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Boundary algebra of a GL_2-web

MASTERARBEIT

zur Erlangung des akademischen Grades

Diplom-Ingenieur/in

Mathematische Computerwissenschaften

eingereicht an der

Technischen Universität Graz

Betreuer/in:
Prof. Dr. Karin Baur

Institut für Mathematik und wissenschaftliches Rechnen
Universität Graz

Graz, Juli 2015

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Acknowledgements

The author wishes to thank first and foremost his supervisor Prof. Dr.Karin Baur for the great help and support she provided.

Contents

1 Introduction

The introducing section of this work is mainly based on ideas taken from Zelevinsky [2007] and Fomin et al. [2007][§ 1].

This thesis deals with the connection of a specific algebraic structure (boundary algebra) with the geometric and combinatorial properties of triangulations. Although the term *cluster algebra* itself is not mentioned below, it is worth giving a brief and informal description, as it is the underlying topic of all further considerations.

Definition 1.1 (Cluster algebra). Cluster algebras are constructively defined commutative rings equipped with a distinguished set of generators (cluster variables) grouped into overlapping subsets (clusters) of the same finite cardinality.

At this point it is necessary to mention that the unusual feature of cluster algebras is that both generators and algebraic relations among them are not given from the outset but are produced by an iterative process of *seed mutations*. These mutations can also be described by matrices.
Among these algebras one finds coordinate rings of many algebraic varieties that play a role in representation theory[1] or the study of total positivity[2] for example. Since its inception by Fomin and Zelevinsky and especially after their second paper concerning the *finite type classification* Fomin and Zelevinsky [2003], the theory of cluster algebras has found a number of exciting connections and applications. One of them are quiver representations[3], which provide a whole amount of essential settings such as *quivers*, *potentials* or *boundary algebras* used in this thesis.
A cluster algebra is built around a combinatorial scaffolding formed by *exchange matrices* which are related to each other by *matrix mutations*. A basic observation by Goncharov[4] showed that this kind of structure arises, when one considers *signed adjacency matrices* associated with triangulations of an orientated surface containing vertices at a fixed set of *marked points* are obsereved. More specifically, matrix mutations arise as transformations of signed adjacency matrices that correspond to *flips* of triangulations.
For a (german) introduction to the connection between polygons and cluster algebras, it is recommended to read Baur [2011].

[1] Barot [2015]
[2] Fomin [2010]
[3] Derksen et al. [2010]
[4] V.V.Fock and Goncharov [2009]

Nice results due to cluster algebras (of finite type) and combinatorial and geometric structures are stated for *generalized associahedrons* (a kind of polytope) in Chapoton et al. [2002].

The used notion of GL_m-webs is a special case of ideal webs which appear in this unpublished[5] work of Goncharov. For a further description and discussion (online) about ideal webs the reader is recommended to have a look at Williams [2014].

[5]Goncharov describes ideal webs at the workshop *Integrability and Cluster Algebras: Geometry and Combinatorics* in August 2014 held by Institute for Computational and Experimental Research in Mathematics (ICERM) at Brown University in Rhode Island, United States. A video of his lecture is available online A.B.Goncharov [2014]

2 General settings

The first part of the thesis deals with the construction of a boundary algebra, starting with a regular convex n-gon. We will consider a triangulation of this polygon and define a GL_m-web of this triangulation. By using a structure called dimer on this web, we can construct a quiver. This leads to a so-called dimer algebra and finally to a boundary algebra of the GL_m-web of the n-gon.

2.1 GL_2-web

Already well known is the definition of the triangulation of a n-gon:

Definition 2.1 (triangulation). A *triangulation* of a regular convex polygon is a subdivision of the n-gon into triangles, where each pair of segments intersect in one of the vertices of the polygon at most.

Remark. A triangulation consists of the $n-$ edges of the n-gon and $n-3$ diagonals of the polygon. Furthermore it is a maximal collection of non-crossing diagonals.

Remark. A special case of triangulation is the so called *fan triangulation*, where each diagonal of the triangulation contains the same vertex of the polygon.

The next step is to define a special kind of partition of each triangle of an arbitrary triangulation of the polygon:

Definition 2.2 (GL_m-web). For a triangle t, we create a web by dividing it with three times $m-1$-lines parallel to the sides of the triangle, where each two of them intersect on the sides. We do this for every triangle and connect the parallel lines. The result is called a GL_m-web.

Figure 1 is an example of a GL_m-web.

This thesis deals with the GL_2-web of the polygon that is created by connecting the midpoints of each triangle of the triangulation of the n-gon.
So from now on let $m = 2$.

2.2 Dimer

We apply the following procedure to each triangle of the triangulation, cf. Figure 2:

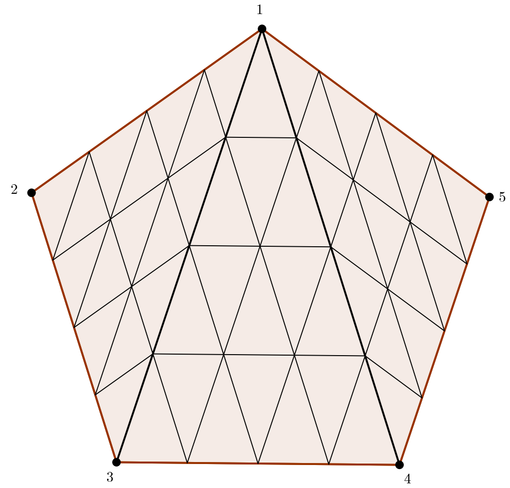

Figure 1: This is a Gl_4 web of the fan triangulation of a pentagon

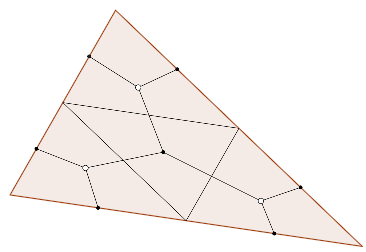

Figure 2: Construction of a dimer for each triangle of a triangulation.

- Put black points on the midpoints of the original sides of the triangle and another one inside the inner triangle of the GL_2-web.
- Put a white point into every other triangle.
- Two points are connected if they differ in color and the points belong to the same triangle or their according triangles have least one side in common.

The resulting object is called *dimer*, Figure 3 shows a dimer (of a GL_2-web) of a fan triangulation of a pentagon.

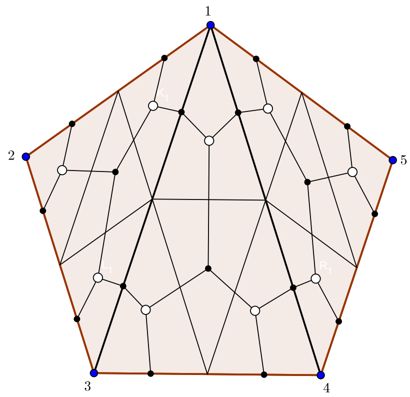

Figure 3: Dimer of a fan triangulation of a pentagon

Remark. Note that the resulting graph is bipartite and splits the original polygon into several areas.

2.3 Quiver

By using the property that the graph is bipartite we want to define an algebraic structure on this dimer. This is done by a so called *quiver*.

Definition 2.3 (Quiver). A quiver is a quadruple $Q = (Q_0; Q_1; s; t)$, where Q_0 is the set of vertices, Q_1 is the set of arrows and s; t are two maps $Q_1 \to Q_0$, assigning the starting vertex and the terminating vertex to each arrow.
A quiver Q is finite if Q_0 and Q_1 are finite sets.

The following step of the construction is based on the same idea used by Baur et al. [2014] for a different type of graph.

The idea is to put a vertex in each area of the dimer that is bounded by the sides of the original polygon and edges between the white and black points. Then each adjacent area is connected by an arrow such that the white point of the dimer

is on the left hand side of the arrow, shown in Figure 4.
We will also use the notion dimer for the associated quiver as the two determine each other.

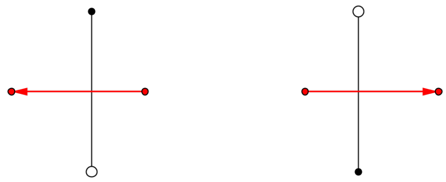

Figure 4: The white point of the dimer is on the left hand side of the arrow

The resulting graph is a *quiver*, in general denoted by Q.

In order to define dimer algebras, we need to introduce the notion of a *quiver with faces*. Given a quiver Q, we write Q_{cyc} for the set of oriented cycles in Q (up to cyclic equivalence).

Definition 2.4 (Quiver with faces). A quiver with faces is a quiver $Q = (Q_0; Q_1; s; t)$ together with a set Q_2 of faces and a map

$$\partial : Q_2 \to Q_{cyc},$$

which assigns to each $F \in Q_2$ its boundary $\partial F \in Q_{cyc}$.

We will always denote a quiver with faces by the same letter Q, regarded now as the tuple $(Q_0; Q_1; s; t; Q_2)$. Again a quiver is called finite if Q_0,Q_1 and Q_2 are finite sets.

Definition 2.5 (Dimer model with boundary). A (finite, oriented) dimer model with boundary is given by a finite quiver with faces $Q = (Q_0; Q_1; s; t; Q_2)$ where Q_2 is written as disjoint union $Q_2 = Q_2^+ \cup Q_2^-$, satisfying the following properties:

(a) the quiver Q has no loops, i.e. no 1-cycles, but 2-cycles are allowed,

(b) all arrows in Q_1 have face multiplicity 1 (boundary arrows) or 2 (internal arrows),

(c) each internal arrow lies in a cycle bounding a face in Q_2^+ and in a cycle bounding a face in Q_2^-,

(d) the incidence graph of Q at each vertex is connected.

Note that Q_2^+ and Q_2^- are the set of all cycles which are oriented counterclockwise and clockwise respectively.

2.4 Dimer algebra and boundary algebra

The upcoming section mainly contains ideas of Baur et al. [2014][p.11-13].

Definition 2.6 (Natural potential W). Let $Q = (Q_0, Q_1, Q_2)$ be the quiver with faces, which leads to a dimer model with boundary. Then there exists a natural potential W by the usual formula

$$W := W_Q := \sum_{\gamma \in Q_2^+} \gamma - \sum_{\gamma \in Q_2^-} \gamma$$

defined up to cyclic equivalence.

Remark (Differentiation of W). Let ∂W be the derivative with respect to all internal arrows α in Q. That means, that if α is part of the positive cycle q and the negative cycle p as shown in Figure 5 the equation

$$\begin{aligned} \frac{\partial W}{\partial \alpha} &: p = q \\ \Leftrightarrow \partial_\alpha(W) &= 0 \\ \Leftrightarrow p &\overset{\alpha}{\cong} q \end{aligned}$$

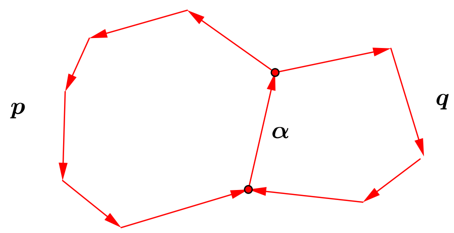

Figure 5: α is part of a positive cylce q and a negative cycle p.

holds. These relations are so-called *F-term* relations. In this thesis we use the last notation for relations obtained by the natural potential W (with respect to the corresponding arrow).

From that we receive a *dimer algebra* which is defined as given:

Definition 2.7 (Dimer algebra). Let $Q = (Q_0, Q_1, Q_2)$ be a dimer model with boundary and let W and ∂W be defined as above. Then the dimer algebra Λ_Q is defined as

$$\Lambda_Q := \mathbb{C}Q/_{\partial W}.$$

Elements of this algebra are linear combinations of the paths of the quiver. The multiplicative operation in this algebra is composition of paths.

As usual, we write e to denote an idempotent of an algebra and in the path algebra $\mathbb{C}Q$, let e_i be the trivial path of length zero at vertex i. It is an idempotent of $\mathbb{C}Q$. Define

$$e_b := e_1 + \ldots + e_t$$

where $1, \ldots, t$ are the boundary vertices of the quiver.

Remark. As part of the main result (Theorem 3.1) we will show that in the case of the quiver of the GL_2-web of an n-gon, $t = 2n$, because Q has $2n$ points on its boundary.

Definition 2.8 (Boundary algebra). The boundary algebra of a dimer model Q with boundary is the spherical subalgebra consisting of linear combinations of paths which have starting and terminating points on the boundary of the quiver (i.e. one of the idempotent elements $e_1, \ldots e_t$):

$$\mathcal{B} := e_b \Lambda_Q e_b,$$

where Λ_Q is the dimer algebra of Q.

Remark. A dimer algebra is a special case of an algebra defined by a quiver Q with commutation relations, that is a quotient $\mathbb{C}Q/\mathcal{I}$, where the ideal I is generated by $\{p_i - q_i : i \in \mathcal{I}\}$ for paths p_i and q_i with the same start and end points for each $i \in I$. Any of those algebras has a couple of elementary properties, in detail:

(a) every path in Q gives a non-zero element of $\mathbb{C}Q/\mathcal{I}$.

This is an immediate corollary of a stronger property that builds on the observation that commutation relations define a natural equivalence relation $\sim$ on the set of paths in Q, generated by requiring that $p \sim q$ if p has a sub-path p_i and q is obtained from p by replacing p_i with q_i, for some $i \in \mathcal{I}$. Then, secondly,

(b) the equivalence classes of $\sim$ form a basis of $\mathbb{C}Q/\mathcal{I}$.

Note that any equivalence class $\overline{p}$ of paths does determine a well-defined element $p + \mathcal{I}$ of $\mathbb{C}Q/\mathcal{I}$ and these elements evidently span $\mathbb{C}Q/\mathcal{I}$. To see that they are independent, observe that there is a well-defined algebra $\mathbb{C}(Q/\sim)$ with base given by the set of equivalence classes of $\sim$, multiplication given by concatenation where possible and zero otherwise, linearly extended. The natural map

$$\pi : \mathbb{C}Q \to \mathbb{C}(Q/\sim) : p \mapsto \overline{p}$$

has each $p_i - q_i$, for $i \in \mathcal{I}$, in its kernel and therefore induces a map $\overline{\pi} : \mathbb{C}Q/\mathcal{I} \to \mathbb{C}(Q/\sim)$, which is the inverse map $\overline{p} \mapsto p + \mathcal{I}$.

Remark. It is obvious that the boundary algebra has a base containing infinitely many elements, because a path composed with a cycle gives a new element. This procedure can be repeated an arbitrary number of times. The important observation is, that every element can be described by $3n$-elements as claimed in Theorem 3.1.

In general 2-cycles (with non-boundary arrows) are not of interest. They will occur because of the definitions of the dimer and the quiver. They can be omitted by using the relations obtained by the natural potential W:
Let α and β be a 2-cycle between the vertices k and l as shown in Figure 6. Furthermore, let p and q be those paths between k and j and j and k respectively, that exist because of the structure of our dimer algebra. By using the relations

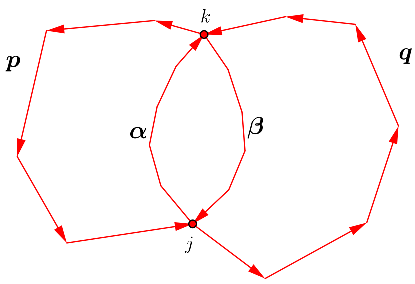

Figure 6: 2-cycle α and β in the dimer algebra.

obtained by the natural potential W we get

$$\alpha \overset{\beta}{\cong} q \overset{\beta}{\cong} p$$
$$\beta \overset{\alpha}{\cong} p \overset{\alpha}{\cong} q$$

and hence α and β can be omitted.

2.5 First result

Definition 2.9 (chordless cycle). A chordless cycle of a quiver Q is a cycle such that the full subquiver on its vertices is also a cycle.

Proposition 2.10. Let Q be the quiver of the GL_2-web of a triangulation. Let k be an arbitrary vertex of Q. Then, up to ∂W, $\gamma_1 = \gamma_2$ for any two chordless cycles γ_1 , γ_2 starting at k.

Proof. First note that Q has $2n$ vertices. We label them anticlockwise so that the vertices of Q near the vertices of the polygon have odd numbers. We will use the special structure of the reduced quiver of the GL_2-web of a triangulation. This structure is stated at Remark 4.2 in case of the fan triangulation and is also true for any triangulation, as Section 5.1 shows that a flip does not change the number of incoming or outgoing arrows for any vertex.
So every odd vertex $2k+1$ is starting and terminating point of a unique chordless cycle u_{2k+1}. The remaining vertices shall be considered in two different cases because the number of occurring cycles differs between inner vertices and boundary vertices.

(1) Let i_k be an inner vertex. The relevant part of the quiver (containing all chordless cycles at i_k) is shown in Figure 7. The outgoing arrows α_1,α_2 and the incoming arrows β_1, β_2 are part of the chordless cycles $\gamma_1, \dots, \gamma_4$,

$$\begin{aligned}\gamma_1 &= \alpha_1 p_1 \beta_1\\ \gamma_2 &= \alpha_2 p_2 \beta_1\\ \gamma_3 &= \alpha_2 p_3 \beta_2\\ \gamma_4 &= \alpha_1 p_4 \beta_2\end{aligned}$$

where $p_1, \dots, p_4$ are paths of at least length 1.
Using the relation obtained by the natural potential W for the arrows α_1,α_2 and β_1 yields

$$\begin{aligned}\gamma_1 = \alpha_1 p_1 \beta_1 &\overset{\alpha_1}{\cong} \alpha_1 p_4 \beta_2 = \gamma_4\\ \gamma_1 = \alpha_1 p_1 \beta_1 &\overset{\beta_1}{\cong} \alpha_2 p_2 \beta_1 = \gamma_2\\ \gamma_2 = \alpha_2 p_2 \beta_1 &\overset{\alpha_2}{\cong} \alpha_2 p_3 \beta_2 = \gamma_3\end{aligned}$$

and hence to

$$\gamma_1 \cong \gamma_2 \cong \gamma_3 \cong \gamma_4$$

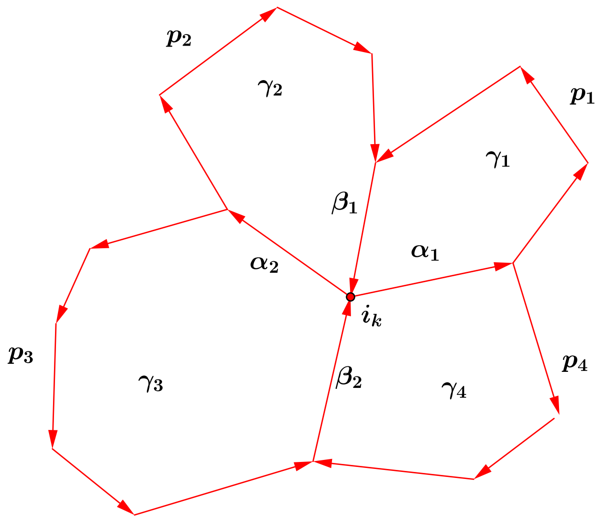

Figure 7: Part of the quiver containing i_k and all chordless cycles at i_k.

which had to be shown. These (equivalent) short cycles will be denoted by u_{i_k}.

(2) Contrary to the first case, one incoming and one outgoing arrow of the even boundary vertex is on the boundary, so one of the cycles (e.g. γ_4) does not exist. The remaining three cycles are equivalent by the same argument as above and the chordless cycles at $2k$ are denoted by u_{2k}.

□

By Proposition 2.10, all chordless cycles at a given vertex are equal and hence it makes sense to refer to any one of them as the cycle at this vertex.

Definition 2.11 (Short cycle u). The chordless cycle u_j at vertex j ($j = 1, \ldots, 2n$ or $j = i_1, \ldots, i_{n-3}$ respectively) is called short cycle at vertex j.

Remark (Notation in remaining sections). The informal notation *use the natural potential W* instead of the exact description *use the relations obtained by the natural potential W* will be used to enhance the readability of this thesis.

3 Main result

This section deals with the description of the boundary algebra for arbitrary large n, starting with any triangulation of the n-gon. The later sections will deal with several steps that are necessary to prove the claimed general properties of the algebra.

Consider the following quiver $\Gamma(n)$ on $2n$ vertices $1, 2, \dots, 2n$ and $3n$ arrows

$$\begin{array}{llll} x_i & : i-1 & \mapsto i & \text{for } i = 1, \dots, 2n \\ z_{2i} & : 2i & \mapsto 2i-2 & \text{for } i = 1, \dots, n \end{array}$$

where we reduce mod $2n$, shown in Figure 8.

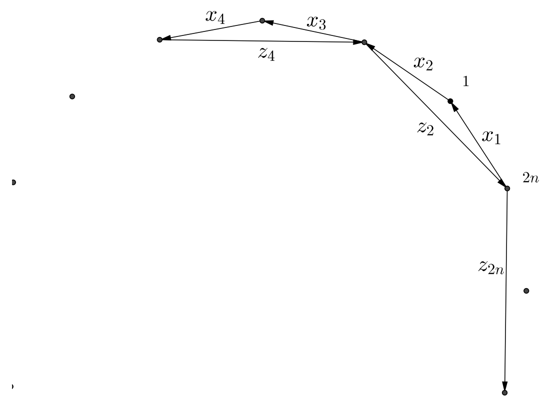

Figure 8: Part of $\mathcal{B}$ of a n-gon.

For $i = 1, \dots, n$ let

$$u_{2i} := x_{2i+1} x_{2i+2} z_{2i+2}$$
$$u_{2k+1} = x_{2k+2} z_{2k+2} x_{2k+1}$$

be the chordless short cycles at the boundary (according to Definition 2.11).
Recall that $\mathcal{B} = e_b \Lambda_Q e_b$ is the boundary algebra obtained from the quiver of a GL$_2$-web of a triangulation of an n-gon, where $e_b = e_1 + \ldots + e_{2n}$ denotes the sum of all idempotents.

Theorem 3.1 (Main Theorem). The quiver of $\mathcal{B}$ is isomorphic to $\Gamma(n)$ subject to the following relations (writing compositions of paths from left to right), for $i = 1, \ldots, n$:

$$\begin{aligned} x_{2i+1}x_{2i+2}z_{2i+2} &= z_{2i}x_{2i-1}x_{2i} \\ u_{2i}^{n-3}x_{2i+1}x_{2i+2} &= \underbrace{z_{2i}z_{2i-2}\ldots z_{2i+4}}_{n-1 \text{ factors}}. \end{aligned}$$

Furthermore the element

$$t := \sum_{i=1}^{n} x_{2i-1}x_{2i}z_{2i} + \sum_{i=1}^{n} x_{2i}z_{2i}x_{2i-1}$$

is central.

Remark. This notation will be used for comfort:

$$x^2 z = zx^2 \text{ (for every even vertex)},$$

where x^2 is an informal notation of the concatenation of 2 successive arrows x. Sometimes the informal notation u for a short cycle u_{2k} is used.

4 Boundary algebra of a fan triangulation

The goal of this chapter is to show, that the fan triangulation for every n-gon leads to the same boundary algebra $\mathcal{B}$. To get an idea of the generial case, we start with the quadrilateral and the pentagon first.

4.1 Boundary algebra for small n

For the 2 smallest non-trivial n-gons ($n = 4, 5$) we will repeat similar steps within this section: Starting with the GL_2-web of the fan triangulation (where without loss of generality each diagonal contains vertex 1), the dimer is constructed and reduced to get rid of the existing 2-cycles. After drawing the associated quiver, the boundary algebra is obtained and it can be seen that indeed it has the claimed structure.

Remark. As before, we will often use the word dimer for the associated quiver, too.

4.1.1 Quadrilateral

For this case every single step is shown, especially the act of reducing. Start with the dimer of the quadrilateral following the construction described above (Figure 9).

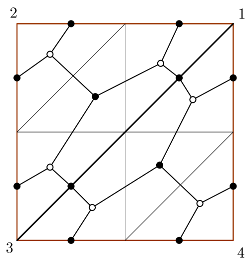

Figure 9: Dimer of a fan triangulation of a quadrilateral

Now the quiver is obtained by putting a vertex in each region and connecting adjacent regions by an arrow. Through this 2-cycles occur, if two regions share more than one (in our case two) edges of the dimer (see the vertices in circles in the figure below). The original dimer can be reduced by consolidating the points which are marked in Figure 10. This reducing step can be done for every n-gon, as often as two regions share more than one edge of the dimer. After reducing, the quiver Q in Figure 11 remains and the boundary algebra $e_b\Lambda_Q e_b$ has to be described in detail. The boundary vertices are numbered from 1 up to 8 and the single internal vertex is marked by i_1. The arrows $x_1, \dots x_8$ and z_4, z_1 are already those, which are claimed in the main theorem in the previous section. Generally, z_{2i} is defined as the path of minimal length from vertex $2i$ to vertex $2i-2$, so the remaining two paths z_6 and z_2 are defined to be the concatenation

$$z_6 = \gamma\delta$$
$$z_2 = \alpha\beta$$

writing paths from left to right.

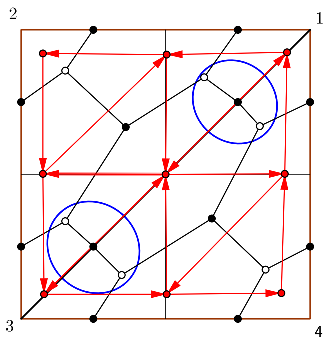

Figure 10: Original quiver of the quadrilateral

What remains to be shown, concerning the structure of the algebra, is that the elements x and z give a basis of the algebra. This means, that every element of the algebra (linear combinations of paths from any vertex $1, \dots, 8$ to another vertex on the boundary) can be described as a combination of concatenations of these elements. Table 1 shows this feature for several paths with starting vertex 1. In analogy one can show that every path of the boundary algebra is a concatenation of these twelve arrows.

Next, the relations

$$u_{2i}^{n-3} x_{2i+1} x_{2i+2} = \underbrace{z_{2i} z_{2i-2} \dots z_{2i+4}}_{n-1 \text{ factors}}$$

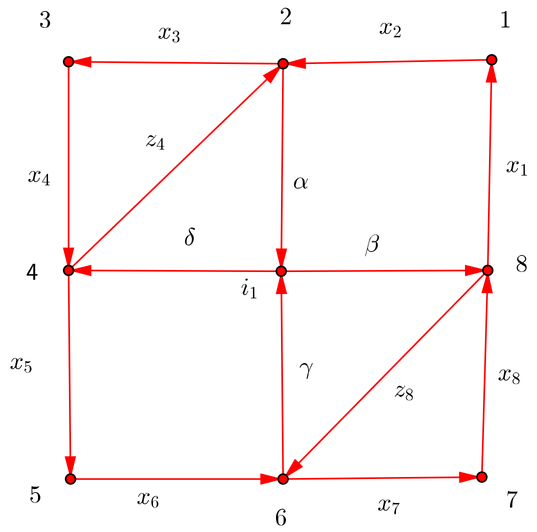

Figure 11: Reduced and labelled quiver of the quadrilateral

are proved. For this the corresponding paths from 8 to 2 are considered first. Secondly the paths from 6 to 8. The relations for the paths 4 to 6 and 2 to 4 can be shown analogously.
We consider the path $z_8z_6z_4$ from 8 to 2:

$$z_8 \underbrace{\gamma\delta}_{z_6} z_4.$$

Now, using the relation defined by β for the path $z_8\gamma$:

$$z_8\gamma \overset{\beta}{\cong} x_1x_2\alpha$$

is obtained and hence

$$z_8 \underbrace{\gamma\delta}_{z_6} z_4 \overset{\beta}{\cong} x_1x_2\alpha\delta z_4$$

and by using the relation for α

$$\delta z_4 \overset{\alpha}{\cong} \beta x_1x_2,$$

starting vertex	sink vertex	path by basis elements
1	2	x_2
1	3	x_2x_3
1	4	$x_2x_3x_4$ or equivalently $x_2z_2z_8z_6$
1	5	$x_2x_3x_4x_5$ or eq. $x_2z_2z_8z_6x_5$
1	6	$x_2x_3x_4x_5x_6$ or eq. $x_2z_2z_8$
1	7	$x_2x_3x_4x_5x_6x_7$ or eq. $x_2z_2z_8x_7$
1	8	$x_2x_3x_4x_5x_6x_7x_8$ or eq. x_2z_2

Table 1: Paths from 1 to every other vertex.

and this equation means

$$z_8 \underbrace{\gamma\delta}_{z_6} z_4 = \underbrace{x_1x_2\alpha\beta}_{u_8} x_1x_2 = u_8x_1x_2$$

which proves the claim.
Similarly, we consider the path $z_6z_4z_2$ from 6 to 8 with

$$\begin{aligned} \underbrace{\gamma\delta}_{z_6} z_4 \underbrace{\alpha\beta}_{z_2} &\overset{\delta}{\cong} \underbrace{\gamma\delta x_5x_6}_{u_5} \gamma\beta \\ &\overset{\gamma}{\cong} \gamma\beta z_8\gamma\beta \\ &\overset{z_8}{\cong} \underbrace{x_7x_8z_8}_{u_6} x_7x_8. \end{aligned}$$

So the relation (here in a loose notation)

$$z^3 = ux^2$$

holds for the quadrilateral case.

4.1.2 Pentagon

Starting with the quiver that can be obtained from the dimer of the fan triangulation of the pentagon and splitting the reduction step into 2 parts, illustrates the idea of the proof for the general case: The first step of reducing the dimer is shown in Figure 13, because of the 2-cycles which occur between the points of the marked parts on the left hand side of the pentagon. Therefore it is possible to reduce the dimer whenever two faces have more than one edge (defined by the black and white points of the dimer) in common by replacing every subgraph of the form shown on left hand side in Figure 12 to the form shown on the right hand side.

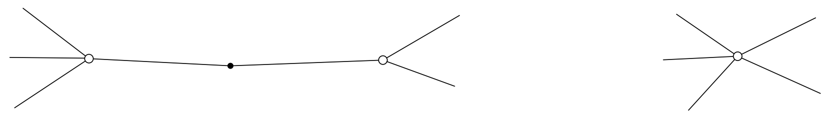

Figure 12: Replace the subgraph on the left hand side to the subgraph of the right hand side. (The number of edges incident with the white points may vary).

We have already shown how this follows from the relations of the potential W (see Section 2.4, in particular, Figure 6) . Note that this is the same reduction step, that can be conducted for the quadrilateral (i.e.$n-1$-gon) as well. The resulting dimer can be reduced a second time (see Figure 14), because the faces above and under the marked points both have two edges in common. (This step didn't occur for the quadrilateral, because it only depends on the new triangle $(1,4,5)$ (i.e. $(1,n-1,n)$) of the triangulation of the pentagon.) Hence one obtains the quiver shown in Figure 15. Using the notation of Figure 15, we now define z_8, z_6, z_2 as follows:

$$\begin{aligned} \beta_1\beta_2 &=: z_8, \text{ (a path from 8 to 6 via } i_2) \\ \gamma_1\gamma_2 &=: z_6, \text{ (a path from 6 to 4 via } i_1) \\ \alpha_1\alpha_2\alpha_3 &=: z_2, \text{ (a path from 2 to 10 passing through } i_1,\ i_2) \end{aligned}$$

and so, z_{2k} is a generator of the paths from $2k$ to $2k-2$ for $k \in [1,5]$, reducing mod 10 if necessary.

Consider the path $z_{10}z_8z_6z_4$ from 10 to 2. By the natural potential W it can

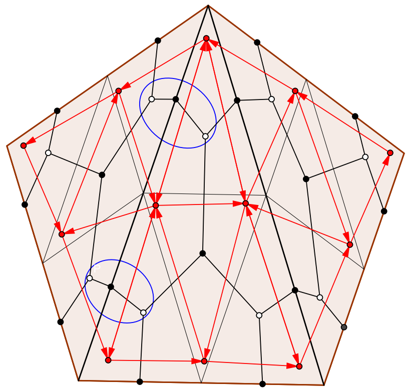

Figure 13: First reduction step of pentagon is indicated.

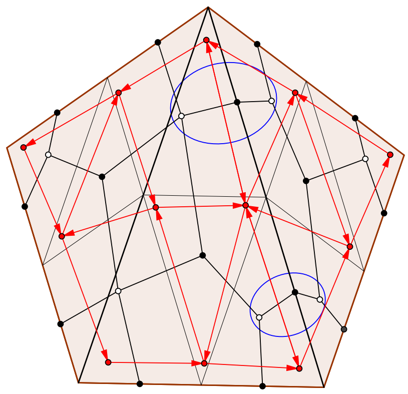

Figure 14: Second step of reduction is indicated.

be seen that

$$\gamma_2 z_4 \overset{\alpha_1}{\cong} \alpha_2\alpha_3 x_1 x_2$$
$$\beta_2\gamma_1 \overset{\alpha_2}{\cong} \alpha_3 x_1 x_2 \alpha_1$$
$$z_{10}\beta_1 \overset{\alpha_3}{\cong} x_1 x_2 \alpha_1 \alpha_2$$

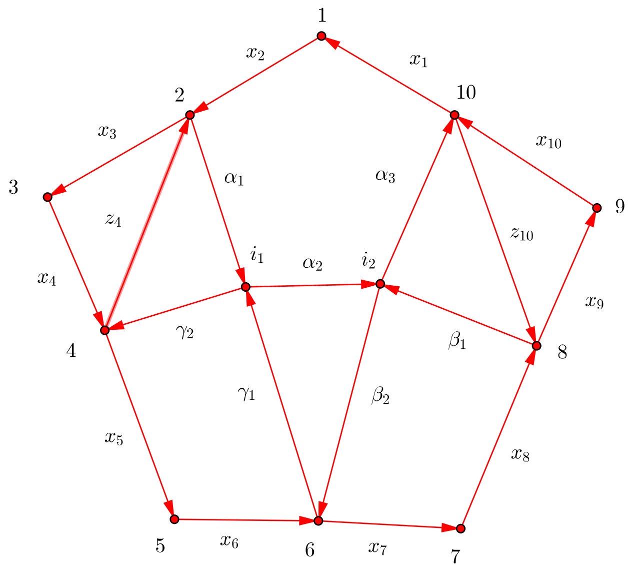

Figure 15: Labeled quiver of the pentagon.

and thus we obtain

$$z_{10}z_8z_6z_4 \cong \underbrace{x_1x_2\alpha_1\alpha_2\alpha_3x_1x_2\alpha_1\alpha_2\alpha_3}_{u_{10}^2}x_1x_2$$

and in informal notation

$$z^4 = u^2x^2$$

as claimed.
For the path from 8 to 2 it has to be shown that the equation also holds. The equations for the remaining paths $2k$ to $2k+2$ can be shown analogously.

As shown in Proposition 2.10, all chordless cycles starting at vertex k are equivalent up to ∂W.

$$\alpha_2\beta_2\gamma_1 \cong \gamma_2z_4\alpha_1 = u_{i_1}$$
$$\beta_2\gamma_1\alpha_2 \cong \alpha_3z_{10}\beta_1 = u_{i_2}.$$

Consider the path $z_8z_6z_4z_2$ from 8 to 10

$$z_8z_6z_4z_2 = \beta_1\beta_2\gamma_1 \underbrace{\gamma_2 z_4 \alpha_1}_{u_{i_1}} \alpha_2\alpha_3$$

Using the relations of the natural potential W:

$$z_{10}\beta_1 \overset{\alpha_3}{\cong} x_1x_2\alpha_1\alpha_2$$
$$\alpha_2\alpha_3x_1x_2 \overset{\alpha_1}{\cong} \gamma_2z_4$$
$$z_4\alpha_1 \overset{\gamma_2}{\cong} x_5x_6\gamma_1$$
$$\gamma_2x_5x_6 \overset{\gamma_1}{\cong} \alpha_2\beta_2$$

we obtain

$$\begin{aligned} z_8z_6z_4z_2 &\cong \beta_1\beta_2\gamma_1u_{i_1}\alpha_2\alpha_3 \\ &\cong \beta_1\ u_{i_2}^2\alpha_3. \end{aligned}$$

Since chordless cycles can be shifted along arrows they contain, we have

$$\beta_1\ u_{i_2}^2\alpha_3 = u_8^2\beta_1\alpha_3$$

and furthermore using the relation

$$\beta_1\alpha_3 \overset{z_{10}}{\cong} x_9x_{10}$$

for the path z_{10} yields

$$\cong \beta_1\ u_{i_2}^2\alpha_3 = u_8^2x_9x_{10} = u^2x^2$$

both in formal and informal notation.
The same idea can be used to show this equality for the other paths mentioned above.

Remark. **??** For an arrow $\gamma = i \to j$, the relation

$$u_i\gamma = \gamma u_j$$

holds.
We use the chordless cycle $u_i = \gamma p$ where p is an appropiate path. Hence

$$u_i\gamma = \gamma p\gamma = \gamma(p\gamma) = \gamma u_j$$

as $p\gamma = u_j$ is a chordless cycle at vertex j.

4.2 Quiver of a fan triangulation

Before describing the boundary algebra, the structure of the quiver of the fan triangulation of an n-gon for arbitrary n has to be determined.

Proposition 4.1. Let Q_F be the reduced dimer model of a fan triangulation of an n-gon, $n \geq 3$. Then Q_F has the following form:
It consists of $2n$ vertices on the boundary, labelled anticlockwise by $1, \ldots, 2n$, and $n-3$ internal vertices labelled $i_1, \ldots, i_{n-3}$.
Furthermore it has $2n+2$ arrows between the boundary vertices

$$\begin{aligned} x_k &: k-1 \to k (\text{taking endpoints} \mod 2n) \\ y_4 &: 4 \to 2 \\ y_{2n} &: 2n \to 2n-2, \end{aligned}$$

and the following internal arrows (where at least the source or the sink is an internal vertex):

$$\begin{aligned} \alpha_0 &: 2 \to i_1 & \alpha_{n-3} &: i_{n-3} \to 2n \\ \alpha_k &: i_k \to i_{k+1} & 1 &\leq k < n-3 \\ \beta_{k-1} &: i_k \to 2k+2 & 1 &\leq k \leq n-3 \\ \gamma_k &: 2k+4 \to i_k & 1 &\leq k \leq n-3 \end{aligned}$$

Proof. The proof is done by induction. We consider the fan where all k diagonals meet at vertex 1. The quivers of the quadrilateral and pentagon (as shown above) have the claimed structure.
Let the described structure be true for n fixed and consider the dimer of the fan triangulation of the $n+1$-gon. Observe that the same reduction steps can be done for the dimer of the $n+1$-gon as for the n-gon, because the only difference between the two triangulations is the additional triangle between 1, n and $n+1$, which does not change the dimer of the former n-gon. Figure 16 only shows the relevant part of the reduced dimer, i.e. the new part obtained from increasing the number of vertices of the polygon.
The white point W_1 exists after reduction for the n-gon.
As the indicated points in Figure 16 show, it is possible to reduce the new web, because the two faces I and II lead to 2-cycles (as they share two sides with the neighbored region), using the relations which follows by the relations of the natural potential W (see Figure 12 . This leads to the reduced dimer shown in Figure 17, where the new quiver (according to the construction rules) is drawn, too. It has the claimed structure.

□

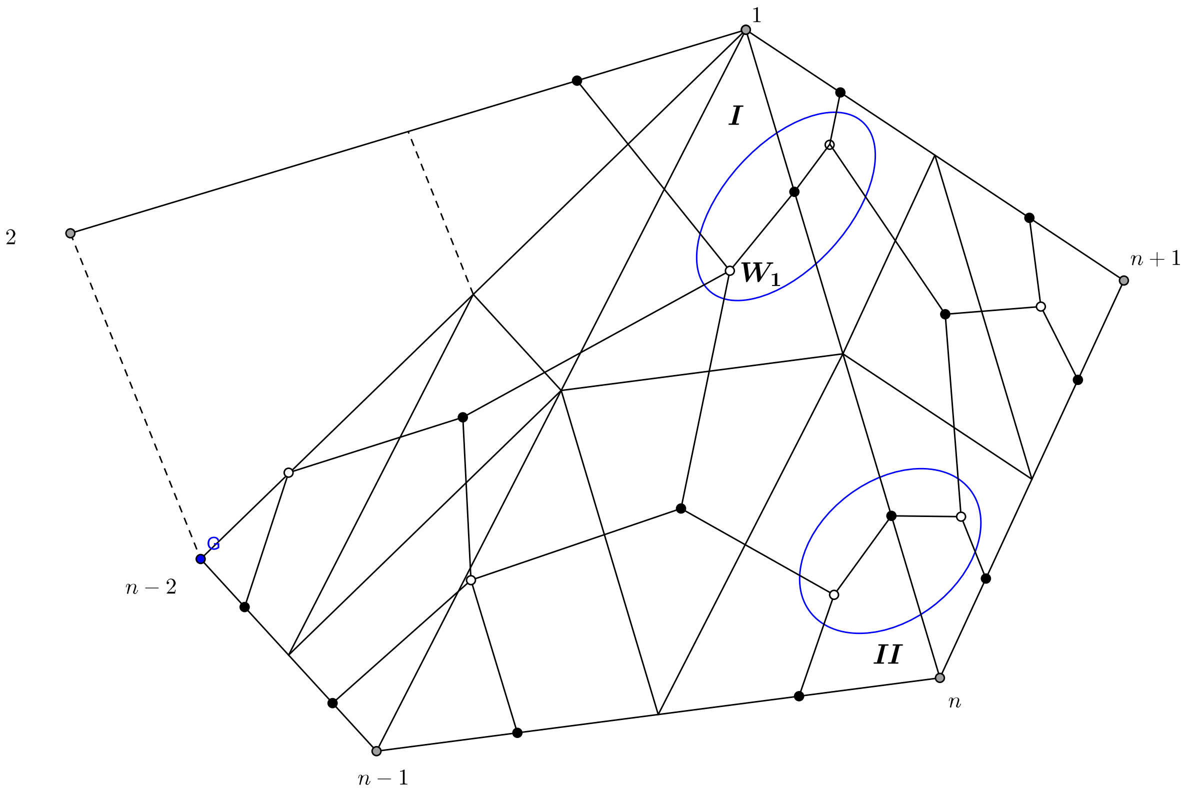

Figure 16: New part of the dimer of the $n+1$-gon.

Remark. Note that the quiver has a nice structure: Every internal point has exactly two incoming and two outgoing arrows, and there is always an oriented triangle $\alpha_k\beta_k\gamma_k$ for $1 \leq k < n-4$ (see Figure 18). This observation is useful for proving the claimed properties of the boundary algebra. Furthermore the boundary vertices with even numbers have two incoming and two outgoing arrows too, whereas the boundary vertices with odd numbers only have one incoming and one outgoing arrow, x_k and x_{k+1} respectively.

4.3 Boundary algebra of a fan triangulation

Knowing the structure of the quiver of a fan triangulation in detail, it‘s possible to describe the boundary algebra of the n-gon.

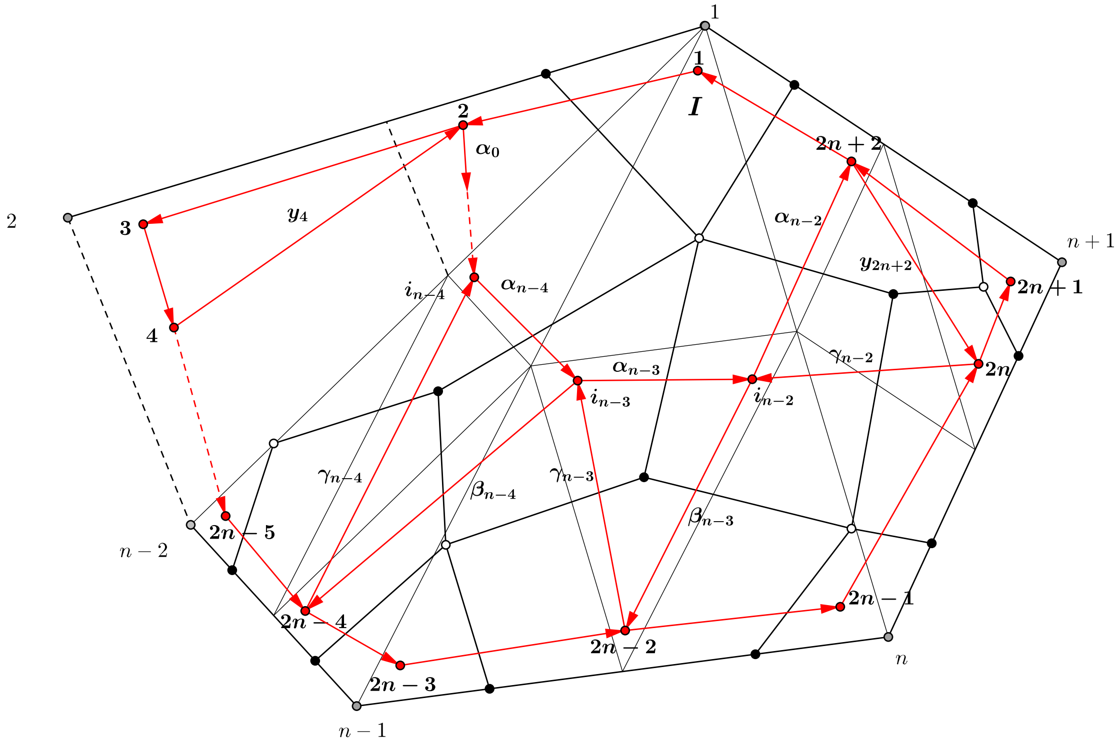

Figure 17: Part of reduced dimer and quiver of $n+1$-gon.

Definition 4.2. We define paths $z_2, \ldots, z_{2n}$ as follows:

$$z_{2k} := \gamma_{k-2}\beta_{k-3} \text{ for } k = 3, \ldots, n-1 \quad (1)$$
$$z_4 := y_4 \quad (2)$$
$$z_2 := \alpha_0\alpha_1 \ldots \alpha_{n-3} \quad (3)$$
$$z_{2n} := y_{2n}. \quad (4)$$

We show that every path between two boundary vertices of Q_F can be expressed as a composition of x_i's and z_{2k}'s.

Lemma 4.3. The x_i, $i = 1, \ldots, 2n$ together with the z_{2k}, $k = 1, \ldots, n$ as defined in 4.2 generate the boundary algebra of the dimer of the fan triangulation.

Proof. It is trivial that all x_i for $i = 1, \ldots, 2n$ and z_4 , z_{2n} as defined in (2) and (4) respectively are generators of the boundary algebras, as they are single arrows. Moreover there are no arrows between $2k$ and $2k-2$ for $k = 3, \ldots, n-1$ (see Proposition 4.1) and so a path between these two boundary vertices must contain of the composition of at least two arcs, as definied in (3).
It remains to show that z_2 as defined in (1), is a generator for paths from 2 to $2n$. For this, consider the path from 2 to $2n$. Let δ_1 be a generator in the dimer algebra for the paths from 2 to i_{k-1} and δ_2 be a generator in the dimer algebra for the paths from i_{k+1} to $2n$.
We claim that the path

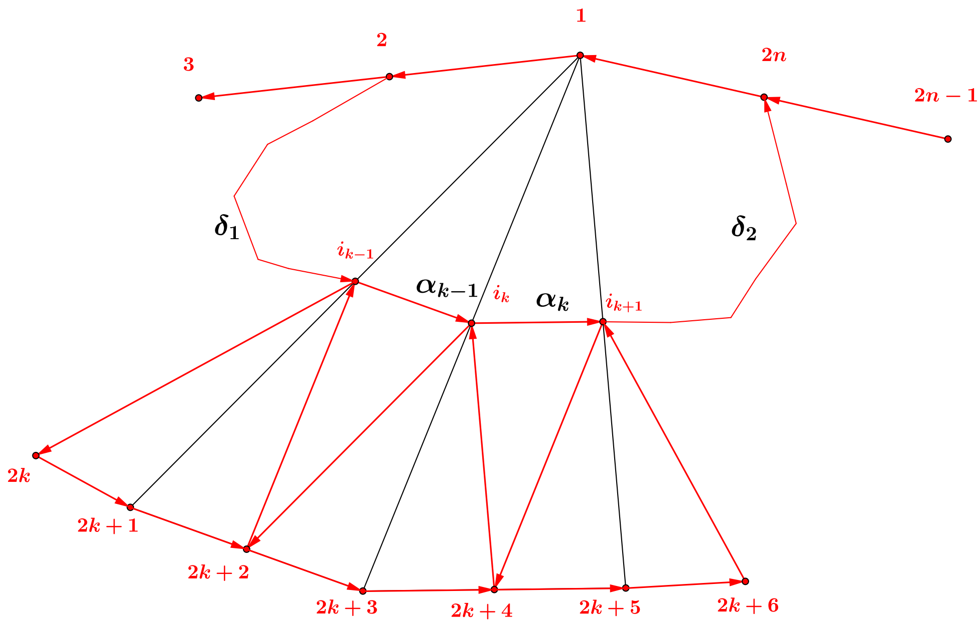

Figure 18: Extract of the path algebra of the n-gon.

$$\delta_1 \alpha_{k-1} \alpha_k \delta_2$$

is a generator of the paths from 2 to $2n$.
If α_{k-1} was not part of the generator, then the only possibility would be that the arrow from i_{k-1} to $2k$ is part of the generator as this is the only other outgoing arrow from i_{k-1}. Now there exist two different cases:

1.) The arrow from $2k$ to $2k+1$ is part of the generator.
Then the generator contains the arrow from $2k+1$ to $2k+2$ and by the relation for γ_{k-1} obtained from the natural potential W the equivalence

$$\beta_{k-1}x_{2k+1}x_{2k+2} \overset{\gamma_k}{\cong} \alpha_{k-1}\beta_k$$

and hence α_{k-1} is part of the path from 2 to $2n$.

2.) The path from $2k$ to $2k+1$ is not part of the generator.
The only possibility is, that we have a path which ends in i_{k-1} again and hence contains a cycle. This is a contradiction to the required property for a generator.

So α_{k-1} is part of a generator of the path from 2 to $2n$.
Analogous ideas applied for α_k guaranty that it is part of a generator and hence z_2 as defined in (1) is a generator.

Now it is clear that the boundary algebra is generated as claimed, because every path can be described by the generators x_i and z_k for $i = 1, \dots 2n$ and $k = 1, \dots, n$ and linear combination of these generators. □

Before the main result of this section is stated a notation that has been used for the description of the boundary algebra shall be recalled:
Remark. The cycle u_{2k+1} for $k \in [1, n-1]$ is defined as follows

$$u_{2k+1} = x_{2k+2}z_{2k+2}x_{2k+1}$$

and additionally

$$\begin{aligned} u_1 &= x_2 z_2 x_{2n} \\ u_{2k} = x_{2k+1}x_{2k+2}z_{2k+2} &= z_{2k}x_{2k-1}x_{2k} \end{aligned}$$

where the last equation holds because of Proposition 2.10.

The last (and main) step of this section is to show, that the relations between the arrows, stated in the previous section are fulfilled for the boundary algebra of the fan triangulation of a polygon. Let Λ be the dimer algebra of Q_F and let $e_b = \sum_{k=1}^{2n} e_b$.
Proposition 4.4. The boundary algebra $e_b\Lambda e_b$ satisfies the following relations:

I.) $u_{2k}^{n-3}x_{2k+1}x_{2k+2} = z_{2k}z_{2k-2}\dots z_{2k+4}$ for $k = 1, \dots, n$

II.) $x_{2k+1}x_{2k+2}z_{2k+2} = z_{2k}x_{2k-1}x_{2k}$.

Proof. Show the first kind of relations by splitting it into 2 cases:

1.) The relation holds for the case $k = n$.

2.) The relation holds for $k \in [1, n-1]$.

Case 1.): We consider

$$z_{2n} z_{2n-2} \dots z_6 z_4.$$

Using the natural potential for α_0, we have

$$\beta_0 z_4 \overset{\alpha_0}{\cong} \alpha_1 \alpha_2 \dots \alpha_{n-3} x_1 x_2.$$

Using the relations for $\alpha_i, i = 1, \dots, n-3$ on $\beta_i \gamma_i$, resp. to $z_{2n}\gamma_{n-3}$, we obtain a path containing repeated cycles,

$$x_1 x_2 (\alpha_0 \dots \alpha_{n-3}) x_1 x_2$$

and since the cycle $u_2 = \alpha_0 \dots \alpha_{n-3} x_1 x_2$ can be shifted along $x_1 x_2$, the last path is equal to

$$u_{2n}^{n-3} x_1 x_2.$$

Case 2.):
Figure 19 shows the considered path. Again, consider the path $z_{2k} \dots z_2 z_{2n} \dots z_{2k+4}$.

Similarly to the argumentation in the first case, using the relation obtained from the natural potential W for the path α_{n-3}

$$z_{2n}\gamma_{n-3} \cong x_1 x_2 \alpha_0 \alpha_1 \dots \alpha_{n-4}$$

and then recursively for the paths α_{n-4} up to α_k (using that we can shift cycles along paths) yields

$$z_{2k} \dots z_{2k+4} \overset{\alpha_0}{\cong} z_{2k} \dots z_2 u_2^{n-k-3} \alpha_0 \alpha_1 \dots \alpha_{k-1} \beta_{k-1}.$$

Iterating this, we get the equivalence

$$z_{2k} z_{2k-2} \dots z_4 z_2 z_{2n} \dots z_{2k+4} \cong \gamma_{k-2} u_{i_{k-2}}^{(n-k-3)+(k-1)} \alpha_{k-2} \alpha_{k-1} \beta_{k-1} \cong \gamma_{k-2} u_{i_{k-2}}^{n-4} \alpha_{k-2} \alpha_{k-1} \beta_{k-1}.$$

Replacing $\gamma_{k-2}\alpha_{k-2}$ in $z_{2k} \dots z_{2k+4}$ thus produces

$$u_{2k}^{n-4 x_{2k+1} x_{2k+2}} \gamma_{k-1} \alpha_{k-1} \beta_{k-1}.$$

The last three arrows in this path form u_{2k+2} and we can shift it to the front of the path (which is then equivalent to u_{2k}) to get the desired result

$$u_{2k}^{n-3} x_{2k+1} x_{2k+2}.$$

□

Thus the main result for the GL_2-web of a fan triangulation for arbitrary large n has been proven .

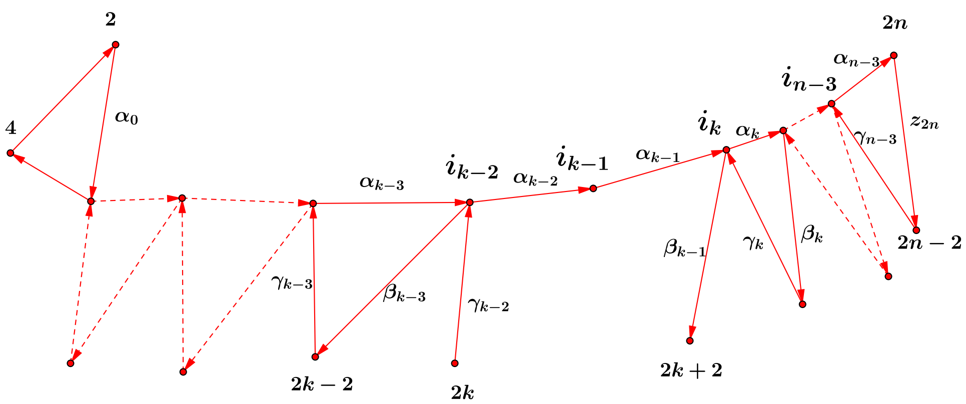

Figure 19: Path $z_{2k} \dots z_{2k+4}$ from $2k$ to $2k+2$.

5 Flips in boundary algebras

This section starts with the definition of a diagonal flip of a triangulation and then shows, that such a flip does not change the structure of the boundary algebra itself. As already shown the boundary algebra of the fan triangulation has the structure given in Theorem 3.1. Together with the main result of this section (Theorem 5.2), this proves that all boundary algebras arising from GL$_2$-webs of triangulations of an n-gon are isomorphic to each other.

Definition 5.1 (Diagonal flip of triangulation.)*.* For a triangulation a diagonal flip is defined as follows. Let (l, j) be a diagonal of the triangulation of the n-gon. Then two triangles l,j,k and l,j,i belong to the triangulation. A flip, as shown in Figure 20,

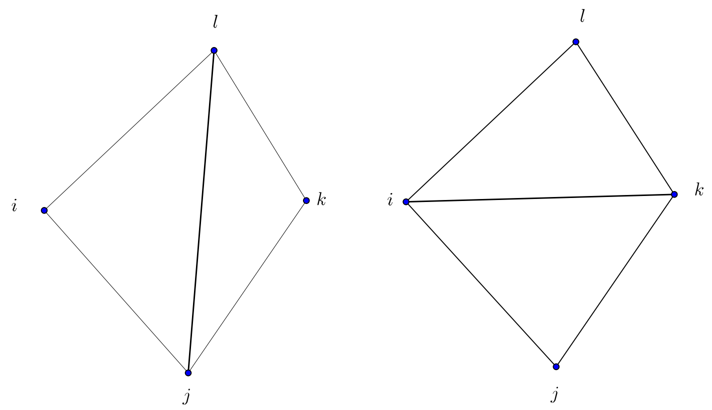

Figure 20: Diagonal flip of a triangulation.

is the removal of the diagonal (l, j) replacing it by the diagonal (i, k).

5.1 Quadrilateral case

There is one unique possibility for a diagonal flip of the quadrilateral. Figure 11 has already shown the quiver obtained by the fan triangulation which contains the diagonal $(1, 3)$. On the other hand, the quiver of the GL_2-web of the triangulation containing the diagonal $(2, 4)$ is shown in Figure 21. Set

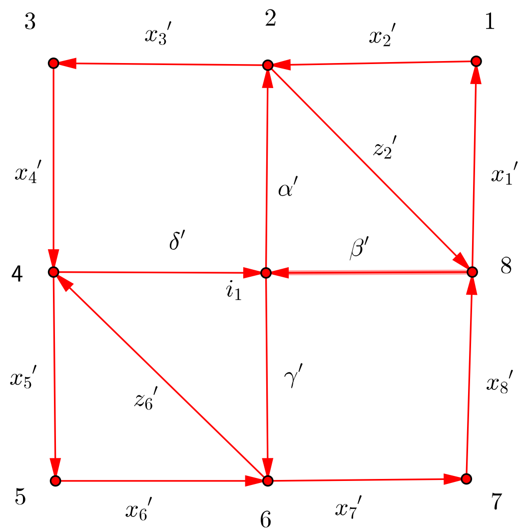

Figure 21: Flip changes the quiver.

$$z_8' := \beta'\gamma'$$
$$z_4' := \delta'\alpha'.$$

Then $z_8'z_6'z_4'$ is a path from 8 to 2.

Using the relations obtained by the natural potential W and the equivalence of cyles shown in Proposition 2.10, we get

$$u_6 = z_6'\delta'\gamma' = x_7'x_8'\beta'\gamma'$$
$$\beta'\alpha' \overset{z_2'}{\cong} x_1'x_2'$$

and finally obtain

$$z_8'z_6'z_4' = u_8x_1'x_2'.$$

Similarly we get the relation $z^3 = ux^2$ for all even vertices. So there is a one-to-one map $z_{2k} \mapsto z'_{2k}$ and (obviously) $x_k \mapsto x'_k$ from the associated quiver of the original fan triangulation to the quiver of the flipped quadrilateral. This idea can be used for the general case as well.

5.2 General flip

Note that a diagonal flip is always a local operation that only changes the structure around vertices corresponding to the edges of the quadrilateral.

Theorem 5.2. Let Q be the quiver of the GL_2-web of the fan triangulation of an n-gon, let Q' be the quiver of the GL_2-web of an arbitrary triangulation of the n-gon, with Λ_Q and $\Lambda_{Q'}$ the corresponding dimer algebras and e_b respectively $e_{b'}$ the sum of the boundary idempotents for Q and for Q' respectively. Then there is an isomorphism

$$e_b \Lambda e_b \cong e_{b'} \Lambda_{Q'} e_{b'}.$$

induced by

$$x_k \mapsto x'_k \text{ for } k \in [1, 2n] \tag{5}$$
$$z_{2k} \mapsto z'_{2i} \text{ for } k \in [1, n]. \tag{6}$$

Remark. At this point several trivial (and useful) statements needed in the proof of Theorem 5.2 are stated.

1.) If the basis elements x_k, z_{2k} of $e_b \Lambda e_b$ are in one to one correspondence to the basis elements x'_k,z'_{2k} in $e_{b'} \Lambda_{Q'} e_{b'}$, then the path algebras are isomorphic.

2.) The pleasant feature of a diagonal flip is, that only the local transformation of the flip has to be considered, because it is a local operation on the n-gon and hence it does not change the rest of the triangulation and hence of the associated quiver.

3.) A well known result in combinatorics is, that every triangulation of a polygon can be reached from any starting triangulation by application of finitely many flips.

Proof. The proof will be done by induction. Starting with the boundary algebra obtained by the quiver of the GL_2-web of the fan triangulation of the polygon, which is already known (see Lemma 4.3) we will show as induction step, that

we can determine the generators of the boundary algebra for the triangulation obtained after flipping one diagonal and prove the statement for this situation. The previous section already showed how a diagonal flip changes the quiver of the quadrilateral.
Let Q_F be the quiver of the GL_2-web of the fan triangulation of the n-gon and $e_b\Lambda_{Q_F}e_b$ be its boundary algebra. By Lemma 4.3 it is known that the x_i, $i = 1, \dots, 2n$ together with the z_{2k}, $k = 1, \dots, n$ generate the boundary algebra of the dimer of the fan triangulation.
As induction basis, we have to show that there is a isomorphism as claimed if a flip of a diagonal of the fan triangulation is performed.

First notice that $x'_k := x_k$ remain basis elements in the flipped boundary algebra, because a diagonal flip does not change the boundary of the quiver of the triangulation. Therefore the first part of the isomorphism, stated in (5) holds. Thus the only thing left to be shown is that the second part of the claimed isomorphism (6) for the z_{2k} is true too.
Without loss of generality consider the flip of the quadrilateral 1,$2i+1$, $2j+1$ and $2k+1$ and the path z_2 (which is the path from 2 to $2n$).

Remark.

- Of course, in case of a fan triangulation for every diagonal the equations $2i+1 = 2j-1 \leftrightarrow i = j-1$ and $2k+1 = 2j+3 \leftrightarrow k = j+1$ hold. As we want to use this argument in the induction step (general case of a flip) too, we will use the general notation of a flip right at the beginning.

- If we are able to show that our definition of z'_2 is a generator for paths from 2 to $2n$ in the new boundary algebra, then the same argument works for any other path containing an arrow of the quadrilateral $(1, 2i+1, 2j+1, 2k+1)$ too as the proof is independent of renaming the vertices.

Figure 22 shows how a flip changes the structure of the quiver. The path from 2 to $2n$ on the left hand side (which is of Q_F) is generated by

$$z_2 := \gamma_1\alpha\beta\gamma_2.$$

as seen in previous chapter.

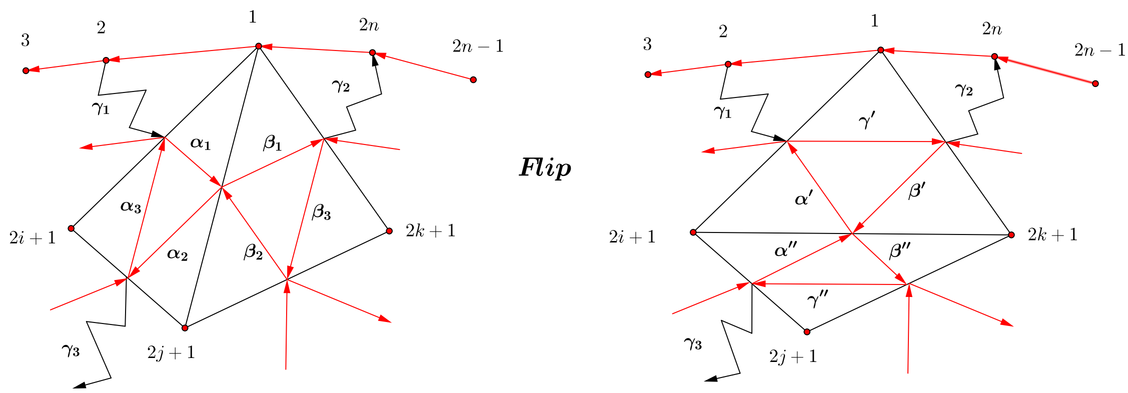

Figure 22: Flip of diagonal $(1, 2j+1)$ to $(2i+1, 2k+1)$ changes the dimer algebra.

So the basis element z_2 in $e_b\Lambda_{Q_F}e_b$ is $\gamma_1\alpha\beta\gamma_2$.
Now consider the the boundary algebra, which is obtained by the quiver Q' of the GL$_2$-web of the triangulation where the diagonal $(1, 2j+1)$ has been flipped to $(2i+1, 2k+1)$ (on the right hand side of figure 22). The new basis elements are $x'_k = x_k$ and z'_{2k} where z'_{2k} is a generator for paths from $2k$ to $2k-2$.
Claim:

$$\gamma_1\gamma'\gamma_2 =: z'_2. \tag{7}$$

Proof of the Claim. Because the flip does not change the rest of the arrows (apart from the ones inside of the quadrilateral $(1, 2i+1, 2j+1, 2k+1)$), γ_1 and γ_2 are still used in the generator for paths from 2 to $2n$. We will proof the claim by contradiction.

The generator for paths from 2 to $2n$ must contain an arrow of the newly arising quadrilateral, otherwise, the path would already have existed in the original triangulation, a contradiction to $\gamma_1\alpha\beta\gamma_2$ being a generator for paths from 2 to $2n$ for the original algebra. Note that γ' is an arrow of the new quadrilateral contained in the generator for paths from 2 to $2n$, if there was another arrow of this quadrilateral used in the generator, it would have to be α' or β'. Consider

first the case where β' is used. If the path of the generator continued using α', it would contain a cycle $(\gamma'\beta'\alpha')$, a contradiction to it being a generator. So assume the path continues with β'': as it would have to end in γ_2, it would still contain a cycle (starting with β').
So assume α' is used in the generator, if the predecessor of α' is β', the path contains the cycle $\gamma'\beta'\alpha'$, a contradiction. If the predecessor of α' is α'', we use the fact that the path start with γ_1 and hence contains a cycle ending with α' (before using γ'), again a contradiction.

So if γ' is contained in the generator, no other arrow in the new quadrilateral is used.

Assume now that γ' is not used in the generator. Then at least one of the arrows α'', γ'', β'', α' or β' have to be used in the generator.

β'. This is the main case and the other ones can be reduced to it. If β' were part of the generator from 2 to $2n$, then the path would have to return to the vertex from where γ_2 starts. Thus there would bea cycle u,a contradiction to being a generator.

α''. If α'' were part of the generator, then by using the natural potential W for α', this part of the path would be equivalent to $\gamma'\beta'$. This case has already been discussed.

α'. If α' were part of the generator a cycle would appear in the path, because the heads of γ_1 and of α' are the same.

β''. If β'' were part of the generator, then either α'' or β' would have to be part of the generator too, which both cannot be the case.

γ''. Finally, if γ'' were part of the generator, the path containing γ_1, γ'' and γ_2 must contain either a cycle containing γ'', or another arrow of the quadrilateral, as the requirement on the path (starting with γ_1 and ending up with γ_2) gives no further posibility. Both cases are a contradiction to being a generator.

Hence we have seen that every path from 2 to $2n$ factors through $z'_2 = \gamma_1\gamma'\gamma_2$.

The same argument can be applied for every z'_{2k} by renaming the quadrilateral.
Each generator, which does not contain an arrow of the quiver of the flipped quadrilateral, remains, $z'_{2k} = z_{2k}$.

It is clear that the relations $(x')^2z' = z'(x')^2$ hold in $\mathcal{B}(\mu Q_F)$ by Proposition 2.11.

It is straightforward to check the relations

$$(u')^{n-3}(x')^2 = (z')^{n-1}.$$

There are basically two possibilities:

(i) A product of the form $(z')^{n-1}$ contains all arrows of the new quadrilateral exactly once.
In this case, the product $(z')^{n-1}$ of the corresponding z's also contain all six arrows of the (original) quadrilateral. These six arrows amount to

$$(u')^2 = u^2.$$

All other arrows involved in $(z)^{n-1}$ remain unchanged in $(z')^{n-1}$.
Hence the two paths are both equal to $x^2 = (x')^2$.

(ii) A product of the form $(z')^{n-1}$ contains 4 (or 5 arrows) of the new quadrilateral, e.g. $\gamma'\beta'\beta''\gamma''$ (e.g. $\beta'\beta''\gamma''\alpha''\alpha'$).
The corresponding product z^{n-1} then contains 5 (or 4) arrows of the original quadrilateral e.g. $\alpha_1\beta_1\beta_3\beta_2\alpha_2$ (e.g. $\beta_3\beta_2\alpha_2\alpha_3$). All other arrows appear in both $(z')^{n-1}$ and in z^{n-1}.
Consider the case with 5 arrows in the new quadrilateral, in notation of Figure 22 we have

$$\beta'\beta''\gamma''\alpha''\alpha' \cong \beta' u\alpha' \cong u\beta'\alpha' \cong u\gamma_2 x_1 x_2\gamma_1$$

and in $\mathcal{B}(Q_F)$, we have

$$\beta_2\alpha_2\alpha_3 \cong \gamma_2 x_1 x_2 \gamma_1\alpha_1\alpha_2\alpha_3 \cong u\gamma_1 x_1 x_2\gamma_1.$$

Hence we have shown the result for a flip of the quiver of the fan triangulation Q_F:

$$\mathcal{B}(Q_F) \cong \mathcal{B}(\mu Q_F),$$

where μ denotes the flip of a arbitrary diagonal.

We will now do an induction on the number of diagonal flips t: For the induction step, we use that we can reach any triangulation of an n-gon by a finite number of diagonal flips. So let Q be the quiver of an arbitrary triangulation, and

$$Q = \mu_t\mu_{t-1}\dots\mu_1 Q_F,$$

where $\mu_1, \dots, \mu_t$ are t flips of diagonals starting with the quiver of the fan triangulation Q_F.

By induction hypothesis, we know that

$$\mathcal{B}(\mu_{t-1} \dots \mu_1 Q_F) \cong \mathcal{B}(Q_F),$$

and it remains to show that

$$\mathcal{B}(\mu_t \mu_{t-1} \dots \mu_1 Q_F) \cong \mathcal{B}(\mu_{t-1} \dots \mu_1 Q_F).$$

We use Figure 22 again, as it also works in general case.

Every element of

$$\mathcal{B}'' := \mathcal{B}(\mu_t \mu_{t-1} \dots \mu_1 Q_F)$$

containing the subpaths γ_1 and γ_2 can be written as an element of

$$\mathcal{B}' := \mathcal{B}(\mu_{t-1} \dots \mu_1 Q_F)$$

analogously as in the induction basis, by replacing $\alpha_1\beta_1$ with γ'. It remains to consider the effect of flips on elements of of $\mathcal{B}'$ containing γ_3. Assume we have such a path starting at 2 that is a generator for $\mathcal{B}'$. It may pass through $\alpha_1\alpha_2$ i.e. be of the form $\gamma_1\alpha_1\alpha_2\gamma_3$. But then we can use the relation w.r.t. α_3, to see that this is equivalent to a path not involving any arrows of the quadrilateral. Hence mutating the diagonal t does not change this path. An analogous argument works for paths involving the arrow outside the quadrilateral predecessing β_3. So we get an isomorphism on the level of paths between $\mathcal{B}'$ and $\mathcal{B}''$.

The relations hold, as described in the case of $\mathcal{B}(\mu Q_F)$-

Furthermore, the relations

I.) $(u_{2n+1}^{n-3})'(x^2)' = (z^{n-1})'$

II.) $(x^2)'z' = z'x'x'$

hold for $e_b'\Lambda_{Q'}e_b'$ by the same arguments as in the induction step. □

Corollary 5.3. Consider the boundary algebra of $\Gamma(n)$ subject to the relations from Theorem 3.1 (i.e. $\mathcal{B}$). Then the element t

$$t := \sum_{i=1}^{n} x_{2i-1}x_{2i}z_{2i} + \sum_{i=1}^{n} x_{2i}z_{2i}x_{2i-1}$$

is a central element of this algebra.

Proof. The element t is a sum of cycles. The element t is the sum of all chordless cycles for all vertices and hence commutes with every element of $\Gamma(n)$.. □

So, as already shown, the structure of the boundary algebra of a quiver Q_F, we receive the general result stated in Theorem 3.1 by using Theorem 5.2.

6 Conclusions

Starting with the GL_2-web of the fan triangulation of a regular polygon and using geometric ideas and the structure of quivers, the dimer D leads to a finite quiver Q and from this a *dimer algebra* $\Lambda_Q = kQ/_{\partial W}$ and finally the *boundary algebra* $\mathcal{B}$ was obtained.
In the thesis, this construction was first realized for the quadrilateral and the pentagon to give an idea about the structure of the boundary algebra. The main result is the description of the boundary algebra, cf. Theorem 3.1. The strategy for proving this was to first establish the result for fan triangulations and then to show that the boundary algebra is invariant under flips of diagonals.

Although the result of isomorphic boundary algebras for arbitrary triangulations of an n-gon does not help answering interesting questions in informatics concerning *flip distances*, it proves a nice new property of GL_2-webs. It would be also interesting to study the boundary algebras of the quiver obtained by a dimer of a GL_m-webs of a triangulation for $m \geq 3$ and associated categories of modules. In particular to see whether such categories have a cluster category structure.

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