_{X'}} \geq c_1^B \|z_h\|_{X'} \qquad \forall z_h \in X_h' \subset X'. \tag{3.6}$$

From Theorem 3.2 we know that this is satisfied as long as the operators are coercive and injective and the discrete spaces are approximating, properties we already required earlier.

The next question is how to translate this composite operator to the discrete level. The problem is, consider for example $A_h\underline{u} = \underline{f}$, that the discrete operator A_h excepts a coefficient vector $\underline{u}$ as input, but its output $\underline{f}$ is a discrete function tested with basis functions of the dual space. We refer to Section 3.1 for the definitions of the matrix and vectors . Hence the output of A_h and the input of B_h do not fit and we have to use auxiliary operators in between. The inverse of the mass matrix, defined as

$$M_h[\ell, k] := \langle \varphi_k, \psi_\ell \rangle \qquad \varphi_k \in X_h, \psi_\ell \in X_h'$$

with bases $\{\varphi_k\}_{k=1}^M$ of X_h and $\{\psi_\ell\}_{\ell=1}^M$ of X_h', has the desired properties. Note that it is no coincidence that the dimensions of both discrete spaces are the same, in fact it is a necessary condition since we need to invert M_h. That this inverse exists is ensured by the following, final, stability condition

$$\sup_{0 \neq w_h \in X_h'} \frac{|\langle v_h, w_h \rangle|}{\|w_h\|_{X'}} \geq c_1^M \|v_h\|_X \qquad \forall v_h \in X_h. \tag{3.7}$$

Theorem 3.5. *[11, Theorem 2.1] Let (3.5)–(3.7) be satisfied, then there exists a constant $c > 0$ independent of the mesh size h such that*

$$\kappa_2(M_h^{-1} B_h M_h^{-\top} A_h) \leq c.$$

Note that the only inverse required to compute the preconditioning matrix

$$C_{A,h}^{-1} = M_h^{-1} B_h M_h^{-\top}.$$

is M_h^{-1}. This can be computed efficiently since M_h is a sparse matrix that is spectrally equivalent to a diagonal matrix.

We have seen that the first two inequalities (3.5)–(3.6) follow for a fine enough mesh if the operators are coercive and injective. The last stability condition (3.7), that only depends on the spaces X_h and X_h', needs more consideration. Here we present two choices for stable pairings in the case that we consider the Sobolev spaces $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$.

The first idea is to use the same discrete space of piecewise linear, globally continuous functions for both continuous spaces. Then the necessary condition of equal dimensions is satisfied. The following lemma shows the inequality (3.7) if the L_2 projection, and therefore the mesh, fits certain criteria.

Lemma 3.6. *[17, Theorem 2.1] For $v \in H^{1/2}(\Gamma)$ the L_2 projection $Q_h : H^{1/2}(\Gamma) \to S_h^1(\Gamma)$ is defined as*

$$\langle Q_h v, w_h \rangle_{L_2(\Gamma)} = \langle v, w_h \rangle_{L_2(\Gamma)} \qquad \forall w_h \in S_h^1(\Gamma).$$

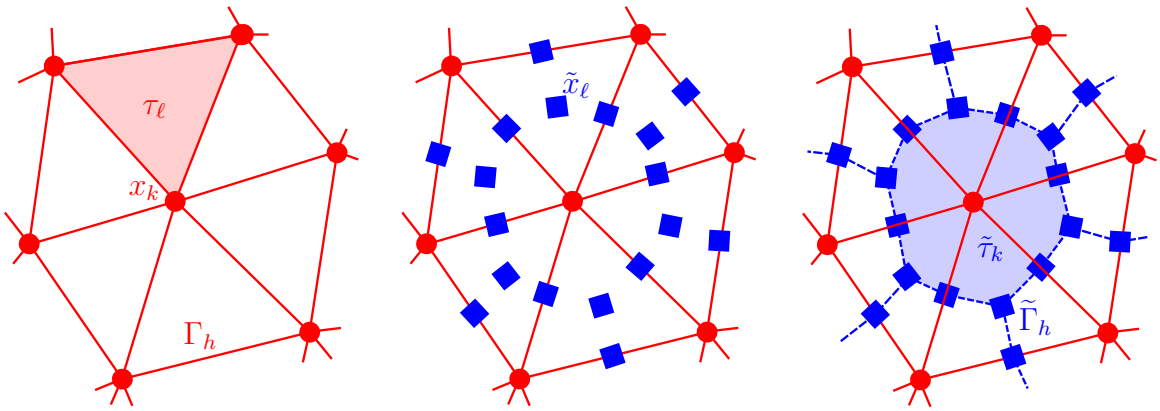


Figure 3.1: Construction of the dual mesh $\widetilde{\Gamma}_h$.

If the L_2 projection is bounded in $H^{1/2}(\Gamma)$, then there exists a $c_M > 0$ satisfying the the stability condition

$$\sup_{0 \neq w_h \in S_h^1(\Gamma)} \frac{|\langle v_h, w_h \rangle_\Gamma|}{\|w_h\|_{H^{-1/2}(\Gamma)}} \geq c_M \|v_h\|_{H^{1/2}(\Gamma)} \qquad \forall v_h \in S_h^1(\Gamma).$$

If we consider a globally quasi–uniform mesh, then the assumptions of Lemma 3.6, i.e. the boundedness of the L_2 projection, is ensured [17, Theorem 1.8]. A second sufficient criteria is to assume only a locally quasi–uniform mesh in combination with a small enough mesh width h [17, Section 2.1]. Thus, the pair $S_h^1(\Gamma) \times S_h^1(\Gamma)$ is stable and can be used for preconditioning.

The second approach keeps the proposed discrete spaces $S_h^1(\Gamma)$ for $H^{1/2}(\Gamma)$ and $S_h^0(\Gamma)$ for $H^{-1/2}(\Gamma)$ but makes use of a dual mesh. The first problem we encounter when using $S_h^1(\Gamma)$ and $S_h^0(\Gamma)$ as introduced in section 3.2 is that the dimension of these spaces are not equal. We have one piecewise constant basis function for each element $\{\tau_\ell\}_{\ell=1}^N$ whereas the basis functions of $S_h^1(\Gamma)$ correlate to the nodes $\{x_k\}_{k=1}^M$.

The solution is to create a new mesh $\widetilde{\Gamma}_h$ of the boundary Γ with M elements $\tilde{\tau}_\ell$ which are no longer triangular. Then we define the space of piecewise constant functions on this new discretization, that is $S_h^0(\widetilde{\Gamma}_h)$.

One way to create such a dual mesh $\widetilde{\Gamma}_h$ as described in [17, Section 2.2] is illustrated in Figure 3.1. The elements of $\widetilde{\Gamma}_h$ are defined by starting from the the primal mesh Γ_h with the vertices x_k. Then we take the the midpoints of edges and triangles, which form the vertices $\tilde{x}_\ell$ of $\widetilde{\Gamma}_h$. The elements $\tilde{\tau}_k \in \widetilde{\Gamma}_h$ of the dual mesh, one for each vertex of the primal mesh x_k, are then constructed by connecting the vertices $\tilde{x}_\ell$. Provided the mesh width h is small enough and the mesh is locally quasi–uniform, this construction leads to a stable pairing $S_h^1(\Gamma_h) \times S_h^0(\widetilde{\Gamma}_h)$ [17, Section 2.2] that can be used for preconditioning.

4 Single homogeneous scatterer

In this chapter we present and discuss several formulations for the model problem of a single homogeneous scatterer, i.e the wave number is constant inside the domain Ω. This simpler model problem not only facilitates the notations and proofs but also changes the characteristics of some formulations compared to the more general case considered in Chapter 5.

The considered approaches are the single trace formulation [12, 21], the Steklov–Poincaré operator formulation [20], the interior Steklov–Poincaré operator formulation [20, 22] and the local multi trace formulation [5, 12].

Each formulation is presented in its own section starting with the motivation and derivation of the operator system. The second step is to ensure unique solvability and at last we construct feasible preconditioning operators for all except the interior Steklov–Poincaré operator formulation. The last section discusses how the discrete Galerkin scheme presented in Section 3.1 can be applied to these formulations.

4.1 Model problem

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain with its boundary $\Gamma := \partial\Omega$. In compliance with the notation in the following chapter we denote the domain as Ω_1 and its complement $\Omega^c = \mathbb{R}^3 \setminus \overline{\Omega}_1$ as Ω_0. In this Chapter the Cauchy data of functions $\tilde{u}_1 \in H^1(\Delta, \Omega_1)$ and $\tilde{u}_0 \in H^1_{\mathrm{loc}}(\Delta, \Omega_0)$ are denoted as

$$\begin{aligned} u_1(x) &:= \gamma_0^{int}\tilde{u}_1, \quad t_1(x) := \gamma_1^{int}\tilde{u}_1 \quad \text{for } x \in \Gamma, \\ u_0(x) &:= \gamma_0^{ext}\tilde{u}_0, \quad t_0(x) := \gamma_1^{ext}\tilde{u}_0 \quad \text{for } x \in \Gamma. \end{aligned}$$

Let κ_0 and κ_1 be given positive constant wave numbers. The Helmholtz transmission problem is to find $\tilde{u}_1 \in H^1(\Omega_1)$ and $\tilde{u}_0 \in H^1_{\mathrm{loc}}(\Omega_0)$ that satisfy partial differential equations

$$-\Delta\tilde{u}_1(x) - \kappa_1^2\tilde{u}_1(x) = 0 \quad \text{for } x \in \Omega_1, \tag{4.1}$$
$$-\Delta\tilde{u}_0(x) - \kappa_0^2\tilde{u}_0(x) = 0 \quad \text{for } x \in \Omega_0 \tag{4.2}$$

and inhomogeneous transmission conditions

$$u_1(x) - u_0(x) = f(x) \quad \text{for } x \in \Gamma, \tag{4.3}$$
$$t_1(x) - t_0(x) = g(x) \quad \text{for } x \in \Gamma \tag{4.4}$$

for given transmission data $f \in H^{1/2}(\Gamma)$ and $g \in H^{-1/2}(\Gamma)$. To ensure unique solvability we have to demand that $\tilde{u}_0$ satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} \int_{|x|=r} \left| \frac{\partial}{\partial n_x} \tilde{u}_0(x) - i\kappa_0 \tilde{u}_0(x) \right|^2 ds_x = 0. \tag{4.5}$$

In [21, Lemma 2.2] it is shown that this model problem (4.1)–(4.5) has a unique solution if the Lipschitz domain Ω_1 is bounded and both wave numbers are greater than zero. From the interior Calderón projection (2.2) we get boundary integral equations for the interior partial differential equation (4.1)

$$(V_{\kappa_1} t_1)(x) - \left(\frac{1}{2} I + K_{\kappa_1}\right) u_1(x) = 0 \quad \text{for } x \in \Gamma, \tag{4.6}$$

$$(D_{\kappa_1} u_1)(x) - \left(\frac{1}{2} I - K'_{\kappa_1}\right) t_1(x) = 0 \quad \text{for } x \in \Gamma. \tag{4.7}$$

In the same way the exterior Calderón projection (2.5) gives us equations for the exterior partial differential equation (4.2)

$$(V_{\kappa_0} t_0)(x) - \left(-\frac{1}{2} I + K_{\kappa_0}\right) u_0(x) = 0 \quad \text{for } x \in \Gamma, \tag{4.8}$$

$$(D_{\kappa_0} u_0)(x) + \left(\frac{1}{2} I + K'_{\kappa_0}\right) t_0(x) = 0 \quad \text{for } x \in \Gamma. \tag{4.9}$$

Combined with the transmission conditions (4.3)–(4.4) we have six equations to find the four unknown traces.

4.2 Single trace formulation

The first approach we study is the single trace formulation as presented in [21] and discussed in [12, 20]. The name already suggests that it includes only one, either the interior or the exterior, Dirichlet and Neumann trace on the boundary Γ. Let $(\tilde{u}_1, \tilde{u}_0) \in H^1(\Omega_1) \times H^1_{\text{loc}}(\Omega_0)$ be a weak solution of (4.1)–(4.5). Then its interior and exterior traces are solutions of the suitable Calderón operators which yields

$$\begin{pmatrix} V_{\kappa_1} & -\frac{1}{2} I - K_{\kappa_1} \\ -\frac{1}{2} I + K'_{\kappa_1} & D_{\kappa_1} \end{pmatrix} \begin{pmatrix} t_1 \\ u_1 \end{pmatrix} = \underline{0}$$

for the interior and

$$\begin{pmatrix} V_{\kappa_0} & \frac{1}{2} I - K_{\kappa_0} \\ \frac{1}{2} I + K'_{\kappa_0} & D_{\kappa_0} \end{pmatrix} \begin{pmatrix} t_0 \\ u_0 \end{pmatrix} = \underline{0}$$

for the exterior domain. By using the transmission conditions (4.3)–(4.4) we can replace the exterior with the interior traces and rewrite the second equation as

$$\begin{pmatrix} V_{\kappa_0} & \frac{1}{2} I - K_{\kappa_0} \\ \frac{1}{2} I + K'_{\kappa_0} & D_{\kappa_0} \end{pmatrix} \begin{pmatrix} t_1 \\ u_1 \end{pmatrix} = \begin{pmatrix} V_{\kappa_0} & \frac{1}{2} I - K_{\kappa_0} \\ \frac{1}{2} I + K'_{\kappa_0} & D_{\kappa_0} \end{pmatrix} \begin{pmatrix} g \\ f \end{pmatrix}.$$

Summing these two equations up yields the single trace formulation, that is find $(t_1, u_1) \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ such that

$$\begin{pmatrix} V_{\kappa_1} + V_{\kappa_0} & -(K_{\kappa_1} + K_{\kappa_0}) \\ K'_{\kappa_1} + K'_{\kappa_0} & D_{\kappa_1} + D_{\kappa_0} \end{pmatrix} \begin{pmatrix} t_1 \\ u_1 \end{pmatrix} = \begin{pmatrix} V_{\kappa_0} & \frac{1}{2}I - K_{\kappa_0} \\ \frac{1}{2}I + K'_{\kappa_0} & D_{\kappa_0} \end{pmatrix} \begin{pmatrix} g \\ f \end{pmatrix}. \tag{4.10}$$

Lemma 4.1. *The single trace operator (4.10) is injective for all* $(\kappa_1, \kappa_0) \in \mathbb{R}^2_+$.

Proof. Let $(u_1, t_1) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ be a solution of the homogeneous system

$$\begin{pmatrix} V_{\kappa_1} + V_{\kappa_0} & -(K_{\kappa_1} + K_{\kappa_0}) \\ K'_{\kappa_1} + K'_{\kappa_0} & D_{\kappa_1} + D_{\kappa_0} \end{pmatrix} \begin{pmatrix} t_1 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

then we have to show that it follows $u_1 = 0$ and $t_1 = 0$. First we define solutions of the interior and exterior Helmholtz equation

$$\begin{aligned} U_1(x) &:= (\widetilde{V}_{\kappa_1} t_1)(x) - (W_{\kappa_1} u_1)(x) \qquad \text{for } x \in \Omega_1, \\ U_0(x) &:= -(\widetilde{V}_{\kappa_0} t_1)(x) + (W_{\kappa_0} u_1)(x) \quad \text{for } x \in \Omega_0. \end{aligned}$$

Both the single layer potential and the double layer potential solve the partial differential equations and satisfy the Sommerfeld radiation condition, thus do U_1 and U_0. Now we apply the trace operators and use that (u_1, t_1) solves the homogeneous system. This gives us for $x \in \Gamma$

$$\begin{aligned} \gamma_0^{int} U_1(x) &= (V_{\kappa_1} t_1)(x) + \left(\frac{1}{2}I - K_{\kappa_1}\right) u_1(x), \\ \gamma_0^{ext} U_0(x) &= -(V_{\kappa_0} t_1)(x) + \left(\frac{1}{2}I + K_{\kappa_0}\right) u_1(x) = (V_{\kappa_1} t_1)(x) + \left(\frac{1}{2}I - K_{\kappa_1}\right) u_1(x), \\ \gamma_1^{int} U_1(x) &= \left(\frac{1}{2}I + K'_{\kappa_1}\right) t_1(x) + (D_{\kappa_1} u_1)(x), \\ \gamma_1^{ext} U_0(x) &= \left(\frac{1}{2}I - K'_{\kappa_0}\right) t_1(x) - (D_{\kappa_0} u_1)(x) = \left(\frac{1}{2}I + K'_{\kappa_1}\right) t_1(x) + (D_{\kappa_1} u_1)(x). \end{aligned}$$

We see that (U_1, U_0) is a solution of the transmission problem (4.1)–(4.5) for homogeneous data $(f, g) = (0, 0)$. From the unique solvability of the model problem we can conclude $U_1 \equiv 0$ in Ω_1 and $U_0 \equiv 0$ in Ω_0 which applied to the Cauchy data gives us

$$V_{\kappa_1} t_1 + \left(\frac{1}{2}I - K_{\kappa_1}\right) u_1 = 0 \quad \text{and} \quad \left(\frac{1}{2}I + K'_{\kappa_1}\right) t_1 + D_{\kappa_1} u_1 = 0 \quad \text{on } \Gamma. \tag{4.11}$$

For the second step we define solutions for the Helmholtz transmission problem for interchanged wave numbers κ_0 and κ_1 by

$$\begin{aligned} \hat{U}_0(x) &:= (\widetilde{V}_{\kappa_0} t_1)(x) - (W_{\kappa_0} u_1)(x) \qquad \text{for } x \in \Omega_1, \\ \hat{U}_1(x) &:= -(\widetilde{V}_{\kappa_1} t_1)(x) + (W_{\kappa_1} u_1)(x) \quad \text{for } x \in \Omega_0. \end{aligned}$$

Taking the Neumann and Dirichlet traces combined with the previous result (4.11) and the assumption that (u_1, t_1) solves the homogeneous system yields on Γ

$$\begin{aligned}
\gamma_0^{int}\hat{U}_0 &= (V_{\kappa_0}t_1) + \left(\frac{1}{2}I - K_{\kappa_0}\right)u_1 = -(V_{\kappa_1}t_1) + \left(\frac{1}{2}I + K_{\kappa_1}\right)u_1 = u_1,\\
\gamma_0^{ext}\hat{U}_1 &= -(V_{\kappa_1}t_1) + \left(\frac{1}{2}I + K_{\kappa_1}\right)u_1 = u_1,\\
\gamma_1^{int}\hat{U}_0 &= \left(\frac{1}{2}I + K'_{\kappa_0}\right)t_1 + (D_{\kappa_0}u_1) = \left(\frac{1}{2}I - K'_{\kappa_1}\right)t_1 - (D_{\kappa_1}u_1) = t_1,\\
\gamma_1^{ext}\hat{U}_1 &= \left(\frac{1}{2}I - K'_{\kappa_1}\right)t_1 - (D_{\kappa_i}u_1) = t_1.
\end{aligned}$$

The pair $(\hat{U}_0, \hat{U}_1)$ solves the homogeneous Helmholtz transmission problem and is therefore equal to zero. From this we get $0 = \gamma_0^{int}\hat{U}_0 = u_1$ and $0 = \gamma_1^{int}\hat{U}_0 = t_1$, hence we have shown the injectivity of the single trace operator. □

Lemma 4.2. *[5, Theorem 2.26] The single trace operator induced by (4.10) is coercive.*

Proof. We define the operator

$$T := \begin{pmatrix} 2V - V_{\kappa_1} - V_{\kappa_0} & -(2K - K_{\kappa_1} - K_{\kappa_0}) \\ 2K' - K'_{\kappa_1} - K'_{\kappa_0} & 2\widetilde{D} - D_{\kappa_1} - D_{\kappa_0} \end{pmatrix}.$$

using the operators from Lemma 2.6. Then T is a composition of compact operators and therefore compact itself. Let $\underline{\varphi} = (t, u)^\top \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ be arbitrary but fixed and denote the single trace operator (4.10) by M, then it holds

$$\begin{aligned}
\Re\left\langle (M+T)\underline{\varphi}, \underline{\varphi}\right\rangle_\Gamma &= 2\Re\left\langle \begin{pmatrix} V & -K \\ K' & \widetilde{D} \end{pmatrix}\begin{pmatrix} t \\ u \end{pmatrix}, \begin{pmatrix} t \\ u \end{pmatrix}\right\rangle_\Gamma \\
&= 2\Re\left\langle Vt, t\right\rangle_\Gamma + 2\Re\left\langle \widetilde{D}u, u\right\rangle_\Gamma.
\end{aligned}$$

For the second equality we exploited that K' is the $L_2(\Gamma)$–adjoint of K and hence the off diagonal blocks cancel one anther out. The coercivity then follows from the ellipticity of $\widetilde{D}$ and V [19]. □

With these properties, namely injectivity and coercivity, we are in a position to apply Lemma 1.2 which gives us unique solvability of the single trace formulation.

Theorem 4.3. *[5, Corollary 2.29] For any given $(\kappa_1, \kappa_0) \in \mathbb{R}^2_+$ and $(f, g) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ there exits the unique solution $(t_1, u_1) \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ of the single trace formulation (4.10).*

The last topic of this section is the construction of a preconditioning operator suited for the preconditioning strategy presented in Section 3.4. Note that the single trace formulation is composed of two matrices originating in the Calderón projection. For each of these matrices we can explicitly construct the inverse using the relations given in Corollary 2.3.

Corollary 4.4. *For $\kappa \in \mathbb{R}$ it holds*

$$\begin{pmatrix} D_\kappa & K'_\kappa \\ -K_\kappa & V_\kappa \end{pmatrix} \begin{pmatrix} V_\kappa & -K_\kappa \\ K'_\kappa & D_\kappa \end{pmatrix} = \frac{1}{4} \begin{pmatrix} I & \\ & I \end{pmatrix}.$$

Since we consider the sum of two such matrices we additionally need the next lemma.

Lemma 4.5. *The following operator system is compact for $(\kappa_1, \kappa_0) \in \mathbb{R}^2_+$*

$$\begin{pmatrix} D_{\kappa_1} & K'_{\kappa_1} \\ -K_{\kappa_1} & V_{\kappa_1} \end{pmatrix} - \begin{pmatrix} D_{\kappa_0} & K'_{\kappa_0} \\ -K_{\kappa_0} & V_{\kappa_0} \end{pmatrix} : H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma).$$

The proof of this lemma follows the proof of Lemma 2.8 for all four boundary integral operators individually. With Corollary 4.4 we can write

$$\begin{pmatrix} D_{\kappa_1} & K'_{\kappa_1} \\ -K_{\kappa_1} & V_{\kappa_1} \end{pmatrix} \begin{pmatrix} V_{\kappa_0} & -K_{\kappa_0} \\ K'_{\kappa_0} & D_{\kappa_0} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} I & \\ & I \end{pmatrix} + \begin{pmatrix} D_{\kappa_1} - D_{\kappa_0} & K'_{\kappa_1} - K'_{\kappa_0} \\ -(K_{\kappa_1} + K_{\kappa_0}) & V_{\kappa_1} V_{\kappa_0} \end{pmatrix} \begin{pmatrix} V_{\kappa_0} & -K_{\kappa_0} \\ K'_{\kappa_0} & D_{\kappa_0} \end{pmatrix}$$

and Lemma 4.5 tells us that the last operator on the right hand side is composed of a bounded and a compact operator. Such a composition is compact since compact operators form a two sided ideal in the space of bounded operators [3]. The same procedure can be done for interchanged wave numbers.

Thus we have found a suitable preconditioning operator

$$B_{\mathrm{STF}} := \begin{pmatrix} D_{\kappa_1} + D_{\kappa_0} & K'_{\kappa_1} + K'_{\kappa_1} \\ -(K_{\kappa_1} + K_{\kappa_0}) & V_{\kappa_1} + V_{\kappa_0} \end{pmatrix}$$

whose application to the single trace operator yields the identity operator plus a compact operator. The only accumulation point of the spectrum of such a operator is one which is beneficial if we apply iterative solver, see [15, Proposition 6.1.8]. The preconditioning operator B_{STF} can be written as

$$B_{\mathrm{STF}} = \begin{pmatrix} & I \\ I & \end{pmatrix} \begin{pmatrix} V_{\kappa_1} + V_{\kappa 0} & -(K_{\kappa_1} + K_{\kappa_0}) \\ K'_{\kappa_1} + K'_{\kappa_0} & D_{\kappa_1} + D_{\kappa_0} \end{pmatrix} \begin{pmatrix} & I \\ I & \end{pmatrix}$$

with an injective and self adjoint switching operator. Thus, coercivity and injectivity of B_{STF} follow from coercivity and injectivity for the single trace formulation (4.10). With this we are able to apply operator preconditioning to the single trace formulation and B_{STF}.

4.3 Steklov–Poincaré operator formulation

The next approach we discuss is the Steklov–Poincaré operator formulation [20] which is named after the Dirichlet to Neumann mapping presented in Section 2.4. For a given Dirichlet function it returns the Neumann datum of the corresponding solution of the

exterior Helmholtz problem. Let $(\tilde{u}_1, \tilde{u}_0) \in H^1(\Omega_1) \times H^1_{\mathrm{loc}}(\Omega_0)$ be a weak solution of (4.1)–(4.5), then the exterior Steklov–Poincaré operator gives us

$$-(S^{ext}_{\kappa_0} u_0)(x) = t_0(x) \quad \text{for } x \in \Gamma.$$

We already know from Lemma 2.11 that this operator is well defined for all wave numbers κ_0. Applying this and (4.7) to the transmission condition on the Neumann trace (4.4) gives us

$$\begin{aligned} g(x) &= t_1(x) - t_0(x) \\ &= (D_{\kappa_1} u_1)(x) + \left(\frac{1}{2} I + K'_{\kappa_1}\right) t_1(x) + (S^{ext}_{\kappa_0} u_0)(x). \end{aligned}$$

Since all operators are linear, the transmission condition (4.3) allows us to express u_0 by u_1 and f which yields

$$g(x) + (S^{ext}_{\kappa_0} f)(x) = (D_{\kappa_1} u_1)(x) + \left(\frac{1}{2} I + K'_{\kappa_1}\right) t_1(x) + (S^{ext}_{\kappa_0} u_1)(x).$$

This in combination with (4.6) results in the Steklov–Poincaré operator formulation to find $(t_1, u_1) \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ as the solution of

$$\begin{pmatrix} V_{\kappa_1} & -(\frac{1}{2} I + K_{\kappa_1}) \\ \frac{1}{2} I + K'_{\kappa_1} & D_{\kappa_1} + S^{ext}_{\kappa_0} \end{pmatrix} \begin{pmatrix} t_1 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 \\ g + S^{ext}_{\kappa_0} f \end{pmatrix}. \tag{4.12}$$

Even though the interior single layer potential may not be injective for all $\kappa_1 \in \mathbb{R}_+$, the resulting system is.

Lemma 4.6. *[20, Lemma 4.2] The operator system (4.12) of the Steklov–Poincaré operator formulation is injective for all wave numbers* $(\kappa_0, \kappa_1) \in \mathbb{R}^2_+$.

Proof. To prove the statement, we have to show that the only solution of the homogeneous system

$$\begin{pmatrix} V_{\kappa_1} & -(\frac{1}{2} I + K_{\kappa_1}) \\ \frac{1}{2} I + K'_{\kappa_1} & D_{\kappa_1} + S^{ext}_{\kappa_0} \end{pmatrix} \begin{pmatrix} t_1 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is the trivial solution. Let (t_1, u_1) be a solution of this system, then

$$U_1(x) := (\widetilde{V}_{\kappa_1} t_1)(x) - (W_{\kappa_1} u_1)(x) \quad \text{for } x \in \Omega_1$$

is a solution of the interior Helmholtz equation. Its Dirichlet trace for $x \in \Gamma$ is

$$\gamma_0^{int} U_1(x) = (V_{\kappa_1} t_1)(x) + \left(\frac{1}{2} I - K_{\kappa_1}\right) u_1(x),$$

which can be simplified to

$$\gamma_0^{int} U_1(x) = u_1(x)$$

since (t_1, u_1) solves the homogeneous system. If we do the same for the Neumann trace, that is applying the trace operator and exploiting that we consider a solution of the homogeneous system, we end up with

$$\gamma_1^{int} U_1(x) = \left(\frac{1}{2} I + K'_{\kappa_1}\right) t_1(x) + (D_{\kappa_1} u_1)(x) = -(S_{\kappa_0}^{ext} u_1)(x) \quad \text{for } x \in \Gamma.$$

The second step is to do similar computations for the exterior domain. First we introduce an exterior Neumann trace $t_0 := -S_{\kappa_0}^{ext} u_1$ and

$$U_0(x) := -(\widetilde{V}_{\kappa_0} t_0)(x) + (W_{\kappa_0} u_1)(x) \quad \text{for } x \in \Omega_0.$$

Since both boundary integral potentials solve the exterior Helmholtz equation, so does U_0. Using the alternative representation $S_{\kappa_0}^{ext} = V_{\kappa_0}^{-1}(\frac{1}{2} I - K_{\kappa_0})$ the Cauchy data for $x \in \Gamma$ read

$$\begin{aligned}
\gamma_0^{ext} U_0(x) &= (V_{\kappa_0} S_{\kappa_0}^{ext} u_1)(x) + \left(\frac{1}{2} I + K_{\kappa_0}\right) u_1(x) = u_1, \\
\gamma_1^{ext} U_0(x) &= -\left(\frac{1}{2} I - K'_{\kappa_0}\right) (S_{\kappa_0}^{ext} u_1)(x) - (D_{\kappa_0}) u_1(x) = -(S_{\kappa_0}^{ext} u_1)(x).
\end{aligned}$$

Note that due to Lemma 2.4 we can use the alternative representation of $S_{\kappa_0}^{ext}$ and remain well defined since V_{κ_0} and $(\frac{1}{2} I - K'_{\kappa_0})$ have the same kernel as V_{κ_0}. Combining all this we know that (U_1, U_0) is a solution of (4.1)–(4.5) for homogeneous transmission conditions $(f, g) = (0, 0)$. Since this problem is uniquely solvable, we can conclude $(U_1, U_0) \equiv (0, 0)$. From this it immediately follows $0 = \gamma_0^{int} U_1 = u_1$ on Γ. Inserting this in the Cauchy traces of U_1 results in

$$V_{\kappa_1} t_1 = 0, \qquad \left(\frac{1}{2} I + K'_{\kappa_1}\right) t_1 = 0.$$

The first equation together with Lemma 2.4 yields

$$\left(\frac{1}{2} I - K'_{\kappa_1}\right) t_1 = 0$$

which further gives us $t_1 = 0$ and thus injectivity of the Steklov–Poincaré operator. □

Lemma 4.7. *[20, Thm 4.3] The operator system (4.12) of the Steklov–Poincaré operator formulation is coercive.*

Proof. We define the operator

$$T_M := \begin{pmatrix} V - V_{\kappa_1} & -(K - K_{\kappa_1}) \\ K' - K'_{\kappa_1} & (\widetilde{D} - D_{\kappa_1}) + T_S \end{pmatrix}$$

using the operators from Lemmata 2.6 and 2.12, then T_M is compact. If we denote the Steklov–Poincaré operator (4.12) by M, then we get for all $\underline{\varphi} = (t, u) \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$

$$\begin{aligned}
\Re \left\langle (M + T_M)\underline{\varphi}, \underline{\varphi} \right\rangle_\Gamma &= \Re \left\langle \begin{pmatrix} V & -(\frac{1}{2} I + K) \\ \frac{1}{2} I + K' & \widetilde{D} + (S_{\kappa_0}^{ext} + T_S) \end{pmatrix} \begin{pmatrix} t \\ u \end{pmatrix}, \begin{pmatrix} t \\ u \end{pmatrix} \right\rangle_\Gamma \\
&= \Re \langle Vt, t \rangle_\Gamma + \Re \left\langle (\widetilde{D} + (S_{\kappa_0}^{ext} + T_S))u, u \right\rangle_\Gamma .
\end{aligned}$$

The statement of the lemma now follows from the ellipticity of V and $\widetilde{D}$ as well as the coercivity of $S_{\kappa_0}^{ext}$, see Lemma 2.12. □

Lemmata 4.6 and 4.7 allow us to apply the Fredholm alternative in form of Lemma 1.2 and thus immediately conclude unique solvability of the Steklov–Poincaré operator formulation.

Theorem 4.8. *[20, Thm 4.3] The system (4.12) has a unique solution* $(u_1, t_1) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ *for any right hand side* $(f, g) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ *and any wave numbers* $(\kappa_0, \kappa_1) \in \mathbb{R}_+^2$.

After ensuring unique solvability, we want to find a preconditioner for the Steklov–Poincaré operator system (4.12). One possible choice for an operator to use for operator preconditioning is the one we already used for the single trace formulation, that is

$$B_{\mathrm{SP}} := \begin{pmatrix} D_{\kappa_1} + D_{\kappa_0} & K'_{\kappa_1} + K'_{\kappa_1} \\ -(K_{\kappa_1} + K_{\kappa_0}) & V_{\kappa_1} + V_{\kappa_0} \end{pmatrix}.$$

This operator has the correct mapping properties, i.e. inverse to those of the Steklov–Poincaré operator formulation (4.12), and is coercive as well as injective as shown for B_{STF}.

4.4 Interior Steklov–Poincaré operator formulation

In this section we consider the interior Steklov–Poincaré operator formulation [20, 22] to emphasize that not all formulations have to be uniquely solvable independent of the wave numbers. For this we start from the Steklov–Poincaré operator formulation

$$\begin{pmatrix} V_{\kappa_1} & -(\frac{1}{2}I + K_{\kappa_1}) \\ \frac{1}{2}I + K'_{\kappa_1} & D_{\kappa_1} + S_{\kappa_0}^{ext} \end{pmatrix} \begin{pmatrix} t_1 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 \\ g + S_{\kappa_0}^{ext} f \end{pmatrix},$$

where we further assume that V_{κ_1} is invertible, i.e. κ_1^2 is not a Dirichlet eigenvalue. Then, see Section 2.4, we define the interior Steklov–Poincaré operator $S_{\kappa_1}^{int} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ as

$$S_{\kappa_1}^{int} := D_{\kappa_1} + \left(\frac{1}{2}I + K'_{\kappa_1}\right) V_{\kappa_1}^{-1} \left(\frac{1}{2}I + K_{\kappa_1}\right).$$

The assumption that V_{κ_1} is invertible further enables us to rewrite the first equation of the Steklov–Poincaré operator formulation as

$$t_1 = V_{\kappa_1}^{-1} \left(\frac{1}{2} + K_{\kappa_1}\right) u_1.$$

Inserting this in the second equation yields the interior Steklov–Poincaré operator formulation to find $u_1 \in H^{1/2}(\Gamma)$ such that

$$(S_{\kappa_1}^{int} + S_{\kappa_0}^{ext}) u_1(x) = g(x) + (S_{\kappa_0}^{ext} f)(x) \quad \text{for } x \in \Gamma. \tag{4.13}$$

Theorem 4.9. *Let $(\kappa_1, \kappa_0) \in \mathbb{R}^2_+$ and $(f, g) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ be given. Assume that κ_1^2 is not a Dirichlet eigenvalue, then $S^{int}_{\kappa_1}$ is well defined and the interior Steklov–Poincaré formulation (4.13) has a unique solution $u_1 \in H^{1/2}(\Gamma)$.*

Proof. To apply the Fredholm alternative in the form of Lemma 1.2 we have to show the injectivity and coercivity of the operator $(S^{int}_{\kappa_1} + S^{ext}_{\kappa_0})$. Coercivity follows from the Lemmata 2.9 and 2.12 which show that both operators individually, and thus their sum, are coercive.

For injectivity let $u_1 \in H^{1/2}(\Gamma)$ be a solution of (4.13) for a homogeneous right hand side, that is

$$(S^{int}_{\kappa_1} + S^{ext}_{\kappa_0})u_1(x) = 0 \quad \text{for } x \in \Omega.$$

We introduce Neumann traces $t_1 = S^{int}_{\kappa_1} u_1$ and $t_0 = -S^{ext}_{\kappa_0} u_1$ with which we can define solutions of the partial differential equation in the interior and exterior domain by

$$\begin{aligned} U_1(x) &:= (\widetilde{V}_{\kappa_1} t_1)(x) - (W_{\kappa_1} u_1)(x) \qquad \text{for } x \in \Omega_1, \\ U_0(x) &:= -(\widetilde{V}_{\kappa_0} t_0)(x) + (W_{\kappa_0} u_1)(x) \quad \text{for } x \in \Omega_0. \end{aligned}$$

From the definition of the Steklov–Poincaré operator we know that (u_1, t_1) are Cauchy traces of a solution of the Helmholtz equation in Ω_1. The same holds true for (u_1, t_0) and the exterior domain Ω_0. Thus, we can conclude for the traces of U_0 and U_1 for $x \in \Gamma$

$$\begin{aligned} \gamma_0^{int} U_1(x) &= u_1(x) \qquad \gamma_0^{ext} U_0(x) = u_1(x), \\ \gamma_1^{int} U_1(x) &= t_1(x) \qquad \gamma_1^{ext} U_0(x) = t_0(x). \end{aligned}$$

Furthermore, since u_1 solves the homogeneous system, for the Neumann traces it holds

$$0 = (S^{int}_{\kappa_1} + S^{ext}_{\kappa_0})u(x) = t_1 - t_0.$$

From this it follows that (U_1, U_0) solves the homogeneous problem (4.1)–(4.5) and is therefore equal to zero. Applying the Dirichlet trace operator yields $0 = \gamma_0^{int} U_1 = u_1$ on Γ and thus injectivity of the operator $(S^{int}_{\kappa_1} + S^{ext}_{\kappa_0})$.

Now we can apply Lemma 1.2 which gives us unique solvability of (4.13). □

The purpose of the interior Steklov–Poincaré operator formulation is to emphasize the problems of interior eigenvalues. We see that, if κ_1^2 is a Dirichlet eigenvalue, $S^{int}_{\kappa_1}$ and thus the formulations (4.13) is not well defined. We do not consider the discretization of this formulation and have no need for a preconditioning operator.

4.5 Local multi trace formulation

The last formulation we discuss is the local multi trace formulation as presented in [5, 12]. The most obvious difference to the other formulations is that we do not eliminate any traces resulting in two Dirichlet and Neumann traces on Γ. As for

the single trace formulation we start by applying the Calderón projection to a weak solution $(\tilde{u}_1, \tilde{u}_0) \in H^1(\Omega_1) \times H^1_{\mathrm{loc}}(\Omega_0)$. First we consider the interior domain and get

$$\begin{pmatrix} V_{\kappa_1} & -\frac{1}{2}I - K_{\kappa_1} \\ -\frac{1}{2}I + K'_{\kappa_1} & D_{\kappa_1} \end{pmatrix} \begin{pmatrix} t_1 \\ u_1 \end{pmatrix} = \underline{0}.$$

Now we use the transmission conditions (4.3) and (4.4) to replace $\frac{1}{2}Iu_1$ and $\frac{1}{2}It_1$ by the exterior traces and transmission data (f, g). This results in

$$\begin{pmatrix} V_{\kappa_1} & -K_{\kappa_1} \\ K'_{\kappa_1} & D_{\kappa_1} \end{pmatrix} \begin{pmatrix} t_1 \\ u_1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} u_0 \\ t_0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} f \\ g \end{pmatrix}.$$

The same proceeding for the exterior domain yields

$$\begin{pmatrix} V_{\kappa_0} & -K_{\kappa_0} \\ K'_{\kappa_0} & D_{\kappa_0} \end{pmatrix} \begin{pmatrix} t_0 \\ u_0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} u_1 \\ t_1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} f \\ g \end{pmatrix}.$$

This local switching operation, it is more apparent what we mean by local for the case of composite scatterers as considered in Chapter 5, is what gives the formulation its name. Combining these two systems results in the local multi trace formulation to find $(t_1, u_1, t_0, u_0) \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ such that

$$\begin{pmatrix} V_{\kappa_1} & -K_{\kappa_1} & & -\frac{1}{2}I \\ K'_{\kappa_1} & D_{\kappa_1} & -\frac{1}{2}I & \\ & \frac{1}{2}I & V_{\kappa_0} & -K_{\kappa_0} \\ \frac{1}{2}I & & K'_{\kappa_0} & D_{\kappa_0} \end{pmatrix} \begin{pmatrix} t_1 \\ u_1 \\ t_0 \\ u_0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} f \\ g \\ f \\ g \end{pmatrix}. \tag{4.14}$$

To discuss the solvability of this system we first ensure injectivity.

Lemma 4.10. *[12, Theorem 4] The local multi trace operator (4.14) is injective.*

Proof. Let $\underline{u} = (t_1, u_1, t_0, u_0) \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ be a solution of the system (4.14) for a homogeneous right hand side $(f, g) = (0, 0)$. To show injectivity we have to show that it follows $\underline{u} = \underline{0}$. First we define solutions of the interior and exterior Helmholtz equation

$$\begin{aligned} U_1(x) &:= (\widetilde{V}_{\kappa_1} t_1)(x) - (W_{\kappa_1} u_1)(x) \qquad \text{for } x \in \Omega_1, \\ U_0(x) &:= -(\widetilde{V}_{\kappa_0} t_0)(x) + (W_{\kappa_0} u_0)(x) \quad \text{for } x \in \Omega_0. \end{aligned}$$

Note that by construction U_0 fulfils the Sommerfeld radiation condition. Taking the traces of these functions and using that $\underline{u}$ solves (4.14) for a homogeneous right hand side yields for $x \in \Gamma$

$$\begin{aligned} \gamma_0^{int} U_1(x) &= (V_{\kappa_1} t_1)(x) + \left(\frac{1}{2}I - K_{\kappa_1}\right) u_1(x) = \frac{1}{2}(u_1(x) + u_0(x)), \\ \gamma_0^{ext} U_0(x) &= -(V_{\kappa_0} t_0)(x) + \left(\frac{1}{2}I + K_{\kappa_0}\right) u_0(x) = \frac{1}{2}(u_1(x) + u_0(x)), \\ \gamma_1^{int} U_1(x) &= \left(\frac{1}{2}I + K'_{\kappa_1}\right) t_1(x) + (D_{\kappa_1} u_1)(x) = \frac{1}{2}(t_1(x) + t_0(x)), \\ \gamma_1^{ext} U_0(x) &= \left(\frac{1}{2}I - K'_{\kappa_0}\right) t_0(x) - (D_{\kappa_0} u_0)(x) = \frac{1}{2}(t_0(x) + t_1(x)). \end{aligned}$$

We see that the pair (U_1, U_0) is a solution of the homogeneous model problem (4.1)–(4.5) and thus it follows from the uniqueness of the solution $U_1 \equiv 0$ and $U_0 \equiv 0$ with the consequence

$$u_1(x) + u_0(x) = 0 \quad \text{and} \quad t_1(x) + t_0(x) = 0 \quad \text{for } x \in \Gamma. \tag{4.15}$$

For the second step of the proof we define a function in Ω_1 using the exterior traces (u_0, t_0) and vice versa

$$\begin{aligned}\hat{U}_0(x) &:= (\widetilde{V}_{\kappa_0} t_0)(x) - (W_{\kappa_0} u_0)(x) \quad \text{for } x \in \Omega_1,\\ \hat{U}_1(x) &:= (\widetilde{V}_{\kappa_1} t_1)(x) - (W_{\kappa_1} u_1)(x) \quad \text{for } x \in \Omega_0.\end{aligned}$$

By construction $\hat{U}_0$ is a solution of the interior Helmholtz equation for the wave number κ_0. Furthermore $\hat{U}_1$ is a solution of the exterior Helmholtz equation for the wave number κ_1 and satisfies the Sommerfeld condition (4.5). As before we apply the trace operators and exploit that $\underline{u}$ solves the homogeneous system (4.14), which gives us

$$\begin{aligned}\gamma_0^{int}\hat{U}_0(x) &= (V_{\kappa_0} t_0)(x) + \left(\frac{1}{2} I - K_{\kappa_0}\right) u_0(x) = \frac{1}{2}(u_0(x) - u_1(x)),\\ \gamma_0^{ext}\hat{U}_1(x) &= (V_{\kappa_1} t_1)(x) - \left(\frac{1}{2} I + K_{\kappa_1}\right) u_1(x) = \frac{1}{2}(u_0(x) - u_1(x)),\\ \gamma_1^{int}\hat{U}_0(x) &= \left(\frac{1}{2} I + K'_{\kappa_0}\right) t_0(x) + (D_{\kappa_0} u_0)(x) = \frac{1}{2}(t_0(x) - t_1(x)),\\ \gamma_1^{ext}\hat{U}_1(x) &= -\left(\frac{1}{2} I - K'_{\kappa_1}\right) t_1(x) + (D_{\kappa_1} u_1)(x) = \frac{1}{2}(t_0(x) - t_1(x)).\end{aligned}$$

Thus, the pair $(\hat{U}_0, \hat{U}_1)$ solves the homogeneous model problem, only this time with interchanged wave numbers. Since the solution for the global problem is unique, it follows $\hat{U}_1 \equiv 0$, $\hat{U}_0 \equiv 0$ and

$$u_0(x) - u_1(x) = 0 \quad \text{as well as} \quad t_0(x) - u_0(x) = 0 \quad \text{for } x \in \Gamma.$$

Combined with (4.15) we get the desired result $\underline{u} = \underline{0}$. □

Lemma 4.11. *[12, Theorem 5] The multi trace operator induced by (4.14) is coercive.*

Proof. First we consider the multi trace operator itself without the additional compact operator. We denote the diagonal blocks as A_1 and A_0 and introduce for the off diagonal block the matrix operator

$$E := \begin{pmatrix} & I \\ I & \end{pmatrix}.$$

Then for any $\underline{\varphi}_i = (t_i, u_i)^\top \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma), i \in \{0, 1\}$ we can write

$$\begin{aligned}\left\langle \begin{pmatrix} A_1 & -\frac{1}{2}E \\ \frac{1}{2}E & A_0 \end{pmatrix} \begin{pmatrix} \underline{\varphi}_1 \\ \underline{\varphi}_0 \end{pmatrix}, \begin{pmatrix} \underline{\varphi}_1 \\ \underline{\varphi}_0 \end{pmatrix} \right\rangle_\Gamma &= \left\langle A_1 \underline{\varphi}_1, \underline{\varphi}_1 \right\rangle_\Gamma - \frac{1}{2}\left\langle E\underline{\varphi}_0, \underline{\varphi}_1 \right\rangle_\Gamma + \frac{1}{2}\left\langle E\underline{\varphi}_1, \underline{\varphi}_0 \right\rangle_\Gamma + \left\langle A_0 \underline{\varphi}_0, \underline{\varphi}_0 \right\rangle_\Gamma\\ &= \left\langle A_1 \underline{\varphi}_1, \underline{\varphi}_1 \right\rangle_\Gamma + \left\langle A_0 \underline{\varphi}_0, \underline{\varphi}_0 \right\rangle_\Gamma.\end{aligned}$$

For the second equation we used that the both terms in the middle cancel one another out. Hence it is enough to show coercivity for the diagonal blocks which is quite similar to what we have done before. For $i \in \{0, 1\}$ we define the compact operator

$$T_{A_i} := \begin{pmatrix} -V_{\kappa_i} + V & K_{\kappa_i} - K \\ -K'_{\kappa_i} + K' & -D_{\kappa_i} + \widetilde{D} \end{pmatrix}$$

which leads to

$$\left\langle (A_i + T_{A_i})\underline{\varphi}_i, \underline{\varphi}_i \right\rangle_\Gamma = \langle V t_i, t_i \rangle_\Gamma + \left\langle \widetilde{D} u_i, u_i \right\rangle_\Gamma \geq c_1^V \|t_i\|^2_{H^{-1/2}(\Gamma)} + c_1^D \|u_i\|^2_{H^{1/2}(\Gamma)}$$

for any $\underline{\varphi}_i = (t_i, u_i)^\top \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$. Note that we made use of the ellipticity of V and $\widetilde{D}$, see Lemma 2.7 or [19]. With this we have shown that A_1 and A_2 are coercive which gives us coercivity of the multi trace operator. □

With these two lemmas, the unique solvability of (4.14) follows from Lemma 1.2.

Theorem 4.12. *[12, Collary 1] The local multitrace formulation (4.14) has a unique solution for all wave numbers $(\kappa_1, \kappa_0) \in \mathbb{R}^2_+$ and given right hand sides $f \in H^{1/2}(\Gamma)$ and $g \in H^{-1/2}(\Gamma)$.*

The next important question is how to construct a preconditioner for this system. The motivation for the preconditioning operator are the inverse operators of the diagonal blocks derived in Corollary 4.4. With those we can set

$$B_{\mathrm{MTF}} = \begin{pmatrix} D_{\kappa_1} & K'_{\kappa_1} & & \\ -K_{\kappa_1} & V_{\kappa_1} & & \\ & & D_{\kappa_0} & K'_{\kappa_0} \\ & & -K_{\kappa_0} & V_{\kappa_0} \end{pmatrix}$$

as the preconditioning operator for the multi trace formulation. Since the diagonal blocks are not coupled, coercivity and injectivity has to be shown for both of them individually. Each block corresponds to a single trace formulation where the interior and exterior wave numbers are the same. Hence the two properties, namely coercivity and injectivity, follow from the Lemmata 4.1 and 4.2. With this we have found a feasible preconditioning operator.

4.6 Boundary element methods for the single homogeneous scatterer

The next step in computing a solution of the model problem or at least an approximate solution is to apply the boundary element method presented in Section 3.1. Since the procedure is very similar for all four formulations, it is only done once for the single trace formulation. For the other formulations we only give the resulting systems of linear equations.

All four formulations fit the Galerkin scheme for coercive operators and we have already established the necessary properties to apply Cea's lemma (Lemma 3.1), namely injectivity and coercivity. The only thing that is missing are suitable discrete spaces for the appearing Sobolev spaces $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$. In Section 3.2 we have already presented approximating discrete spaces for both, i.e.

$$S_h^0(\Gamma_h) = \operatorname{span}\{\varphi_\ell^0\}_{\ell=1}^N \subset H^{-1/2}(\Gamma_h) \quad \text{and} \quad S_h^1(\Gamma_h) = \operatorname{span}\{\varphi_k^1\}_{k=1}^M \subset H^{1/2}(\Gamma_h).$$

If we apply the procedure described in Section 3.1 to the single trace formulation (4.10) we get the complex system of linear equations

$$\begin{pmatrix} V_{\kappa_1,h} + V_{\kappa_0,h} & -(K_{\kappa_1,h} + K_{\kappa_0,h}) \\ K_{\kappa_1,h}^\top + K_{\kappa_0,h}^\top & D_{\kappa_1,h} + D_{\kappa_0,h} \end{pmatrix} \begin{pmatrix} \underline{t}_1 \\ \underline{u}_1 \end{pmatrix} = \begin{pmatrix} V_{\kappa_0,h} & \frac{1}{2}M_h - K_{\kappa_0,h} \\ \frac{1}{2}M_h^\top + K_{\kappa_0,h}^\top & D_{\kappa_0,h} \end{pmatrix} \begin{pmatrix} \underline{g} \\ \underline{f} \end{pmatrix}.$$

The matrices $K_{\kappa,h}^\top$ and $M_h^\top$ denote the transpose matrices of $K_{\kappa,h}$ and M_h without taking the conjugate of the elements. The unknown vectors $\underline{u}_1 \in \mathbb{C}^M$ and $\underline{t}_1 \in \mathbb{C}^N$ correspond to discrete functions $(t_{1,h}, u_{1,h}) \in S_h^0(\Gamma) \times S_h^1(\Gamma)$ via the representations

$$t_{1,h} = \sum_{\ell=1}^M \underline{t}_1[\ell]\varphi_\ell^0 \quad \text{and} \quad u_{1,h} = \sum_{k=1}^N \underline{u}_1[k]\varphi_k^1.$$

In the same manner the vectors $\underline{f} \in \mathbb{C}^M$ and $\underline{g} \in \mathbb{C}^N$ are the coefficients of the functions f and g. If the given functions do not belong to the discrete spaces, then the vectors contain the coefficients of approximations $f_h \in S_h^1(\Gamma)$ and $g_h \in S_h^1(\Gamma)$, see the discussion at the end of Section 3.1. The matrices for $i \in \{0,1\}$ are given by

$$\begin{aligned}
V_{\kappa_i,h} \in \mathbb{C}^{N\times N}: &\quad V_{\kappa_i,h}[\ell,k] := \left\langle V_{\kappa_i}\varphi_k^0, \varphi_\ell^0 \right\rangle_\Gamma && k,\ell = 1,\dots,N;\\
K_{\kappa_i,h} \in \mathbb{C}^{N\times M}: &\quad K_{\kappa_i,h}[\ell,k] := \left\langle K_{\kappa_i}\varphi_k^1, \varphi_\ell^0 \right\rangle_\Gamma && k = 1,\dots,M;\ \ell = 1,\dots,N;\\
D_{\kappa_i,h} \in \mathbb{C}^{M\times M}: &\quad D_{\kappa_i,h}[\ell,k] := \left\langle D_{\kappa_i}\varphi_k^1, \varphi_\ell^1 \right\rangle_\Gamma && k,\ell = 1,\dots,M;\\
M_h \in \mathbb{R}^{N\times M}: &\quad M_h[\ell,k] := \left\langle \varphi_k^1, \varphi_\ell^0 \right\rangle_\Gamma && k = 1,\dots,M;\ \ell = 1,\dots,N.
\end{aligned}$$

With these matrix representations of the discrete boundary integral operators it is easy to write down the systems of linear equation we get for the remaining formulations. These are the Steklov–Poincaré operator formulation

$$\begin{pmatrix} V_{\kappa_1,h} & -(\frac{1}{2}M_h + K_{\kappa_1,h}) \\ \frac{1}{2}M_h^\top + K_{\kappa_1,h}^\top & D_{\kappa_1,h} + S_{\kappa_0,h}^{ext} \end{pmatrix} \begin{pmatrix} \underline{t}_1 \\ \underline{u}_1 \end{pmatrix} = \begin{pmatrix} \underline{0} \\ \underline{g} + S_{\kappa_0,h}^{ext}\underline{f} \end{pmatrix}$$

with the discretization of the Steklov–Poincaré operator as discussed in Section 3.3, the interior Steklov–Poincaré operator formulation

$$(S_{\kappa_1,h}^{int} + S_{\kappa_0,h}^{ext})\underline{u}_1 = \underline{g} + S_{\kappa_0,h}^{ext}\underline{f}$$

and the local multi trace formulation

$$\begin{pmatrix} V_{\kappa_1,h} & -K_{\kappa_1,h} & & -\frac{1}{2}M_h \\ K_{\kappa_1 h}^\top & D_{\kappa_1,h} & -\frac{1}{2}M_h^\top & \\ & \frac{1}{2}M_h & V_{\kappa_0,h} & -K_{\kappa_0,h} \\ \frac{1}{2}M_h^\top & & K_{\kappa_0,h}^\top & D_{\kappa_0,h} \end{pmatrix} \begin{pmatrix} \underline{t}_1 \\ \underline{u}_1 \\ \underline{t}_0 \\ \underline{u}_0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \underline{f} \\ \underline{g} \\ \underline{g} \\ \underline{f} \end{pmatrix}.$$

Remark 4.13. *In [19] convergence results were derived for Dirichlet and Neumann boundary value problems under some regularity assumptions for the boundary integral operators. Here we give a short summary what order of convergence can be expected by combining these result.*

For a globally quasi–uniform (3.3) mesh let $t_1 \in H^s_{pw}(\Gamma)$ *and* $u_1 \in H^\rho(\Gamma)$ *be the unique solutions of the model problem (4.1)–(4.5) with* $s \in [0,1]$ *and* $\rho \in [s+1,2]$. *Let* $(u_{1,h}, t_{1,h}) \in S^1_h(\Gamma) \times S^0_h(\Gamma)$ *be the unique discrete solution of either the Steklov–Poincaré operator formulation, the single trace formulation or the local multi trace formulation. If the boundary integral operators fulfil some regularity condition, see [19, Lemma 12.2] and [19, Theorem 12.8], then there hold the error estimates*

$$\begin{aligned} \|u_1 - u_{1,h}\|_{L_2(\Gamma)} &\leq ch^\rho \left\{ |u_1|_{H^\rho(\Gamma)} + |t_1|_{H^s(\Gamma)} \right\}, \\ \|t_1 - t_{1,h}\|_{L_2(\Gamma)} &\leq ch^s \left\{ |u_1|_{H^\rho(\Gamma)} + |t_1|_{H^s_{pw}(\Gamma)} \right\}. \end{aligned}$$

We have already introduced suitable preconditioning operators to apply the preconditioning strategy presented in Section 3.4. To do so the spaces have to satisfy the stability condition (3.7)

$$\sup_{0 \neq w_h \in \Pi_h} \frac{|\langle v_h, w_h \rangle_\Gamma|}{\|w_h\|_{H^{-1/2}(\Gamma)}} \geq c_1^M \|v_h\|_{H^{1/2}(\Gamma)} \qquad \forall v_h \in X_h$$

with discrete spaces $\Pi_h \subset H^{-1/2}(\Gamma_N)$ and $X_h \subset H^{1/2}(\Gamma_N)$. Unfortunately, this condition is not satisfied if we choose the discrete spaces as above. At the end of Section 3.4 we have presented two possible choices to solve this problem, the use of a dual mesh and the approximation of $H^{-1/2}(\Gamma)$ by piecewise linear, globally continuous functions.

To summarize we have shown that all four formulations have unique solutions, in the continuous as well as the discrete setting, and are compatible with operator preconditioning. For an application of these results we refer to Chapter 6 on numerical results.

5 Composite scatterer

In Chapter 4 we considered scattering at a homogeneous structure, that means the wave number κ was constant inside of Ω. This chapter discusses the first step towards a more general problem by allowing piecewise constant wave numbers. An equivalent formulation of this model is to consider a domain Ω that is composed of disjoint subdomains Ω_i such that κ is equal to a constant κ_i in each subdomain.

Section 5.1 presents the model problem as well as notations that are used throughout this chapter. In the Sections 5.2–5.5 we present and discuss the same formulations as in Chapter 4. Those are the single trace formulation [12, 21], the interior Steklov–Poincaré operator formulation [20, 22], the Steklov–Poincaré operator formulation [20] and the local multi trace formulation [5, 12]. Section 5.6 gives remarks at how to apply the boundary element method presented in Chapter 3.

5.1 Model problem

As a model problem we consider the case that Ω is composed of two subdomains. This model suffices to show the differences to the problem considered in Chapter 4 and keeps the notation as simple as possible. Before going into detail of the model problem, we establish the notations for this chapter. Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain that admits the decomposition

$$\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2, \quad \Omega_1 \cap \Omega_2 = \emptyset$$

with two Lipschitz domains Ω_1 and Ω_2 as shown in Figure 5.1. The exterior domain is defined as $\Omega_0 := \mathbb{R}^3 \setminus \overline{\Omega}$. The boundaries, the intersections thereof and the skeleton

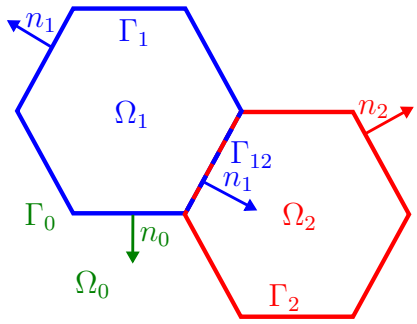


Figure 5.1: Sketch of the model geometry $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$.

boundary are denoted as

$$\begin{aligned}\Gamma_i &:= \partial\Omega_i && i = 0,1,2,\\ \Gamma_{ij} &:= \Gamma_i \cap \Gamma_j && i,j = 0,1,2; i \neq j,\\ \Gamma_S &:= \bigcup_{i=0}^{2} \Gamma_i.\end{aligned}$$

Note that there is a symmetry regarding the interfaces between two domains, that is $\Gamma_{ij} = \Gamma_{ji}$. Next we consider a triple of functions $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_0) \in H^1(\Delta, \Omega_1) \times H^1(\Delta, \Omega_2) \times H^1(\Delta, \Omega_0)$ and define

$$\begin{aligned}u_i(x) &:= \gamma_0^{int}\tilde{u}_i(x) \qquad t_i(x) := \gamma_1^{int}\tilde{u}_i(x) && x \in \Gamma_i,\ i \in \{1,2\},\\ u_0(x) &:= \gamma_0^{ext}\tilde{u}_0(x) \qquad t_0(x) := \gamma_1^{ext}\tilde{u}_0(x) && x \in \Gamma_0,\\ u_{ij}(x) &:= u_{i|\Gamma_{ij}}(x) && x \in \Gamma_{ij}, i,j \in \{0,1,2\},\ i \neq j,\\ t_{ij}(x) &:= t_{i|\Gamma_{ij}}(x) && x \in \Gamma_{ij}, i,j \in \{0,1,2\},\ i \neq j.\end{aligned}$$

With these notations we are able to describe the Helmholtz transmission problem. As in Chapter 4 we assume that all wave numbers κ_0, κ_1 and κ_2 are real and non negative. Let $(f, g) \in H^{1/2}(\Gamma_0) \times H^{-1/2}(\Gamma_0)$ be given jumps conditions on the exterior boundary Γ_0. On the boundary between the two scatterers we enforce homogeneous transmission conditions. Then the problem is to find $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_0) \in H^1(\Omega_1) \times H^1(\Omega_2) \times H^1_{\mathrm{loc}}(\Omega_0)$ as the solution of

$$\begin{aligned}-\Delta\tilde{u}_i(x) - \kappa_i^2\tilde{u}_i(x) &= 0 && x \in \Omega_i,\ i = 0,1,2, && (5.1)\\ u_{12}(x) - u_{21}(x) &= 0 && x \in \Gamma_{12}, && (5.2)\\ u_{10}(x) - u_{01}(x) &= f(x) && x \in \Gamma_{10}, && (5.3)\\ u_{20}(x) - u_{02}(x) &= f(x) && x \in \Gamma_{20}, && (5.4)\\ t_{12}(x) + t_{21}(x) &= 0 && x \in \Gamma_{12}, && (5.5)\\ t_{10}(x) - t_{01}(x) &= g(x) && x \in \Gamma_{10}, && (5.6)\\ t_{20}(x) - t_{02}(x) &= g(x) && x \in \Gamma_{20}. && (5.7)\end{aligned}$$

The plus sign in (5.5) compensates for the fact that the two interior Neumann traces point in opposite directions. To ensure uniqueness of the solution, $\tilde{u}_0$ has to satisfy the Sommerfeld radiation condition

$$\lim_{r\to\infty} \int_{|x|=r} \left| \frac{\partial}{\partial n_x}\tilde{u}_0(x) - i\kappa_0\tilde{u}_0(x) \right|^2 ds_x = 0. \tag{5.8}$$

With this it follows from [21, Lemma 2.2] that the model problem is uniquely solvable.

5.2 Single trace formulation

In this section we present a more general case of the single trace formulation [21] from Section 4.2. Recall that in Section 4.2 we expressed the exterior traces by using the

interior traces and the transmission conditions. This way, we only had to consider one pair of Dirichlet and Neumann traces on the boundary, thus the name single trace formulation.

To apply the same idea for the composite scatterer the trace spaces have to be considered not on a closed surface but on the skeleton boundary Γ_S, those spaces were presented in Section 1.5. The Dirichlet trace belongs to $H^{1/2}(\Gamma_S)$, the space of all functions whose restriction to Γ_i is in $H^{1/2}(\Gamma_i)$ for $i \in \{0,1,2\}$. Since we can only keep one of the two traces, we have to dismiss one on each interface. In this thesis we keep the traces coming form Ω_1 on Γ_1 and the traces coming from Ω_2 on Γ_{02}. The space $H^{-1/2}(\Gamma_S)$ for Neumann traces is a bit more involved since we have to take care of the orientation on Γ_{12} but follows the same ideas, for details see Section 1.5.

Let $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_0) \in H^1(\Omega_1) \times H^1(\Omega_2) \times H^1_{\mathrm{loc}}(\Omega_0)$ be a weak solution for the model problem (5.1)–(5.8). The transmission conditions (5.2)–(5.7) tell us that there exists a pair $(u,t) \in H^{1/2}(\Gamma_S) \times H^{-1/2}(\Gamma_S)$ such that

$$\begin{aligned} u_{|\Gamma_i} &= u_i, & \mathbb{L}_i t &= t_i, & &\text{on } \Gamma_i, i \in \{1,2\},\\ u_{|\Gamma_0} &= u_0 + f, & \mathbb{L}_0 t &= t_0 + g, & &\text{on } \Gamma_0. \end{aligned}$$

where the operator $\mathbb{L}_i$ is the restriction of t to the local Neumann trace and takes care of the orientation, see Section 1.5. The goal is to use these relations to replace the traces of $\tilde{u}$ with (u,t). Since the triple $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_0)$ solves the local partial differential equations, its Cauchy data can be plugged into the Calderón projections (2.2) and (2.5). Let $i \in \{1,2\}$, then this reads

$$\begin{pmatrix} V_{\kappa_i} & -\frac{1}{2}I - K_{\kappa_i} \\ -\frac{1}{2}I + K'_{\kappa_i} & D_{\kappa_i} \end{pmatrix} \begin{pmatrix} t_i \\ u_i \end{pmatrix} = \underline{0}$$

while for the exterior domain we get

$$\begin{pmatrix} V_{\kappa_0} & \frac{1}{2}I - K_{\kappa_0} \\ \frac{1}{2}I + K'_{\kappa_0} & D_{\kappa_0} \end{pmatrix} \begin{pmatrix} t_0 \\ u_0 \end{pmatrix} = \underline{0}.$$

We can rewrite this in terms of $(u,t) \in H^{1/2}(\Gamma_S) \times H^{-1/2}(\Gamma_S)$ as

$$\begin{aligned} \begin{pmatrix} V_{\kappa_0} & \frac{1}{2}I - K_{\kappa_0} \\ \frac{1}{2}I + K'_{\kappa_0} & D_{\kappa_0} \end{pmatrix} \begin{pmatrix} \mathbb{L}_0 t \\ u_{|\Gamma_0} \end{pmatrix} &= \begin{pmatrix} V_{\kappa_0} & \frac{1}{2}I - K_{\kappa_0} \\ \frac{1}{2}I + K'_{\kappa_0} & D_{\kappa_0} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix},\\ \begin{pmatrix} V_{\kappa_1} & -\frac{1}{2}I - K_{\kappa_1} \\ -\frac{1}{2}I + K'_{\kappa_1} & D_{\kappa_1} \end{pmatrix} \begin{pmatrix} \mathbb{L}_1 t \\ u_{|\Gamma_1} \end{pmatrix} &= \underline{0},\\ \begin{pmatrix} V_{\kappa_2} & -\frac{1}{2}I - K_{\kappa_2} \\ -\frac{1}{2}I + K'_{\kappa_2} & D_{\kappa_2} \end{pmatrix} \begin{pmatrix} \mathbb{L}_2 t \\ u_{|\Gamma_2} \end{pmatrix} &= \underline{0}. \end{aligned}$$

Each of these equations is satisfied in the sense of $H^{1/2}(\Gamma_i)$ or $H^{-1/2}(\Gamma_i)$ on the respective boundary. Let $(v,\tau) \in H^{1/2}(\Gamma_S) \times H^{-1/2}(\Gamma_S)$, then we can test those three

systems with the appropriate restriction of (v, τ). First, we only consider the identity operators, here summing up yields

$$\frac{1}{2}\left\langle \begin{pmatrix} u_{|\Gamma_0} \\ \mathbb{L}_0 t \end{pmatrix}, \begin{pmatrix} \mathbb{L}_0 \tau \\ v_{|\Gamma_0} \end{pmatrix} \right\rangle_{\Gamma_0} - \frac{1}{2}\left\langle \begin{pmatrix} u_{|\Gamma_1} \\ \mathbb{L}_1 t \end{pmatrix}, \begin{pmatrix} \mathbb{L}_1 \tau \\ v_{|\Gamma_1} \end{pmatrix} \right\rangle_{\Gamma_1} - \frac{1}{2}\left\langle \begin{pmatrix} u_{|\Gamma_2} \\ \mathbb{L}_2 t \end{pmatrix}, \begin{pmatrix} \mathbb{L}_2 \tau \\ v_{|\Gamma_2} \end{pmatrix} \right\rangle_{\Gamma_2} = 0.$$

To see that this equality holds we have to split the duality pairings up and use the definition of $H^{-1/2}(\Gamma_S)$. Of particular importance is the relation $\mathbb{L}_2 t = -\mathbb{L}_1 t$ on Γ_{12}. If we sum up the three Calderón projections and use this equality we get the single trace formulation to find $(u, t) \in H^{1/2}(\Gamma_S) \times H^{-1/2}(\Gamma_S)$ such that

$$\sum_{i=0}^{2} \left\langle \begin{pmatrix} V_{\kappa_i} & -K_{\kappa_i} \\ K'_{\kappa_i} & D_{\kappa_i} \end{pmatrix} \begin{pmatrix} \mathbb{L}_i t \\ u_{|\Gamma_i} \end{pmatrix}, \begin{pmatrix} \mathbb{L}_i \tau \\ v_{|\Gamma_i} \end{pmatrix} \right\rangle_{\Gamma_i} = \left\langle \begin{pmatrix} V_{\kappa_0} & \frac{1}{2}I - K_{\kappa_0} \\ \frac{1}{2}I + K'_{\kappa_0} & D_{\kappa_0} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} \mathbb{L}_0 \tau \\ v_{|\Gamma_0} \end{pmatrix} \right\rangle_{\Gamma_0} \tag{5.9}$$

holds for all $(v, \tau) \in H^{1/2}(\Gamma_S) \times H^{-1/2}(\Gamma_S)$. Another way to derive this formulation would be to start form the local multi trace formulation (5.18) and use functions in $H^{1/2}(\Gamma_S) \times H^{-1/2}(\Gamma_S)$ as test and trial functions. Then all off diagonal blocks cancel one another out, for details see [12].

The proof of injectivity is more complicated than for the case of a single homogeneous scatterer. We can no longer interchange the wave numbers to create a new Helmholtz transmission problem in $\mathbb{R}^3$ since the complementary domains overlap if we consider more than two domains. Hence we need a different tool to show that the solutions of the local partial differential equations are equal to zero which is given by Rellich's lemma.

Lemma 5.1 (Rellich)**.** *[6, Lemma 2.11] Let $\Omega \subset \mathbb{R}^3$ be a bounded open set. If a solution $u \in H^1_{loc}(\Omega^c)$ of the exterior Helmholtz equation further satisfies*

$$\lim_{r\to\infty} \int_{|x|=r} |u(x)|^2 \, dx = 0$$

then it holds $u = 0$ in Ω^c.

With this we can show injectivity for the single trace formulation and Steklov–Poincaré operator formulation.

Lemma 5.2. *[5, Lemma 2.28] The single trace operator induced by (5.9) is injective.*

Proof. Let $(u, t) \in H^{1/2}(\Gamma_S) \times H^{-1/2}(\Gamma_S)$ be a solution of the homogeneous problem

$$\sum_{i=0}^{2} \left\langle \begin{pmatrix} V_{\kappa_i} & -K_{\kappa_i} \\ K'_{\kappa_i} & D_{\kappa_i} \end{pmatrix} \begin{pmatrix} \mathbb{L}_i t \\ u_{|\Gamma_i} \end{pmatrix}, \begin{pmatrix} \mathbb{L}_i \tau \\ v_{|\Gamma_i} \end{pmatrix} \right\rangle_{\Gamma_i} = 0 \qquad \forall (v, \tau) \in H^{1/2}(\Gamma_S) \times H^{-1/2}(\Gamma_S),$$

then we have to show that u and t are equal to zero.

First we define solutions of the local partial differential equations that fulfil the Sommerfeld radiation condition by

$$\begin{aligned} U_i(x) &:= (\widetilde{V}_{\kappa_i} \mathbb{L}_i t)(x) - (W_{\kappa_i} u_{|\Gamma_i})(x) \qquad \text{for } x \in \Omega_i, i \in \{1, 2\}, \\ U_0(x) &:= -(\widetilde{V}_{\kappa_0} \mathbb{L}_0 t)(x) + (W_{\kappa_0} u_{|\Gamma_0})(x) \quad \text{for } x \in \Omega_0. \end{aligned}$$

Taking the Dirichlet and Neumann traces for one of the interior solution yields on Γ_i

$$\gamma_0^{int} U_i = (V_{\kappa_i} \mathbb{L}_i t) + \left(\frac{1}{2} I - K_{\kappa_i}\right) u_{|\Gamma_i},$$
$$\gamma_1^{int} U_i = \left(\frac{1}{2} I + K'_{\kappa_i}\right) \mathbb{L}_i t + (D_{\kappa_i} u_{|\Gamma_i}).$$

In the same way we get on Γ_0

$$\gamma_0^{ext} U_0 = -(V_{\kappa_0} \mathbb{L}_0 t) + \left(\frac{1}{2} I + K_{\kappa_0}\right) u_{|\Gamma_0},$$
$$\gamma_1^{ext} U_0 = \left(\frac{1}{2} I - K'_{\kappa_0}\right) \mathbb{L}_0 t - (D_{\kappa_0} u_{|\Gamma_0}).$$

As always we want to show that these three function satisfy homogeneous transmission conditions. For this let $(v, \tau) \in H^{1/2}(\Gamma_S) \times H^{-1/2}(\Gamma_S)$ be test functions that vanish on all interfaces except Γ_{0i} for $i \in \{1, 2\}$. Then we get

$$\left\langle \gamma_0^{int} U_i - \gamma_0^{ext} U_0, \mathbb{L}_0 \tau \right\rangle_{\Gamma_{0i}} = \left\langle V_{\kappa_i} \mathbb{L}_i t - K_{\kappa_i} u_{|\Gamma_i} + V_{\kappa_0} \mathbb{L}_0 t - K_{\kappa_0} u_{|\Gamma_0}, \mathbb{L}_0 \tau \right\rangle_{\Gamma_{0i}} = 0.$$

For the first equality we used that, since $u \in H^{1/2}(\Gamma_S)$, the identities cancel one another out. Note that by definition on Γ_{0i} it holds $\mathbb{L}_i \tau = \mathbb{L}_0 \tau$. Thus, we can use that (u, t) solves the homogeneous system (5.9) which gives us the second equality. Similar arguments for the Neumann trace yield

$$\left\langle \gamma_1^{int} U_i - \gamma_1^{ext} U_0, v_{|\Gamma_0} \right\rangle_{\Gamma_{0i}} = \left\langle K'_{\kappa_i} \mathbb{L}_i t + D_{\kappa_i} u_{|\Gamma_i} + K'_{\kappa_0} \mathbb{L}_0 t + D_{\kappa_0} u_{\Gamma_0}, v_{\Gamma_0} \right\rangle_{\Gamma_{0i}} = 0.$$

Next we consider the remaining interface Γ_{12} with appropriate test functions. Keep in mind that on Γ_{12} it holds $\mathbb{L}_1 \tau = -\mathbb{L}_2 \tau$ as well as $v_{|\Gamma_1} = v_{|\Gamma_2}$ and hence

$$\begin{aligned}\left\langle \gamma_0^{int} U_1 - \gamma_0^{int} U_2, \mathbb{L}_1 \tau \right\rangle_{\Gamma_{12}} &= \left\langle V_{\kappa_1} \mathbb{L}_1 t - K_{\kappa_1} u_{|\Gamma_1}, \mathbb{L}_1 \tau \right\rangle_{\Gamma_{12}} + \left\langle V_{\kappa_2} \mathbb{L}_2 t - K_{\kappa_2} u_{|\Gamma_2}, \mathbb{L}_2 \tau \right\rangle_{\Gamma_{12}} \\ &= 0, \\ \left\langle \gamma_1^{int} U_1 + \gamma_1^{int} U_2, v_{|\Gamma_1} \right\rangle_{\Gamma_{12}} &= \left\langle K'_{\kappa_1} \mathbb{L}_1 t + D_{\kappa_1} u_{\Gamma_1} + K'_{\kappa_2} \mathbb{L}_2 t + D_{\kappa_2} u_{|\Gamma_2}, v_{|\Gamma_1} \right\rangle_{\Gamma_{12}} = 0.\end{aligned}$$

We see that (U_0, U_1, U_2) solves the model problem (5.1)–(5.8) for homogeneous transmission data. Therefore, all three functions must be zero and, by taking the Cauchy traces, we can conclude for $i \in \{1, 2\}$

$$0 = -(V_{\kappa_0} \mathbb{L}_0 t) + \left(\frac{1}{2} I + K_{\kappa_0}\right) u_{|\Gamma_0}, \qquad 0 = (V_{\kappa_i} \mathbb{L}_i t) + \left(\frac{1}{2} I - K_{\kappa_i}\right) u_{|\Gamma_i}, \tag{5.10a}$$
$$0 = \left(\frac{1}{2} I - K'_{\kappa_0}\right) \mathbb{L}_0 t - (D_{\kappa_0} u_{|\Gamma_0}), \qquad 0 = \left(\frac{1}{2} I + K'_{\kappa_i}\right) \mathbb{L}_i t + (D_{\kappa_i} u_{|\Gamma_i}). \tag{5.10b}$$

For the second part of the proof, we define solutions of the Helmholtz equation for κ_i in Ω_i^c for $i \in \{0, 1, 2\}$. Similar to before, these functions are defined as

$$\hat{U}_i(x) := -(\widetilde{V}_{\kappa_i} \mathbb{L}_i t)(x) + (W_{\kappa_i} u_{|\Gamma_i})(x) \quad \text{for } x \in \Omega_i^c, i \in \{1, 2\},$$
$$\hat{U}_0(x) := (\widetilde{V}_{\kappa_0} \mathbb{L}_0 t)(x) - (W_{\kappa_0} u_{|\Gamma_0})(x) \quad \text{for } x \in \Omega_0^c.$$

With the same reasoning as in the first step we can conduct

$$\gamma_0^{ext}\hat{U}_1 = \gamma_0^{int}\hat{U}_0 \quad \text{and} \quad \gamma_1^{ext}\hat{U}_1 = \gamma_1^{int}\hat{U}_0 \quad \text{on } \Gamma_{10}, \tag{5.11a}$$

$$\gamma_0^{ext}\hat{U}_2 = \gamma_0^{int}\hat{U}_0 \quad \text{and} \quad \gamma_1^{ext}\hat{U}_2 = \gamma_1^{int}\hat{U}_0 \quad \text{on } \Gamma_{20}, \tag{5.11b}$$

$$\gamma_0^{ext}\hat{U}_1 = \gamma_0^{ext}\hat{U}_2 \quad \text{and} \quad \gamma_1^{ext}\hat{U}_1 = -\gamma_1^{ext}\hat{U}_2 \quad \text{on } \Gamma_{12}. \tag{5.11c}$$

Our goal is to apply Lemma 5.1 (Rellich) for $\hat{U}_1$ and $\hat{U}_2$ so we have to show that their L_2 norm taken on the surface of a sphere vanishes as the radius goes to infinity. Let $r > 0$ large enough so that $\Omega \subset B_r(0)$, then we consider Greens first formula [15, Section 2.7] in $\Omega_i^c \cap B_r(0)$. For Ω_i^c with $i \in \{1, 2\}$ and Ω_0^c this gives us

$$\begin{aligned}\int_{|x|=r} \gamma_0^{ext}\hat{U}_i \overline{\frac{\partial}{\partial n_r}\hat{U}_i} ds_x &= \int_{B_r(0)\cap\Omega_i^c} \left|\nabla\hat{U}_i\right| - \kappa_i^2 \left|\hat{U}_i\right| dx - \int_{\Gamma_i} \gamma_0^{ext}\hat{U}_i \overline{\gamma_1^{ext}\hat{U}_i} ds_x, \\ 0 &= \int_{\Omega_0^c} \left|\nabla\hat{U}_0\right| - \kappa_0^2 \left|\hat{U}_0\right| dx - \int_{\Gamma_0} \gamma_0^{int}\hat{U}_0 \overline{\gamma_1^{int}\hat{U}_0} ds_x.\end{aligned}$$

Since by definition all considered functions solve the local partial differential equations, the integrals in the domain are equal to zero. Summing up and taking the imaginary part yields

$$-\Im\left(\sum_{i=1}^{2}\int_{|x|=r} \gamma_0^{ext}\hat{U}_i \overline{\frac{\partial}{\partial n_r}\hat{U}_i} ds_x\right) = \Im\left(-\int_{\Gamma_0} \gamma_0^{int}\hat{U}_0 \overline{\gamma_1^{int}\hat{U}_0} ds_x + \sum_{i=1}^{2}\int_{\Gamma_i} \gamma_0^{ext}\hat{U}_i \overline{\gamma_1^{ext}\hat{U}_i} ds_x\right).$$

The right hand side of this equation vanishes due to the relations (5.11) we derived earlier. Hence, we finally get

$$-\Im\left(\sum_{i=1}^{2}\int_{|x|=r} \gamma_0^{ext}\hat{U}_i \overline{\frac{\partial}{\partial n_r}\hat{U}_i} ds_x\right) = 0.$$

Recall that the exterior solutions $\hat{U}_1$ and $\hat{U}_2$ satisfy the Sommerfeld radiation condition (5.8), summing those up gives us

$$\begin{aligned}0 &= \lim_{r\to\infty} \sum_{j=1}^{2}\int_{|x|=r} \left|\frac{\partial}{\partial n_x}\hat{U}_j - i\kappa_j\hat{U}_j\right|^2 ds_x \\ &= \lim_{r\to\infty} \sum_{j=1}^{2}\left[\int_{|x|=r} \left|\frac{\partial}{\partial n_x}\hat{U}_j\right| ds_x + \kappa_j^2 \int_{|x|=r} \left|\hat{U}_j\right| ds_x - 2\kappa_j \Im\left(\int_{|x|=r} \gamma_0^{ext}\hat{U}_j \overline{\gamma_1^{ext}\hat{U}_j}\right)\right].\end{aligned}$$

We have already shown that the last integral vanishes. Since all other integrals are positive, we can conclude

$$\lim_{r\to\infty} \int_{|x|=r} \left|\hat{U}_i\right|^2 ds_x = 0 \qquad \text{for } i = 1, 2$$

which allows us to apply Lemma 5.1 for $\hat{U}_1$ and $\hat{U}_2$. This gives us $(\hat{U}_1, \hat{U}_2) = (0, 0)$ and from the relations (5.11) it follows $\hat{U}_0 = 0$. So the Dirichlet and Neumann traces

of $(\hat{U}_0, \hat{U}_1, \hat{U}_2)$ are zero as well, on the other hand they have the representation

$$0 = (V_{\kappa_0}\mathbb{L}_0 t) + \left(\frac{1}{2}I - K_{\kappa_0}\right) u_{|\Gamma_0}, \qquad 0 = -(V_{\kappa_i}\mathbb{L}_i t) + \left(\frac{1}{2}I + K_{\kappa_i}\right) u_{|\Gamma_i},$$
$$0 = \left(\frac{1}{2}I + K'_{\kappa_0}\right) \mathbb{L}_0 t + (D_{\kappa_0} u_{|\Gamma_0}), \qquad 0 = \left(\frac{1}{2}I - K'_{\kappa_i}\right) \mathbb{L}_i t - (D_{\kappa_i} u_{|\Gamma_i}).$$

This in combination with the result of the first part (5.10) finally gives us $(u, t) = (0, 0)$ and thus the injectivity of the single trace operator.

□

Lemma 5.3. *[5, Theorem 2.26] The single trace operator induced by (5.9) is coercive.*

Proof. First we define a compact operator for each subdomain $i \in \{0, 1, 2\}$ by

$$T_i = \begin{pmatrix} V - V_{\kappa_i} & -K + K_{\kappa_i} \\ K' - K'_{\kappa_i} & \widetilde{D} - D_{\kappa_i} \end{pmatrix},$$

the compactness follows from Lemma 2.6. Let the diagonal blocks of the single trace operator be denoted by A_i, then the same procedure as in the proof of Lemma 4.2 gives us for $(u, t) \in H^{1/2}(\Gamma_S) \times H^{-1/2}(\Gamma_S)$

$$\begin{aligned} \sum_{i=0}^{2} \left\langle (A_i + T_i) \begin{pmatrix} \mathbb{L}_i t \\ u_{|\Gamma_i} \end{pmatrix}, \begin{pmatrix} \mathbb{L}_i t \\ u_{|\Gamma_i} \end{pmatrix} \right\rangle_{\Gamma_i} &\geq c \sum_{i=0}^{2} \left(\left\| u_{|\Gamma_i} \right\|^2_{H^{1/2}(\Gamma_i)} + \|\mathbb{L}_i t\|^2_{H^{-1/2}(\Gamma_i)} \right) \\ &\geq \tilde{c}(\|u\|^2_{H^{1/2}(\Gamma_S)} + \|t\|^2_{H^{-1/2}(\Gamma_S)}). \end{aligned}$$

The last thing we have to show is the compactness of the combined operator. This follows from the compactness of T_i, the boundedness of the localization operators and the fact that the space of compact operator is a two–sided operator ideal in the space of bounded operators [3]. □

Lemmata 5.2 and 5.3 provide everything we need to apply Lemma 1.2 (the Fredholm alternative) which gives us unique solvability.

Theorem 5.4. *The single trace formulation (5.9) has a unique solution for all given right hand sides $(f, g) \in H^{1/2}(\Gamma_0) \times H^{-1/2}(\Gamma_0)$ and wave numbers $\kappa_i \in \mathbb{R}_+$, $i \in \{0, 1, 2\}$.*

In Section 1.5 we discussed that it is not obvious if and how the duality pairing between $H^{1/2}(\Gamma_S)$ and $H^{-1/2}(\Gamma_S)$ can be expressed for arbitrary functions. Hence we are not able to compute the mass matrix which prevents us from applying operator preconditioning as presented in Section 3.4. Therefore we do not derive an operator B_{STF} as we did for the homogeneous scatterer in Section 4.2.

5.3 Interior Steklov–Poincaré operator formulation

Here we first present the interior Steklov–Poincaré operator formulation [22] in the case that none of the wave numbers κ_i corresponds to a eigenvalue of the Laplace operator. Then, as shown in Section 2.4, all Steklov–Poincaré operators, interior as well as exterior, are well defined which facilitates the formulation considerably. As for the single trace formulation we make use of skeleton trace spaces and refer to Section 1.5 for details.

Let $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_0) \in H^1(\Omega_1) \times H^1(\Omega_2) \times H^1_{\mathrm{loc}}(\Omega_0)$ be a weak solution of the model problem (5.1)–(5.8). The local problem, for example for Ω_1, reads

$$\begin{aligned} -\Delta\tilde{u}_1 - \kappa_1^2\tilde{u}_1 &= 0 && \text{in } \Omega_1,\\ u_1 &= u_2 && \text{on } \Gamma_{12},\\ u_1 &= u_0 + f && \text{on } \Gamma_{10},\\ t_1 &= -t_2 && \text{on } \Gamma_{12},\\ t_1 &= t_0 + g && \text{on } \Gamma_{10}. \end{aligned}$$

Now we express the normal derivatives t by means of the Dirichlet datum and the Dirichlet to Neumann mappings, i.e $S^{int}_{\kappa_i} u_i = t_i$ and $-S^{ext}_{\kappa_0} u_0 = t_0$. Then by definition of the Steklov–Poincaré operator the traces belong to a solution of the local partial differential equation and it can thus be omitted. At this point we need do find $u_i \in H^{1/2}(\Gamma_i), i \in \{0,1,2\}$ that solve on Γ_1

$$\begin{aligned} u_1 &= u_2 && \text{on } \Gamma_{12},\\ u_1 &= u_0 + f && \text{on } \Gamma_{10},\\ S^{int}_1 u_1 &= -S^{int}_2 u_2 && \text{on } \Gamma_{12},\\ S^{int}_1 u_1 &= -S^{ext}_0 u_0 + g && \text{on } \Gamma_{10}. \end{aligned}$$

Next we enforce continuity across the boundaries, that is find $u \in H^{1/2}(\Gamma_S)$ instead of local traces u_i. For the interior boundary Γ_{12}, this fits the given transmission conditions. On the boundary Γ_0 we have to decide if the restriction of u should be the interior or exterior trace. The other can be expressed by means of the given transmission conditions. We consider the restriction of u to Γ_0 to be the interior Dirichlet trace. The original Dirichlet datum can then be obtained by

$$u_1 = u_{|\Gamma_1}, \quad u_2 = u_{|\Gamma_2} \quad \text{and} \quad u_0 = u_{|\Gamma_0} - f.$$

With this the problem is to find $u \in H^{1/2}(\Gamma_S)$ such that

$$\begin{aligned} S^{int}_1 u_{|\Gamma_1} + S^{int}_2 u_{|\Gamma_2} &= 0 && \text{on } \Gamma_{12},\\ S^{int}_1 u_{|\Gamma_1} + S^{ext}_0 u_{|\Gamma_0} &= S^{ext}_0 f + g && \text{on } \Gamma_{10},\\ S^{int}_1 u_{|\Gamma_2} + S^{ext}_0 u_{|\Gamma_0} &= S^{ext}_0 f + g && \text{on } \Gamma_{20}. \end{aligned}$$

By multiplying this with a test function $v \in H^{1/2}(\Gamma_S)$, integrating and summing up, we get the interior Steklov–Poincaré operator variational formulation. Find $u \in H^{1/2}(\Gamma_S)$ as the solution of

$$\left\langle S_0^{ext} u_{|\Gamma_0}, v_{|\Gamma_0} \right\rangle_{\Gamma_0} + \sum_{i=1}^{2} \left\langle S_i^{int} u_{|\Gamma_i}, v_{|\Gamma_i} \right\rangle_{\Gamma_i} = \left\langle S_0^{ext} f + g, v_{|\Gamma_0} \right\rangle_{\Gamma_0} \quad \forall v \in H^{1/2}(\Gamma_S). \tag{5.12}$$

Under the assumption that the interior wave numbers do not belong to eigenvalues of the Laplace operator, this formulation is uniquely solvable as is stated in the following theorem.

Theorem 5.5. *[22, Section 7.2] Let $(\kappa_0, \kappa_1, \kappa_2) \in \mathbb{R}^3_+$ be given wave numbers such that neither κ_1^2 nor κ_2^2 are Dirichlet or Neumann eigenvalues. Then there exists a unique solution $u \in H^{1/2}(\Gamma_S)$ of the interior Steklov–Poincaré formulation (5.12) for any $(f, g) \in H^{1/2}(\Gamma_0) \times H^{-1/2}(\Gamma_0)$.*

Proof. In order to apply Lemma 1.2, we have to show injectivity and coercivity of the operator induced by (5.12). Coercivity is a direct consequence of the assumption concerning the wave numbers. Lemma 2.12 and Lemma 2.9 tell us all three appearing Dirichlet to Neumann mappings are coercive with compact operators T_{S_i}, $i \in \{0, 1, 2\}$. With these compact operators it holds for any $v \in H^{1/2}(\Gamma_S)$

$$\begin{aligned}
\Re \left\langle (S_0^{ext} + T_{S_0}) v_{|\Gamma_0}, v_{|\Gamma_0} \right\rangle_{\Gamma_0} + \sum_{i=1}^{2} \Re \left\langle (S_i^{int} + T_{S_i}) v_{|\Gamma_i}, v_{|\Gamma_i} \right\rangle_{\Gamma_i} & \\
\geq & c_0 \left\| v_{|\Gamma_0} \right\|^2_{H^{1/2}(\Gamma_0)} + \sum_{i=1}^{2} c_i \left\| v_{|\Gamma_i} \right\|^2_{H^{1/2}(\Gamma_i)} \\
\geq & c \|v\|^2_{H^{1/2}(\Gamma_S)}.
\end{aligned}$$

This concludes the proof of coercivity.

To show injectivity let $u \in H^{1/2}(\Gamma_S)$ be a solution of the homogeneous problem (5.12), this is equivalent to

$$\begin{aligned}
S_1^{int} u_{|\Gamma_1} + S_2^{int} u_{|\Gamma_2} &= 0 \quad \text{on } \Gamma_{12}, \\
S_1^{int} u_{|\Gamma_1} + S_0^{ext} u_{|\Gamma_0} &= 0 \quad \text{on } \Gamma_{10}, \\
S_2^{int} u_{|\Gamma_2} + S_0^{ext} u_{|\Gamma_0} &= 0 \quad \text{on } \Gamma_{20},
\end{aligned}$$

we now have to show that $u = 0$. We start by defining Neumann traces for each domain

$$t_0 := -S_{\kappa_0}^{ext} u_{|\Gamma_0}, \qquad t_1 := S_{\kappa_1}^{int} u_{|\Gamma_1}, \qquad t_2 := S_{\kappa_2}^{int} u_{|\Gamma_2}.$$

Since u solves the homogeneous problem (5.12) we immediately get the relations

$$t_0 = t_1 \quad \text{on } \Gamma_{10}, \qquad t_0 = t_2 \quad \text{on } \Gamma_{20}, \qquad t_1 = -t_2 \quad \text{on } \Gamma_{12}.$$

With those traces solutions of the local Helmholtz equations are given by

$$\begin{aligned} U_0 &:= -(\widetilde{V}_{\kappa_0} t_0) + (W_{\kappa_0} u_{|\Gamma_0}) && \text{in } \Omega_0, \\ U_1 &:= (\widetilde{V}_{\kappa_1} t_1) - (W_{\kappa_1} u_{|\Gamma_1}) && \text{in } \Omega_1, \\ U_2 &:= (\widetilde{V}_{\kappa_2} t_2) - (W_{\kappa_2} u_{|\Gamma_2}) && \text{in } \Omega_2. \end{aligned}$$

We see that the triple (U_1, U_2, U_0) solves the model problem (5.1)–(5.8) for homogeneous transmission conditions, for more details see the first part of the proof of Lemma 5.2. Since the model problem is uniquely solvable, it follows $U_i = 0$ in Ω_i for $i \in \{0, 1, 2\}$. By applying the Dirichlet trace operator we get $u = 0$ on Γ_S and hence injectivity.

This provides us with everything we need to apply Lemma 1.2 which concludes the proof. □

5.4 Steklov–Poincaré operator formulation

In this section we present the Steklov–Poincaré operator formulation [20] without restrictions to the wave numbers $\kappa_i \in \mathbb{R}_+$. Then it follows from Section 2.4 that only the exterior Steklov–Poincaré operator is well defined, coercive and injective.

As for the single trace formulation we use the skeleton trace spaces from Section 1.5 and restrict our considerations to one Dirichlet and Neumann trace on each interface. This means for a solution $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_0) \in H^1(\Omega_1) \times H^1(\Omega_2) \times H^1_{\mathrm{loc}}(\Omega_0)$ of (5.1)–(5.8) we consider $(u, t) \in H^{1/2}(\Gamma_S) \times H^{-1/2}(\Gamma_S)$ satisfying

$$\begin{aligned} u_{|\Gamma_i} &= u_i, & \mathbb{L}_i t &= t_i, & i &\in \{1, 2\}, \\ u_{|\Gamma_0} &= u_0 + f, & \mathbb{L}_0 t &= t_0 + g. \end{aligned}$$

The transmission conditions (5.2)–(5.7) ensure that such functions exist. Let $i \in \{1, 2\}$, then the Neumann transmission condition on the interface Γ_{i0} can be rewritten as

$$\begin{aligned} g &= t_i - t_0 \\ &= \left(\frac{1}{2} I + K'_{\kappa_i}\right) t_i + D_{\kappa_i} u_i + S^{ext}_{\kappa_0} u_0 \\ &= \left(\frac{1}{2} I + K'_{\kappa_i}\right) \mathbb{L}_i t + D_{\kappa_i} u_{|\Gamma_i} + S^{ext}_{\kappa_0} u_{|\Gamma_0} - S^{ext}_{\kappa_0} f. \end{aligned}$$

While the first equality is the given transmission condition (5.6)–(5.7), the second equality follows from the interior Calderón projection (2.2) on Γ_i and the definition of the exterior Steklov–Poincaré operator. For the last step we replaced the local traces (u_i, t_i) by the skeleton traces using the relations described earlier. This holds true for the interfaces Γ_{10} and Γ_{20}, now consider the remaining boundary part Γ_{12}. The second line of the interior Calderón projection allows us to reformulate the Neumann transmission condition as

$$\left(\frac{1}{2} I + K'_{\kappa_1}\right) \mathbb{L}_1 t + D_{\kappa_1} u_{|\Gamma_1} = \mathbb{L}_1 t = -\mathbb{L}_2 t = -\left(\frac{1}{2} I + K'_{\kappa_2}\right) \mathbb{L}_2 t - D_{\kappa_2} u_{|\Gamma_2}.$$

Testing these three equations with the appropriate restriction of $v \in H^{1/2}(\Gamma_S)$ yields

$$\left\langle S_{\kappa_0}^{ext} u_{|\Gamma_0}, v_{|\Gamma_0} \right\rangle_{\Gamma_0} + \sum_{i=1}^{2} \left\langle \left(\frac{1}{2} I + K'_{\kappa_i} \right) \mathbb{L}_i t, v_{|\Gamma_i} \right\rangle_{\Gamma_i} = \left\langle g + S_{\kappa_0}^{ext} f, v_{\Gamma_i} \right\rangle_{\Gamma_i}, \tag{5.13}$$

which is the first equation of the Steklov–Poincaré operator formulation.

For the second equation we consider the interior Calderón operator (2.2) on Γ_i for $i \in \{1, 2\}$, in particular the first line

$$V_{\kappa_i} \mathbb{L}_i t - \left(\frac{1}{2} I + K_{\kappa_i} \right) u_{|\Gamma_i} = 0$$

where we already switched to skeleton traces (u, t). This equation is satisfied in the sense of $H^{1/2}(\Gamma_i)$. Testing with $\tau \in H^{-1/2}(\Gamma_S)$ and adding up yields

$$\sum_{i=1}^{2} \left\langle V_{\kappa_i} \mathbb{L}_i t - \left(\frac{1}{2} I + K_{\kappa_i} \right) u_{|\Gamma_i}, \mathbb{L}_i \tau \right\rangle_{\Gamma_i} = 0.$$

Together with (5.13) this is the Steklov–Poincaré operator formulation to find $(u, t) \in H^{1/2}(\Gamma_S) \times H^{-1/2}(\Gamma_S)$ such that

$$\left\langle S_{\kappa_0}^{ext} u_{|\Gamma_0}, v_{|\Gamma_0} \right\rangle_{\Gamma_0} + \sum_{i=1}^{2} \left\langle \begin{pmatrix} V_{\kappa_i} & -\left(\frac{1}{2} I + K_{\kappa_i} \right) \\ \frac{1}{2} I + K'_{\kappa_i} & D_{\kappa_i} \end{pmatrix} \begin{pmatrix} \mathbb{L}_i t \\ u_{|\Gamma_i} \end{pmatrix}, \begin{pmatrix} \mathbb{L}_i \tau \\ v_{|\Gamma_i} \end{pmatrix} \right\rangle_{\Gamma_i} = \left\langle g + S_{\kappa_0}^{ext} f, v_{|\Gamma_0} \right\rangle_{\Gamma_0} \tag{5.14}$$

is satisfied for all $(v, \tau) \in H^{1/2}(\Gamma_S) \times H^{-1/2}(\Gamma_S)$.

Lemma 5.6. *The Steklov–Poincaré operator induced by (5.14) is injective for all wave numbers $\kappa_i \in \mathbb{R}_+$, $i \in \{0, 1, 2\}$.*

Proof. The proof of the statement follows the proof of injectivity for the single trace formulation, Lemma 5.2. For this reason we skip some steps that are identical and just give the results.

We have to show that if $(u, t) \in H^{1/2}(\Gamma_S) \times H^{-1/2}(\Gamma_S)$ is a solution of the homogeneous Steklov–Poincaré operator formulation

$$\left\langle S_{\kappa_0}^{ext} u_{|\Gamma_0}, v_{|\Gamma_0} \right\rangle_{\Gamma_0} + \sum_{i=1}^{2} \left\langle \begin{pmatrix} V_{\kappa_i} & -\left(\frac{1}{2} I + K_{\kappa_i} \right) \\ \frac{1}{2} I + K'_{\kappa_i} & D_{\kappa_i} \end{pmatrix} \begin{pmatrix} \mathbb{L}_i t \\ u_{|\Gamma_i} \end{pmatrix}, \begin{pmatrix} \mathbb{L}_i \tau \\ v_{|\Gamma_i} \end{pmatrix} \right\rangle_{\Gamma_i} = 0,$$

it follows that u and t are equal to zero.

For the first step we define functions and show that these solve the homogeneous model problem. We introduce an exterior Neumann trace $t_0 := -S_{\kappa_0}^{ext} u_{|\Gamma_0}$ and with this the functions

$$\begin{aligned} U_i(x) &:= (\widetilde{V}_{\kappa_i} \mathbb{L}_i t)(x) - (W_{\kappa_i} u_{|\Gamma_i})(x) \quad \text{for } x \in \Omega_i, i \in \{1, 2\}, \\ U_0(x) &:= -(\widetilde{V}_{\kappa_0} t_0)(x) + (W_{\kappa_0} u_{|\Gamma_0})(x) \quad \text{for } x \in \Omega_0. \end{aligned}$$

The idea is now to show that these functions satisfy homogeneous transmission conditions. To do this we take the Cauchy traces and apply test functions that vanish everywhere except on one interface Γ_{ij}. With this we get the desired result that

$$\begin{aligned}
\gamma_0^{ext}U_0 &= \gamma_0^{int}U_1 \quad \text{and} \quad \gamma_1^{ext}U_0 = \gamma_1^{int}U_1 \quad \text{on } \Gamma_{10},\\
\gamma_0^{ext}U_0 &= \gamma_0^{int}U_2 \quad \text{and} \quad \gamma_1^{ext}U_0 = \gamma_1^{int}U_2 \quad \text{on } \Gamma_{20},\\
\gamma_0^{int}U_1 &= \gamma_0^{ext}U_2 \quad \text{and} \quad \gamma_1^{int}U_1 = -\gamma_1^{int}U_2 \quad \text{on } \Gamma_{12}.
\end{aligned}$$

For a more detailed description of the procedure please see the proof of Lemma 5.2. Since the model problem only has one solution it follows that U_i as well as their Cauchy traces are zero for $i \in \{0,1,2\}$. On the other hand we have a representation of these Cauchy traces by means of (u,t), this gives us for $i \in \{1,2\}$

$$0 = \gamma_0^{ext}U_0 = u_{|\Gamma_0} \quad \text{on } \Gamma_0, \tag{5.15a}$$

$$0 = \gamma_0^{int}U_i = (V_{\kappa_i}\mathbb{L}_i t) + \left(\frac{1}{2}I - K_{\kappa_i}\right) u_{|\Gamma_i} \quad \text{on } \Gamma_i, \tag{5.15b}$$

$$0 = \gamma_1^{int}U_i = \left(\frac{1}{2} + K'_{\kappa_i}\right)\mathbb{L}_i t + (D_{\kappa_i}u_{|\Gamma_i}) \quad \text{on } \Gamma_i. \tag{5.15c}$$

For the second part we define functions on the complement of each subdomain. We already know $u_{|\Gamma_0} = 0$ and therefore $t_0 = -S_{\kappa_0}^{ext}u_{|\Gamma_0} = 0$ so we set $\hat{U}_0(x) = 0$ for $x \in \Omega_0^c$. For the both interior domains we define

$$\hat{U}_i(x) := -(\widetilde{V}_{\kappa_i}\mathbb{L}_i t)(x) + (W_{\kappa_i}u_{|\Gamma_i})(x) \quad \text{for } x \in \Omega_i^c, i \in \{1,2\}.$$

Taking the traces and applying the previous results (5.15) yields

$$\begin{aligned}
\gamma_0^{ext}\hat{U}_i &= -(V_{\kappa_i}\mathbb{L}_i t) + \left(\frac{1}{2}I + K_{\kappa_i}\right) u_{|\Gamma_i} = u_{|\Gamma_i},\\
\gamma_1^{ext}\hat{U}_i &= \left(\frac{1}{2}I - K'_{\kappa_i}\right)\mathbb{L}_i t - (D_{\kappa_i}u_{|\Gamma_I}) = \mathbb{L}_i t.
\end{aligned}$$

As in the proof of Lemma 5.2 we want to apply Lemma 5.1 (Rellich). To do this we apply Greens formula [15, Section 2.7] to $B_r(0) \cap \Omega_i^c$ for $r > 0$ large enough and get

$$\int_{|x|=r} \gamma_0^{ext}\hat{U}_i \overline{\frac{\partial}{\partial n_r}\hat{U}_i} ds_x = \int_{B_r(0)\cap\Omega_i^c} \left|\nabla\hat{U}_i\right| - \kappa_i^2\left|\hat{U}_i\right| dx - \int_{\Gamma_i} \gamma_0^{ext}\hat{U}_i\overline{\gamma_1^{ext}\hat{U}_i} ds_x.$$

The integral in the domain vanishes since $\hat{U}_i$ solves the partial differential equation. Summing up and taking the imaginary part gives us

$$\Im\left(\sum_{i=1}^{2}\int_{|x|=r} \gamma_0^{ext}\hat{U}_i\overline{\frac{\partial}{\partial n_r}\hat{U}_i} ds_x\right) = -\Im\left(\sum_{i=1}^{2}\int_{\Gamma_i} \gamma_0^{ext}\hat{U}_i\overline{\gamma_1^{ext}\hat{U}_i} ds_x\right) = 0.$$

The last equality holds because on Γ_{i0} the Dirichlet trace of $\hat{U}_i$ is equal to $u_{|\Gamma_0}$ and thus zero. On Γ_{12} the two Neumann traces have opposing signs so the integrals cancel

one another out. As in the proof of Lemma 5.2 we combine this with the Sommerfeld radiation condition and can conclude

$$\lim_{r\to\infty}\int_{|x|=r}\left|\hat{U}_i\right|ds_x = 0 \qquad \text{for } i=1,2.$$

This allows us to apply Lemma 5.1 (Rellich) which gives us $\hat{U}_i = 0$ for $i \in \{1,2\}$. Together with the previous results $\gamma_0^{ext}\hat{U}_i = u_{|\Gamma_i}$ and $\gamma_1^{ext}\hat{U}_i = \mathbb{L}_i t$ this finally gives us $(u,t) = (0,0)$ which completes the proof. □

Lemma 5.7. *The Steklov–Poincaré operator induced by (5.14) is coercive for all wave numbers $\kappa_i \in \mathbb{R}_+$, $i \in \{0,1,2\}$.*

Proof. Since the exterior Steklov–Poincaré operator is coercive, see Lemma 2.12, there exists a compact operator $T_S : H^{1/2}(\Gamma_0) \to H^{-1/2}(\Gamma_0)$ and a constant $c > 0$ satisfying

$$\Re\left\langle (S_{\kappa_0}^{ext} + T_S)v, v\right\rangle_{\Gamma_0} \geq c\|v\|^2_{H^{1/2}(\Gamma_0)} \qquad \forall v \in H^{1/2}(\Gamma_0).$$

With this we define for $\underline{\varphi} = (t,u)^\top \in H^{-1/2}(\Gamma_S)\times H^{1/2}(\Gamma_S)$ and $\underline{\psi} = (\tau,v)^\top \in H^{-1/2}(\Gamma_S)\times H^{1/2}(\Gamma_S)$ the sesquilinear form

$$t(\underline{\varphi},\underline{\psi}) := \left\langle T_S\, u_{|\Gamma_0}, v_{|\Gamma_0}\right\rangle_{\Gamma_0} + \sum_{i=1}^{2}\left\langle \begin{pmatrix} V_{\Omega_i} - V_{\kappa_i} & -(K_{\Omega_i} - K_{\kappa_i}) \\ K'_{\Omega_i} - K'_{\kappa_i} & D_{\Omega_i} - D_{\kappa_i}\end{pmatrix}\begin{pmatrix}\mathbb{L}_i t \\ u_{|\Gamma_i}\end{pmatrix}, \begin{pmatrix}\mathbb{L}_i \tau \\ v_{|\Gamma_i}\end{pmatrix}\right\rangle_{\Gamma_i}.$$

The boundary integral operators with a domain as subindex denote the Laplace boundary integral operators in this domain, i.e V_{Ω_i} is the Laplace single layer boundary integral operator in Ω_i. It follows from Lemma 2.6 and the compactness of T_S that $t(\cdot,\cdot)$ is a composition of compact and bounded operators and therefore compact itself. Denote the Steklov–Poincaré operator (5.14) by M and let $\underline{\varphi} = (t,u)^\top \in H^{-1/2}(\Gamma_S)\times H^{1/2}(\Gamma_S)$ be arbitrary but fixed, then we get

$$\begin{aligned}
\Re[\left\langle M\underline{\varphi},\underline{\varphi}\right\rangle + t(\underline{\varphi},\underline{\varphi})] &= \Re\left\langle (S_{\kappa_0}^{ext} + T_S)u_{|\Gamma_0}, u_{|\Gamma_0}\right\rangle_{\Gamma_0} \\
&\quad + \sum_{i=1}^{2}\Re\left\langle \begin{pmatrix} V_{\Omega_i} & -\left(\frac{1}{2}I + K_{\Omega_i}\right) \\ \frac{1}{2}I + K'_{\Omega_i} & \widetilde{D}_{\Omega_i}\end{pmatrix}\begin{pmatrix}\mathbb{L}_i t \\ u_{|\Gamma_i}\end{pmatrix}, \begin{pmatrix}\mathbb{L}_i t \\ u_{|\Gamma_i}\end{pmatrix}\right\rangle_{\Gamma_i} \\
&= \Re\left\langle (S_{\kappa_0}^{ext} + T_S)u_{|\Gamma_0}, u_{|\Gamma_0}\right\rangle_{\Gamma_0} + \sum_{i=1}^{2}\Re\left\langle \begin{pmatrix} V_{\Omega_i}\mathbb{L}_i t \\ \widetilde{D}_{\Omega_i}u_{|\Gamma_i}\end{pmatrix}, \begin{pmatrix}\mathbb{L}_i t \\ u_{|\Gamma_i}\end{pmatrix}\right\rangle_{\Gamma_i} \\
&\geq c\left\|u_{|\Gamma_0}\right\|^2_{H^{1/2}(\Gamma_0)} + \tilde{c}\sum_{i=1}^{2}\left(\left\|u_{|\Gamma_i}\right\|^2_{H^{1/2}(\Gamma_i)} + \|\mathbb{L}_i t\|^2_{H^{-1/2}(\Gamma_i)}\right) \\
&\geq \hat{c}(\|u\|^2_{H^{1/2}(\Gamma_S)} + \|t\|^2_{H^{-1/2}(\Gamma_S)}).
\end{aligned}$$

For the second equality we used that K_{Ω_i} is the adjoint of K'_{Ω_i} and those terms cancel each other out. The rest follows from the ellipticity of V_{Ω_i} and $\widetilde{D}_{\Omega_i}$ and the coercivity of $S_{\kappa_0}^{ext}$. Note that for the Neumann datum we only have the norm on the interior

boundaries. Lemma 1.5 tells us that we can estimate those by the norm on Γ_0 to get the norm on all boundaries and further on the skeleton trace.

Since this inequality holds for all $\underline{\varphi} \in H^{-1/2}(\Gamma_S) \times H^{1/2}(\Gamma_S)$, we have shown coercivity. □

These two lemmata enable us to apply the Fredholm alternative (Lemma 1.2) which gives us unique solvability of the formulation.

Theorem 5.8. *The Steklov–Poincaré operator formulation (5.14) has a unique solution for all given right hand sides* $(f, g) \in H^{1/2}(\Gamma_0) \times H^{-1/2}(\Gamma_0)$ *and non negative wave numbers* $\kappa_i \in \mathbb{R}_+$, $i \in \{0, 1, 2\}$.

Since the Steklov–Poincaré operator formulation uses the same skeleton trace spaces as the single trace formulation, we encounter the same issues regarding the duality pairing discussed at the end of Section 1.5. Since this means that this formulation is not compatible with operator preconditioning, we do not consider and operator of opposite order B_{SP} for this formulation.

5.5 Local multi trace formulation

The last formulation we discuss is the local multi trace formulation presented in [5, 12] which considers local traces for each domain instead of global skeleton traces. That way we have to consider two Dirichlet and Neumann traces on each interface, hence the name multi trace formulation. One advantage of this formulation is that the duality pairing is just the sum of local L_2 type duality parings, therefore we can apply operator preconditioning for this formulation.

Let $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_0) \in H^1(\Omega_1) \times H^1(\Omega_2) \times H^1_{\mathrm{loc}}(\Omega_0)$ be a solution of (5.1)–(5.8), then their Cauchy traces fulfil the Calderón projections, for example for the exterior domain

$$\underline{0} = \begin{pmatrix} V_{\kappa_0} & -K_{\kappa_0} \\ K'_{\kappa_0} & D_{\kappa_0} \end{pmatrix} \begin{pmatrix} t_0 \\ u_0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} u_0 \\ t_0 \end{pmatrix}.$$

The idea is to exchange the identity operator with the interior traces using the transmission conditions. The main difference to the homogeneous scatterer in Chapter 4 is that Ω_0 now has interfaces with two domains. Therefore, we first have to split the traces into two parts, one for Γ_{10} and one for Γ_{20}. Then we can use the local transmission conditions to get functions defined on the interfaces which can be extended by zero such that we end up with functions defined on the whole of Γ_0. Unfortunately, these functions are in general not in $H^{1/2}(\Gamma_0)$ or $H^{-1/2}(\Gamma_0)$ respectively. Here is where we make us of the piecewise defined spaces presented in Section 1.4. The mapping described above is summarized in the bounded operators

$$\mathbb{X}_{ij} : H^{1/2}(\Gamma_i) \to H^{1/2}_{pw}(\Gamma_j), \qquad \mathbb{X}_{ij} u_i = \begin{cases} u_{ij} & \text{on } \Gamma_{ij} \\ 0 & \text{else} \end{cases},$$

$$\mathbb{Y}_{ij} : H^{-1/2}(\Gamma_i) \to H^{-1/2}_{pw}(\Gamma_j), \quad \mathbb{Y}_{ij} t_i = \begin{cases} t_{ij} & \text{on } \Gamma_{ij} \\ 0 & \text{else} \end{cases}.$$

To describe the splitting we additionally need the indicator function χ_{ij} which is one on the boundary Γ_{ij} and zero everywhere else. With this we can come back to the exterior Calderón projection and write

$$\begin{aligned}\underline{0} &= \begin{pmatrix} V_{\kappa_0} & -K_{\kappa_0} \\ K'_{\kappa_0} & D_{\kappa_0} \end{pmatrix} \begin{pmatrix} t_0 \\ u_0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} u_0 \\ t_0 \end{pmatrix} \\ &= \begin{pmatrix} V_{\kappa_0} & -K_{\kappa_0} \\ K'_{\kappa_0} & D_{\kappa_0} \end{pmatrix} \begin{pmatrix} t_0 \\ u_0 \end{pmatrix} + \frac{1}{2}\chi_{01} \begin{pmatrix} u_0 \\ t_0 \end{pmatrix} + \frac{1}{2}\chi_{02} \begin{pmatrix} u_0 \\ t_0 \end{pmatrix} \\ &= \begin{pmatrix} V_{\kappa_0} & -K_{\kappa_0} \\ K'_{\kappa_0} & D_{\kappa_0} \end{pmatrix} \begin{pmatrix} t_0 \\ u_0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \mathbb{X}_{10}(u_1 - f) \\ \mathbb{Y}_{10}(t_1 - g) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \mathbb{X}_{20}(u_2 - f) \\ \mathbb{Y}_{20}(t_2 - g) \end{pmatrix}.\end{aligned}$$

For the last equation we used the given transmission conditions. Bringing all known data to one side and the rest to the other yields

$$\begin{pmatrix} V_{\kappa_0} & -K_{\kappa_0} \\ K'_{\kappa_0} & D_{\kappa_0} \end{pmatrix} \begin{pmatrix} t_0 \\ u_0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \mathbb{X}_{10}u_1 \\ \mathbb{Y}_{10}t_1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \mathbb{X}_{20}u_2 \\ \mathbb{Y}_{20}t_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} f \\ g \end{pmatrix}.$$

When we do the same computations for an interior domain Ω_i with $i \in \{1, 2\}$, we have to take care of the change in sign for the normal derivative on Γ_{12}. Let $i, j \in \{1, 2\}$ and $i \neq j$, then it holds

$$\begin{aligned}\underline{0} &= \begin{pmatrix} V_{\kappa_i} & -K_{\kappa_i} \\ K'_{\kappa_i} & D_{\kappa_i} \end{pmatrix} \begin{pmatrix} t_i \\ u_i \end{pmatrix} - \frac{1}{2} \begin{pmatrix} u_i \\ u_i \end{pmatrix} \\ &= \begin{pmatrix} V_{\kappa_i} & -K_{\kappa_i} \\ K'_{\kappa_i} & D_{\kappa_i} \end{pmatrix} \begin{pmatrix} t_i \\ u_i \end{pmatrix} - \frac{1}{2}\chi_{i0} \begin{pmatrix} u_i \\ t_i \end{pmatrix} - \frac{1}{2}\chi_{ij} \begin{pmatrix} u_i \\ t_i \end{pmatrix} \\ &= \begin{pmatrix} V_{\kappa_i} & -K_{\kappa_i} \\ K'_{\kappa_i} & D_{\kappa_i} \end{pmatrix} \begin{pmatrix} t_i \\ u_i \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \mathbb{X}_{0i}(u_0 + f) \\ \mathbb{Y}_{0i}(t_0 + g) \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \mathbb{X}_{ji}u_j \\ \mathbb{Y}_{ji}(-t_j) \end{pmatrix}.\end{aligned}$$

So we have found that the traces of $(\tilde{u}_1, \tilde{u}_2, \tilde{u}_0)$ solve the system

$$\begin{pmatrix} V_{\kappa_0} & -K_{\kappa_0} & & \frac{1}{2}\mathbb{X}_{10} & & \frac{1}{2}\mathbb{X}_{20} \\ K'_{\kappa_0} & D_{\kappa_0} & \frac{1}{2}\mathbb{Y}_{10} & & \frac{1}{2}\mathbb{Y}_{20} & \\ & -\frac{1}{2}\mathbb{X}_{01} & V_{\kappa_1} & -K_{\kappa_1} & & -\frac{1}{2}\mathbb{X}_{21} \\ -\frac{1}{2}\mathbb{Y}_{01} & & K'_{\kappa_1} & D_{\kappa_1} & \frac{1}{2}\mathbb{Y}_{21} & 0 \\ & -\frac{1}{2}\mathbb{X}_{02} & & -\frac{1}{2}\mathbb{X}_{12} & V_{\kappa_2} & -K_{\kappa_2} \\ -\frac{1}{2}\mathbb{Y}_{02} & & \frac{1}{2}\mathbb{Y}_{12} & & K'_{\kappa_2} & D_{\kappa_2} \end{pmatrix} \begin{pmatrix} t_0 \\ u_0 \\ t_1 \\ u_1 \\ t_2 \\ u_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} f \\ g \\ \mathbb{X}_{01}f \\ \mathbb{Y}_{01}g \\ \mathbb{X}_{02}f \\ \mathbb{Y}_{02}g \end{pmatrix}.$$

From what we have seen earlier, we know that the first line is satisfied in the sense of $H^{1/2}_{pw}(\Gamma_0)$, the second in the sense of $H^{-1/2}_{pw}(\Gamma_0)$ and so forth. Therefore, when we want to rewrite this as a variational formulation, we have to make use of the proper dual spaces presented in Section 1.4. To facilitate further reading we define the proper spaces as

$$H(\Gamma) := H^{-1/2}(\Gamma_0) \times H^{1/2}(\Gamma_0) \times H^{-1/2}(\Gamma_1) \times H^{1/2}(\Gamma_1) \times H^{-1/2}(\Gamma_2) \times H^{1/2}(\Gamma_2), \tag{5.16}$$

$$\Phi(\Gamma) := \widetilde{H}^{-1/2}_{pw}(\Gamma_0) \times \widetilde{H}^{1/2}_{pw}(\Gamma_0) \times \widetilde{H}^{-1/2}_{pw}(\Gamma_1) \times \widetilde{H}^{1/2}_{pw}(\Gamma_1) \times \widetilde{H}^{-1/2}_{pw}(\Gamma_2) \times \widetilde{H}^{1/2}_{pw}(\Gamma_2). \tag{5.17}$$

From the definition of these spaces in Section 1.4 it follows $\Phi(\Gamma) \subset H(\Gamma)$, Lemma 1.4 tells us that $\Phi(\Gamma)$ is even a dense subspace of $H(\Gamma)$. With this the local multi trace formulation is to find $(t_0, u_0, t_1, u_1, t_2, u_2)^\top \in H(\Gamma)$ such that

$$\left\langle \begin{pmatrix} V_{\kappa_0} & -K_{\kappa_0} & & \frac{1}{2}\mathbb{X}_{10} & & \frac{1}{2}\mathbb{X}_{20} \\ K'_{\kappa_0} & D_{\kappa_0} & \frac{1}{2}\mathbb{Y}_{10} & & \frac{1}{2}\mathbb{Y}_{20} & \\ & -\frac{1}{2}\mathbb{X}_{01} & V_{\kappa_1} & -K_{\kappa_1} & & -\frac{1}{2}\mathbb{X}_{21} \\ -\frac{1}{2}\mathbb{Y}_{01} & & K'_{\kappa_1} & D_{\kappa_1} & \frac{1}{2}\mathbb{Y}_{21} & 0 \\ & -\frac{1}{2}\mathbb{X}_{02} & & -\frac{1}{2}\mathbb{X}_{12} & V_{\kappa_2} & -K_{\kappa_2} \\ -\frac{1}{2}\mathbb{Y}_{02} & & \frac{1}{2}\mathbb{Y}_{12} & & K'_{\kappa_2} & D_{\kappa_2} \end{pmatrix} \begin{pmatrix} t_0 \\ u_0 \\ t_1 \\ u_1 \\ t_2 \\ u_2 \end{pmatrix}, \begin{pmatrix} \tau_0 \\ v_0 \\ \tau_1 \\ v_1 \\ \tau_2 \\ v_2 \end{pmatrix} \right\rangle_\Gamma = \left\langle \frac{1}{2} \begin{pmatrix} f \\ g \\ \mathbb{X}_{01} f \\ \mathbb{Y}_{01} g \\ \mathbb{X}_{02} f \\ \mathbb{Y}_{02} g \end{pmatrix}, \begin{pmatrix} \tau_0 \\ v_0 \\ \tau_1 \\ v_1 \\ \tau_2 \\ v_2 \end{pmatrix} \right\rangle_\Gamma \tag{5.18}$$

holds for all $(\tau_0, v_0, \tau_1, v_1, \tau_2, v_2)^\top \in \Phi(\Gamma)$. To show unique solvability of this formulation we prove that all prerequisites for Theorem 1.3 are fulfilled.

Lemma 5.9. *[12, Theorem 9] The local multi trace operator induced by (5.18) is injective in the sense of Theorem 1.3.*

Proof. Let M be the local multi trace operator induced by (5.18), then we have to show for $\underline{\varphi} = (t_0, u_0, t_1, u_1, t_2, u_2) \in H(\Gamma)$ that from

$$\left\langle M\underline{\varphi}, \underline{\psi} \right\rangle_\Gamma = 0 \qquad \forall \underline{\psi} \in \Phi(\Gamma)$$

it follows $\underline{\varphi} = \underline{0}$. The proof is split into two parts.

For the first step we use $\underline{\varphi}$ to define solutions of the Helmholtz equation by

$$\begin{aligned} U_i(x) &:= (\widetilde{V}_{\kappa_i} t_i)(x) - (W_{\kappa_i} u_i)(x) && \text{for } x \in \Omega_i, i \in \{1,2\}, \\ U_0(x) &:= -(\widetilde{V}_{\kappa_0} t_0)(x) + (W_{\kappa_0} u_0)(x) && \text{for } x \in \Omega_0. \end{aligned}$$

Our goal is to show that these functions solve the homogeneous model problem. Hence we consider for $i, j \in \{1,2\}$ their Dirichlet and Neumann traces and exploit that $\underline{\varphi}$ solves the homogeneous multi trace formulation

$$\begin{aligned} \gamma_0^{int} U_i &= (V_{\kappa_i} t_i) + \left(\frac{1}{2} I - K_{\kappa_i}\right) u_i = \frac{1}{2} u_i + \frac{1}{2} \mathbb{X}_{0i} u_0 + \frac{1}{2} \mathbb{X}_{ji} u_j, \\ \gamma_1^{int} U_i &= \left(\frac{1}{2} I + K'_{\kappa_i}\right) t_i + (D_{\kappa_i} u_i) = \frac{1}{2} t_i + \frac{1}{2} \mathbb{Y}_{0i} t_0 - \frac{1}{2} \mathbb{Y}_{ji} t_j, \\ \gamma_0^{ext} U_0 &= -(V_{\kappa_0} t_0) + \left(\frac{1}{2} I + K_{\kappa_0}\right) u_0 = \frac{1}{2} u_0 + \frac{1}{2} \mathbb{X}_{10} u_1 + \frac{1}{2} \mathbb{X}_{20} u_2, \\ \gamma_1^{ext} U_0 &= \left(\frac{1}{2} I - K'_{\kappa_0}\right) t_0 - (D_{\kappa_0} u_0) = \frac{1}{2} t_0 + \frac{1}{2} \mathbb{Y}_{10} t_1 + \frac{1}{2} \mathbb{Y}_{20} t_2. \end{aligned}$$

Note that we are only allowed to use these relations if the test function belongs to $\Phi(\Gamma)$. Thus we choose for a test function the trace of a smooth function $\varphi \in D(\mathbb{R}^3)$ whose support is compactly embedded in one interface Γ_{ij}, then the trace can be

extended by zero to a function in $\Phi(\Gamma)$, Let φ be such a function for the interface Γ_{0i}, $i \in \{1,2\}$, then this yields

$$\left\langle \gamma_0^{int} U_i - \gamma_0^{ext} U_0, \varphi \right\rangle_{\Gamma_{i0}} = \frac{1}{2} \left\langle u_i + \mathbb{X}_{0i} u_0 - u_0 - \mathbb{X}_{i0} u_i, \varphi \right\rangle_{\Gamma_{i0}} = 0,$$
$$\left\langle \gamma_1^{int} U_i - \gamma_1^{ext} U_0, \varphi \right\rangle_{\Gamma_{i0}} = \frac{1}{2} \left\langle t_i + \mathbb{Y}_{0i} t_0 - t_0 - \mathbb{Y}_{i0} t_i, \varphi \right\rangle_{\Gamma_{i0}} = 0.$$

For the interior interface Γ_{12} we get

$$\left\langle \gamma_0^{int} U_1 - \gamma_0^{int} U_2, \varphi \right\rangle_{\Gamma_{12}} = \frac{1}{2} \left\langle u_1 + \mathbb{X}_{21} u_2 - u_2 - \mathbb{X}_{12} u_1, \varphi \right\rangle_{\Gamma_{12}} = 0,$$
$$\left\langle \gamma_1^{int} U_1 + \gamma_1^{int} U_2, \varphi \right\rangle_{\Gamma_{12}} = \frac{1}{2} \left\langle t_1 - \mathbb{Y}_{21} t_2 + t_2 - \mathbb{Y}_{12} t_1, \varphi \right\rangle_{\Gamma_{12}} = 0.$$

We see that the triple (U_0, U_1, U_2) solves the homogeneous model problem (5.1)–(5.8) and thus is equal to zero. Applying this to the Cauchy data of (U_0, U_1, U_2) concludes the first part of this proof and gives us

$$u_i = -\mathbb{X}_{0i} u_0 - \mathbb{X}_{ji} u_j \qquad t_i = -\mathbb{Y}_{0i} t_0 + \mathbb{Y}_{ji} t_j \qquad i, j \in \{1,2\}, i \neq j, \tag{5.19a}$$
$$u_0 = -\mathbb{X}_{10} u_1 - \mathbb{X}_{20} u_2 \qquad t_0 = -\mathbb{Y}_{10} t_1 - \mathbb{Y}_{20} t_2. \tag{5.19b}$$

For the second step we define solutions of the local partial differential equations on the complementary domains

$$\hat{U}_i(x) := -(\widetilde{V}_{\kappa_i} t_i)(x) + (W_{\kappa_i} u_i)(x) \quad \text{for } x \in \Omega_i^c, i \in \{1,2\},$$
$$\hat{U}_0(x) := (\widetilde{V}_{\kappa_0}) - (W_{\kappa_0} u_0)(x) \quad \text{for } x \in \Omega_0^c.$$

Recall that for the case of a homogeneous scatterer, i.e Lemma 4.10, this procedure gave us two functions that we combined to a function defined on $\mathbb{R}^3$. This is no longer possible since the complements of Ω_i are overlapping if we consider more than two domains. Hence we construct a multi–valued function defined on sheets of $\mathbb{R}^3$, one sheet $\mathbb{R}^3_i$ for each domain, that are connected via the skeleton boundary Γ_S. We define such a function $\hat{U}^\sigma : \mathbb{R}^{3\times 3} \to \mathbb{C}$ as

$$\hat{U}^\sigma := \{\sigma_i \hat{U}_i : \Omega_i^c \subset \mathbb{R}^3_i \to \mathbb{C}, i = 0,1,2\}$$

with signs $\sigma_i \in \{\pm 1\}$ that can be chosen for each domain Ω_i^c individually. The idea is to show that $\hat{U} = 0$. To do this we apply the unique continuation principle for solutions of the homogeneous Helmholtz equation [21, Section 2]. It states that a solution of the Helmholtz equation is uniquely determined by its values on any open ball inside its domain. This also holds for solutions on the complementary domains, which multivalued solutions are.

In order to apply this principle we consider the Cauchy traces of $\hat{U}_i$, as we did in

the first step, and get

$$\begin{aligned}
\gamma_0^{ext}\hat{U}_i &= -(V_{\kappa_i}t_i) + \left(\frac{1}{2}I + K_{\kappa_i}\right)u_i = \frac{1}{2}u_i - \frac{1}{2}\mathbb{X}_{0i}u_0 - \frac{1}{2}\mathbb{X}_{ji}u_j,\\
\gamma_1^{ext}\hat{U}_i &= \left(\frac{1}{2}I - K'_{\kappa_i}\right)t_i - (D_{\kappa_i}u_i) = \frac{1}{2}t_i - \frac{1}{2}\mathbb{Y}_{0i}t_0 + \frac{1}{2}\mathbb{Y}_{ji}t_j,\\
\gamma_0^{int}\hat{U}_0 &= (V_{\kappa_0}t_0) + \left(\frac{1}{2}I - K_{\kappa_0}\right)u_0 = \frac{1}{2}u_0 - \frac{1}{2}\mathbb{X}_{10}u_1 - \frac{1}{2}\mathbb{X}_{20}u_2,\\
\gamma_1^{int}\hat{U}_0 &= \left(\frac{1}{2}I + K'_{\kappa_0}\right)t_0 + (D_{\kappa_0}u_0) = \frac{1}{2}t_0 - \frac{1}{2}\mathbb{Y}_{10}t_1 - \frac{1}{2}\mathbb{Y}_{20}t_2.
\end{aligned}$$

With this we can consider the jump of $\hat{U}^\sigma$ on an interface Γ_{ij}. Let φ be a smooth test function with compact support on Γ_{i0}

$$\begin{aligned}
\left\langle [\gamma_0\hat{U}^\sigma]_{\Gamma_{i0}}, \varphi \right\rangle_{\Gamma_{i0}} &= \left\langle \sigma_i\gamma_0^{ext}\hat{U}_i - \sigma_0\gamma_0^{int}\hat{U}_0, \varphi \right\rangle_{\Gamma_{i0}}\\
&= \sigma_i\frac{1}{2}\left\langle u_i - \mathbb{X}_{0i}u_0, \varphi\right\rangle_{\Gamma_{ij}} - \sigma_0\frac{1}{2}\left\langle u_0 - \mathbb{X}_{i0}u_i, \varphi\right\rangle_{\Gamma_{ij}},\\
\left\langle [\gamma_1\hat{U}^\sigma]_{\Gamma_{i0}}, \varphi \right\rangle_{\Gamma_{i0}} &= \left\langle \sigma_i\gamma_1^{ext}\hat{U}_i - \sigma_0\gamma_1^{ext}\hat{U}_0, \varphi \right\rangle_{\Gamma_{i0}}\\
&= \sigma_i\frac{1}{2}\left\langle t_i - \mathbb{Y}_{0i}t_0, \varphi\right\rangle_{\Gamma_{i0}} - \sigma_0\frac{1}{2}\left\langle t_0 - \mathbb{Y}_{i0}t_i, \varphi\right\rangle_{\Gamma_{i0}}.
\end{aligned}$$

We see that these jumps are zero if the domains have opposite signs, i.e. $\sigma_i = -\sigma_0$. The same holds true on the interior interface Γ_{12}. Let $\sigma_0 = 1$, then it is obvious that for a domain as sketched in Figure 5.1 we cannot choose the signs in such a way that all neighbouring domains have opposing signs.

For a permutation of σ_1 and σ_2 we set $\hat{U}^{(\sigma_1,\sigma_2)}$ as the corresponding function. Then, since for all permutations $\hat{U}^{(\sigma_1,\sigma_2)}$ has a zero jump on two interfaces and all functions $\hat{U}^{(\sigma_1,\sigma_2)}$ coincide on Ω_0^c, the unique continuation principle states that it holds

$$\hat{U}^{(-,-)} = \hat{U}^{(-,+)} = \hat{U}^{(+,+)} \quad \text{in } \Omega_0^c \cup \Omega_1^c \cup \Omega_2^c.$$

Then from the definition of $\hat{U}^{(\sigma_1,\sigma_2)}$ it follows

$$\hat{U}_i = -\hat{U}_i \quad \text{in } \Omega_i^c$$

and hence $\hat{U}_i = 0$. With this the interior jump is always zero and $\hat{U}^{(-,-)}$ is a solution of the homogeneous Helmholtz problem which gives us $\hat{U}^{(-,-)} = 0$ and in particular $\hat{U}_0 = 0$ in Ω_0^c. So it follows for the Cauchy traces of $(\hat{U}_0, \hat{U}_1, \hat{U}_2)$ that

$$\begin{aligned}
u_i &= \mathbb{X}_{0i}u_0 + \mathbb{X}_{ji}u_j \qquad & t_i &= \mathbb{Y}_{0i}t_0 - \mathbb{Y}_{ji}t_j \qquad & i,j \in \{1,2\}, i \neq j,\\
u_0 &= \mathbb{X}_{10}u_1 + \mathbb{X}_{20}u_2 \qquad & t_0 &= \mathbb{Y}_{10}t_1 + \mathbb{Y}_{20}t_2.
\end{aligned}$$

Together with the previous results (5.19) we can conclude $u_i = -u_i$ and $t_i = -t_i$ on Γ_i and thus $(u_i, t_i) = (0,0)$ for $i \in \{0,1,2\}$. To summarize we have shown $\underline{\varphi} = \underline{0}$, this concludes the proof of injectivity for the local multi trace operator. □

Lemma 5.10. *[12, Theorem 10] The local multi trace operator induced by (5.18) is* $\Phi(\Gamma)$*–coercive in the sense of Theorem 1.3.*

Proof. Let $M : H(\Gamma) \to \Phi(\Gamma)'$ be the local multi trace operator induced by (5.18). We have to show that there exists a compact operator $T : H(\Gamma) \to H(\Gamma)'$ and a constant $c > 0$ such that

$$\Re\left\langle (M+T)\underline{\psi}, \underline{\psi}\right\rangle \geq c\left\|\underline{\psi}\right\|^2_{H(\Gamma)} \qquad \forall \underline{\psi} \in \Phi(\Gamma).$$

Let $\underline{\psi} = (\tau_0, v_0, \tau_1, v_1, \tau_2, v_2)^\top \in \Phi(\Gamma)$, then we get

$$\left\langle M\underline{\psi}, \underline{\psi}\right\rangle = \sum_{i=0}^{2} \left\langle \begin{pmatrix} V_{\kappa_i} & -K_{\kappa_i} \\ K'_{\kappa_i} & D_{\kappa_i} \end{pmatrix} \begin{pmatrix} \tau_i \\ v_i \end{pmatrix}, \begin{pmatrix} \tau_i \\ v_i \end{pmatrix} \right\rangle_{\Gamma_i} \tag{5.20a}$$

$$+ \frac{1}{2} \sum_{i=1}^{2} \left\langle \begin{pmatrix} \mathbb{X}_{i0} v_i \\ \mathbb{Y}_{i0} \tau_i \end{pmatrix}, \begin{pmatrix} \tau_0 \\ v_0 \end{pmatrix} \right\rangle_{\Gamma_0} - \frac{1}{2} \sum_{j=1}^{2} \left\langle \begin{pmatrix} \mathbb{X}_{0j} v_0 \\ \mathbb{Y}_{0j} \tau_0 \end{pmatrix}, \begin{pmatrix} \tau_j \\ v_j \end{pmatrix} \right\rangle_{\Gamma_j} \tag{5.20b}$$

$$+ \frac{1}{2} \left\langle \begin{pmatrix} -\mathbb{X}_{12} v_1 \\ \mathbb{Y}_{12} \tau_1 \end{pmatrix}, \begin{pmatrix} \tau_2 \\ v_2 \end{pmatrix} \right\rangle_{\Gamma_2} + \frac{1}{2} \left\langle \begin{pmatrix} -\mathbb{X}_{21} v_2 \\ \mathbb{Y}_{21} \tau_2 \end{pmatrix}, \begin{pmatrix} \tau_1 \\ v_1 \end{pmatrix} \right\rangle_{\Gamma_1}. \tag{5.20c}$$

First we consider the non diagonal part, that is (5.20b)–(5.20c). From the definition of the switching operators we conclude

$$\langle \mathbb{X}_{ij} v_i, \tau_j \rangle_{\Gamma_j} = \langle v_i, \mathbb{Y}_{ji} \tau_j \rangle_{\Gamma_i} = \overline{\langle \mathbb{Y}_{ji} \tau_j, v_i \rangle_{\Gamma_i}}.$$

Thus, we can write

$$\sum_{i=1}^{2} \left\langle \begin{pmatrix} \mathbb{X}_{i0} v_i \\ \mathbb{Y}_{i0} \tau_i \end{pmatrix}, \begin{pmatrix} \tau_0 \\ v_0 \end{pmatrix} \right\rangle_{\Gamma_0} = \sum_{i=1}^{2} \overline{\left\langle \begin{pmatrix} \mathbb{Y}_{0i} \tau_0 \\ \mathbb{X}_{0i} v_0 \end{pmatrix}, \begin{pmatrix} v_i \\ \tau_i \end{pmatrix} \right\rangle_{\Gamma_i}},$$

$$\left\langle \begin{pmatrix} -\mathbb{X}_{12} v_1 \\ \mathbb{Y}_{12} \tau_1 \end{pmatrix}, \begin{pmatrix} \tau_2 \\ v_2 \end{pmatrix} \right\rangle_{\Gamma_2} = \overline{\left\langle \begin{pmatrix} -\mathbb{Y}_{21} \tau_2 \\ \mathbb{X}_{21} v_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ \tau_1 \end{pmatrix} \right\rangle_{\Gamma_1}}.$$

When we use this relation in (5.20b) and (5.20c), we see that everything but (5.20a) is purely imaginary. Hence, it holds

$$\Re\left\langle M\underline{\psi}, \underline{\psi}\right\rangle = \sum_{i=0}^{2} \Re\left\langle \begin{pmatrix} V_{\kappa_i} & -K_{\kappa_i} \\ K'_{\kappa_i} & D_{\kappa_i} \end{pmatrix} \begin{pmatrix} \tau_i \\ v_i \end{pmatrix}, \begin{pmatrix} \tau_i \\ v_i \end{pmatrix} \right\rangle_{\Gamma_i}.$$

The remaining operator is the same operator we already considered for the single trace formulation. Therefore coercivity follows as in the proof of Lemma 5.3. □

Theorem 5.11. *[12, Theorem 11] The local multi trace formulation (5.18) has a unique slution* $\underline{u} \in H(\Gamma)$ *for any given right hand side* $(f, g) \in H^{1/2}(\Gamma_0) \times H^{-1/2}(\Gamma_0)$ *and non negative wave numbers* $\kappa_i \in \mathbb{R}_+$*,* $i \in \{0, 1, 2\}$*.*

Proof. The statement of the theorem follows from Theorem 1.3 if we can ensure all required properties. We start by setting $X := H(\Gamma)$, Lemma 1.4 shows that $\Pi := \Phi(\Gamma)$

is a feasible choice for a dense subset. The sesquilinear form $a(\cdot,\cdot)$ is of course induced by the multi trace operator (5.18) for which Lemmata 5.9 and 5.10 show injectivity and coercivity. Continuity follows from the continuity of all individual operators in the appropriate space $H(\Gamma) \times \Phi(\Gamma)$. Thus the multi trace formulation has a unique solution satisfying the stability estimate

$$\sum_{i=0}^{2} \left(\|u_i\|^2_{H^{1/2}(\Gamma_i)} + \|t_i\|^2_{H^{-1/2}(\Gamma_i)} \right) \leq c \left(\|f\|^2_{H^{1/2}(\Gamma_0)} + \|g\|^2_{H^{-1/2}(\Gamma_0)} \right).$$

□

In contrast to the formulations using skeleton trace spaces the local multi trace formulation suits the preconditioning strategy outlined in Section 3.4. In fact, finding formulations for an arbitrary number of subdomains that can be easily preconditioned was one of the main motivations for deriving the local multi trace formulation [12]. The major difference is that the appearing duality pairings are simply the local L_2 type duality pairing on the closed boundaries Γ_i for $i \in \{0,1,2\}$. The preconditioning operator can be derived using the same procedure as in Section 4.5 which yields

$$B_{\mathrm{MTF}} := \begin{pmatrix} D_{\kappa_0} & K'_{\kappa_0} & & & & \\ -K_{\kappa_0} & V_{\kappa_0} & & & & \\ & & D_{\kappa_1} & K'_{\kappa_1} & & \\ & & -K_{\kappa_1} & V_{\kappa_1} & & \\ & & & & D_{\kappa_2} & K'_{\kappa_2} \\ & & & & -K_{\kappa_2} & V_{\kappa_2} \end{pmatrix}.$$

5.6 Boundary element method for the composite scatterer

This section discusses how the boundary element method presented in Chapter 3 can be applied in the case of a composite scatterer. Since the formulations make use of different spaces, this section is split into two parts. First we explain how to handle skeleton trace spaces which corresponds to the Steklov–Poincaré operator formulation and the single trace formulation. The second part addresses the multi trace formulation and its discretization.

For the first part we discuss all formulations that make use of the skeleton trace spaces $H^{1/2}(\Gamma_S)$ and $H^{-1/2}(\Gamma_S)$. Let Γ_h be a mesh of the skeleton boundary Γ_S that resolves the splitting into different domains. The space of piecewise linear, globally continuous functions $S^1_h(\Gamma_S)$ is then the same as for a single domain

$$S^1_h(\Gamma_S) := \operatorname{span}\{\varphi^1_k\}_{k=1}^M \quad \text{with} \quad \varphi^1_k(x) := \begin{cases} 1 & x = x_k \\ 0 & x = x_\ell, \ell \neq k \\ \text{linear} & \text{else} \end{cases}.$$

For the space of piecewise constant functions we follow the definition of $H^{-1/2}(\Gamma_S)$ and define a product space

$$S_T^0(\Gamma_S) := \prod_{i=0}^{2} S_h^0(\Gamma_i).$$

Recall that functions in $H^{-1/2}(\Gamma_S)$ match up to the sign on each interface Γ_{ij} in the sense of $H^{-1/2}(\Gamma_{ij})$. In the case of piecewise constant discontinuous functions we can enforce this equality even pointwise. Let $t_h \in S_T^0(\Gamma_S)$ be composed of $t_h = (t_{h,0}, t_{h,1}, t_{h,2})$, then we define

$$S_h^0(\Gamma_S) := \{t_h \in S_T^0(\Gamma_S) | t_{h,0} = t_{h,i} \text{ on } \Gamma_{0i} \text{ for } i \in \{1,2\}; t_{h,1} = -t_{h,2} \text{ on } \Gamma_{12}\}$$

With these two discrete spaces one can show the same approximation properties as in the case of one domain, i.e. $S_h^1(\Gamma_S)$ approximates $H^{1/2}(\Gamma_S)$ and $S_h^0(\Gamma_S)$ approximates $H^{-1/2}(\Gamma_S)$. Hence we can apply the standard Galerkin scheme described in Section 4.6 which again leads to systems of linear equations and the same convergence results as for the case of a single scatterer. For the reasons discussed in Section 1.5 it is not clear how to interpret the duality pairing $\langle \cdot, \cdot \rangle_{\Gamma_S}$ for arbitrary functions in $S_h^1(\Gamma_S) \times S_h^0(\Gamma_S)$. This prevents us from computing the global mass matrix we would need for operator preconditioning. Thus we do not discuss preconditioning strategies for these formulations.

The second part of this section considers the local multi trace formulation for the case that we have a family of meshes Γ_h that resolve the decomposition of Ω. From the definition of the space $H(\Gamma)$ (5.16) it is clear that its discrete space has to be defined as

$$S_h^{\mathrm{MTF}}(\Gamma_h) := S_h^0(\Gamma_0) \times S_h^1(\Gamma_0) \times S_h^0(\Gamma_1) \times S_h^1(\Gamma_1) \times S_h^0(\Gamma_2) \times S_h^1(\Gamma_2).$$

Note that S_h^{MTF} is not a subspace of $\Phi(\Gamma)$ as can be seen if we consider a basis function $\varphi_k^1 \in S_h^1(\Gamma_i)$ belonging to a junction point of all three domains. If φ_k^1 should belong to $H_{pw}^{1/2}(\Gamma_i)$ then it has to be zero in the junction point and thus everywhere. Such problems do not occur for the Neumann spaces, i.e. $S_h^0(\Gamma_i) \subset H_{pw}^{-1/2}(\Gamma_i)$. Nonetheless we use S_h^{MTF} as the discrete test and trial space for $\Phi(\Gamma)$ and see that every expression in the local multi trace formulation (5.18) stays well defined. With these discrete spaces we can derive a system of linear equations and hence a discrete solution.

Unfortunately, unique solvability and approximating properties of the solution do not follow as for the other formulations since the local multi trace formulation does not fit the standard Fredholm theory. In particular the BBL condition (3.2) has to be shown. The critical part in order to prove this, see [12, Remark 12], is to show that an inequality of the form

$$\|\tau_h\|_{\widetilde{H}_{pw}^{-1/2}(\Gamma_i)} \leq c_i(1 + \log h)\|\tau_h\|_{H^{-1/2}(\Gamma_i)}$$

holds for all $\tau_h \in S_h^0(\Gamma_i)$. In [12] the unique solvability of the discrete system was proven for the two dimensional case but there are no results yet for three dimensional domains.

Theorem 5.12. *[12, Theorem 13] Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and the discrete space $S_h^{MTF}(\Gamma_h)$ be given as above. Assume that the inverse of the local multi trace operator (5.18) maps $H^1(\Gamma_0) \times \cdots \times L_2(\Gamma_2)$ into itself continuously, then there exists a n_0 such that the discrete inf–sup condition (3.2) is satisfied for all $n > n_0$.*

This concludes the part regarding unique solvability of the discrete problem. Regarding preconditioning everything discussed for the single scatterer in Section 4.6 holds true since the duality is only between the spaces $H^{1/2}(\Gamma_i)$ and $H^{-1/2}(\Gamma_i)$ for $\{0, 1, 2\}$. Hence piecewise linear, globally continuous functions for all spaces or piecewise linear, globally continuous functions on a primal mesh and piecewise constant functions on a dual mesh provide stable pairings.

6 Numerical examples

In this chapter we use numerical examples to test and compare the three presented formulations. To improve readability, in particular of the tables and graphs, we introduce the abbreviations Sp for the Steklov–Poincaré operator formulation, Stf for the single trace formulation and Mtf for the local multi trace formulation. The implementation was done using BEM++[1], a C++/Python boundary element library primarily developed by the University College London [23]. Since the boundary element method results in dense matrices we used an adaptive cross approximation [1] library to reduce computation time and memory usage. The meshes were created using the mesh generator Gmsh[2].

The first two sections of this chapter check correctness of the implementation by showing that the orders of convergence match with those given in Remark 4.13. Sections 6.3 and 6.4 discuss how the number of iteration steps required to compute the solution depends on the degrees of freedom, the wave numbers and preconditioning. In Section 6.5 we consider thin domains and their influence on the iterative solver.

6.1 Convergence study - Sphere

The first and most important task is to ensure that the computed solutions approximate the analytical solution independent of the considered wave numbers. To do this we consider a case where we know solutions for the local Helmholtz equations and use these to construct transmission data f and g.

Of particular interest are wave numbers that correspond to Dirichlet or Neumann eigenvalues, so–called spurious modes. For the unit sphere $\Omega_1 = B_1(0) \subset \mathbb{R}^3$ these eigenvalues are well known [10, Section 3.3] and are given by the roots of certain Bessel functions $J_\ell(x)$.

$$\hat{\kappa}^2 \text{ is a Dirichlet eigenvalue for } \Omega_1 \Leftrightarrow \hat{\kappa} \text{ is a positive root of } J_{\ell+1/2}(\kappa) \text{ for } \ell \in \mathbb{N}.$$
$$\hat{\kappa}^2 \text{ is a Neumann eigenvalue for } \Omega_1 \Leftrightarrow \hat{\kappa} \text{ is a positive root of } J'_{\ell+1/2}(\kappa) \text{ for } \ell \in \mathbb{N}.$$

The first couple of wave numbers $\hat{\kappa}_D$ and $\hat{\kappa}_N$, which belong to Dirichlet or Neumann

[1] www.bempp.org
[2] http://geuz.org/gmsh

eigenvalues, are:

$$\begin{aligned}
\hat{\kappa}_{D,1} &\approx 3.14159\ldots & \hat{\kappa}_{N,1} &\approx 1.16556\ldots \\
\hat{\kappa}_{D,2} &\approx 4.49341\ldots & \hat{\kappa}_{N,1} &\approx 2.46054\ldots \\
\hat{\kappa}_{D,3} &\approx 5.76369\ldots & \hat{\kappa}_{N,1} &\approx 3.63280\ldots \\
\hat{\kappa}_{D,4} &\approx 6.28319\ldots & \hat{\kappa}_{N,1} &\approx 4.60422\ldots \\
\hat{\kappa}_{D,5} &\approx 6.98793\ldots & \hat{\kappa}_{N,1} &\approx 6.02929\ldots
\end{aligned}$$

The next step is to compute analytical solutions, which we can use as a reference for the computed discrete solutions. Let $\Omega_1 = B_1(0) \subset \mathbb{R}^3$ be the unit sphere, then, for any $(\kappa_1, \kappa_0) \in \mathbb{R}^2_+$, solutions of the Helmholtz equation are given by

$$\begin{aligned}
u_1(x) &:= \cos(\kappa_1 x_1) && \text{for } x \in \Omega_1, \\
u_0(x) &:= \frac{e^{i\kappa_0|x|}}{|x|} && \text{for } x \in \Omega_0.
\end{aligned}$$

Note that the exterior solution fulfils the Sommerfeld radiation condition (4.5), hence (u_1, u_0) solves the model problem (4.1)–(4.5) with the transmission data

$$f(x) := u_1(x) - u_0(x) \quad \text{and} \quad g(x) := n(x) \cdot (\nabla u_1(x) - \nabla u_0(x)) \quad \text{for } x \in \Gamma.$$

Now we are able to compute approximate solution for the problem described above and study their behaviour as the mesh size decreases. This is described by the order of convergence s that tells us if the mesh size h gets smaller, the error decreases like h^s. For the numerical examples we give the estimated order of convergence (eoc)

$$\text{eoc} := \frac{\log \|u - u_{h_n}\| - \log \left\|u - u_{h_{n+1}}\right\|}{\log h_n - \log h_{n+1}},$$

which is compared to the theoretical results from Remark 4.13. Instead of the mesh size h, we use the number of triangles N to describe the different levels of meshes. Note that in the three–dimensional case these characteristics behave as

$$h \sim \sqrt{\frac{1}{N}}.$$

We consider a mesh with N triangles and M nodes that approximates the unit sphere $\Omega_1 \approx B_1(0) \subset \mathbb{R}^3$. First we consider the wave numbers $\kappa_0 = 1$ and $\kappa_1 = 2.46054$, which means κ_1^2 is close to a Neumann eigenvalue. The results for this scenario are given in Tables 6.1–6.3. Tables 6.4–6.6 contain the results for $\kappa_0 = 1$ and $\kappa_1 = 3.14159$, which corresponds to a Dirichlet eigenvalue.

In both cases all three formulations yield the results suggested by Remark 4.13, i.e. quadratic convergence for the Dirichlet datum and linear convergence for the Neumann datum. Moreover, the results of the three formulations are virtually identical, even more so if we consider finer meshes. Although further results are not included, an increase of the absolute error has been observed if the interior wave number increases. This modification, however, does not affect the order of convergence.

N	M	$\|u_1 - u_{1,h}\|_{L_2(\Gamma)}$	eoc	$\|t_1 - t_{1,h}\|_{L_2(\Gamma)}$	eoc
48	26	1.858e-01	-	1.090e+00	-
192	98	5.358e-02	1.79	5.543e-01	0.98
768	386	1.293e-02	2.05	2.648e-01	1.07
3072	1538	3.196e-03	2.02	1.308e-01	1.02
12288	6146	7.952e-04	2.01	6.524e-02	1.00
theory			2		1

Table 6.1: SP, sphere, $\kappa_0 = 1.0$, $\kappa_1 = 2.46054$

N	M	$\|u_1 - u_{1,h}\|_{L_2(\Gamma)}$	eoc	$\|t_1 - t_{1,h}\|_{L_2(\Gamma)}$	eoc
48	26	1.862e-01	-	1.089e+00	-
192	98	5.390e-02	1.79	5.540e-01	0.98
768	386	1.298e-02	2.05	2.647e-01	1.07
3072	1538	3.200e-03	2.02	1.308e-01	1.02
12288	6146	7.957e-04	2.01	6.524e-02	1.00
theory			2		1

Table 6.2: STF, sphere, $\kappa_0 = 1.0$, $\kappa_1 = 2.46054$

N	M	$\|u_1 - u_{1,h}\|_{L_2(\Gamma)}$	eoc	$\|t_1 - t_{1,h}\|_{L_2(\Gamma)}$	eoc
48	26	1.858e-01	-	1.090e+00	-
192	98	5.373e-02	1.79	5.545e-01	0.97
768	386	1.295e-02	2.05	2.648e-01	1.07
3072	1538	3.198e-03	2.02	1.308e-01	1.02
12288	6146	7.954e-04	2.01	6.524e-02	1.00
theory			2		1

Table 6.3: MTF, sphere, $\kappa_0 = 1.0$, $\kappa_1 = 2.46054$

N	M	$\|u_1 - u_{1,h}\|_{L_2(\Gamma)}$	eoc	$\|t_1 - t_{1,h}\|_{L_2(\Gamma)}$	eoc
48	26	2.431e-01	-	1.223e+00	-
192	98	8.963e-02	1.44	7.404e-01	0.72
768	386	2.090e-02	2.10	3.396e-01	1.12
3072	1538	5.101e-03	2.03	1.658e-01	1.03
12288	6146	1.265e-03	2.01	8.243e-02	1.01
theory			2		1

Table 6.4: SP, sphere, $\kappa_0 = 1.0$, $\kappa_1 = 3.14159$

N	M	$\|u_1 - u_{1,h}\|_{L_2(\Gamma)}$	eoc	$\|t_1 - t_{1,h}\|_{L_2(\Gamma)}$	eoc
48	26	2.443e-01	-	1.223e+00	-
192	98	9.025e-02	1.44	7.403e-01	0.72
768	386	2.098e-02	2.10	3.395e-01	1.12
3072	1538	5.108e-03	2.04	1.658e-01	1.03
12288	6146	1.266e-03	2.01	8.243e-02	1.01
theory			2		1

Table 6.5: STF, sphere, $\kappa_0 = 1.0$, $\kappa_1 = 3.14159$

N	M	$\|u_1 - u_{1,h}\|_{L_2(\Gamma)}$	eoc	$\|t_1 - t_{1,h}\|_{L_2(\Gamma)}$	eoc
48	26	2.442e-01	-	1.224e+00	-
192	98	9.026e-02	1.44	7.407e-01	0.72
768	386	2.098e-02	2.11	3.396e-01	1.13
3072	1538	5.107e-03	2.04	1.658e-01	1.03
12288	6146	1.266e-03	2.01	8.243e-02	1.01
theory			2		1

Table 6.6: MTF, sphere, $\kappa_0 = 1.0$, $\kappa_1 = 3.14159$

6.2 Convergence study - Cube

For the second considered domain, the unit cube $\Omega_1 = (0,1)^3$, we have to shift the pole of the exterior solution from the origin to some point $\tilde{x}$ inside of Ω_1. Let $\tilde{x} \in \Omega_1$, then solutions of the local partial differential equations are given by

$$u_1(x) := \cos(\kappa_1 x_1) \quad \text{for } x \in \Omega_1,$$
$$u_0(x) := \frac{e^{i\kappa_0|x-\tilde{x}|}}{|x-\tilde{x}|} \quad \text{for } x \in \Omega_0.$$

For our computations we choose the pole of u_0 to be the center of the unit cube $\tilde{x} = (0.5, 0.5, 0.5)$.

Similar to the unit sphere we can compute the eigenvalues for a cuboid explicitly [10, Section 3.1]. Let $(n_1, n_2, n_3) \in \mathbb{N} \setminus \{0\}$, then

$$u_\lambda(x) := \sin(n_1\pi x_1)\sin(n_2\pi x_2)\sin(n_3\pi x_3)$$

is a Dirichlet eigenfunction with the eigenvalue $\lambda = \pi^2(n_1^2 + n_2^2 + n_3^2)$ and the corresponding wave number $\kappa = \sqrt{\lambda}$. For the same eigenvalue a Neumann eigenfunction is given by

$$u_\mu(x) := \cos(n_1\pi x_1)\cos(n_2\pi x_2)\cos(n_3\pi x_3).$$

Note that additional Neumann eigenfunctions and eigenvalues can be found if we allow n_i to be zero as long as at least one n_i is greater than zero.

If a wave number belongs to a Dirichlet eigenvalue, it automatically also belongs to a Neumann eigenvalues, thus we do not consider Neumann eigenvalues separately.

The exterior wave number is set to be $\kappa_0 = 1.0$ and the interior wave number is chosen close to the smallest wave number that belongs to a Dirichlet and a Neumann eigenvalue $\kappa_1 = 5.4414 \approx \sqrt{3}\pi$. Preconditioning is of no concern in this section, hence we can use boundary elements of lowest order, that are $S_h^1(\Gamma)$ for $H^{1/2}(\Gamma)$ and $S_h^0(\Gamma)$ to approximate $H^{-1/2}(\Gamma)$.

The results for the unit cube are given in the tables 6.7–6.9. As for the sphere, we get the maximal possible convergence rate we expect from Remark 4.13. The discontinuity of the normal derivative across the edges has no effect on the order of convergence since the regularity is only required piecewise, i.e. on the faces of the cube. As opposed to the unit sphere we see slight differences in the errors between the formulations, indicating that their accuracy could vary for more complex geometries,.

N	M	$\|u_1 - u_{1,h}\|_{L_2(\Gamma)}$	eoc	$\|t_1 - t_{1,h}\|_{L_2(\Gamma)}$	eoc
72	38	4.369e-01	-	2.154e+00	-
288	146	1.008e-01	2.12	1.315e+00	0.71
1152	578	2.067e-02	2.29	4.912e-01	1.42
4608	2306	4.735e-03	2.13	1.647e-01	1.58
18432	9218	1.133e-03	2.06	7.818e-02	1.07
theory			2		1

Table 6.7: SP, cube, $\kappa_0 = 1.0$, $\kappa_1 = 5.4414$

N	M	$\|u_1 - u_{1,h}\|_{L_2(\Gamma)}$	eoc	$\|t_1 - t_{1,h}\|_{L_2(\Gamma)}$	eoc
72	38	3.889e-01	-	1.595e+00	-
288	146	9.967e-02	1.96	9.499e-01	0.75
1152	578	2.075e-02	2.26	3.759e-01	1.34
4608	2306	4.800e-03	2.11	1.474e-01	1.35
18432	9218	1.153e-03	2.06	6.856e-02	1.10
theory			2		1

Table 6.8: STF, cube, $\kappa_0 = 1.0$, $\kappa_1 = 5.4414$

N	M	$\|u_1 - u_{1,h}\|_{L_2(\Gamma)}$	eoc	$\|t_1 - t_{1,h}\|_{L_2(\Gamma)}$	eoc
72	38	4.092e-01	-	2.222e+00	-
288	146	9.658e-02	2.08	1.372e+00	0.70
1152	578	2.040e-02	2.24	5.027e-01	1.45
4608	2306	4.707e-03	2.12	1.709e-01	1.56
18432	9218	1.129e-03	2.06	7.224e-02	1.24
theory			2		1

Table 6.9: MTF, cube, $\kappa_0 = 1.0$, $\kappa_1 = 5.4414$

6.3 Preconditioning and iterations - Sphere

In this section we discuss how the number of elements and the wave numbers influence the number of iteration steps required for a given accuracy. Additionally, we apply the preconditioners derived in Chapter 4 to see how they perform in comparison to the original system.

The transmission data (f, g) are provided by the analytical solutions presented in Section 6.1. As discrete test and trial spaces we use piecewise linear, globally continuous functions for $H^{1/2}(\Gamma)$ as well as $H^{-1/2}(\Gamma)$. Lemma 3.6 tells us that this is a stable pair, i.e. the stability condition (3.7) is satisfied, and we can apply the preconditioning strategy presented in Section 3.4. All systems are solved using GMRES to a relative accuracy of `1e-7`.

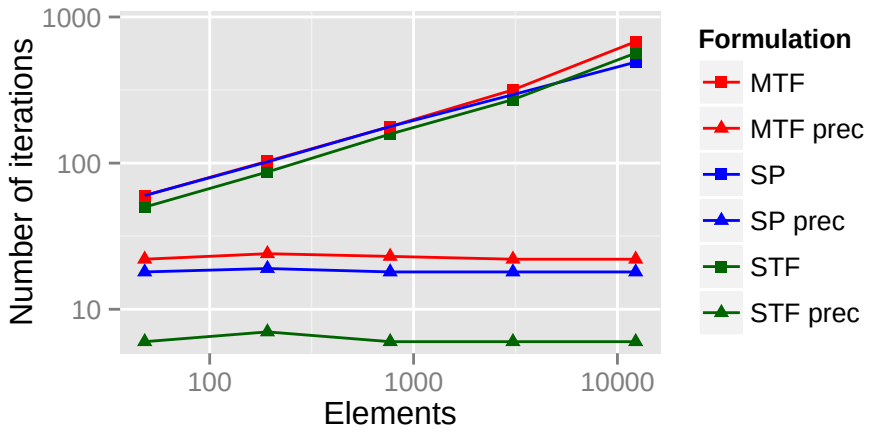


Figure 6.1: Sphere, $\kappa_0 = 1.0$, $\kappa_1 = 2.0$

First we compare the original formulations and the preconditioned systems for fixed wave numbers. Note that we used a logarithmic scale for the number of iteration steps and number of elements. Figure 6.1 shows the results for a small jump of the wave numbers from $\kappa_0 = 1$ to $\kappa_1 = 2$. As one would expect, the number of iteration steps required increases with the number of elements in the mesh. The preconditioned systems do not show this behaviour as their iterations are virtually constant at $(22, 16, 6)$ for the local multi trace formulation, the Steklov–Poincaré operator formulation and the single trace formulation.

When we look at Figure 6.2, which shows the results for a larger jump from $\kappa_0 = 1$ to $\kappa_1 = 6.5$, two points are of interest. The first point is that the numbers of required iteration steps are larger than in the previous example. The single and multi trace formulation without preconditioning even abort because the maximum of 1000 iteration steps is reached. The second noticeable point is that the preconditioning strategy seems to be more efficient for finer meshes, giving us decreasing iteration numbers if the number of elements increase.

The reason for this could be that the theory only allows statements concerning

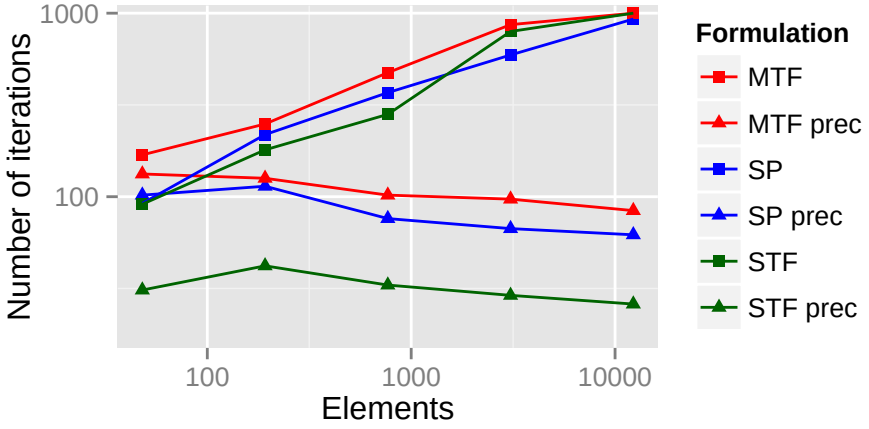


Figure 6.2: Sphere, $\kappa_0 = 1.0$, $\kappa_1 = 6.5$

asymptotic behaviours. This means that the results, like the boundedness of the condition numbers of the preconditioned systems, are only guaranteed if the mesh size h is small enough. What qualifies as small enough can generally not be determined in advance and does of course depend on the parameters such as the wave numbers.

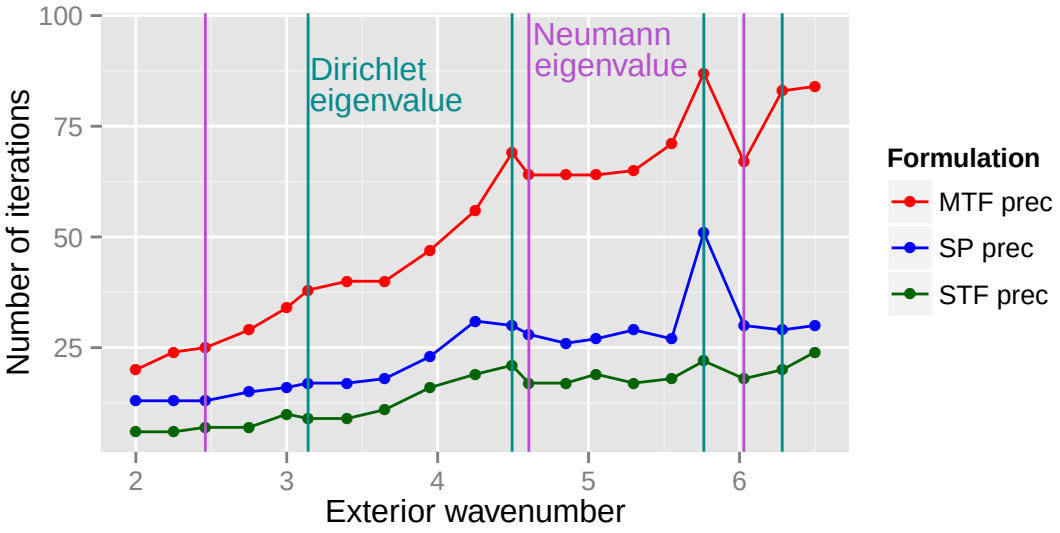


Figure 6.3: Sphere, 3072 elements, $\kappa_1 = 1.0$

Since preconditioning proved useful, we only consider the preconditioned formulations further and study how the number of required iteration steps depends on the wave numbers. We consider a mesh with 3072 elements and one variable wave number whereas the second wave number is set to 1.0. Figure 6.3 shows the results for a variable exterior wave number and Figure 6.4 for a variable interior wave number. Additionally, we mark wave numbers that are close to critical wave numbers with

vertical lines.

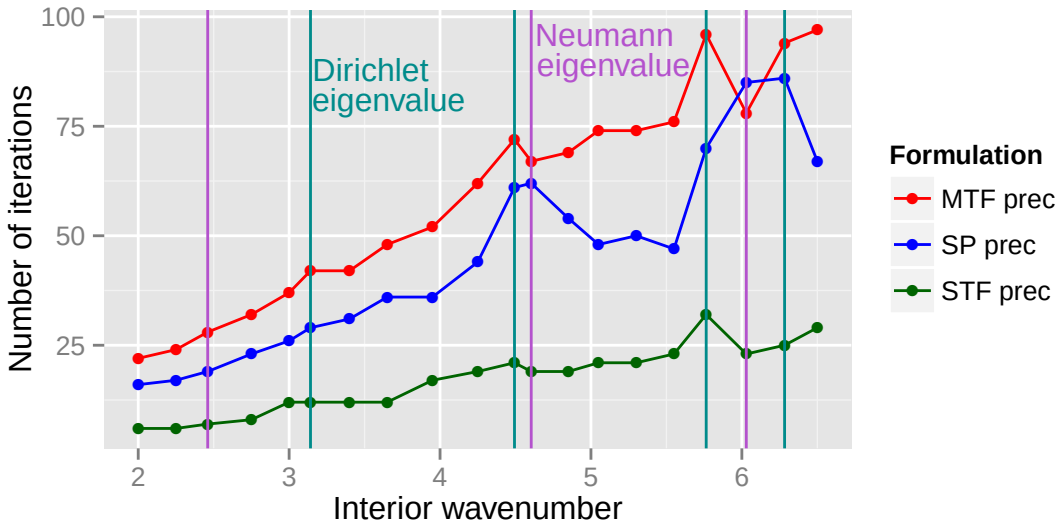


Figure 6.4: Sphere, 3072 elements, $\kappa_0 = 1.0$

We see that the required iteration steps increase with the wave numbers. This could possibly be counteracted if we consider a finer mesh since preconditioning seems to be more efficient in that case. Furthermore, eigenvalues of the Laplace operator apparently have some influence on the convergence of the GMRES algorithm. This behaviour is most apparent for the Steklov–Poincaré operator formulation in Figure 6.4. These spikes in iterations seem to be more distinctive for higher wave numbers, which indicates that the mesh is not fine enough for the preconditioners to perform optimally.

6.4 Preconditioning and iterations - Cube

This section is a continuation of Section 6.3, with the difference being that the considered domain is the unit sphere $\Omega_1 = (0,1)^3$. Once more we discuss how the number of degrees of freedom, the wave numbers and the use of preconditioners influence the required iteration steps.

The analytical solution is the same function we used in Section 6.2 to study the order of convergence. As discrete test and trial spaces we use piecewise linear, globally continuous functions $S_h^1(\Gamma_h)$ for $H^{1/2}(\Gamma)$. To discretize $H^{-1/2}(\Gamma_h)$ we use piecewise constant functions defined on the dual grid $S_h^0(\widetilde{\Gamma}_h)$ as presented at the end of Section 3.4. This is a stable pair, i.e. the stability condition (3.7) is satisfied, and we can apply the preconditioning strategy presented in Section 3.4. All systems are solved using GMRES with a relative accuracy of `1e-7`.

Figure 6.5 shows the results comparing the number of triangles N and the required iteration steps. Similar to the results in Section 6.3, all three formulations are ill con-

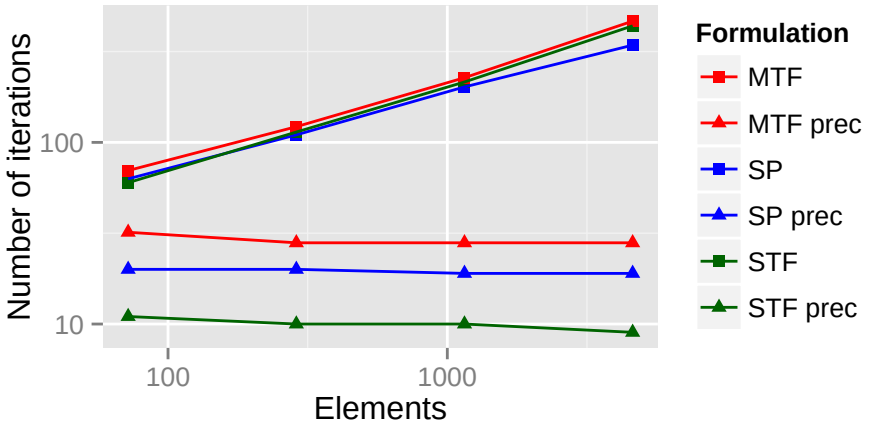


Figure 6.5: Cube, $\kappa_0 = 1.0$, $\kappa_1 = 2.0$

ditioned which leads to an increase in iterations. No such behaviour can be observed for the preconditioned formulation as they have virtually constant iteration steps for all meshes.

The second numerical experiment, shown in Figure 6.6, shows the influence of the interior wave number on the required iteration steps. Since preconditioning proved useful, only preconditioned formulations are considered for this example. The behaviour mimics the one observed for the sphere, meaning the number of iterations increases with the interior wave number κ_1. An additional effect of wave numbers corresponding to eigenvalues can not be observed, possibly because the values of κ_1 are too small.

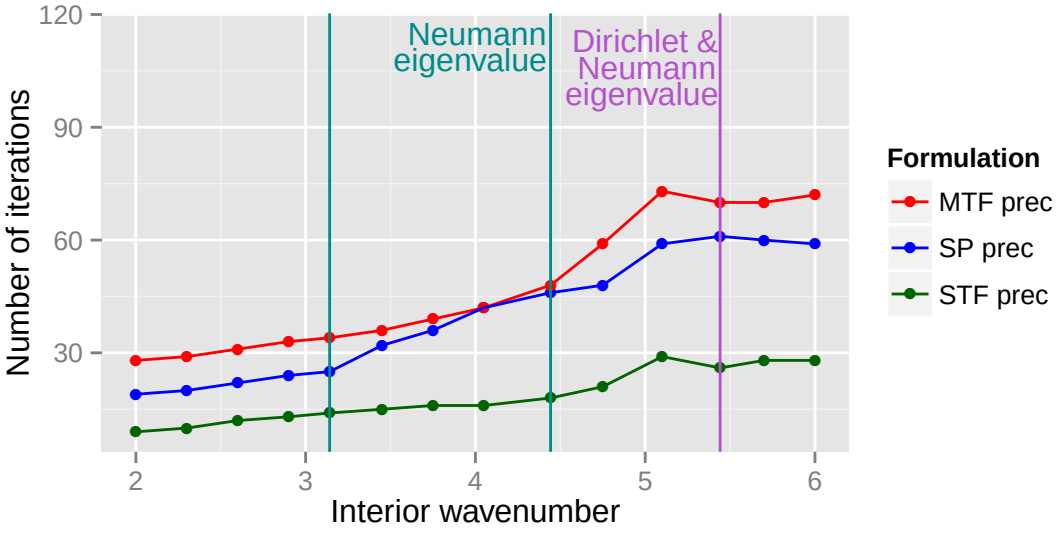


Figure 6.6: Cube, 4608 elements, $\kappa_0 = 1.0$

6.5 Thin geometries - Cuboid

The last example we consider is a cuboid with a variable height ℓ, i.e $\Omega_1 = (0,1) \times (0,1) \times (0,\ell)$. We are interested if the thickness of the geometry affects the number of required iteration steps.

For the analytical solution we consider

$$\begin{aligned} u_1(x) &:= \cos(\kappa_1 x_1) \quad \text{for } x \in \Omega_1, \\ u_0(x) &:= 0 \qquad\qquad\quad \text{for } x \in \Omega_0 \end{aligned}$$

with the wave numbers $\kappa_1 = 2$ and $\kappa_0 = 1$. Since the geometry is different for every ℓ, we fix the mesh size $h = 0.04$ and generate an according mesh for each ℓ. The remaining set-up is the same as in Section 6.4.

Table 6.10 shows the results for $\ell \in [0.04, 0.4]$ with step size 0.04. We see that for the preconditioned systems the number of required iterations steps stay virtually constant. When we consider the original systems, we notice a slight increase for (STF,MTF) or decrease (SP) of iterations from the thickest to the thinnest cuboid.

Recall from Sections 6.3–6.4 that more degrees of freedom lead to an increase in required iteration steps. This means that, if we would compute the thinnest geometry with the same amount of element as the thickest geometry (5988), the iterations would increase significantly for the formulations without preconditioning. Hence, it seams that at least the single trace formulation and the multi trace formulation are affected by the thinness of the geometry. To summarize, this numerical example indicates that the preconditioned formulations are not affected by thin geometries whereas for the original systems the required iteration steps increase.

ℓ	N	M	MTF	MTF prec	SP	SP prec	STF	STF prec
0.04	3784	1894	771	25	307	16	762	10
0.08	3808	1906	689	25	298	16	542	10
0.12	4134	2069	669	26	309	17	513	10
0.16	4408	2206	651	26	328	17	508	10
0.20	4692	2348	655	24	338	16	509	9
0.24	4942	2473	626	28	349	18	507	11
0.28	5326	2665	636	24	360	17	513	9
0.32	5488	2746	646	25	371	17	522	9
0.36	5862	2933	674	25	383	17	532	9
0.40	5988	2996	677	25	393	17	537	9

Table 6.10: $\Omega_1 = (0,1) \times (0,1) \times (0,\ell)$, $h \approx 0.04$, $\kappa_0 = 1.0$, $\kappa_1 = 2.0$

Conclusion

This thesis presented four direct formulations to solve the Helmholtz transmission problem. For three formulations unique solvability was shown independent of the wave numbers whereas we had to exclude spurious modes for the interior Steklov–Poincaré operator formulation. This formulation was therefore neglected in further considerations. A Galerkin scheme was presented for discretization, which resulted in optimal orders of convergence for all formulations. Since the systems of linear equations were ill conditioned, the formulations were studied regarding their compatibility with operator preconditioning. Only the local multi trace formulation proved compatible for composite structures whereas we had to restrict ourself to single scatterers for the other formulations. Finally, we confirmed the theoretical results with numerical examples. The next paragraphs give some topics for subsequent studies.

Operator preconditioning for the single trace formulation and the Steklov–Poincaré operator formulation is only applicable in the special case of a homogeneous scatterer. The local multi trace formulation is compatible with operator preconditioning for arbitrary number of domains. It is still an open question if this restriction can be eliminated, which would give us preconditioners for all formulations and composite structures.

The influence of wave numbers on the number of required iterations was only considered briefly and large wave number were excluded completely. From our numerical experiments we can already detect a common weakness of all three formulations, which is that larger wave numbers lead to ill conditioned systems. Operator preconditioning can help with this problem but needs a fine enough mesh to reach its full potential. The larger the wave numbers are, the smaller the mesh size h has to be.

This and the consideration of more complex geometries, for example from real life applications, demand larger meshes with more elements. In order to keep computation time within reasonable bounds, parallel computation and fast boundary element method techniques should be considered more thoroughly.

The last proposed topic is to consider other partial differential equations. Of particular interest are the Maxwell equations, which can be used to model electromagnetic scattering. Research regarding this topic has been conducted in [5, 22], where all formulations except the Steklov–Poincaré operator formulation are discussed for the Maxwell case.

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