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The Line Graph of Every Locally Finite 6-Edge-Connected Graph with Finitely Many Ends is Hamiltonian

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Abstract

Homeomorphic images of the unit circle in the Freudenthal compactification of a graph can be seen as an infinite analogue of finite graph theoretic circles. In this sense the main result of the present thesis constitutes a partial generalisation of Thomassen's result that every finite 4-edge-connected graph has a Hamiltonian line graph. We show that every locally finite 6-edge-connected graph with finitely many ends has a Hamiltonian line graph. In the proof of this result we will encounter an auxiliary result which may be interesting on its own behalf, namely that every locally finite $2k$-edge-connected graph with finitely many ends has $k-1$ edge disjoint end faithful spanning trees.

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1 Introduction

The Hamiltonian problem, that is, to decide whether or not a given graph contains a spanning circle is probably one of the most popular problems in graph theory. Although many sufficient conditions for the Hamiltonicity of a finite graph are known, there are still numerous open problems and unsettled conjectures in connection with the Hamiltonian problem (see [25] for an overview).

1.1 Infinite Graphs and Hamiltonicity

Despite the popularity of the Hamiltonian problem in finite graphs Hamiltonicity of infinite graphs has not received much attention for a long time, partly due to the absence of suitable concepts for Hamilton cycles in infinite graphs.

In some publications [27, 31, 40, 42, 45] spanning rays or double rays were considered to be the infinite analogon to finite Hamiltonian cycles. This approach has yielded several results, but it obviously does not allow Hamiltonicity results for any graph with more than two ends.

Surprisingly the solution to this problem is topological rather than combinatorial. In 2004 Diestel and Kühn [11, 15, 16] proposed to use topological concepts for circles, paths and trees in infinite graphs. Considering the amount of follow-up publications [2, 3, 4, 5, 6, 7, 12, 20, 21, 22, 23, 24, 29, 44] it can be said that their work was groundbreaking for future research on infinite graphs.

As for circles, they suggested to use topological circles, i.e., homeomorphic images of the unit circle S^1 in the Freudenthal compactification of G as a generalization of finite circles in graphs. Consequently a Hamilton circle can be defined as a topological circle containing all vertices of a graph.

Using this notion of Hamiltonian circles some well known Hamiltonicity results for finite graphs could be extended to locally finite graphs. Bruhn and Yu [7] found a partial generalization of a theorem of Tutte [48] stating that every finite 4-connected planar graph is Hamiltonian.

Theorem 1.1 (Bruhn and Yu [7]). *Let G be a locally finite 6-connected planar graph with finitely many ends. Then G has a Hamilton circle.*

Georgakopoulos [22] generalized a result by Fleischner [18] to arbitrary locally finite graphs. Previously Thomassen [45] had extended this result to single ended locally finite graphs.

Theorem 1.2 (Georgakopoulos [22]). *If G is a locally finite 2-connected graph, then G^2 has a Hamilton circle.*

In the same paper he also proves an extension of a theorem of Karaganis [32] and Sekianina [42].

Theorem 1.3 (Georgakopoulos [22]). *If G is a connected locally finite graph, then G^3 has a Hamilton circle.*

1.2 The Conjectures of Thomassen and Georgakopoulos

A well known unsettled conjecture associated with the finite Hamiltonian problem due to Thomassen [46] relates the connectivity of a line graph to the property of Hamiltonicity.

Conjecture 1 (Thomassen [46]). Every finite 4-connected line graph has a Hamilton cycle.

Thomassen made this conjecture motivated by the following observation which he stated in [46] without a proof.

Theorem 1.4 (Thomassen [46]). *The line graph of every finite* 4*-edge-connected graph has a Hamiltonian cycle.*

In Section 4.1 we will give a short proof sketch for this result which obviously constitutes a special case of Thomassen's conjecture.

Motivated by Thomassen's conjecture there have been serveral other results connecting connectivity and Hamiltonicity of line graphs. Below there is a brief overview of the most important results related to Conjecture 1.

Theorem 1.5 (Zhan [49]). *Every finite* 7*-connected line graph is Hamiltonian.*

Theorem 1.6 (Lai [35]). *Every finite* 4*-connected line graph of a planar graph is Hamiltonian.*

Theorem 1.7 (Kriesell [33]). *Every finite* 4*-connected line graph of a claw free graph is Hamiltonian.*

Theorem 1.8 (Lai et al. [36]). *Every finite* 3*-connected essentially* 11*-connected line graph is Hamiltonian.*

Apart from a partial extension of Theorem 1.4 by Brewster and Funk [1] none of these results have been extended to locally finite graphs so far.

The proofs of Theorems 1.4 and 1.5 rely on finding a spanning Eulerian subgraphs of a graph G in order to show that its line graph is Hamiltonian, which may be the key for generalizing these results to infinite graphs. Locally finite Eulerian graphs are better understood than their Hamiltonian counterparts in the sense that the following necessary and sufficient condition for the existence of a topological Euler tour is easy to verify.

Theorem 1.9 (Diestel and Kühn [15]). *A locally finite graph G admits a topological Euler tour if and only if every finite cut in G is even.*

Georgakopoulos [22] proved a result which comes in handy when constructing a Hamiltonian cycle in the line graph of a Eulerian graph, since a Hamiltonian cycle (unlike an Euler tour) needs to be injective at ends.

Theorem 1.10 (Georgakopoulos [22]). *If a locally finite multigraph has a topological Euler tour, then it also has one that is injective at ends.*

Based on this fact he made the following conjecture to which the main results of this thesis are related.

Conjecture 2 (Georgakopoulos [21]). The line graph of every locally finite 4-edge-connected graph has a Hamiltonian cycle.

1.3 Outline of the Results of this Thesis

Motivated by Georgakopoulos' work a partial generalization of Theorem 1.4 has been given by Brewster and Funk [1] recently.

Theorem 1.11 (Brewster and Funk [1]). *Let G be a locally finite, 6-edge-connected graph with finitely many ends all of which are thin. Then the linegraph of G is Hamiltonian.*

One of the main outcomes of this thesis will be a further extension of this result to graphs with both thin and thick ends.

Theorem 1.12. *Every locally finite 6-edge-connected graph with finitely many ends has a Hamiltonian line graph.*

The proof of Theorem 1.12 will among other things require one of the following two statements concerning edge disjoint end faithful spanning trees of $2k$-edge-connected graphs which also may be interesting on their own behalf.

Theorem 1.13. *Let G be a locally finite $2k$-edge-connected graph with finitely many ends. Then there exist edge disjoint topological spanning trees $T^1, T^2, \dots, T^k$ such that $T^i \cup T^j$ is an end faithful connected spanning subgraph of G for every $i \neq j$.*

Theorem 1.14. *Let G be a locally finite $2k$-edge-connected graph with finitely many ends. Then G has $k-1$ edge disjoint end faithful spanning trees.*

2 Graph Theoretic Definitions and Facts

The set $\mathbb{N}$ of natural numbers does not contain 0. Throughout this thesis (V, E) denotes a graph with a non empty vertex set V and edge set $E \subseteq V \times V = V^2$ where every edge $e \in E$ contains exactly two vertices. If $G = (V, E)$ we define $V(G) = V$ and $E(G) = E$. We will write $x \in G$ instead of $x \in V(G)$ or $x \in E(G)$.

A vertex $v \in V$ and an edge $e \in E$ are called incident if $v \in e$. Two vertices $u, v \in V$ are called adjacent if there is an edge to which both u nd v are incident. We allow multiple edges (i.e., E can be a multiset) but no loops (i.e., edges incident to only one vertex).

We call $H = (V', E')$ a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$ such that every edge in E' is only incident to vertices in V'. In the case that $V' = V$ the graph H is called a spanning subgraph of G.

Edges will be seen not only as sets of two vertices but as homeomorphic copies of the unit interval where 0 and 1 map to vertices and the image of $(0, 1)$ is disjoint with all other edges. This definition allows us to see a graph as a topological object, namely a 1-complex. We say that an edge $e \in E$ with $e(0) = u$ and $e(1) = v$ connects u and v and denote it by uv. This notation may be ambigous if there is more than one such edge but as all of these edges are equivalent for our purpose this will not be a problem.

A u-v-path is a sequence $u = v_1 e_1 v_2 e_2 \ldots e_{k-1} v_k = v$ where all $v_i \in V$ and $e_i = v_i v_{i+1} \in E$. We call a path simple if all v_i are different for $1 \leq i \leq k-1$. A u-u-path is called a cycle, a simple cycle is a circle. A one sided infinite path is called a ray, a two sided infinite path a double ray.

We will call a graph connected if there is a u-v-path for every pair of vertices $u, v \in V$.

2.1 Eulerian and Hamiltonian Cycles

Definition 2.1. Let $G = (V, E)$ be a finite graph. A cycle containing every edge exactly once is called a *Eulerian cycle*. G is called *Eulerian* if G is connected and contains a Eulerian cycle.

The notion goes back to Euler [17] who, inspired by the seven bridges of Königsberg, proved that it is not possible to find a walk through the city of Königsberg crossing every bridge exactly once. His result that a finite graph admits a Eulerian cycle if and only if all vertices have even degree is nowadays a standard result in graph theory. The proof is short and straightforward and we will omit it as most readers probably are familiar with the result. Those who are interested in a proof can find one in [13].

Definition 2.2. Let $G = (V, E)$ be a finite graph. A cycle containing every vertex exactly once is called a *Hamiltonian cycle*. G is called *Hamiltonian* if it contains a Hamiltonian cycle. Note that a Hamiltonian cycle always has to be a circle.

Hamiltonian cycles are named after Sir William Rowan Hamilton who invented a game which involves finding a Hamiltonian cycle in the edge graph of the dodecahedron.

As there is an easy and natural solution to the Eulerian problem one might expect that the same also is true for the Hamiltonian problem. This is, however, not the case which might be a reason for the continuing interest in the problem.

Remark. Since the graph theoretic definition of a cycle only allows it to contain finitely many vertices and edges both of these definitions make no sense in infinite graphs. In Section 3.2 we will introduce concepts for cycles that can be used to extend the notions and some results concerning Eulerian and Hamiltonian cycles to infinite graphs.

2.2 Linegraphs

Definition 2.3. Let $G = (V, E)$ be a graph. Then the *line graph* of G is defined as

$$L(G) = \big(E, \{ef \mid e, f \in E, e \cap f \neq \emptyset\}\big),$$

i.e., the edges of G are the vertices of $L(G)$ with two of them being adjacent if they have a vertex in common.

The following results connecting Eulerian subgraphs of a graph to a Hamiltonian cycle in its line graph are due to Chartrand [10]. The proofs are all straightforward but will be given for the convenience of the reader.

Proposition 2.4 (Chartrand [10])**.** *The line graph of a finite Eulerian graph is Hamiltonian.*

Proof. Let $e_1, e_2, \ldots, e_k$ be the sequence of edges of a Eulerian tour in G. Then obviously e_i and e_{i+1} have one common endpoint and thus the corresponding vertices are adjacent in $L(G)$. Since every edge appears exactly once in an Eulerian cycle $e_1, e_2, \ldots, e_k$ is the sequence of vertices of a Hamiltonian cycle in $L(G)$. □

Proposition 2.5 (Chartrand [10])**.** *Let $G = (V, E)$ be a finite graph. If G has a spanning Eulerian subgraph H then $L(G)$ is Hamiltonian.*

Proof. Let $v_1 e_1 v_2 e_2 \ldots v_k e_k v_1$ be a Eulerian cycle in H. Define

$$E_i = \{e \in E \mid e \notin H, v_i \in e, \forall j < i \colon v_j \notin e\}\,.$$

Consider the sequence $E_1', e_1, E_2', e_2, \ldots, E_k', e_k$ where E_i' stands for an arbitrary sequence of the edges in E_i.

Obviously every edge of G (and thus every vertex of $L(G)$) appears exactly once in the sequence. Furthermore every edge has at least one vertex in common with the next edge of the sequence. Thus it is the vertex sequence of a Hamiltonian cycle in $L(G)$. □

In a similar manner we can prove one implication of the following result by Harary and Nash-Williams [30].

Proposition 2.6 (Harary and Nash-Williams [30]). *Let $G = (V, E)$ be a finite graph. $L(G)$ is Hamiltonian if and only if there is a cycle in G that uses at least one vertex of each edge.*

Proof. The proof of the backward implication works exactly like the proof of Proposition 2.5.

Now assume that $L(G)$ is Hamiltonian and let $e_1 \dots e_k$ be the sequence of vertices of $L(G)$ used by a Hamiltonian cycle in $L(G)$. We may without loss of generality assume that there are at least two non-parallel edges and that e_k and e_1 are not parallel. Hence there is only one vertex in $e_k \cap e_1$. Denote this vertex by v_1.

Now let

$$r_1 = \begin{cases} \min \{r \geq 1 \mid v_1 \notin e_r \cap e_{r+1}\} & \text{if the set is not empty} \\ k & \text{otherwise.} \end{cases}$$

Since e_{r_1} is incident with v_1 there is exactly one vertex in $e_{r_1} \cap e_{r_1+1}$. Denote this vertex by v_2. Now define

$$r_2 = \begin{cases} \min \{r > r_1 \mid v_2 \notin e_r \cap e_{r+1}\} & \text{if the set is not empty} \\ k & \text{otherwise.} \end{cases}$$

Let v_3 be the unique vertex in $e_{r_2} \cap e_{r_2+1}$.

Continue inductively until $r_s = k$. By construction of the sequence there is an edge (namely e_{r_i}) connecting v_i and v_{i+1} for $1 \leq i \leq s - 1$. Furthermore $e_k = v_s v_1$ or $v_s = v_1$ since e_k has to be incident to v_s and v_1 by construction. Either way we obtain a cycle containing exactly the vertices v_i.

Every edge e_r is incident with v_s for $r_s \leq r < r_{s+1}$ and hence the cycle contains at least one vertex of every edge. □

2.3 Cuts and Connectivity

Definition 2.7. Let $G = (V, E)$ be a connected graph.

- A set $S \subseteq E$ is called an *edge cut*, if $G \setminus S$ is not connected.
- A set $S \subseteq V$ is called a *vertex cut*, if $G \setminus S$ is not connected.
- A cut S *seperates two vertices u and v* if they lie in different components of $G \setminus S$. In this case S is called a u-v-cut.

Definition 2.8. For vertices $u, v \in V$ define the *local edge connectivity* $\kappa'_G(u, v)$ as the minimal cardinality of a u-v-edge cut. The *(global) edge connectivity* of a graph is defined as

$$\kappa'(G) = \min_{u,v \in V} \kappa'_G(u, v).$$

Note that $\kappa'(G)$ is the minimal number of edges we have to remove to disconnect the graph.

The *local vertex connectivity* $\kappa_G(u,v)$ is the minimal cardinality of a u-v-vertex cut. The *(global) vertex connectivity* of a graph is

$$\kappa(G) = \min_{u,v \in V} \kappa_G(u,v).$$

In the rest of this thesis the term "cut" will refer to an edge cut. If vertex cuts are meant it will be explicitely mentioned. Furthermore, we will sometimes write $\kappa(u,v)$ and $\kappa'(u,v)$ instead of $\kappa_G(u,v)$ and $\kappa'_G(u,v)$ if G is clear from the context.

Proposition 2.9. *Let $G = (V,E)$ be a graph and $u, v \in V$. If S is a minimal u-v-cut with respect to inclusion, then $G \setminus S$ has exactly two components.*

Proof. Let C_u and C_v be the components of $G \setminus S$ containing u and v respectively. Assume there is a third component C. If there are edges from C to both C_u and C_v in S, then we can obtain a smaller u-v-cut by deleting all C-C_u-edges from S. Otherwise C is connected only to one of C_u and C_v and we get a smaller u-v-cut by deleting all edges from S that are adjacent to C. $\square$

Note that a minimal cut with respect to inclusion does not necessarily have to be a cut of minimal cardinality, but a cut of minimal cardinality is always a minimal cut with respect to inclusion.

The next result is due to Menger [38], see [13] for three different proofs of the theorem.

Theorem 2.10 (Menger)**.** *Let $G = (V,E)$ be a graph and $u, v \in V$. Then*

- $\kappa(u,v) \geq k$ *if and only if there are k independent u-v-paths,*
- $\kappa'(u,v) \geq k$ *if and only if there are k edge disjoint u-v-paths.*

2.4 Contractions of Vertex Sets

Definition 2.11. Let $G = (V,E)$ be a graph and $U \subseteq V$. Then

$$G/U = (V', E')$$

where V' is obtained from V by replacing the set U by a new vertex x_U and E' is obtained from E by replacing all endpoints in U by x_U and deleting loops. The resulting graph is called the *contraction of U in G* (see Figure 2.1).

Note that we can define an injective function $\phi\colon E' \to E$ in the following way:

- An edge $e = xy$, that does not have x_U as an endpoint, maps to an edge xy in the original graph.
- If $e = x_U y$ then $\phi(e) = uy$ with $u \in U$.

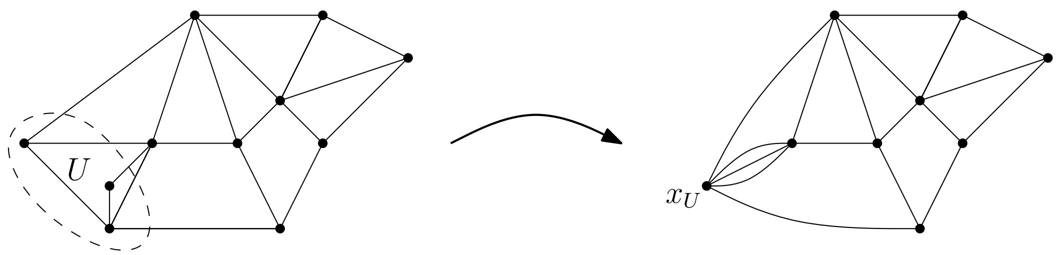

Figure 2.1: Contraction of the set U to a single vertex x_U

If it is clear from the context, we will write e instead of $\phi(e)$.

The following propositions show that contracting sets of vertices does not affect cuts of minimal cardinality. In particular, edge connectivity can only increase under contractions.

Proposition 2.12. *Let $G = (V, E)$ be a graph and $u, v \in V$. If S is a u-v-cut of minimal cardinality and $u \in U \subseteq V$, then S is a x_U-v-cut of minimal cardinality in G/U.*

Proof. Assume there was a smaller x_U-v-cut S' in G/U. Then there is no path from x_U to v in $(G/U) \setminus S'$, which implies that there is no path from U to v in $G \setminus S'$. Hence S' is also a u-v-cut in G, and thus S was not minimal. $\square$

Proposition 2.13. *Let $G = (V, E)$ be a graph and $u, v \in V$. If S is a u-v-cut of minimal cardinality and $u, v \notin W \subseteq V$, then S is a u-v-cut of minimal cardinality in G/W.*

Proof. Assume there was a smaller u-v-cut S' in G/W. Then there is no path from u to v in $(G/W) \setminus S'$, which implies that there is no path from u to v in $G \setminus S'$. Hence S' is also a u-v-cut in G, and thus S was not minimal. $\square$

Proposition 2.14. *Let $G = (V, E)$ be a graph and $W \subseteq V$. If G is k-edge connected then so is G/W.*

Proof. Combine Propositions 2.12 and 2.13. $\square$

2.5 Induced Subgraphs and Minors

Definition 2.15. Let $G = (V, E)$ be a graph and $U \subseteq V$.

- The *subgraph induced by U in G* is defined as $G[U] = (U, E \cap U^2)$.
- The *minor induced by U in G* is the graph obtained from G by contracting every component of $G \setminus U$ to a vertex and denoted by $G\{U\}$.

(see Figure 2.2)

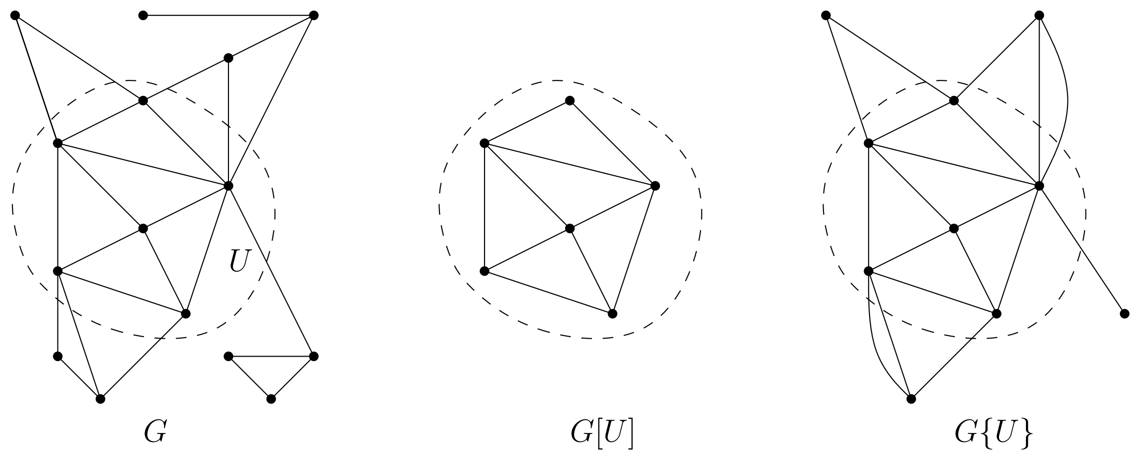

Figure 2.2: Induced subgraph and minor of U in G

It is clear from the definition that $G[U]$ is a subgraph of $G\{U\}$. It's also obvious that for a subset $U' \subseteq U$ the equations

$$(G[U])\,[U'] = G[U']$$

and

$$(G\{U\})\{U'\} = G\{U'\}$$

hold. Also note that if $G \setminus U$ is connected then $G\{U\} = G/(V \setminus U)$.

Definition 2.16. Let $G = (V, E)$ be a graph, $U \subseteq V$, and $H = (V', E')$ a subgraph of G. The *restriction of H to U* is defined as the subgraph of $G\{U\}$ that contains exactly the edges of H and is denoted by $H|_U$.

Remark. Obviously $H|_U$ also depends on the graph G. Throughout this thesis, however, there will always be only one sensible choice for G at any time (which will mostly be denoted by G or G_n for convenience).

It is another easily observed fact that if H is a connected spanning subgraph of G then $H|_U$ is a connected spanning subgraph of $G\{U\}$, since in that case $U \cap V' = U$ and, by Proposition 2.14, contracting does not decrease connectivity.

2.6 Tree Packing

Definition 2.17. Let $G = (V, E)$ be a graph. A *spanning tree packing of G* is a set

$$\mathcal{T} = \left\{T^1, T^2, \ldots, T^k\right\}$$

such that

- every T^i is a spanning tree of G and

- the T^i are pairwise edge disjoint.

The *spanning tree packing number* of G is the maximal cardinality of a spanning tree packing of G and will be denoted by $\tau(G)$.

Theorem 2.18 (Tutte [47], Nash-Williams [39])**.** *Let $G = (V, E)$ be a finite graph. For a partitioning $\mathcal{V}$ of V denote by $E(\mathcal{V})$ the set of edges that connect different sets in $\mathcal{V}$. Then G admits a spanning tree packing of cardinality k if and only if*

$$|E(\mathcal{V})| \geq k(|\mathcal{V}| - 1)$$

for every partitioning $\mathcal{V}$ of V.

A proof for Theorem 2.18 can be found in the original publications in [13]. We will state two well known implications of the result mentioned in [8]. Alternative proofs for both of these corollaries will be obtained in Section 5.1.

The first result will be useful for the proof of Thomassen's Theorem in Section 4.1.

Corollary 2.19. *Let $G = (V, E)$ be a finite graph. If $\kappa'(G) \geq 2k$ then $\tau(G) \geq k$.*

Proof. Let

$$\mathcal{V} = \{V_1, V_2, \ldots, V_r\}$$

be an arbitrary partitioning of V. Now consider the graph $G' = (V', E')$ obtained from G by contracting every set V_i to a vertex v_i. Since G is $2k$-edge-connected, G' also is $2k$-edge connected, and the number of edges in G' obviously equals $|E(\mathcal{V})|$. Now

$$|E'| = \frac{1}{2}\sum_{i=1}^{r} \deg(v_r) \geq \frac{1}{2}\sum_{i=1}^{r} 2k = rk \geq k(|\mathcal{V}| - 1)$$

and Theorem 2.18 completes the proof. □

The second corollary, sometimes attributed to Catlin [8, 9] characterizes the edge connectivity of a graph by the spanning tree packing number of certain subgraphs. The proof we will give can be found in [41].

Corollary 2.20. *Let G be a finite graph.*

1. *$\kappa'(G) \geq 2k$ if and only if $\tau(G \setminus F) \geq k$ for every set F of at most k edges.*

2. *$\kappa'(G) \geq 2k + 1$ if and only if $\tau(G \setminus F) \geq k$ for every set F of at most $k + 1$ edges.*

Proof. 1. "$\Leftarrow$" If we remove k edges we still get k edge disjoint spanning trees. Thus we have to remove at least another k edges in order to disconnect the graph. So the minimal cardinality of a cut has to be at least $2k$.

"⇒" Let $\mathcal{V} = \{V_1, V_2, \dots, V_r\}$ be an arbitrary partition of V and $F \subseteq E$ a set of at most k edges.

As in the proof of Corollary 2.19 denote by $G' = (V', E')$ the graph that we get from G by contracting every set V_i to a vertex v_i. Furthermore, let $F' \subseteq E'$ be the set of edges in E' that have been obtained from edges in F.

It is an easy observation that $|F'| \leq |F| \leq k$ and that $|E' \setminus F'|$ corresponds to $|E(\mathcal{V})|$ in the graph $G \setminus F$. Since G' is $2k$-edge-connected we obtain

$$\begin{aligned}
|E' \setminus F'| &= \frac{1}{2}\sum_{v \in V'} \deg(v) - k \\
&= \frac{1}{2}\sum_{i=1}^{r} \deg(v_r) - k \\
&\geq \frac{1}{2}\sum_{i=1}^{r} 2k - k \\
&= rk - k \\
&= k(|\mathcal{V}| - 1).
\end{aligned}$$

2. "⇐" If we remove $k+1$ edges we still get k edge disjoint spanning trees. Thus we have to remove at least another k edges in order to disconnect the graph. So the minimal cardinality of a cut has to be at least $2k+1$.

"⇒" Let E' be a set of $k+1$ edges and $e' \in E'$. Then $G \setminus e'$ is $2k$-edge-connected and thus

$$(G \setminus e') \setminus (E \setminus e') = G \setminus E$$

has k edge disjoint spanning trees. □

2.7 Extending Spanning Trees of Induced Minors

In the proofs of the main results of this thesis we will need to create spanning tree packings of induced minors out of spanning tree packings of smaller induced minors. The results in this section show how this can be done.

Proposition 2.21. *Let $G = (V, E)$ be a graph, $U \subseteq V$ and H a spanning subgraph of G. Furthermore let $U_1, U_2, \dots, U_k$ be the vertex sets of the components of $G \setminus U$ and assume that $H|_U$ is a spanning tree of $G\{U\}$. Then $H[U_i]$ is a spanning tree of $G[U_i]$ for all $1 \leq i \leq k$ if and only if H is a spanning tree of G.*

Proof. "⇒" We have to show that H is connected and does not contain a cycle.

First assume that H is not connected. $H|_U$ is connected, hence a cut in G separating two components of H has to contain at least one edge in $G[U_i]$ for some i. However, as $H[U_i]$ is connected no two different components of H can intersect with the same U_i. Thus H has to be connected.

Now assume that H contains a cycle. For any two vertices x, y on this cycle $\kappa'_H(x, y) \geq 2$ holds. If the cycle intersects with U then we would by Propositions 2.12 and 2.13 obtain a pair of vertices with edge connectivity at least 2 in $H|_U$. This would imply that $H|_U$ contains a cycle, a contradiction to $H|_U$ being a tree.

If the cycle intersects with more than one U_i it also has non-empty intersection with U since U seperates the U_i. Thus all vertices of the cycle have to be contained in the same set U_i which is a contradiction to $H[U_i]$ being a tree. Hence H cannot contain a cycle at all.

"$\Leftarrow$" A cycle in $H[U_i]$ would also be a cycle in H, hence the $H[U_i]$ have to be cycle free.

Assume that $H[U_i]$ is not connected. Then the components of $H[U_i]$ have to be connected by a path in $H \setminus H[U_i]$. This path, however, starts and ends in the same vertex of $H|_U$ and thus constitutes a cycle in $H|_U$ which is impossible. □

Remark. Let $U_1, U_2, \ldots, U_k$ be the vertex sets of the components of $G \setminus U$. If we have spanning subgraphs H_U of $G\{U\}$ and H_i of $G[U_i]$ for $1 \leq i \leq k$ then there is a unique spanning subgraph H of G satisfying

- $H|_U = H_U$ and
- $H[U_i] = H_i$.

Proposition 2.22. *Let $G = (V, E)$ be a graph, $U \subseteq V$ and let $U_1, U_2, \ldots, U_k$ be the vertex sets of the components of $G \setminus U$. Given spanning trees H_U of $G\{U\}$ and H_i of $G[U_i]$ there is a unique spanning tree H of G satisfying*

- *$H|_U = H_U$ and*
- *$H[U_i] = H_i$.*

Proof. Combine Proposition 2.21 with the above remark. □

Proposition 2.23. *Let $G = (V, E)$ be a graph, H_1, H_2 subgraphs of G and for $U \subseteq V$ denote by $\mathcal{V}(U)$ the set containing all vertex sets of components of $G \setminus U$. The following statements are equivalent:*

(1) H_1 and H_2 are edge disjoint.

(2) For every $U \subseteq V$ the pairs $(H_1|_U, H_2|_U)$ and $(H_1[U'], H_2[U'])_{U' \in \mathcal{V}(U)}$ are edge disjoint.

(3) For some $U \subseteq V$ the pairs $(H_1|_U, H_2|_U)$ and $(H_1[U'], H_2[U'])_{U' \in \mathcal{V}(U)}$ are edge disjoint.

Proof. The implication (1) $\Rightarrow$ (2) follows directly from the definitions of $H|_U$ and $H[U']$, (2) $\Rightarrow$ (3) is trivial.

For the proof of (3) $\Rightarrow$ (1) we assume that H_1 and H_2 are not edge disjoint. Let e be an edge in $H_1 \cap H_2$. If e connects two vertices that both lie in U' for some $U' \in \mathcal{V}(U)$ then $e \in H_1[U'] \cap H_2[U']$ Otherwise $e \in H_1|_U \cap H_2|_U$. In both cases (3) does not hold. □

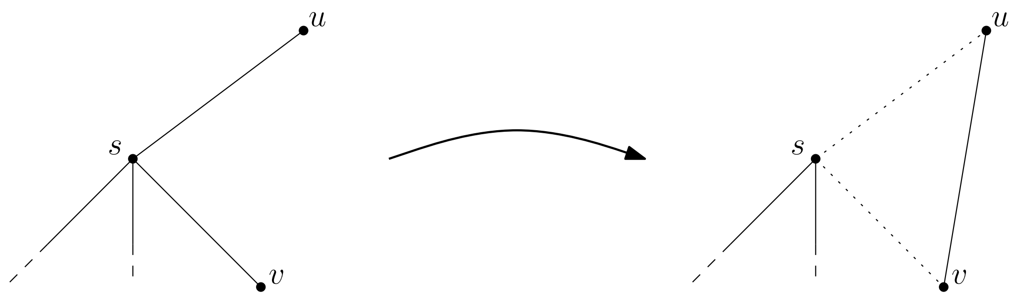

Figure 2.3: Splitting off $\{us, vs\}$

Proposition 2.24. *Let $G = (V, E)$ be a graph, $U \subseteq V$ and let $U_1, U_2, \ldots, U_k$ be the vertex sets of the components of $G \backslash U$. If we have spanning tree packings $\mathcal{T}_U = \{T_U^1, T_U^2, \ldots, T_U^r\}$ of $G\{U\}$ and $\mathcal{T}_i = \{T_i^1, T_i^2, \ldots, T_i^r\}$ of $G[U_i]$ then there is a unique spanning tree packing $\mathcal{T} = \{T^1, T^2, \ldots, T^r\}$ of G satisfying*

- $T^j|_U = T_U^j$ *and*
- $T^j[U_i] = H_i^j$

for $1 \leq j \leq r$.

Proof. Follows from Propositions 2.22 and 2.23. □

2.8 Splitting Off Edges

Definition 2.25. Let $G = (V, E)$ be a graph and let s be a vertex of degree ≥ 2. If u and v are neighbours of s *splitting off the pair of edges* $\{us, vs\}$ means deleting these two edges and replacing them by a new edge uv (see Figure 2.3).

The inverse operation of splitting off is called *pinching*, i.e., pinching a set of edges $E' \subseteq E$ at a vertex w means replacing every edge $uv \in E'$ by the two edges uw and vw. Note that if $w \notin V$ we need to add w to V first. For convenience we will omit the brackets if $E' = \{e\}$.

The following theorem by Mader [37] will be an important ingredient to the proofs of the main results of this thesis. A bridge here means a cut of cardinality 1, i.e., an edge whose removal disconnects the graph.

Theorem 2.26 (Mader [37])**.** *Let $G = (V, E)$ be a connected graph, $s \in V$ not incident to any bridges in G and $\deg(s) \neq 3$. Then we can find a pair of edges incident to s that can be split off such that the local edge connectivity remains unchanged for all $x, y \in V \setminus \{s\}$.*

3 Infinite Graphs and Topology

3.1 The Freudenthal Compactification

3.1.1 Definition and Elementary Facts

This section will contain a short introduction to the Freudenthal compactification (often also refered to as end compactification) of a graph. For a more extensive introduction to the topic see [13].

Freudenthal [19] was the first to introduce the idea of compactifying a graph by its ends, i.e., to use the directions along which a sequence of vertices can tend to infinity for the compactification. The commonly used combinatoric description of ends as equivalence classes of rays that we will focus on has been introduced by Halin [26] independently of Freudenthal's work. The two definitions fail to coincide in a non locally finite setting and there are different approaches to define the end space of general infinite graphs (see [34], [14] and [16]). In locally finite graphs, however, all of these approaches are equivalent to the definitions of Freudenthal and Halin. Hence for the purpose of this thesis the following definitions will be sufficient since it will only deal with locally finite graphs.

Definition 3.1. Let $G = (V, E)$ be a locally finite graph. A *ray* in G is a one sided infinite path. An infinite subpath of a ray is called a *tail* of this ray.

We say that a finite set U of edges *separates* two rays γ_1 and γ_2 if there are different components C_1 and C_2 of $G \setminus U$ such that some tail of γ_i lies in C_i for $i \in \{1, 2\}$.

Now we define an equivalence relation $\sim$ on the set of rays by

$$\gamma_1 \sim \gamma_2 \Leftrightarrow \text{There is no finite set of edges separating } \gamma_1 \text{ and } \gamma_2$$

and call the equivalence classes of rays with respect to $\sim$ the *ends* of G. The set of all ends of G is denoted by $\Omega(G)$. If G is clear from the context we will sometimes write Ω instead of $\Omega(G)$.

Remark.

- It is easy to see that $\sim$ is indeed an equivalence relation. If the graph is locally finite it is also straightforward to check that two rays are equivalent if and only if they cannot be separated by a finite number of vertices.

  Furthermore two rays are equivalent if and only if there are infinitely many disjoint paths connecting the two rays. As a consequence two rays are equivalent if and only if there is a third ray meeting both of them infinitely often.

  When proving equivalence of rays we will sometimes use one of the above conditions without explicitly mentioning it.

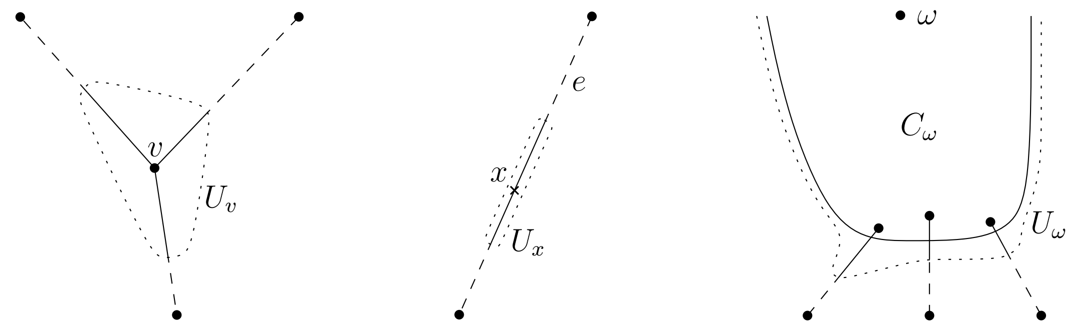

Figure 3.1: Basic open neighbourhoods in the end topology

- If one ray has a tail that lies completely in a component C of $G \setminus U$ for some finite set U of edges then so do all equivalent rays. Hence we may think of the end lying in this component.

In the next step we will define a topology on $G \cup \Omega$ by defining basic open neighbourhoods of every point in the set. Figure 3.1 illustrates these open neighbourhoods. Note that the boundary is not included in any of the neighbourhoods. A bit more formally speaking:

- For a vertex v the basic open neighbourhoods of v consist of v and an open half edge for every edge incident to v.
- For an inner point x of an edge e the basic open neighbourhoods are exactly the open intervals on e containing x.
- For an end ω and a finite set S of edges denote by C_ω the component of $G \setminus S$ in which ω lies. Open neighbourhoods of ω consist of such a set C_ω (including all ends that lie in this component) plus a half edge of every $e \in S$.

Definition 3.2. The open neighbourhoods defined above form a basis of a topology τ on $G \cup \Omega$. The topological space $(G \cup \Omega, \tau)$ is called the *Freudenthal compactification* or *end compactification* of G and will be denoted by $\overline{G}$.

It is a well known fact that for a locally finite connected graph G the space $\overline{G}$ is compact and metrizable (see [13] for proofs). Among other things this implies that $\overline{G}$ is a complete metric space and a normal and thus also Hausdorff topological space. All of this also holds if G is locally finite and has finitely many components. If it has infinitely many components then $\overline{G}$ is still a complete metric space but obviously it is not compact anymore.

The sequence of vertices of a ray converges to the end in which the ray lies. More generally, a sequence of vertices has an end ω as an accumulation point if and only if there is an infinite comb with all teeth in the sequence whose spine lies in ω. This follows easily from the definition of basic open neighbourhoods of an end.

Another obvious implication of the definition of the basic open sets is that G as a subspace of $\overline{G}$ has the same topology as if we consider it a 1-complex.

3.1.2 End Faithful Subgraphs

Proposition 3.3. *Let $G = (V, E)$ be a graph and $H = (V', E')$ be a subgraph of G. Let ι be the embedding of H in G, i.e., $\forall x \in H\colon \iota(x) = x$. Then there is a unique continuous function $\overline{\iota}\colon \overline{H} \to \overline{G}$ such that $\overline{\iota}|_H = \iota$.*

Proof. Suppose we have such a function $\overline{\iota}$. Let $(v_n)_{n\in\mathbb{N}}$ be the sequence of vertices of a ray γ converging to an end $\omega \in \overline{H}$ and let ω' be the end of G defined by γ. Then

$$\overline{\iota}(\omega) = \overline{\iota}\left(\lim_{n\to\infty} v_n\right) = \lim_{n\to\infty} \overline{\iota}\,(v_n) = \lim_{n\to\infty} v_n = \omega'.$$

Thus there can be at most one continuous function with the desired properties.

For the proof of existence we have to check whether the only possible $\overline{\iota}$, that is

$$\overline{\iota}(x) = \begin{cases} x & \text{if } x \in G \\ \omega & \text{if } x \in \Omega(H) \text{ and a ray converging to } x \text{ in } \overline{H} \text{ converges to } \omega \text{ in } \overline{G} \end{cases}$$

is well defined and continuous.

It is well defined because two rays that are equivalent in H certainly are equivalent in G. Thus the end ω in the above definition does not depend on the choice of the ray.

Preimages of basic open neighbourhoods of points in G are basic open neighbourhoods of points in H and thus open.

Now let $\omega \in \Omega(G)$ and let O be a basic open neighbourhood of ω. Then O consists of a component of $G \setminus S$ and a half edge of every $e \in S$ for a finite cut S. The preimage of O consists the half edges in $S \cap E'$ and some components of $H \setminus S$. Now obviously every point in $\overline{\iota}^{-1}(O)$ has an open neighbourhood that is completely contained in $\overline{\iota}^{-1}(O)$ and thus $\overline{\iota}^{-1}(O)$ is open. □

Definition 3.4. Let $G = (V, E)$ be a graph and H a subgraph of G. Then H is said to be *end faithful* if for the function $\overline{\iota}$ from the previous proposition $\overline{\iota}|_{\Omega(H)}\colon \Omega(H) \to \Omega(G)$ is injective.

Obviously H is an end faithful subgraph of G if and only if any two rays of H that are equivalent in G are also equivalent in H. In the special case that H is connected the following proposition provides another necessary and sufficient condition for H being end faithful.

Proposition 3.5. *Let G be a graph and let H be a connected subgraph of G. Then H is end faithful if and only if $\overline{\iota}$ is a homeomorphism from $\overline{H}$ onto the closure of H in $\overline{G}$.*

Proof. The backward implication is trivial because a homeomorphism is injective.

For the forward implication recall that $\overline{H}$ is known to be compact and that $\overline{G}$ is Hausdorff. The function $\overline{\iota}$ is injective and continuous.

The following two well known facts from topology (see for example [43]) complete the proof.

- Every injective continuous function from a compact space to a Hausdorff space is a homeomorphism onto its image.
- Given a continuous function from a compact space to a Hausdorff space the image of the closure of a set is the closure of the image of the same set. □

In particular this result implies that a connected spanning subgraph H of a connected graph G is end faithful if and only if $\bar{\iota}|_{\Omega(H)}$ is a homeomorphism between the end spaces of G and H.

3.2 Topological Paths, Circles and Trees in Infinite Graphs

In this section we will describe some of the topological concepts concepts that Diestel and Kühn[11, 15, 16] utilized to extend some graph theoretical notions from finite to infinite graphs. For a more extensive introduction to the topic also see [13].

Definition 3.6. Let $G = (V, E)$ be a locally finite graph and denote by $\overline{G}$ the Freudenthal compactification of G and by Ω the set of its ends.

- A *topological path* is a continuous (but not necessarily injective) map from the closed unit interval $[0, 1]$ to $\overline{G}$.

  Given $U_1, U_2 \subseteq V$ we say that a topological path connects U_1 and U_2 if it maps 0 to $u_1 \in U_1$ and 1 to $u_2 \in U_2$. In this case it is called a topolocical U_1-U_2-path. For convenience we will omit the brackets for topological $\{u_1\}$-$\{u_2\}$-paths.
- An *arc* is a homeomorphic image of the closed unit interval $[0, 1]$ in $\overline{G}$, i.e., the image of an injective topological path.
- A *topological ray* a homeomorphic image of the half open unit interval $[0, 1)$ in $\overline{G}$.
- A *topological circle* is a homeomorphic image of the unit circle C^1 in $\overline{G}$.

  The *circuit* associated with a circle C is defined as $C \cap G$.
- A *topological tree* is a path-connected subspace of $\overline{G}$ that does not contain a topological cirle.

  A *topological spanning tree* of G is a topological tree that contains all vertices and all ends of G and every edge of which it contains an inner point. Note that if T is a topological spanning tree then $T \cap G$ is a spanning forest of G.

The next two statements about topological paths and arcs are basic results from topology. The first result deals with the composition of paths. The proof is short and straightforward and can be found in most topology textbooks, see [43] for example.

Proposition 3.7. *If A is the image of a topological a-x-path and B is the image of a topological x-b-path then $A \cup B$ is the image of a topological a-b-path.*

The second proposition can be used to create arcs out of paths. The proof is more involved than the proof of Proposition 3.7 and can be found in [28].

Proposition 3.8. *If A is the image of a topological path then A is arcwise connected, i.e., for any two points $a, b \in A$ there is an arc in A connecting a and b.*

The next proposition assures that there is no non-constant topological path whose image consists only of ends. Note that the requirement that G is locally finite is crucial since the statement is not true for arbitrary countable graphs. See [16] for a non locally finite counterexample.

Proposition 3.9. *Let $G = (V, E)$ be a locally finite graph and let $P \subseteq \overline{G}$ be the image of a topological path. If P contains at least two points of $V \cup \Omega$ then P contains at least one vertex.*

Proof. Assume that P contains no vertices and denote by $\omega_1, \omega_2 \in P$ two distinct ends.

ω_1 and ω_2 can be separated by removing a finite set of vertices, say V'. This implies that $\overline{G} \setminus V'$ is the disjoint union of two open sets containing ω_1 and ω_2 respectively which implies that there is no topological ω_1-ω_2-path in $\overline{G} \setminus V'$.

Thus P has to contain at least one vertex in V', a contradiction to P containing no vertex at all. □

Remark. Note that the above proposition also implies that in a locally finite graph every topological circle has to contain at least one vertex.

Proposition 3.10. *Let T be a tree. Then its Freudenthal compactification $\overline{T}$ does not contain a topological circle.*

Proof. A topological circle in $\overline{T}$ has to contain at least one end since a topological circle containing no end at all is a graph theoretical circle.

If there was a topological circle containing only one end then there would be two disjoint rays converging to that end. Since this is not possible in a tree a possible circle has to contain at least two ends.

A topological circle containing at least two ends ω_1, ω_2 can be decomposed into two disjoint arcs connecting those ends. Let v be a vertex of T such that ω_1 and ω_2 lie in different components of $T \setminus V$ (such a vertex exists because T is a tree). Then every ω_1-ω_2-arc contains v and thus there cannot be a pair of disjoint ω_1-ω_2-arcs. □

3.2.1 End Faithful Spanning Trees and Topological Spanning Trees

Definition 3.11. Let G be a graph and let $T \subseteq \overline{G}$ be a topological spanning tree of G. We call T *end faithful* if $T \cap G$ is an end faithful subgraph of G.

The following result provides some sufficient conditions for end faithful topological spanning trees of a locally finite graph. In [16] Diestel and Kühn obtain a similar result.

They also show that the statement does not remain true for arbitrary countable graphs by giving a non locally finite counterexample, i.e., an end faithful spanning tree whose closure contains a topological circle.

Proposition 3.12. *Let $G = (V, E)$ be a locally finite graph and let T be a subspace of $\overline{G}$ containing all ends of G. The following statements are equivalent.*

1. *$T \cap G$ is an end faithful spanning tree.*
2. *$T \cap G$ is a spanning tree and T is a topological spanning tree.*
3. *T contains a (graph theoretical) path between any two vertices but no topological circle.*

Proof. $1 \Rightarrow 2$: As every graph theoretical path in $T \cap G$ is also a topological path in T we only have to prove that T does not contain a topological circle.

Since $T \cap G$ is a connected and end faithful subgraph of G we know that T is homeomophic to the Freudenthal compactification $\overline{T \cap G}$ by Proposition 3.5. By Proposition 3.10 there is no topological circle in $\overline{T \cap G}$.

$2 \Rightarrow 1$: The graph $T \cap G$ is a spanning tree. Hence it is sufficient to show that any two rays in $T \cap G$ which are equivalent in G are also equivalent in $T \cap G$.

Assume that the converse holds, i.e., there is an end ω and two rays belonging to different ends of $T \cap G$ which are both contained in ω. Since $T \cap G$ is connected there is a finite path in $T \cap G$ connecting the two rays. This path obviously does not contain the end ω and it's an easy observation that we can construct a topological circle in T out of this path and the two rays, a contradiction to T being a topological spanning tree.

$2 \Leftrightarrow 3$: The forward implication is trivial, for the backward implication observe that every graph theoretical path (circle) also is a topological path (circle). □

Proposition 3.13. *Let G be a locally finite graph and let T be a spanning tree of G. Then T is end faithful if and only if $T \cup \Omega(G)$ is a topological spanning tree of G.*

Proof. Follows from $1 \Leftrightarrow 2$ in Proposition 3.12. □

Proposition 3.14. *Let $G = (V, E)$ be a locally finite graph with finitely many ends and let T be a topological spanning tree of G. Then T is end faithful if and only if $T \cap G$ spanning tree of G.*

Proof. The backward implication follows from $2 \Rightarrow 1$ in Proposition 3.12.

For the forward implication assume that $T \cap G$ is an end faithful spanning subgraph of G. Clearly $T \cap G$ is circle free. So the only thing left to prove is that $T \cap G$ is connected.

Assume that u and v are vertices that lie in different components of $T \cap G$. Since T is a topological spanning tree there is a u-v-arc A in T. This arc has to contain at least one end ω. There are only finitely many ends so there is a neighbourhood of ω in A that contains no other ends.

This neighbourhood can be decomposed into two disjoint rays γ_1, γ_2 converging to ω. Since T is end faithful these two rays are equivalent in $T \cap G$. So there is a path $P \subseteq T$ connecting γ_1 and γ_2 and $P \cup \gamma_1 \cup \gamma_2$ contains a topological circle, a contradiction to T being a topological spanning tree. □

Remark. The requirement that G has finitely many ends is crucial. There are examples of graphs with infinitely many ends such that the statement does not hold.

3.2.2 Limits of Subgraphs of Induced Minors

Sometimes it can be useful to know what the limit of a sequence of graphs looks like. This is especially the case if we want to deduct properties of the limit graph from properties of the elements of the sequence. The following result will particularly be relevant for the proofs of Theorems 1.13 and 1.14 where we will construct topological spanning trees as limits of spanning trees of finite contractions.

Proposition 3.15. *Let $G = (V, E)$ be a locally finite graph and let V_n be a sequence of finite subsets of V such that $\lim_{n\to\infty} V_n = V$. Let $G_n = G\{V_n\}$. If T_n is a spanning tree of G_n and $T_{n+1}|_{G_n} = T_n$ for every $n \in \mathbb{N}$ then $T := \lim_{n\to\infty} T_n$ is a topological spanning tree of G.*

Proof. First we show that there is a topological u-v-path P for every pair $u, v \in V$. We will get this path as a limit of paths P_n in T_n. Let P_1 be a u-v-path in T_1 that uses every contracted vertex at most once. We may choose a parametrisation such that the preimage of every contracted vertex under P_1 is an interval of length $\varepsilon_1 > 0$. Since there are only finitely many contracted vertices we can always achieve this property by simple homotopic transformations.

Now construct P_{n+1} out of P_n by replacing every contracted vertex by a path that lies completely in the component of $G_{n+1} \setminus V_n$ corresponding to this vertex (recall that T_{n+1} induces a spanning tree in each of these components by Proposition 2.21). We can do so by only modifying the preimages of contracted vertices. Again we can assume that P_{n+1} uses every contracted vertex at most once and choose the parametrization of P_{n+1} such that the preimage of every contracted vertex under P_{n+1} is an interval of length $\varepsilon_{n+1} > 0$.

Now define $P(x) = P_{n_0}(x)$ if the sequence $P_n(x)$ is constant from n_0 on. If there is no such n_0 then the sequence $P_n(x)$ contains only contracted vertices. Since the sequence does not stay in any bounded subset of the graph all of its accumulation points have to be ends. There cannot be more than one accumulation point because any two distinct ends are separated by a finite set of vertices and thus will eventually be contained in different components of $G \setminus V_n$. Hence $P_n(x)$ converges to a unique end ω and we define $P(x) = \omega$.

To check continuity of P consider the preimages of basic open sets. The preimages of basic open neighbourhoods of vertices and inner points of edges are open due the fact that $P = P_n$ on those neighbourhoods for some n.

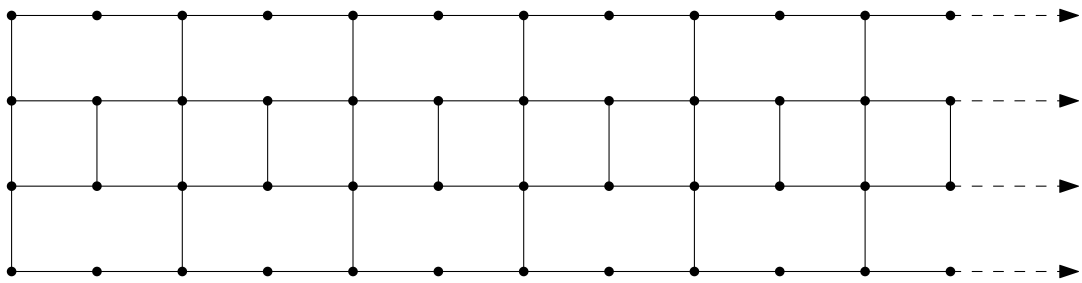

Figure 3.2: A non-Hamiltonian graph

Let O be an open neighbourhood of an end, that is, a component C of $G \setminus E'$ for some finite set $E' \subseteq E$ plus an open half edge for every $e \in E'$. There is an index n such that V_n contains all endpoints of edges in E'. All points in $[0,1]$ that are mapped to a component of $G \setminus V_n$ by P_n will be mapped to the same component by P_m for every $m > n$. This holds for components contained in O as well as for components disjoint to O. Hence the preimage of O under P_n and under P are the same and thus the preimage of O under P is open.

The construction of a topological path between a vertex and an end or between two ends works analogously, the only difference is that we need to construct paths connecting a vertex in V to a contracted vertex or two contracted vertices respectively.

Finally assume that T contains a topological circle C. By Proposition 3.9 this circle has to contain at least one vertex, say v. Then v is included in every V_n from some index n_0 on. The restriction of C to H_n is a circle for every $n \geq n_0$, a contradiction to H_n being a tree. □

As we have seen, the limit of a suitable sequence of spanning trees is a topological spanning tree. If we were able to prove that something similar holds for Hamiltonian cycles it would certainly help extending Hamiltonicity results from finite to locally finite graphs.

It has been proposed to construct Hamilton cycles as limits of Hamilton cycles in the graphs $G\{V_n\}^*$ obtained from finite contractions $G\{V_n\}$ by connecting any two vertices in V_n that are connected to the same contracted vertex. The next example, however, shows that there are non-Hamiltonian graphs such that whenever V_n is finite $G\{V_n\}^*$ is Hamiltonian. Furthermore the graph in the example allows an infinite increasing sequence of subsets V_n of V whose limit is V such that $G\{V_n\}$ is Hamiltonian for every n.

Example. Let G be the graph shown in Figure 3.2.

Claim 1. G is not Hamiltonian.

In a possible Hamilton cycle every vertex has to have degree two. Hence such a cycle has to contain all of the upper and lower ray, i.e., two rays converging to the single end ω of the graph.

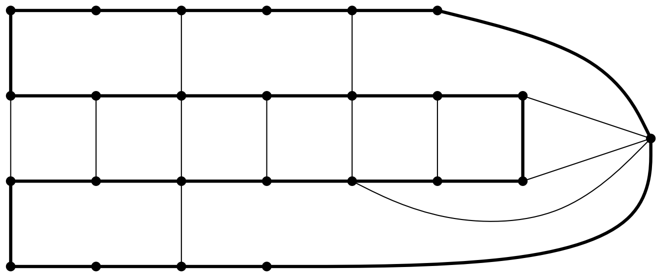

Figure 3.3: A possible finite contraction of the graph in Figure 3.2. The bold edges form a Hamilton cycle.

The rest of the cycle has to span all vertices of the two rays inbetween. So this part of the cycle contains at least one more ray converging to ω. This however implies that the cycle has to pass through ω at least twice and thus is not injective.

Claim 2. There is a sequence $(V_n)_{n\in\mathbb{N}}$ of subsets of V whose limit is V such that every $G\{V_n\}$ is Hamiltonian.

Consider the contraction and the Hamiltonian cycle in it that is shown in Figure 3.3. In a similar way we can find Hamiltonian cycles in infinitely many contractions $G\{V_n\}$. We only need to ensure that the subpaths of the two middle rays contained in V_n have the same length.

Claim 3. Whenever $V_n \subseteq V$ is finite $G\{V_n\}^*$ is Hamiltonian.

By Claim 2 we can find $V_n \subseteq V_m \subseteq V$ such that $G\{V_m\}$ is Hamiltonian. We will now use a Hamiltonian cycle in $G\{V_m\}$ to construct a Hamiltonian cycle in $G\{V_n\}^*$.

Let $v_0, v_1, v_2, \ldots, v_{k-1}, v_k = v_0$ be the vertex sequence of a Hamilton cycle in $G\{V_m\}$. We may without loss of generality assume that $v_0 \in V_n$. Now let $P = v_0$ and perform the following steps until all vertices of V_n are contained in P:

Let v_i be the last vertex that has been added to P

- If $v_{i+1} \in V_n$ then append v_{i+1} to P.
- If $v_{i+1} \notin V_n$ denote by C the vertex set of the component of $G\{V_m\} \setminus V_n$ to which $v_i + 1$ belongs. Let v_C be the contracted vertex corresponding to C and let $j_0 = \min\{j > i \mid v_j \notin C\}$. Obviously $v_{j_0} \in V_n$ because if this was not the case it would belong to C (note that there is an edge connecting v_{j_0} and C).
  - If v_C is not contained in P so far then append v_C and v_{j_0} to P.
  - Otherwise only append v_{j_0} to P.

The list P contains all vertices of $G\{V_n\}^*$ because the cycle in $G\{V_m\}$ contained all vertices of $G\{V_m\}$ and $G\{V_n\} = (G\{V_m\})\{V_n\}$ has the same vertex set as $G\{V_n\}^*$. There is an edge between two consecutive vertices in the list P by the definition of $G\{V_n\}^*$. Hence P is the vertex list of a Hamilton cycle in $G\{V_n\}^*$.

4 Thomassen's Theorem

4.1 A Proof Sketch

In this section we will give a short proof sketch of Thomassen's Theorem and discuss briefly what needs to be done in order to make a similar proof work for infinite graphs. But first recall the statement of the theorem:

Theorem 1.4 (Thomassen [46]). *The line graph of every finite* 4*-edge-connected graph has a Hamiltonian cycle.*

Proof Sketch. Basically the proof consists of three steps:

Step 1: By Corollary 2.19 a 4-edge-connected graph G has two edge disjoint spanning trees T^1 and T^2.

Step 2: From T^1 and T^2 we can construct a spanning Eulerian subgraph H of G as follows:

For every edge e of T^1 there is precisely one circle in $T^2 \cup \{e\}$, the fundamental circle of e with respect to T^2. Let $V(H) = V(G)$ and let $E(H)$ consist of all edges of T^1 and of those edges in T^2 that lie in an odd number of such fundamental circles.

Then H is connected because T^1 is a subgraph of H and it can easily be verified that every vertex has even degree in H.

Step 3: Apply Proposition 2.5 to obtain a Hamilton cycle in $L(G)$. □

In order to make this proof work for locally finite graphs as well we first of all need to find a suitable generalization of Corollary 2.19 to make sure that Step 1 works for locally finite graphs.

As for Step 2 we need to ensure that we can define H like we did in the finite case, i.e., that it can not happen that an edge is contained in infinitely many fundamental circles. We also need to check whether the construction yields a spanning Eulerian subgraph in the locally finite case.

In order to make Step 3 work in a locally finite setting we need to extend Proposition 2.5 to locally finite graphs. It will be crucial that the spanning Eulerian subgraph that has been constructed in Step 2 does not contain any end more than twice. Otherwise Step 3 is bound to fail since a Hamilton circle contains exactly two topological rays to every end and the end spaces of a graph and its line graph are the same.

4.2 Topological Spanning Trees

As mentioned before we will first of all need a suitable generalization of Corollary 2.19. Stein [44] proved the following natural extension of the theorem of Tutte and Nash-Williams to locally finite graphs from which we can deduce a generalization of the corollary.

Theorem 4.1 (Stein [44])**.** *Let $G = (V, E)$ be a locally finite graph. For a partitioning $\mathcal{V}$ of V denote by $E(\mathcal{V})$ the set of edges that connect different sets in $\mathcal{V}$. Then G has k edge disjoint topological spanning trees if and only if $|E(\mathcal{V})| \geq k(|\mathcal{V}| - 1)$ for every partitioning $\mathcal{V}$ of V .*

Corollary 4.2. *Let G be a $2k$-edge-connected locally finite graph. Then G has k edge disjoint topological spanning trees.*

Furthermore Diestel and Kühn [16] showed that Step 2 generalizes verbatim to topological spanning trees. Note that the fundamental circles with respect to a topological spanning tree are actually topological circles.

Theorem 4.3 (Diestel and Kühn [16])**.** *Let G be a locally finite graph and let T be a topological spanning tree of G. Then every edge is only contained in finitely many fundamental circles with respect to T.*

Proposition 4.4. *Let G be a locally finite graph and let T^1, T^2 be topological spanning trees of G. Let H be the subgraph of G obained by Step 2, i.e., H contains all edges of T^1 and those edges of T^2 that are in an odd number of fundamental circles of edges in T^1.*

Then H contains an even number of edges in S for every finite cut S in G.

Proof. Observe that $e \in T^1$ is contained in an odd number (namely one) of fundamental circles. Denote by K_S the set of fundamental circles that contain edges in S for some finite cut S. Note that every such circle contains an even number of vertices in S.

$$|S \cap H| \equiv \sum_{e \in S} \sum_{\substack{K \in K_S \\ e \in K}} 1 \equiv \sum_{K \in K_S} \sum_{e \in S \cap K} 1 \equiv \sum_{K \in K_S} |K \cap S| \mod 2$$

because all of the sums are finite. For every circle $|K \cap S|$ is even, so $|S \cap H|$ is also even. □

However promising substituting topological spanning trees for ordinary spanning trees may look, it is not sufficient to prove an Thomassen's theorem for locally finite graphs. The following example illustrates why.

Example. Figure 4.1 shows a 4-edge-connected graph G and two edge disjoint topological spanning trees of G. When we apply Step 2 for these two topological spanning trees we obtain the subgraph H shown in Figure 4.2. Obviously there are 5 non-equivalent rays in H that converge to the same end in G and thus H can not be used to construct a Hamiltonian cycle in $L(G)$.

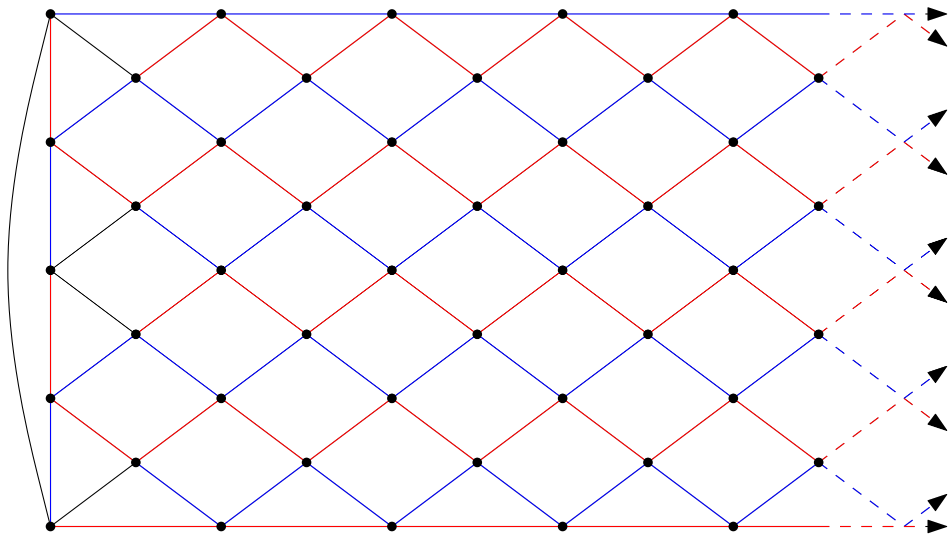

Figure 4.1: Two edge disjoint topological spanning trees in a locally finite 4-edge-connected graph.

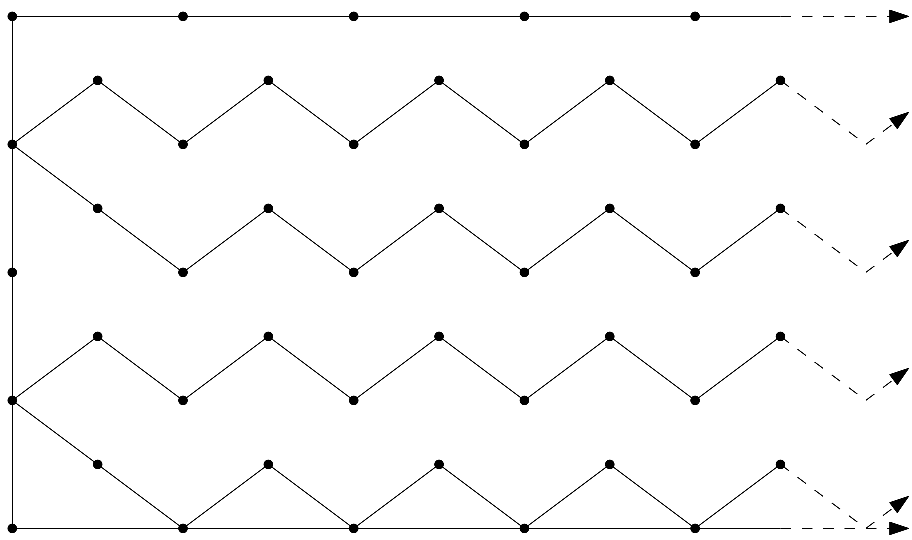

Figure 4.2: The subgraph obtained by Step 2 from the two topological spanning trees in Figure 4.1.

4.3 An Additional Restriction

We have seen that finding edge disjoint topological spanning trees is not sufficient for a generalizaion of the proof of Theorem 1.4 to locally finite graphs. However, things look brighter if one of the two topological spanning trees fulfills the additional requirement that its intersection with G is an ordinary spanning tree of G as the results in this section will show.

Proposition 4.5. *Let G be a locally finite graph and let T and T' be two edge disjoint topological spanning trees of G such that $T \cap G$ is an ordinary spanning tree of G. Then G has an end faithful spanning subgraph that admits a topological Euler tour.*

Proof. For and edge e in T denote by K_e the fundamental circle of e with respect to T'. Let H be the subgraph of G containing all edges of T and those edges of T' that are contained in an odd number of K_e. By Theorem 4.3 this is well defined because no edge is contained in infinitely many fundamental circles.

To show that H admits a topological Euler tour we need to show that every cut in H is either even or infinite. For this purpose let S be a cut in H. Then we can find a cut S' in G such that the vertex sets of the components of $G \setminus S'$ are precisely vertex sets of the components of $H \setminus S$.

- If S' is finite then by Proposition 4.4 the cut S is even.
- If S' is infinite then we claim that $S \cap T \subseteq S$ is infinite as well.

  Assume that this was not the case. Then there is an infinite sequence of edges $x_i y_i \in S' \setminus T$. We may without loss of generality assume that the sequence x_i is convergent, otherwise choose a convergent subsequence. The limit is an end ω because the sequence consists of infinitely many different vertices and thus eventually abandons every bounded subset of V. Clearly the sequence y_i converges to the same limit as the sequence x_i.

  Since $T \cap G$ is connected we can find an infinite comb in $T \cap G$ with all teeth in the set $\{x_i \mid i \in \mathbb{N}\}$. The spine of this comb lies in ω. We also can find an infinite comb in $T \cap G$ with all teeth in the set $\{y_i \mid i \in \mathbb{N}\}$ whose spine also lies in ω.

  The sets of teeth of the two combs lie in different components of $T \setminus S$. So if $S \cap T$ is finite the spines have tails lying in different components of $T \setminus S$. This implies that they are not equivalent in T, a contradiction to T being end faithful.

This proves that H is Eulerian.

Now all that is left to show is that H is end faithful. For this purpose let γ be an arbitrary ray in H and let $(x_i)_{i \in \mathbb{N}}$ be the sequence of vertices of γ. There is an infinite comb in T with all teeth in γ and the spine of this comb lies in the same end as γ. So every ray in H is equivalent in H to a ray in T. Since T is end faithful so is H. $\square$

The next result was obtained by Brewster and Funk [1] and among other things it implies that the line graph of a graph with an end faithful spanning Eulerian subgraph contains a topological Hamilton cycle. A closed dominating trail here means a closed topological path which contains at least one endpoint of each edge. The proof is based on Georgakopoulos' [22] proof of a very similar result.

Proposition 4.6 (Brewster and Funk [1]). *Let $G = (V, E)$ be a locally finite graph. If $\overline{G}$ contains a closed dominating trail which is injective on the ends of G then $L(G)$ is Hamiltonian.*

So in order to prove that the line graph of a graph G is Hamiltonian it is sufficient to show that G has two edge disjoint topological spanning trees one of which yields an ordinary spanning tree under intersection with G.

5 Main Results

5.1 Edge Disjoint Spanning Trees with Restrictions

Definition 5.1. Let $G = (V, E)$ be a graph, $v \in V$ and let a and b be vertices or inner points of edges of G. An a-b-arc is called an *a-b-bypass of v* if it does not contain v.

The set of all spanning tree packings $\mathcal{T}$ of G of cardinality k such that there are l trees in $\mathcal{T}$ whose union contains an a-b-bypass of v will be denoted by $\mathfrak{T}_G^{k,l}(a, b, v)$.

Definition 5.2. For a graph $G = (V, E)$ and $u \in V$ denote by E_u the set of edges incident to u, i.e., $E_u = \{e \in E \mid u \in e\}$.

The following lemma constitutes the main result of this section. It will be one of the main ingredients for the proof of Theorem 1.13 and consequently also for the proofs of Theorems 1.12 and 1.14.

Lemma 5.3. *Let $G = (V, E)$ be a finite $2k$-edge-connected graph and let $u, v \in V$ be such that E_u is a u-v-cut of minimal cardinality. Let a and b be vertices or inner points of edges of $G \setminus \{u\}$. If there is an a-b-bypass of v in $G \setminus \{u\}$ then*

$$\mathfrak{T}_{G\setminus\{u\}}^{k,2}(a, b, v) \neq \emptyset. \tag{$*$}$$

Remark. Note that in particular Lemma 5.3 implies that $G \setminus \{u\}$ has a spanning tree packing of cardinality k since for $a = b \neq v$ an a-b-bypass of v always exists. If $G \setminus \{u\}$ consists of v only then the existence of such a spanning tree packing is trivial.

We will prove the statement of Lemma 5.3 by induction on the number of vertices. Before doing so however, let's take a look at some consequences of this result.

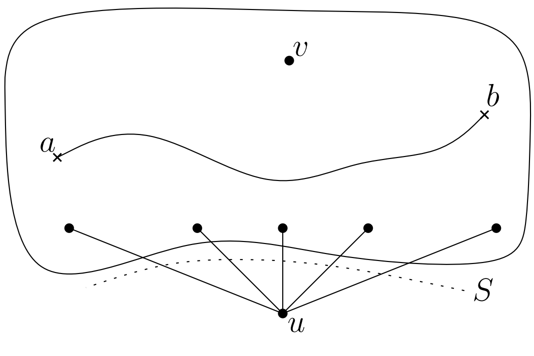

Figure 5.1: Situation in Lemma 5.3. Note that a and b do not necessarily need to be vertices.

5.1.1 Some Corollaries

The first corollary we will deduce from Lemma 5.3 will have its main application in the proof of Theorem 1.14.

Corollary 5.4. *Let $G = (V, E)$ be a finite $2k$-edge-connected graph, $k \geq 2$, and let $u, v \in V$ be such that E_u is a u-v-cut of minimal cardinality. Let a and b be vertices of $G \setminus \{u\}$. If there is an a-b-arc that does not contain u and v then*

$$\mathfrak{T}^{k-1,1}_{G\setminus\{u\}}(a, b, v) \neq \emptyset.$$

Proof. By Lemma 5.3 we can find

$$\mathcal{T} = \left\{T^1, T^2, \ldots, T^k\right\} \in \mathfrak{T}^{k,2}_{G\setminus\{u\}}(a, b, v).$$

We may without loss of generality assume that $T^1 \cup T^2$ contains an a-b-bypass of v. Since both a and b are vertices every a-b-arc is a simple graph theoretical path and thus cycle free.

Every acyclic subgraph of $T^1 \cup T^2$ can be extended to a spanning tree T' of $T^1 \cup T^2$ which also is a spanning tree of $G \setminus \{u\}$. Clearly T' contains the same a-b-bypass of v as $T^1 \cup T^2$ and T' and T^i are edge disjoint for $i > 2$ because $T' \subseteq T^1 \cup T^2$. So

$$\mathcal{T}' := \left\{T', T^3, \ldots, T^k\right\} \in \mathfrak{T}^{k-1,1}_{G\setminus\{u\}}(a, b, v). \qquad \square$$

Remark. A similar proof can be given if a and b are inner points of edges e_a and e_b as long as at least one of the edges is not incident to v. If both e_a and e_b are incident to v then an a-b-bypass of v induces a circle and hence such a bypass cannot be contained in a tree.

The next corollary is very similar to the corollaries of Theorem 2.18 in Section 2.6 that deal with $2k$-edge-connected graphs. In fact we can derive alternative proofs for Corollaries 2.19 and 2.20 from Corollary 5.5.

Corollary 5.5. *Let $G = (V, E)$ be a finite $2k$-edge-connected graph, let $u, v \in V$ and let S be a u-v-cut of minimal cardinality. Then there is a spanning tree packing $\mathcal{T}$ of cardinality k of G such that every tree $T \in \mathcal{T}$ contains exactly one edge in S.*

Proof. Let C_u and C_v be the vertex sets of the two components of $G \setminus S$ in which u and v lie respectively (by Proposition 2.9 these are the only components). Consider the graph G/C_u and denote by u' the vertex obtained from C_u in G/C_u.

By Proposition 2.12 the cut S is a u'-v-cut of minimal cardinality and since

$$S = \{e \in E(G/C_u) \mid u' \in e\}$$

we can apply Lemma 5.3 to obtain k edge disjoint spanning trees of

$$(G/C_u) \setminus \{u'\} = G[C_v].$$

Analogously we get k edge disjoint spanning trees of $G[C_u]$.

By connecting a spanning tree of $G[C_v]$ and a spanning tree of $G[C_u]$ with an edge in S we obtain a spanning tree of G that uses exactly one edge of S. There are at least $2k$ edges in S and we have k edge disjoint spanning trees of $G[C_u]$ and $G[C_v]$ respectively. This allows to construct the desired spanning tree packing and completes the proof. □

As mentioned before this result can be used to obtain alternative proofs for Corollaries 2.19 and 2.20 of Theorem 2.18.

Corollary 2.19. *Let $G = (V, E)$ be a finite graph. If $\kappa'(G) \geq 2k$ then $\tau(G) \geq k$.*

Proof. Corollary 5.5 guarantees the existence of k edge disjoint spanning trees with certain restrictions. □

Corollary 2.20. *Let G be a finite graph.*

1. *$\kappa'(G) \geq 2k$ if and only if $\tau(G \setminus F) \geq k$ for every set F of at most k edges.*
2. *$\kappa'(G) \geq 2k + 1$ if and only if $\tau(G \setminus F) \geq k$ for every set F of at most $k + 1$ edges.*

Proof.

1. "$\Leftarrow$" If we remove k edges we still get k edge disjoint spanning trees. Thus we have to remove at least another k edges in order to disconnect the graph. So the minimal cardinality of a cut has to be at least $2k$.

   "$\Rightarrow$" Let $F \subseteq$ be a set of k edges and denote by G' the graph obtained from G by pinching F at a vertex $v \notin V$.

   G' is $2k$-edge connected because every cut contains at least $2k$ edges:

   - For any two vertices $a, b \in V(G)$ there are $2k$ edge disjoint a-b-paths inherited from G. So the cardinality of any cut that separates a and b is at least $2k$.
   - The only cut which does not disconnect two vertices in $V(G)$ is the cut E_v which contains $2k$ edges.

   It follows immediately that S has to be a v-x-cut of minimal cardinality for an arbitrary vertex $x \in V$ because it is a cut of cardinality $2k$ in a $2k$-edge-connected graph. Hence by Corollary 5.5 we can find k edge disjoint trees each of which only uses one edge of S. This implies that v is a leaf in all of the trees. Thus we can remove all edges incident to v to obtain k edge disjoint spanning trees of $G \setminus F$.

2. The second part of the proof can be copied verbatim from the proof of Corollary 2.20 in Section 2.6. □

5.1.2 The Proof of Lemma 5.3

Before starting to prove Lemma 5.3 let's recall its statement.

Lemma 5.3. *Let $G = (V, E)$ be a finite $2k$-edge-connected graph and let $u, v \in V$ be such that E_u is a u-v-cut of minimal cardinality. Let a and b be vertices or inner points of edges of $G \setminus \{u\}$. If there is an a-b-bypass of v in $G \setminus \{u\}$ then*

$$\mathfrak{T}^{k,2}_{G\setminus\{u\}}(a, b, v) \neq \emptyset. \tag{$*$}$$

Proof of Lemma 5.3. As mentioned earlier we will prove the lemma by induction on the number of vertices. If G is a graph on two vertices then $G \setminus \{u\}$ consists only of v. So there cannot be two points a and b as claimed in the condition of Lemma 5.3. Hence induction starts at $|V| = 3$.

Let G be a $2k$-edge-connected graph on three vertices u, v and w and let E_u be a u-v-cut of minimal cardinality.

A spanning tree of $G \setminus \{u\}$ consists of a vw-edge. So in order to obtain a spanning tree packing of cardinality k we need to ensure that there are at least k such edges. By Menger's Theorem 2.10 there are as many edge disjoint u-v-paths as edges in E_u. In particular there is a vw-edge for every uw-edge and since $\deg(w) \geq 2k$ there are at least k edges connecting v and w.

Now a and b can be either inner points of vw-edges or equal to w. Either way, there are two vw-edges whose union contains an a-b-bypass of v. We can select these two edges to form trees of the spanning tree packing.

For the induction step we may assume that $|E_u| > \kappa'(G)$ because the lemma holds for G if and only if it holds for the graph obtained from G by adding a bundle of parallel uv-edges since those edges are irrelevant for the statement of the lemma.

Now let S be a cut such that $|S| = \kappa'(G)$. The cut S does not separate u and v as E_u is a u-v-cut of minimal cardinality and $|E_u| > \kappa'(G)$. Denote by C and C' the vertex sets of the components of $G \setminus S$ and assume without loss of generality that $u, v \in C$.

We will distinguish the following two cases in both of which we will show that $(*)$ holds:

Case 1: $|C'| > 1$,

Case 2: $|C'| = 1$, i.e., $C' = \{w\}$ for some $w \in V$.

In case 1 consider the graph G/C'. Denote by $x_{C'}$ the vertex in G/C' that has been obtained by contracting the set C' and define $a' = x_{C'}$ if a lies in $G[C']$ and $a' = a$ otherwise. Analogously define b' from b.

We now claim that

(1) $G[C']$ has k edge disjoint spanning trees,

(2) $\mathfrak{T}^{k,2}_{(G/C')\setminus\{u\}}(a', b', v) \neq \emptyset$ and

(3) the statement $(*)$ follows from (1) and (2).

So let us first prove (1). For this purpose denote by x_C the vertex that corresponds to C in G/C. It can easily be seen that $G[C'] = (G/C) \setminus x_C$. Moreover G/C is $2k$-edge-connected by Proposition 2.14 and the set of edges incident to x_C in G/C is a cut of minimal cardinality in G/C by Proposition 2.12. As G/C has strictly less vertices than G we can by induction hypothesis find k edge disjoint spanning trees of $(G/C) \setminus \{x_C\}$. This proves (1).

Next we prove (2). Clearly G/C' has strictly less vertices than G. From Proposition 2.14 it follows that G/C' is $2k$-edge connected and by Propositions 2.12 and 2.13 the set of edges incident to u in G/C' is a u-v-cut of minimal cardinality in this graph. The a-b-bypass of v in $G \setminus \{u\}$ corresponds to an a'-b'-bypass of v in $(G/C') \setminus \{u\}$. So $\mathfrak{T}^{k,2}_{(G/C')\setminus\{u\}}(a', b', v) \neq \emptyset$ by the induction hypothesis. This proves (2).

In order to prove (3) let $\mathcal{T}_{C'} = \left\{T^1_{C'}, T^2_{C'}, \ldots, T^k_{C'}\right\}$ be a spanning tree packing of $G[C']$ and let $\mathcal{T}_{G/C'} = \left\{T^1_{G/C'}, T^2_{G/C'}, \ldots, T^k_{G/C'}\right\} \in \mathfrak{T}^{k,2}_{(G/C')\setminus\{u\}}(a', b', v)$. Since we can permute the trees in the packing freely we may without loss of generality assume that $T^1_{G/C'} \cup T^2_{G/C'}$ contains an a'-b'-bypass of v.

If a lies in $G[C']$ we may without loss of generality assume that a lies in $T^1_{C'}$ because

- if a is a vertex it is contained in every $T^i_{C'}$,
- if a is an inner point of an edge contained in some $T^i_{C'}$ we can permute the trees and
- if a is an inner point of an edge e not contained in any of the $T^i_{C'}$ we can modify $T^1_{C'}$ by adding e and removing an arbitrary edge of the circle that has been closed by doing so.

For the same reasons we may assume that if b lies in $G[C']$ and if b is not an inner point of an edge of $T^1_{C'}$ then b lies in $T^2_{C'}$.

Now let T^i be the subgraph of G that is obtained by replacing $x_{C'}$ in $T^i_{G/C'}$ by $T^i_{C'}$. We claim that $\mathcal{T} = \{T^i \mid 1 \leq i \leq k\} \in \mathfrak{T}^{k,2}_G(a, b, v)$.

By Propositions 2.21 and 2.23 $\mathcal{T}$ is a spanning tree packing. Thus it suffices to prove that $T^1 \cup T^2$ contains an a-b-bypass of v.

- If both a and b lie in $G[C']$ let $x \in C'$ be an arbitrary vertex. Since x is a vertex it is contained in every $T^i_{C'}$. By our choice of $\mathcal{T}_{C'}$ it holds that $a, b \in T^1_{C'} \cup T^2_{C'}$. Hence we can find an a-x-arc and a x-b-arc in $T^1_{C'} \cup T^2_{C'}$.

  The union of these two arcs clearly contains an a-b-arc. This arc does not contain v because $v \notin C'$, so it is an a-b-bypass of v in $T^1 \cup T^2$.

- If only one of a and b, say a, is contained in $G[C']$ then $T^1_{G/C'} \cup T^2_{G/C'}$ contains a $x_{C'}$-b-bypass of v. This implies that there is a vertex $x \in C'$ such that $T^1 \cup T^2$ contains a x-b-bypass B of v.

  For the same reason as before there is an a-x-arc A in $T^1_{C'} \cup T^2_{C'}$.

  The union $A \cup B$ contains an a-b-arc which is an a-b-bypass of v because v is not contained in either of A and B.

- If both a and b are not contained in $G[C']$ there is an a-b-bypass of v in $T^1_{G/C'} \cup T^2_{G/C'}$.

  If this bypass does not contain $x_{C'}$ then it is also an a-b-bypass of v in $T^1 \cup T^2$ and we are done.

  So assume that it does contain $x_{C'}$. In this case there are vertices $x_a, x_b \in U$ such that $T^1 \cup T^2$ contains an a-x_a-bypass A and a x_b-b-bypass B of v. Furthermore there is a x_a-x_b-arc A' in $T^1_{C'}$ since $T^1_{C'}$ is connected and both x_a and x_b are vertices and thus contained in $T^1_{C'}$.

  Clearly $A \cup B \cup A'$ contains an a-b-bypass of v.

This completes the proof of (3) and thus $(*)$ holds in case 1.

Now consider case 2, i.e., assume that $C' = \{w\}$. If $\deg(w)$ is odd then G is $(2k+1)$-edge-connected, so removing an arbitrary edge from G will leave it $2k$-edge-connected. In order to be able to apply the induction hypothesis we want to select the edge that we delete in a way that E_u remains a u-v-cut of minimal cardinality after deleting it.

To decide which edge to delete choose $\kappa'_G(u, v)$ edge disjoint u-v-paths. The total number of edges incident to w used by these paths has to be even as a path that enters w via one edge has to leave the vertex via another one. Thus there is at least one such edge, say e, which is not used by any of the paths. Hence E_u will still be a u-v-cut of minimal cardinality after e has been removed since there are still $\kappa'_G(u, v)$ edge disjoint u-v-paths.

Now assume that $\deg w$ is even (possibly after deleting an edge incident to w). By Mader's Theorem 2.26 we can split off all edges incident to w in pairs without changing the local edge connectivity of any pair of vertices in $V \setminus \{w\}$. In particular the set of edges incident to u remains a u-v-cut of minimal cardinality throughout this procedure. Denote the graph that we obtain by $H = (V_H, E_H)$. Now define a multiset V' over V_H (i.e., V' consists of vertices but may contain the same vertex more than once) and $E' \subseteq E_H$ as follows:

- Add a copy of $v' \in V_H$ to V' for every $v'w$-edge that has been split off with a uw-edge. Add another copy of v' if a $v'w$-edge has been deleted in order to make $\deg(w)$ even.

- Add $e' \in E_H$ to E' if it has been created by splitting off a pair of edges none of which is incident to u.

It is an easy observation that $G \setminus \{u\}$ is obtained from $H \setminus \{u\}$ by pinching all edges in E' at w and adding a $v'w$-edge for every $v' \in V'$.

We now claim that

(4) whenever $\mathcal{T}_H = \{T^1_H, T^2_H, \dots, T^k_H\}$ is a spanning tree packing of $H \setminus \{u\}$ and e_1, e_2 are edges of $G \setminus \{u\}$ that did not result from pinching an edge in $\bigcup_{i=3}^k T^i_H$ then there is a spanning tree packing $\mathcal{T} = \{T^1, T^2, \dots, T^k\}$ of $G \setminus \{u\}$ such that

   a) for every $e \in E \cap E_H$ it holds that $e \in T^i_H$ if and only if $e \in T^i$ and

b) e_1 and e_2 are contained in $T^1 \cup T^2$.

The first step in the proof of (4) is to turn $\mathcal{T}_H$ into a tree packing $\mathcal{T} = \{T^1, T^2, \ldots, T^k\}$ of $G \setminus \{u\}$ by the following pinching procedure.

Begin with $T^i = T_H^i$ for all i then pinch one edge in E' after the other at w and modify the T^i as follows (let $e = xy$ be the next edge to be pinched).

(A) If e belongs to none of the T_H^i we do not modify any T^i.

(B) If $e \in T_H^i$ and no edges incident to w have been added to T^i so far then we remove e from T^i and add both xw and yw to T^i in order to replace e.

(C) If $e \in T_H^i$ and we have already added edges to T^i before we also remove e from T^i. In this case adding both xw and yw to T^i would result in a circle containing w because there is either a x-w-path or a y-w-path in T^i that does not use e. Hence we only add the edge which is not contained in this circle to T^i.

Note that the T^i remain trees in every one of these steps and that no $e \in E \setminus E'$ is removed from or added to T^i. So after pinching all of E' according to (A), (B) and (C) we obtain k edge disjoint trees each of which spans V_H.

Still some of the trees may not contain w. Before dealing with this problem, however, we will take care of e_1 and e_2.

The edge e_1 can only be contained in T^i for some i if it has been added to T^i applying (B) or (C). Since e_1 did not result from pinching an edge of T^i for $i > 2$ this implies that either $e_1 \in T^1 \cup T^2$ or e_1 is not contained in any of the T^i at all. Assume $e_1 \notin T^1 \cup T^2$.

- If $w \notin T^1$ adding e_1 to T^1 makes T^1 a spanning tree of $G \setminus \{u\}$.
- If T^1 is a spanning tree of $G \setminus \{u\}$ already then adding e_1 to it completes a circle that contains w. Remove the other edge incident to w in this circle from T^1 to obtain a tree again.

So we may assume that $e_1 \in T^1 \cup T^2$, without loss of generality assume that $e_1 \in T^1$. If $e_2 \notin T^1 \cup T^2$ we can use the same procedure as above to obtain $e_2 \in T^2$. Since in this case only T^2 is modified e_1 remains in T^1.

Finally we need to ensure that all of the trees contain w. For this purpose define

$$L = \{T^i \in \mathcal{T} \mid w \notin T^i\}$$
$$L' = \{e \in E_w \mid u \notin e \text{ and } \forall T^i \in \mathcal{T} : e \notin T^i\}$$

and let $l = |L|$ and $l' = |L'|$. All that is left to prove is that $l' - l \geq 0$ because in this case we can add w to each tree in L using an edge in L'.

Note that $l' - l$ is invariant under the above procedure for adding e_1 and e_2 to $T^1 \cup T^2$ because if $w \notin T^1$ both values decrease by one while in the case that $w \in T^1$ they are constant.

So it is sufficient to show that $l' - l \geq 0$ held before these modifications. At that time there is an edge incident to w in $G \setminus \{u\}$ which is not used by any tree for every vertex in V' and for every edge in E' for which (B) is not applied. Since (B) is applied exactly once per tree containing w this implies that

$$\begin{aligned} l' &= |L'| \\ &= |V'| + (|E'| - |\mathcal{T} \setminus L|) \\ &= (|V'| + |E'|) - (|\mathcal{T}| - |L|) \\ &\geq k - (k - l) \\ &= l. \end{aligned}$$

This completes the proof of (4). We will now distinguish the following subcases of case 2 in each of which we will apply the induction hypothesis and (4) to show that $(*)$ holds. Note that we can apply the induction hypothesis to H because H has strictly less vertices than G and the set of edges incident to u is a u-v-cut of minimal cardinality in H.

Case 2a: both a and b lie in H and there is an a-b-bypass of v in $H \setminus \{u\}$,

Case 2b: both a and b lie in H and there is no a-b-bypass of v in $H \setminus \{u\}$,

Case 2c: a lies in H and b is an inner point of an edge that has been split off to generate an edge $e' \in E'$,

Case 2d: a lies in H and b is an inner point of a $v'w$-edge for $v' \in V'$,

Case 2e: both a and b lie on edges incident or are equal to to w.

Clearly these cases exhaust all possibilities where a and b could lie. So all we need to show is that $(*)$ holds in every one of them.

In case 2a we can apply the induction hypothesis to find $\mathcal{T}_H = \left\{T_H^1, T_H^2, \ldots, T_H^k\right\} \in \mathfrak{T}^{k,2}_{H\setminus\{u\}}(a, b, v)$. We may without loss of generality assume that there is an a-b-bypass A of v in $T_H^1 \cup T_H^2$. If A contains no edge of E' then it is also an a-b-bypass of v in the spanning tree packing $\mathcal{T}$ obtained by (4).

So assume that there is at least one edge in E' that is used by A. Denote by e_a the first edge in E' that we pass through when we traverse A starting at a. Clearly e_a is contained in $T^1 \cup T^2$. It is also easy to see that for one endpoint a' of e_a there is an a-a'-subarc A_a of A that does not contain any inner point of an edge in E'. Analogously define e_b and find a b-b'-subarc A_b of A that does not contain any inner point of an edge in E' for an endpoint b' of e_b.

By (4) we can find a spanning tree packing $\mathcal{T}$ of $G \setminus \{u\}$ such that there are two trees in $\mathcal{T}$ whose union contains A_a, A_b and the edges $a'w$ and $b'w$. Clearly the union of the two paths and the two edges constitutes an a-b-bypass of v completing the proof of case 2a.

Next consider case 2b. Since there is no a-b-bypass of v in $H \setminus \{u\}$ every such bypass in $G \setminus \{u\}$ used w. Thus such a bypass in $G \setminus \{u\}$ induces an a-a'-bypass and a b-b'-bypass of v in $H \setminus \{u\}$ where a' and b' are vertices in V' or inner points of edges in E'.

We claim that in this case

(5) there is a spanning tree packing $\mathcal{T}_H = \{T_H^1, T_H^2, \dots, T_H^k\}$ of H such that $T_H^1 \cup T_H^2$ contains an a-a'-bypass and a b-b'-bypass of v.

By the induction hypothesis we can find a spanning tree packing $\mathcal{T}_a = \{T_a^1, T_a^2, \dots, T_a^k\}$ of $H \setminus \{u\}$ such that $T_a^1 \cup T_a^2$ contains an a-a'-bypass of v. We can also find a spanning tree packing $\mathcal{T}_b = \{T_b^1, T_b^2, \dots, T_b^k\}$ of $H \setminus \{u\}$ such that $T_b^1 \cup T_b^2$ contains a b-b'-bypass of v.

Since there is no a-b-bypass of v in $H \setminus \{u\}$ we know that v is a cut vertex of $H \setminus \{u\}$ and that a and b lie in different components of $H \setminus \{u, v\}$. Denote by C_a the set of vertices of the component in which a lies and let $C_b = V_H \setminus (C_a \cup \{u, v\})$. It is easy to see that $T_a^i \{C_a\}$ is a spanning tree of $(H \setminus \{u\}) \{C_a\}$ and that $T_b^i \{C_b\}$ is a spanning tree of $(H \setminus \{u\}) \{C_b\} = (H \setminus \{u\})[V_H \setminus C_a]$.

This in particular implies that there is only one component of $H \setminus (C_a \cup \{u\})$ and (5) follows from Proposition 2.24.

Now let $\mathcal{T}_H$ be a spanning tree packing of H as claimed in (5) and let A be an a-a'-bypass of v in $T_H^1 \cup T_H^2$.

- If A contains an inner point of an edge in E' let e_a be the first edge in E' that we pass through when we traverse A starting at a. Let a'' be an endpoint of that edge such that A contains an a-a'' subarc A' that does not use inner points of any edge in E'. In this case an $a''w$-edge results from pinching $e_a \in T_H^1 \cup T_H^2$
- Otherwise $a' \in V'$ holds. Let $a'' = a'$ and $A' = A$. In this case there is an $a''w$-edge which did not result from pinching any edge, in particular not from pinching an edge in T^i for $i > 2$.

Analogously define b'' and B' from a b-b'-bypass B of v.

Now we can by (4) find a spanning tree packing $\mathcal{T} = \{T^1, T^2, \dots, T^k\}$ of $G \setminus \{u\}$ such that $T^1 \cup T^2$ contains A', B' and the edges $a''w$ and $b''w$. Clearly the union of the two paths and the two edges constitutes an a-b-bypass of w in $T^1 \cup T^2$. This completes the proof in case 2b.

Next let us turn to case 2c. In this case again the a-b-bypass of v in $G \setminus \{u\}$ becomes an a-a'-bypass in $H \setminus \{u\}$ where a' is a vertex in V' or an inner point of an edge in E'. By the induction hypothesis we can find a spanning tree packing $\mathcal{T}_H = \{T_H^1, T_H^2, \dots, T_H^k\}$ of $H \setminus \{u\}$ such that $T_H^1 \cup T_H^2$ contains an a-a'-bypass A of v. Analogously to case 2b define a'' and an a-a''-subarc A' of A.

Recall that in case 2c the point b is an inner point of an edge that has been split off to create an edge $e' \in E'$. We claim that

(6) we can chose $\mathcal{T}_H$ in a way that $e' \in T_H^1 \cup T_H^2$.

If a and e' lie in the same component of $H \setminus \{u, v\}$ we may assume that a' has been an inner point of e' in the first place and thus $e' \in T_H^1 \cup T_H^2$ holds.

So assume that a and e' lie in different components. In this case we can choose a vertex c in the component in which e' lies. Since there is a c-b'-bypass of v in $H \setminus \{u\}$

for every inner point b' of e' we can apply (5) to find a spanning tree packing with the desired properties.

This proves (6) and thus we can assume that the edge $b''w$ of which b is an inner point did not result from splitting off an edge in T^i for $i > 2$.

By (4) we can find a spanning tree packing $\mathcal{T} = \{T^1, T^2, \dots, T^k\}$ of $G \setminus \{u\}$ such that $T^1 \cup T^2$ contains A' and the edges $a''w$ and $b''w$ whose union contains an a-b-bypass of v which completes the proof of case 2c.

In case 2d once again apply the induction hypothesis to find a spanning tree packing $\mathcal{T}_H = \{T_H^1, T_H^2, \dots, T_H^k\}$ of $H \setminus \{u\}$ such that $T_H^1 \cup T_H^2$ contains an a-a'-bypass A of v where a' is a vertex in V' or an inner point of an edge in E'. As in the previous two cases define a'' and an a-a''-subarc A' of A.

Now we can apply (4) to find a spanning tree packing $\mathcal{T} = \{T^1, T^2, \dots, T^k\}$ of $G \setminus \{u\}$ such that $T^1 \cup T^2$ contains A' and the edges $a''w$ and bw which clearly proves $(*)$ in case 2d.

In case 2e apply the induction hypothesis to find k edge disjoint spanning trees of $H \setminus \{u\}$. If the edges $a'w$ and $b'w$ on which a and b lie were created by pinching some edges in E' we may permute the trees such that these edges lie in $T^1 \cup T^2$ or in none of the trees at all. We then apply (4) to find a spanning tree packing $\mathcal{T} = \{T^1, T^2, \dots, T^k\}$ of $G \setminus \{u\}$ such that $T^1 \cup T^2$ contains both of $a'w$ and $b'w$. This proves that $(*)$ holds in case 2e.

Since there are no more cases left it also completes the induction step and thus the proof of Lemma 5.3. $\square$

5.2 Gaps, Bridges and End Faithful Spanning Trees

Definition 5.6. Let $G = (V, E)$ be a graph. A non-decreasing sequence $(V_n)_{n \in \mathbb{N}}$ of subsets of V is called *exhausting* if $\lim_{n \to \infty} V_n = V$.

Definition 5.7. Let $G = (V, E)$ be a locally finite graph and $(V_n)_{n \in \mathbb{N}}$ an exhausting sequence of subsets of V. Define $G_n = G\{V_n\}$, i.e., G_n is the graph obtained from G by contracting each component of $G \setminus V_n$ to a single vertex.

- Given a spanning tree T_n of G_n we call a pair of components of $T_n[V_n]$ a *gap of T_n in G_n*.

  If (C_1, C_2) is a gap and $u \in C_1$ and $v \in C_2$ we say that the gap *separates* u and v.

- Assume that T_k is a spanning tree of G_k and that $T_{k+1}|_{V_k} = T_k$ for every $k \in \mathbb{N}$ and let $m < n$ be natural numbers.

  - A gap (C_1, C_2) of T_n *extends* a gap (C'_1, C'_2) of T_m if the vertex set of C'_i is a subset of the vertex set of C_i for $i \in \{1, 2\}$.

    Note that a gap (C'_1, C'_2) of T_m cannot be extended by multiple gaps in T_n. Hence we will not distinguish between a gap and its extension. This allows to talk about gaps in the sequence T_k.

  - A gap (C_1, C_2) of T_m *terminates in* V_n if it is not extended by a gap of T_n. Note that in this case C_1 and C_2 both are contained in the same component of $T_n[V_n]$ and thus connected by a path in T_n that does not use any contracted vertex.
  - A gap that does not terminate is called a *persistent gap* in the sequence T_k.

- Given a gap $\Gamma = (C_1, C_2)$ of T_n we say that a tree $T'_n \subseteq G_n$ *bridges the gap* Γ in V_n if there is a path P from C_1 to C_2 in $T_n \cup T'_n$ which does not use any contracted vertex. The set of edges in $P \cap T'_n$ is called a Γ-bridge in T'_n.

  A sequence T'_n bridges a gap Γ in a sequence T_n infinitely often if there are arbitrarily large sets of disjoint Γ-bridges in T'_n as $n \to \infty$. Note that this can only be the case if Γ is a persistent gap.

Proposition 5.8. *Let $G = (V, E)$ be a locally finite graph and $(V_n)_{n\in\mathbb{N}}$ an exhausting sequence of finite subsets of V. Furthermore let T_n be a sequence of spanning trees of $G\{V_n\}$ such that $T_n|_{V_{n-1}} = T_{n-1}$. If the sequence T_n contains no infinite gaps then $T := \lim_{n\to\infty} T_n$ is an end faithful spanning tree of G.*

Proof. By Proposition 3.12 it is sufficient to show that T contains a (graph theoretical) u-v-path for every pair $u, v \in V$ and does not contain a topological circle.

By Proposition 3.15, T is a topological spanning tree and thus it does not contain a topological circle.

To show that T is connected consider a pair u, v of vertices. Then there is an index $n_0 \in \mathbb{N}$ such that $u, v \in V_{n_0}$. This implies that there is a u-v-path in T_{n_0}. However, this path might use some contracted vertices and thus it is not necessarily a path in T. If this is the case, let Γ be the gap separating u and v. Now there is n_1 such that Γ terminates in G_{n_1}. Then in T_{n_1} there is a u-v-path that does not use any contracted vertex and is thus also a u-v-path in T. □

Proposition 5.9. *Let $G = (V, E)$ be a locally finite graph and $(V_n)_{n\in\mathbb{N}}$ an exhausting sequence of finite subsets of V. Furthermore let T_n, T'_n be sequences of spanning trees of $G\{V_n\}$ such that $T_n|_{V_{n-1}} = T_{n-1}$ and $T'_n|_{V_{n-1}} = T'_{n-1}$. If the sequence T'_n bridges every persistent gap of the sequence T_n infinitely often then $H := T \cup T'$ is an end faithful connected spanning subgraph of G where $T = \lim_{n\to\infty} T_n$ and $T' = \lim_{n\to\infty} T'_n$.*

Proof. The graph H is connected because any pair of components of H would constitute an infinite gap in the sequence T_n that is not bridged.

So we only need to show that any two rays γ_1 and γ_2 in H belonging to the same end ω of G are equivalent in H. Assume that γ_i belongs to an end ω_i of H for $i \in \{1, 2\}$ and that $\omega_1 \neq \omega_2$.

Let n be big enough that ω_1 and ω_2 lie in different components C^1_H and C^2_H of $H \setminus V_n$. Note that C^1_H and C^2_H both are subsets of the same component C_G of $G \setminus V_n$ because γ_1 and γ_2 converge to the same end of G. For $i \in \{1, 2\}$ denote by $v_i \in C^i_H$ a vertex of γ_i such that all consecutive vertices lie in C^i_H as well (see Figure 5.2).

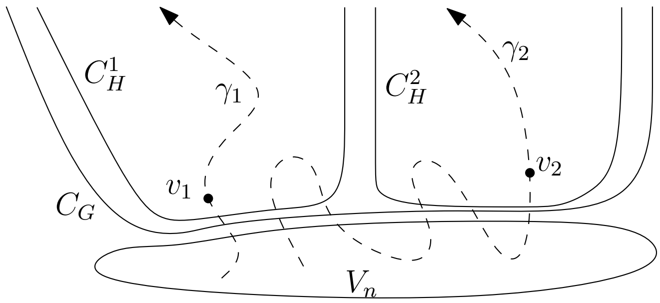

Figure 5.2: Situation in the proof of Proposition 5.9

If v_1 and v_2 belonged to the same component of T then the unique path in T connecting v_1 and v_2 would have to use vertices in V_n. Hence this path would correspond a circle in T_n. So v_1 and v_2 belong to different components of T and there is an infinite gap Γ that separates v_1 and v_2.

We know that Γ is bridged infinitely often, i.e., there are arbitrarily large sets $\mathcal{P}$ of paths in H connecting v_1 and v_2 such that

$$\forall P_1, P_2 \in \mathcal{P} \colon P_1 \cap P_2 \cap E(T') = \emptyset.$$

If a path $P \in \mathcal{P}$ has non-empty intersection with V_n it has to use at least one edge of T' with one endpoint in V_n because otherwise $P\mid_{V_n}$ would be a cycle in T_n. Hence the number of such paths is bounded by the number of edges with an endpoint in V_n. So if $|\mathcal{P}|$ is large enough then there is a path $P^* \in \mathcal{P}$ that connects v_1 and v_2 such that P^* contains no vertex of V_n.

Consequently v_1 and v_2 lie in the same component of $H \setminus V_n$, a contradiction. □

5.3 Compatible Cuts and a Partitioning of the Vertex Set

Definition 5.10. Let G be a graph, let $x, x_1, x_2, \ldots, x_k$ be vertices or ends of G and let S_i be a x-x_i-cut for $1 \le i \le k$. Let C_i be the component of $G \setminus S_i$ in which x_i lies. The set $\{S_1, \ldots, S_k\}$ is said to be *compatible* if no S_i contains an edge that connects two points in C_j for $1 \le i, j \le k$, otherwise it is said to be *incompatible*.

Definition 5.11. Let $G = (V, E)$ be a graph, $x \in V \cup \Omega$ and let $Y \subseteq (V \cup \Omega) \setminus \{x\}$. Define

$$\mathfrak{C}_x^Y = \{S \mid S \text{ is a } x\text{-}y\text{-cut of minimal cardinality for some } y \in Y\}$$

and denote by $\mathfrak{S}_x^Y$ the power set of $\mathfrak{C}_x^Y$, i.e., the elements of $\mathfrak{S}_x^Y$ are sets of cuts.

Now we define a binary relation $\sqsubset$ on $\mathfrak{S}_x^Y \times \mathfrak{S}_x^Y$. We say that $\mathcal{S} \sqsubset \mathcal{U}$ if

(D1) $|\mathcal{S}| \le |\mathcal{U}|$,

(D2) $S \subseteq \bigcup_{U \in \mathcal{U}} U$ for every $S \in \mathcal{S}$ and

(D3) the component of $G \setminus \bigcup_{S \in \mathcal{S}} S$ in which x lies is exactly the component of $G \setminus \bigcup_{U \in \mathcal{U}} U$ in which x lies.

Remark. Clearly $\mathcal{S} \sqsubset \mathcal{U}$ and $\mathcal{U} \sqsubset \mathcal{S}$ implies that $\bigcup_{S \in \mathcal{S}} S = \bigcup_{U \in \mathcal{U}} U$. It does however not imply that $\mathcal{S} = \mathcal{U}$.

It is also an easily observed fact that the relation $\sqsubset$ is transitive and that whenever $U \in \mathfrak{C}_x^Y$ then $\mathcal{S} \sqsubset \mathcal{U}$ implies that $\mathcal{S} \cup \{U\} \sqsubset \mathcal{U} \cup \{U\}$.

Proposition 5.12. *Let $G = (V, E)$ be a locally finite graph, $x \in V \cup \Omega$ and let $Y \subseteq (V \cup \Omega) \setminus \{x\}$. Let $\mathcal{U} \in \mathfrak{S}_x^Y$ be finite. Then there is a compatible set $\mathcal{S} \in \mathfrak{S}_x^Y$ such that $\mathcal{S} \sqsubset \mathcal{U}$.*

Proof. We will prove Proposition 5.12 by induction on $|\mathcal{U}|$. For $|\mathcal{U}| = 1$ there is nothing to show because a set of one cut is always compatible.

For $|\mathcal{U}| > 1$ let $U \in \mathcal{U}$ and apply the induction hypothesis to $\mathcal{U} \setminus \{U\}$ to obtain a compatible set $\mathcal{S}' \sqsubset \mathcal{U} \setminus \{U\}$ of cuts. By the above remark $\mathcal{S}' \cup \{U\} \sqsubset \mathcal{U}$. Now distinguish the following two cases.

- If $|\mathcal{S}'| < |\mathcal{U}| - 1$ we can apply the induction hypothesis again to $\mathcal{S}' \cup \{U\}$ to obtain $\mathcal{S} \sqsubset \mathcal{S}' \cup \{U\}$ where $\mathcal{S}$ is compatible. From the above remark it follows that $\mathcal{S} \sqsubset \mathcal{U}$ which completes the proof.

- So now assume that $|\mathcal{S}'| = |\mathcal{U}| - 1$. Choose a cut $S_1 \in \mathfrak{C}_x^Y$ fulfilling $\mathcal{S}' \cup \{S_1\} \sqsubset \mathcal{U}$ with the property that the number of cuts $S' \in \mathcal{S}'$ that are incompatible with S_1 is minimal. Note that there is such a cut because $\mathcal{S}' \cup \{U\} \sqsubset \mathcal{U}$.

  If S_1 is compatible with all cuts in $\mathcal{S}'$ we are done. So assume that there is a cut $S_2 \in \mathcal{S}'$ such that S_1 and S_2 are incompatible. For $i \in \{1, 2\}$ let $y_i \in Y$ be such that S_i is a x-y_i-cut of minimal cardinality.

  For the next step of the proof we will need some definitions which are explained in Figure 5.3. Denote by C_0 the component of $G \setminus (S_1 \cup S_2)$ in which x lies. Let C_i be the component of $G \setminus S_i$ in which y_i lies. Let A_1 be the set of edges connecting $C_1 \setminus C_2$ to C_0 and let B_1 be the set of edges connecting $C_1 \cap C_2$ to $C_2 \setminus C_1$. Analogously define A_2 and B_2. Let C be the set of edges between $C_1 \setminus C_2$ and $C_2 \setminus C_1$ and let D be the set of edges connecting C_0 and $C_1 \cap C_2$.

  From the definitions it is clear that A_1, A_2, B_1, B_2, C and D are pairwise disjoint and that $S_i = A_i \cup B_i \cup C \cup D$. It is also clear that for (D1) to (D3) it does not matter if an edge of B_1, B_2 or C is contained in any cut.

  We will now define a new cut which – depending on where y_1 and y_2 lie – can either be used to replace S_1 and S_2 so that we can apply the induction hypothesis again or contradicts the assumption that the number of cuts $S' \in \mathcal{S}'$ that are incompatible with S_1 is minimal.

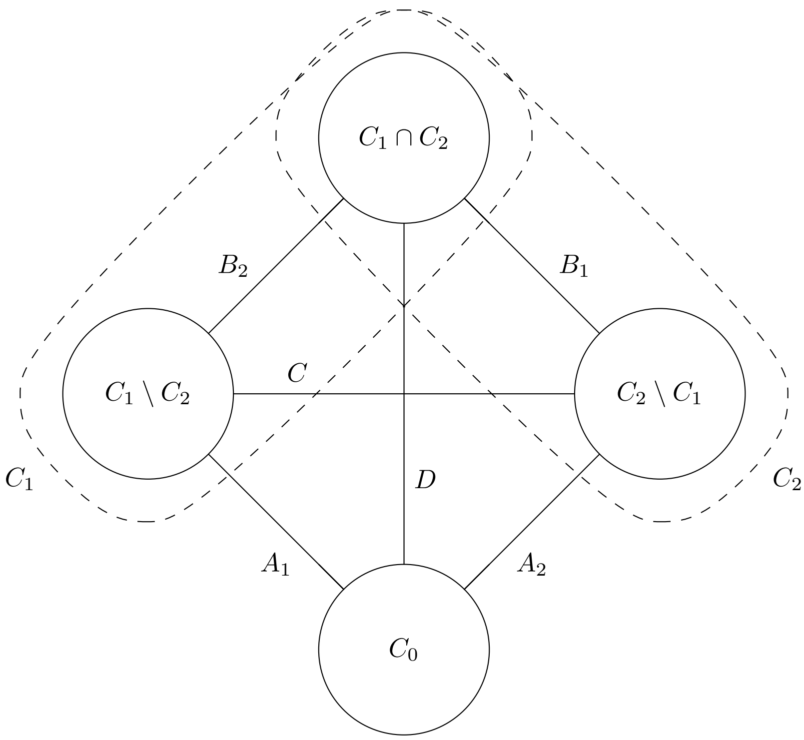

Figure 5.3: Schematic drawing of the sets used in the proof of Proposition 5.12

- Suppose that y_1 is contained in $C_1 \cap C_2$. Since S_i is a x-y_i-cut of minimal cardinality it follows that

$$\begin{aligned} |A_1| + |B_1| + |C| + |D| \leq |B_1| + |B_2| + |D| \quad &\Rightarrow \quad |A_1| + |C| \leq |B_2|, \\ |A_2| + |B_2| + |C| + |D| \leq |A_1| + |A_2| + |D| \quad &\Rightarrow \quad |B_2| + |C| \leq |A_1|. \end{aligned}$$

  This implies that $|C| = 0$ and $|A_1| = |B_2|$. Hence

$$|A_2| + |B_2| + |C| + |D| = |A_1| + |A_2| + |D|$$

  and thus the cut defined by $S^* := A_1 \cup A_2 \cup D$ is a x-y_2-cut of minimal cardinality. It is easy to see that $(\mathcal{S}' \setminus S_2) \cup S^* \sqsubset \mathcal{U}$ and since $(\mathcal{S}' \setminus S_2) \cup S^*$ has strictly less elements than $\mathcal{U}$ we can apply the induction hypothesis to find a compatible set $\mathcal{S} \sqsubset (\mathcal{S}' \setminus S_2) \cup S^* \sqsubset \mathcal{U}$.

- For $y_2 \in C_1 \cap C_2$ an analogous argument to the previous case works.

- Finally assume that $y_1 \in C_1 \setminus C_2$ and $y_2 \in C_2 \setminus C_1$. Then

$$\begin{aligned} |A_1| + |B_1| + |C| + |D| \leq |A_1| + |B_2| + |C| \quad &\Rightarrow \quad |B_1| + |D| \leq |B_2|, \\ |A_2| + |B_2| + |C| + |D| \leq |A_2| + |B_1| + |C| \quad &\Rightarrow \quad |B_2| + |D| \leq |B_1|. \end{aligned}$$

  This in particular implies that $|D| = 0$ and $|B_1| = |B_2|$ and thus

$$|A_1| + |B_1| + |C| + |D| = |A_1| + |B_2| + |C|.$$

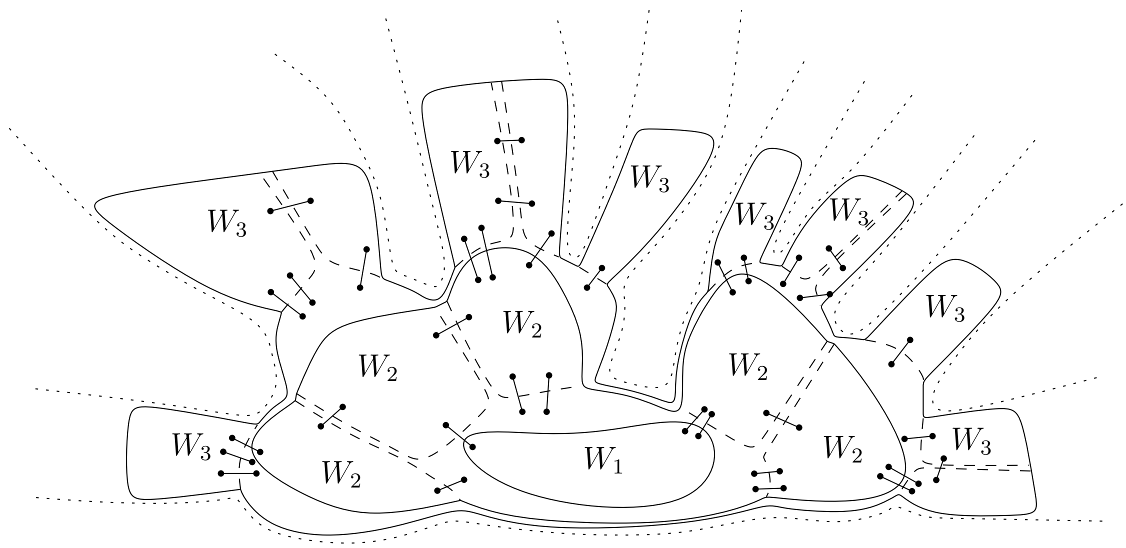

Figure 5.4: Possible cuts and vertex sets in Lemma 5.13. Solid lines mark the boundaries of the vertex sets while dashed lines mark cuts

So the cut $S_1^* := A_1 \cup B_2 \cup C$ is a x-y_1-cut of minimal cardinality.

Clearly $\mathcal{S}' \cup \{S_1^*\} \sqsubset \mathcal{S}' \cup \{S_1\} \sqsubset \mathcal{U}$. Whenever $S' \in \mathcal{S}'$ and S_1 are compatible it can easily be seen that S' and S_1^* are compatible as well (recall that S' and S_2 are compatible). The cuts S_1^* and S_2 are compatible while S_1 and S_2 are incompatible. So the number of cuts in $\mathcal{S}'$ that are incompatible to S_1^* is strictly smaller than the number of cuts that are incompatible to S_1. This contradicts S_1 minimizing that number. ☐

Lemma 5.13. *Let $G = (V, E)$ be a locally finite graph. Then there is an exhausting sequence $(W_n)_{n\in\mathbb{N}}$ of finite connected subsets of V and a sequence $(\mathcal{S}_n)_{n\in\mathbb{N}}$ of finite compatible sets of finite cuts such that*

- *any two points in $W_{n+1} \setminus W_n$ that are connected by a path in $G \setminus W_n$ are connected by a path in $G[W_{n+1} \setminus W_n]$,*
- *the component of $G \setminus \bigcup_{S\in\mathcal{S}_n} S$ that contains W_n is finite,*
- *every $S \in \mathcal{S}_n$ is a ω-x_n-cut of minimal cardinality in G/W_n for some end ω of G where x_n denotes the unique contracted vertex in G/W_n and*
- *every $S \in \mathcal{S}_n$ is contained in $G[W_{n+1}]$.*

Proof. Start with an arbitrary finite connected set W_1. Now perform the following steps for every $n \in \mathbb{N}$:

- For every end ω of G let S_ω be an ω-W_n-cut of minimal cardinality. Clearly all cuts S_ω are finite. The component C of $G \setminus \bigcup_{\omega\in\Omega} S_\omega$ that contains W_n is finite because it does not contain any end. Hence there is a finite subset O of Ω such that the component of $G \setminus \bigcup_{\omega\in O} S_\omega$ that contains W_n coincides with C.

  Consider the graph G/W_n and apply Proposition 5.12 to find a compatible set $\mathcal{S}_n$ of cuts each of which is an ω-W_n-cut of minimal cardinality for some $\omega \in O$.

- Since $\mathcal{S}_n$ is a finite set of finite cuts we can find a finite connected set W_{n+1} of vertices such that $G[W_{n+1}]$ contains every $S \in \mathcal{S}_n$. It is an easy observation that we can enlarge the set W_{n+1} so that it contains all neighbours of W_n and that $G[W_{n+1} \setminus W_n]$ contains a path between any two vertices that are connected by a path in $G \setminus W_n$.

Clearly the required properties hold for the sequences W_n and S_n obtained by this construction. □

Remark. There is always exactly one finite component of $G \setminus \bigcup_{S\in\mathcal{S}_n} S$, namely the one containing W_n. All other components have to be infinite because they must contain an end.

Also note that $G \setminus \bigcup_{n\in\mathbb{N}} \bigcup_{S\in\mathcal{S}_n} S$ has only finite components because every vertex is eventually contained in V_n. Every such component is bounded by edges of one cut in $\mathcal{S}_i$ and finitely many cuts in $\mathcal{S}_{i+1}$ for some $i \in \mathbb{N}$.

Let C be the vertex set of one such component and consider the graph $G\{C\}$. Let u be the contracted vertex corresponding to the component of $G \setminus C$ in which W_1 lies. Then the set of edges incident to u in $G\{C\}$ is a u-v-cut of minimal cardinality for some other contracted vertex v of $G\{C\}$. Note the connection to Lemma 5.3.

If G has only finitely many ends we can choose W_1 big enough that there is only one end in every component of $G \setminus W_n$. In this case there are only two contracted vertices in $G\{C\}$ for every component C that does not contain W_1. This in particular implies that there is only one possible choice for the contracted vertex v in the above paragraph. We will exploit this fact in the proofs of Theorems 1.13 and 1.14.

Lemma 5.14. *Let $G = (V, E)$ be a locally finite graph with finitely many ends. Then there is a partition*

$$V = V_0 \uplus \biguplus_{\substack{\omega\in\Omega\\ n\in\mathbb{N}}} V_n^\omega$$

such that (let $V_0^\omega = V_0 \forall\omega$)

(P1) every component of $G \setminus V_0$ contains exactly one end,

(P2) each of the graphs $G[V_n^\omega]$ is finite and connected,

(P3) ω lies in $\biguplus_{n\geq m} V_n^\omega$ for every end ω and $m \in \mathbb{N}$,

(P4) apart from the edges inside the graphs $G[V_n^\omega]$ there are only edges from V_n^ω to V_{n-1}^ω and to V_{n+1}^ω present in G and

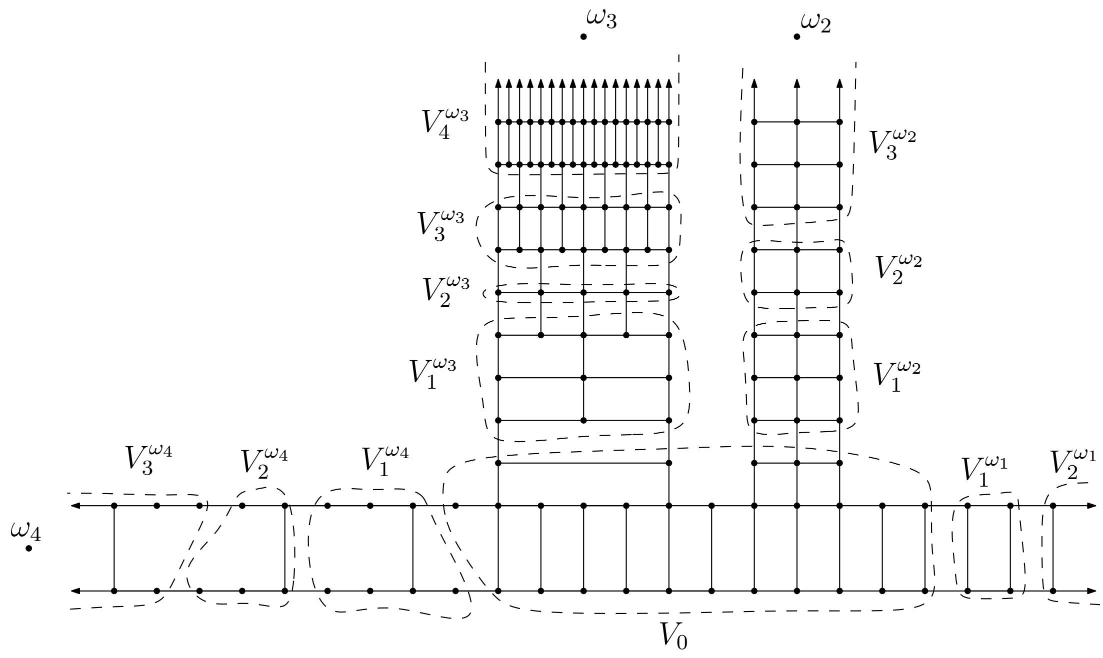

Figure 5.5: A possible decomposition of the vertex set of a graph with 4 ends as in Lemma 5.14

(P5) the set S_n^ω of edges between V_{n-1}^ω and V_n^ω constitutes an ω-V_{n-1}^ω-cut of minimal cardinality for all ω and all n.

(see Figure 5.5)

Proof. If we choose W_1 in Lemma 5.13 big enough that no component of $G \setminus W_1$ contains more than one end then the vertex sets of the components of $G \setminus \bigcup_{n\in\mathbb{N}} \bigcup_{S\in\mathcal{S}_n} S$ form a partition with the desired properties. ☐

5.4 Proof of the Main Results

Theorem 1.13. *Let G be a locally finite $2k$-edge-connected graph with finitely many ends. Then there exist edge disjoint topological spanning trees $T^1, T^2, \ldots, T^k$ such that $T^i \cup T^j$ is an end faithful connected spanning subgraph of G for every $i \neq j$.*

Proof. Let

$$V = V_0 \uplus \biguplus_{\substack{\omega\in\Omega\\ n\in\mathbb{N}}} V_n^\omega$$

be a partitioning of the vertex set as in Lemma 5.14 and define

$$V_n = V_0 \cup \bigcup_{\substack{\omega\in\Omega\\ i<n}} V_i^\omega$$

Obviously V_n is an exhausting sequence of finite subsets of V. We will now construct a sequence $\mathcal{T}_n = \{T_n^1, T_n^2, \ldots, T_n^k\}$ of spanning tree packings of $G_n = G\{V_n\}$ such that

(1) $T_{n+1}^i \mid_{V_n} = T_n^i$ and

(2) for $i \neq j$ every persistent gap in T_n^i is bridged infinitely often by T_n^j.

Then $T^i = \lim_{n\to\infty} T_n^i$ is a topological spanning tree by Proposition 3.15. Whenever $i \neq j$ the edge sets of T^i and T^j are disjoint because T_n^i and T_n^j are edge disjoint for every $n \in \mathbb{N}$ and by Proposition 5.9 the graph $T^i \cup T^j$ is an end faithful connected spanning subgraph of G.

Regarding the construction of the sequence we will start by selecting an arbitrary spanning tree packing $\mathcal{T}_1 = \{T_1^1, T_1^2, \ldots, T_1^k\}$ of G_1. Such a spanning tree packing exists because G_1 can be obtained from the $2k$-edge-connected graph G by a finite sequence of contractions and is thus $2k$-edge-connected.

We will now inductively define $\mathcal{T}_{n+1}$ from $\mathcal{T}_n$ by selecting a spanning tree packing $\mathcal{T}_n^\omega$ of $G\{V_n^\omega\}\setminus\{u\}$ for every ω where u denotes the vertex that resulted from contracting the component in which V_0 lies. Such a spanning tree packing always exists by Lemma 5.3. Now Proposition 2.24 can be applied to find a spanning tree packing of G_{n+1} for which (1) holds.

In order to take care of (2) we will bridge one possible gap in every step of the construction process. For this purpose let $v_1, v_2, v_3, \ldots$ be an enumeration of the vertices of G and choose a function

$$\varphi\colon \mathbb{N} \to \mathbb{N}^2 \times \{(i,j) \mid 1 \leq i,j \leq k \text{ and } i \neq j\}$$

such that $\varphi^{-1}(s)$ is infinite for every s. This function will be used to make sure that every persistent gap is bridged infinitely often in the following way: if $\varphi(n) = (n_1, n_2, i, j)$ and there is a gap Γ in T_n^i that separates v_{n_1} and v_{n_2} we construct a Γ-bridge in $T_{n+1}^j \setminus V_n$. So if the gap Γ is persistent there will be arbitrarily large sets of disjoint Γ-bridges since $\varphi^{-1}(n_1, n_2, i, j)$ is infinite.

All that is left to show now is that we can construct such a Γ-bridge, i.e., that we can choose the spanning tree packings of $G\{V_n^\omega\} \setminus \{u\}$ accordingly. For this purpose let P be the unique path in T_n^i that connects v_{n_1} and v_{n_2}. Note that $v_{n_1}, v_{n_2} \in V_n$ and that P uses at least one contracted vertex because Γ separates v_{n_1} and v_{n_2}.

Now for every end ω consider the graph $G\{V_n^\omega\} \setminus \{u\}$ and select a spanning tree packing as follows.

- If the corresponding contracted vertex in G_n is not contained in P choose an arbitrary spanning tree packing $\mathcal{T}_n^\omega$ of cardinality k.

- If it is contained in P let v^ω be the contracted vertex corresponding to the component in which ω lies and denote by a^ω and b^ω the vertices in V_n^ω that are incident to edges used by P (see Figure 5.6).

  The graph $G\{V_n^\omega\}$ is $2k$-edge-connected and $\{e \in E(G\{V_n^\omega\}) \mid u \in e\}$ is a u-v^ω-cut of minimal cardinality. Furthermore $G[V_n^\omega]$ is connected, so in particular there is an a^ω-b^ω-bypass of v^ω in $G\{V_n^\omega\} \setminus \{u\}$.

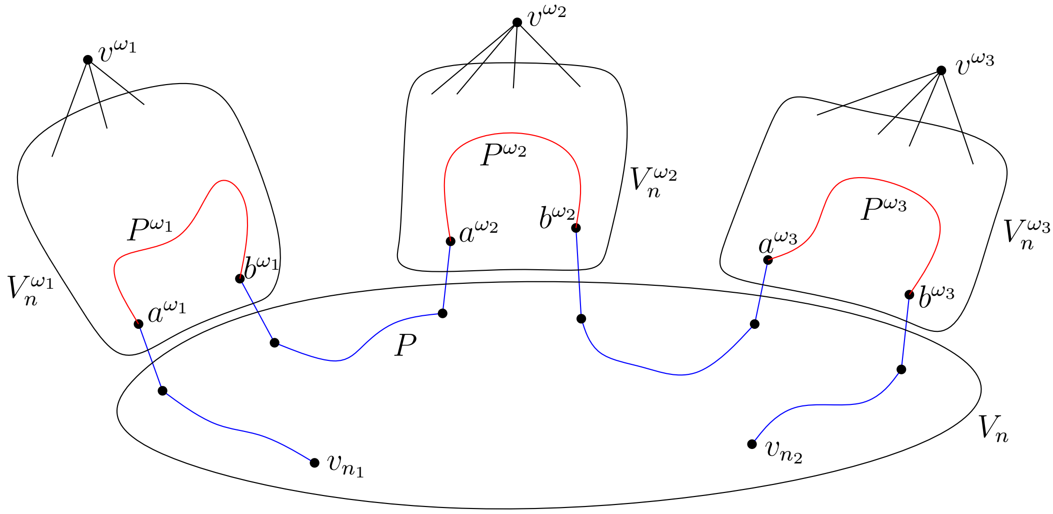

Figure 5.6: Construction of a Γ-bridge. The blue parts of the path correspond to the path P in T_n^i while every red path is a v^ω-bypass in $T_n^{\omega,i} \cup T_n^{\omega,j}$ for some ω.

So Lemma 5.3 can be used to find a spanning tree packing $\mathcal{T}_n^\omega = \{T_n^{\omega,1}, \ldots, T_n^{\omega,1}\}$ of $G\{V_n^\omega\} \setminus \{u\}$ such that $T_n^{\omega,i} \cup T_n^{\omega,j}$ contains an a^ω-b^ω-bypass P^ω of v^ω.

Assume that we have chosen the spanning tree packings $\mathcal{T}_n^\omega$ as described above and applied Proposition 2.24 to obtain a spanning tree packing $\mathcal{T}_{n+1} = \{T_{n+1}^1, \ldots, T_{n+1}^k\}$ of G_{n+1}. It is immediate that the union P^* of P and all of the P^ω is a v_{n_1}-v_{n_2}-path in $T_{n+1}^i \cup T_{n+1}^j$ that does not use any contracted vertex. So P^* contains a Γ-bridge in T_{n+1}^j. All edges of this bridge are contained in $T_{n+1}^j \setminus V_n$ because $P^* |_{V_n}$ is a path in T_n^i. □

Theorem 1.14. *Let G be a locally finite $2k$-edge-connected graph with finitely many ends. Then G has $k-1$ edge disjoint end faithful spanning trees.*

Proof. Construct a sequence of spanning tree packings as we did in the proof of Theorem 1.13.

This time however, instead of Lemma 5.3 we will use Corollary 5.4 to obtain a^ω-b^ω-paths that are contained in T_{n+1}^i and consequently the v_{n_1}-v_{n_2}-path P^* is completely contained in T_{n+1}^i. So if at some point in the construction process there is a gap separating v_{n_1} and v_{n_2} it will eventually terminate.

Now use Proposition 5.8 to conclude that the limit of the spanning tree packings is an end faithful spanning tree packing of G. □

Theorem 1.12. *Every locally finite 6-edge-connected graph with finitely many ends has a Hamiltonian line graph.*

Proof. By Propositions 4.5 and 4.6 it is sufficient to show that the graph G in question has two topological spanning trees T_1, T_2 such that $T_1 \cap G$ is an ordinary spanning tree of G. Any of Theorems 1.13 and 1.14 provides a way to find such a pair of topological spanning trees. □

6 Conclusion

6.1 Summary

We have showed that under certain conditions high edge connectivity of a graph implies that its line graph is Hamiltonian. More precisely we proved that the line graph of every locally finite 6-edge-connected graph with only finitely many ends contains a Hamilton cycle extending a result by Brewser and Funk [1] which additionally required all ends of the graph to be thin.

In the proof of this result we encountered some interesting auxiliary results related to spanning tree packings in both finite and infinite graphs.

As for finite graphs we proved that a finite $2k$-edge-connected graph G always admits a spanning tree packing of cardinality k such that every tree in the packing uses only one edge in a given u-v-cut S of minimal cardinality where u and v are arbitrary vertices of G. Furthermore we can choose this packing in a way that there are two trees in it whose union contains a path that does not contain v between two given points a and b in the graph if such a path exists in $G \setminus S$.

Based on this result we proved two results concerning spanning tree packings of infinite graphs. We showed that every locally finite $2k$-edge-connected graph G with finitely many ends admits a topological spanning tree packing of cardinality k such that the union of any two distinct topological trees in the packing is an end faithful connected spanning subgraph of G. Finally, under the same conditions we proved the existence of an end faithful spanning tree packing of G of cardinality $k-1$.

6.2 Possible Directions for Further Research

Although Theorem 1.12 provides a sufficient condition for Hamiltonicity in locally finite line graphs there is still room for improvements. Comparing the result to Georgakopoulos' conjecture which motivated it there are two obvious drawbacks. Firstly the graph has to be 6-edge-connected instead of (as conjectured) 4-edge-connected and secondly the proof does not work for graphs with infinitely many ends.

A closer look at the proofs of the main results reveils that there are not many steps that need to be improved in order to deal with these two issues. In fact a suitable refinement of one of Lemma 5.3 and Corollary 5.4 could provide a solution to both of them.

As for edge connectivity, it is obvious that the only step in the proofs that depends on the graph being 6-edge-connected is when we apply Lemma 5.3 or Corollary 5.4 respectively. So it comes as no surprise that an improvement of one of these two results, i.e., a similar result for 4-edge-connected graphs would be sufficient to deal with the connectivity problem. As there are still many possible decompositions of the vertex set fulfilling Lemma 5.14 it might not even be necessary to achieve such a result for all 4-edge-connected graphs. One could also prove that there is a decomposition of the vertex set where every part meets certain requirements and then prove that a result similiar to Lemma 5.3 or Corollary 5.4 can be obtained for the minors that are induced by the parts.

As mentioned before there is another refinement of one of the two results which could be used to extend the proof to graphs with infinitely many ends. This follows from the observation that Lemma 5.13 still gives rise to a decomposition of the vertex set if the graph in question has infinitely many ends. The only problem that we need to overcome is that there may be more than one contraced vertex in the induced minors that are used for the proof, i.e., we need to construct edge disjoint spanning trees containing paths which avoid more than one vertex in these induced minors. If this could be done for some decomposition the proofs of the main results could also be shown to hold for graphs with infinitely many ends.

Finally one might ask whether or not the main results remain true for non-locally finite graphs. However, it will probably take a completely different approach to settle this question since many of the proofs rely on the graph being locally finite.

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