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Orthogeodesic Point Set Embeddings of Outerplanar Graphs

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Abstract

Let G be a simple undirected n-vertex graph and let $P \subset \mathbb{R}^2$ be a general point set of size m, that is, a set of points where each two points have distinct x- and distinct y-coordinates. We ask whether an orthogeodesic embedding of G in P exists, that is, an embedding where the edges have minimal L^1-length and are drawn on the grid of horizontal and vertical lines induced by the points of P. More generally, we ask for a minimal m such that any n-vertex graph of a certain subclass admits an orthogeodesic embedding in every general point set of size m. Besides arbitrary orthogeodesic embeddings, we investigate L-shaped embeddings, that is, orthogeodesic embeddings where each edge is drawn with at most one bend.

The first result we state is that outerplanar graphs with maximum degree 4 or greater and planar graphs with maximum degree 3 or greater do not admit an orthogeodesic embedding in diagonal point sets, that is, a subclass of point sets. According to these observations, we focus on outerplanar graphs with maximum degree 3 or less and on trees with maximum degree 4 or less. The core of this thesis is the analysis for trees and caterpillars.

For trees and caterpillars we give an improvement for most of the bounds currently known. In particular, we provide a sub-quadratic upper bound on the number of points needed, such that any n-tree with maximum degree 4 or less admits a planar L-shaped embedding in every point set of that size. By introducing the saturation-property for trees, we manage to improve this sub-quadratic upper bound for certain trees, where the improved bound depends on the saturation of the given tree. Also, we state an analogous result for point sets which fulfill certain properties.

By making use of probability theory, we prove the following result on planar L-shaped embeddings of trees: There exists a quasilinear upper bound on the number of points needed such that any tree with maximum degree 3 can be embedded with probability at least $\frac{1}{2}$ in a point set that is chosen uniformly at random. Moreover, we generalize this result to trees with maximum degree 4.

Zusammenfassung

Sei G ein einfacher, ungerichteter Graph mit n Knoten und sei $P \subset \mathbb{R}^2$ eine allgemeine Punktmenge der Größe m, also eine Menge von Punkten, sodass je zwei Punkte unterschiedliche x- und y-Koordinaten haben. Wir beschäftigen uns mit der Frage, ob eine orthogeodätische Einbettung von G in P existiert, also eine Einbettung, in der Kanten minimale L^1-Länge haben und auf dem Gitter von horizontalen und vertikalen Linien gezeichnet sind, welches von den Punkten aus P induziert wird. Darüber hinaus fragen wir nach der kleinsten Zahl m, sodass jeder Graph einer bestimmten Teilklasse mit n Knoten eine orthogeodätische Einbettung in jeder allgemeinen Punktmenge der Größe m besitzt. Neben beliebigen orthogeodätischen Einbettungen beschäftigen wir uns auch mit L-Form-Einbettungen, also orthogeodätische Einbettungen mit höchstens einem Knick pro Kante.

Als Erstes zeigen wir, dass es für außerplanare Graphen mit Maximalgrad größer gleich 4 und planare Graphen mit Maximalgrad größer gleich 3 keine orthogeodätische Einbettung in diagonalen Punktmengen gibt. Diagonale Punktmengen sind eine Teilklasse von Punktmengen. Aufgrund dieser Beobachtungen beschränken wir uns auf außerplanare Graphen mit Maximalgrad kleiner gleich 3 und auf Bäume mit Maximalgrad kleiner gleich 4. Im Zentrum dieser Arbeit steht die Analyse für Bäume und Caterpillars.

Für Bäume und Caterpillars verbessern wir die momentan besten oberen Schranken. Im Speziellen beweisen wir eine sub-quadratische obere Schranke für die Anzahl der benötigten Punkte, sodass jeder Baum mit n Knoten und Maximalgrad kleiner gleich 4 eine planare Einbettung in jeder Punktmenge dieser Größe besitzt. Durch die Einführung der Sättigungs-Eigenschaft für Bäume können wir diese obere Schranke für manche Bäume noch weiter verbessern, wobei diese Schranke von der Sättigung des jeweiligen Baumes abhängt. Ein analoges Ergebnis können wir auch für Punktmengen erzielen, welche bestimmte Eigenschaften erfüllen.

Weiters haben wir uns der Wahrscheinlichkeitstheorie bedient, um folgendes Ergebnis für planare L-Form-Einbettungen von Bäumen zu beweisen: Es gibt eine quasilineare obere Schranke für die Anzahl der benötigten Punkte,

sodass jeder Baum mit Maximalgrad 3 in einer zufällig gewählten Punktmenge mit Wahrscheinlichkeit $\frac{1}{2}$ eingebettet werden kann. Des Weiteren können wir dieses Resultat auf Bäume mit Maximalgrad 4 erweitern.

Acknowledgements

I would like to express my sincere gratitude to my supervisors Oswin Aichholzer and Thomas Hackl for their great support and for spending many hours on discussions. Also, I want to thank Thomas for reading through some earlier versions of this thesis and giving me feedback several times. This iterative writing process significantly enhanced the quality of this thesis.

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Last but not least, I want to thank my parents Ernestine and Wilfried for supporting me during the last years.

Contents

Chapter 1

Introduction

This thesis is about embeddings of graphs in point sets under certain restrictions. First of all, we give a short introduction to graph theory in order to avoid ambiguities. Afterwards, we head to point sets, to embeddings of graphs in point sets and state different versions of the embedding-problem that are considered throughout the whole thesis.

The following definitions are analogous to the book Combinatorial Optimization by Korte and Vygen [19], whereas we make use of some simplifications. A **simple undirected graph** is a triple (G, V, ψ) where V and E are finite sets and $\psi : E \to \{X \subseteq V : |X| = 2\}$ is an injective mapping. Since we neither consider directed graphs nor non-simple graphs in this thesis, we simply call them **graphs**. Furthermore, we can also write (V, E) for a graph where $E \subseteq \{X \subseteq V : |X| = 2\}$. The set V is called the set of **vertices** and E is called the set of **edges**. Two vertices u and v with $\{u, v\} \in E$ are said to be **adjacent**. Vertices adjacent to v are said to be the **neighbors** of v.

A graph is said to be a **tree** if each pair of vertices is connected by a unique path where a **path** is a sequence of distinct vectices $v_0, \dots, v_k$ with v_{i-1} and v_i being adjacent for any $1 \leq i \leq k$. A vertex v in a tree is called **leaf** if it has at most one neighbor, otherwise v is said to be an **inner vertex**. In this thesis, we denote a tree with maximum degree at most k as **k-tree**. Note that by rooting a k-tree, the root r can have up to k children and all of the other vertices can have up to $k-1$ children. A vertex u is said to be a **child** of v if u and v are adjacent and the r-u-path is longer than the r-v-path. Keep in mind that k-trees must not be confused with k-ary trees, where a **k-ary tree** is a tree that can be rooted in a way such that every vertex - including the root - has at most k children.

A tree is said to be a **caterpillar** if a path is left by the removal of all leaves. This path is called the **spine**. Analogously to k-trees, we denote a caterpillar as **k-caterpillar** if it has a maximum degree of at most k.

A subset $J \subset \mathbb{R}^2$ is said to be a **polygonal arc** with **endpoints** p and q if J can be written as the union of a finite number of straight-line segments and if J is the image of an injective continuous function $\phi : [0,1] \to \mathbb{R}^2$ with $\phi(0) = p$ and $\phi(1) = q$. We denote a point on a polygonal arc where two line segments meet as a **bend**. An **embedding** of a graph $G = (V, E)$ is a tuple (ν, μ) where $\nu : V \to \mathbb{R}^2$ is an injective mapping of the vertices into the plane and μ maps every edge $e = \{u, v\}$ to a polygonal arc with endpoints $\nu(u)$ and $\nu(v)$. In an embedding, we identify a vertex v with its image $\nu(v)$ and an edge e with $\mu(e)$. An embedding is said to be **planar** if the interior of every edge neither intersects other edges nor contains vertices. If a graph has a planar embedding, the graph is said to be **planar**. Otherwise, the graph is said to be **nonplanar**. We remark that in [10] the authors used to denote any embedding - planar or not - as **nonplanar** embedding. For means of compatibility, we will also use this notation in our thesis but the reader should keep in mind that a better description would have been "not-necessarily-planar embedding".

A graph G is said to be **outerplanar** if there exists a planar embedding of G, such that every vertex lies on the boundary of the unbounded region of the embedding [5]. Note that every tree is outerplanar by definition.

Given a graph $G = (V, E)$, a set of points $P \subset \mathbb{R}^2$, and an embedding of G with $\nu(V) \subseteq P$, we call this embedding an **embedding of G in P**. Cabello [3] has shown that deciding, whether there exists a planar straight-line embedding of an n-vertex graph in a point set of size n, is an NP-complete problem, even though every planar graph admits a straight-line embedding according to Fáry's Theorem [4,9]. We also remark that there is a linear-time algorithm for finding a planar (not necessarily straight-line) embedding of a given graph or deciding that it is not planar [13,19]. In particular, for every planar n-vertex graph a straight-line embedding can be found in linear time such that every vertex has positive integer coordinates each smaller or equal to $n - 2$ [32]. Gritzmann et al. have proven that an n-graph admits a straight line embedding in every point set of size n if and only if it is outerplanar [1,26].

Kaufmann and Wiese [18] considered a relaxation of the straight-line embeddings where the edges are allowed to have very few bends. They proved that every planar graph admits a planar embedding in every point set such that every edge has at most two bends. Furthermore, they proved that it is NP-complete to decide, whether a graph admits a planar embedding in a point set where every edge has at most one bend.

An embedding of G in P is said to be **orthogeodesic** if it fulfills the following conditions [10,17]:

1. every edge (embedded as a polygonal arc) has minimal L^1-length, that is, the length of every edge is equal to the Manhattan distance of its connected vertices,
2. for every point along any edge there is a point in P with the same x- or y-coordinate, and
3. all edges connected to a vertex enter from distinct directions.

Note that according to the last condition, each vertex can have at most 4 neighbors. By definition, every edge in an orthogeodesic embedding is a finite chain of axis-parallel line segments. Note that by this definition, an edge in a planar orthogeodesic embedding might contain points of P in which no vertices are embedded. Figure 1.1 gives an illustration.

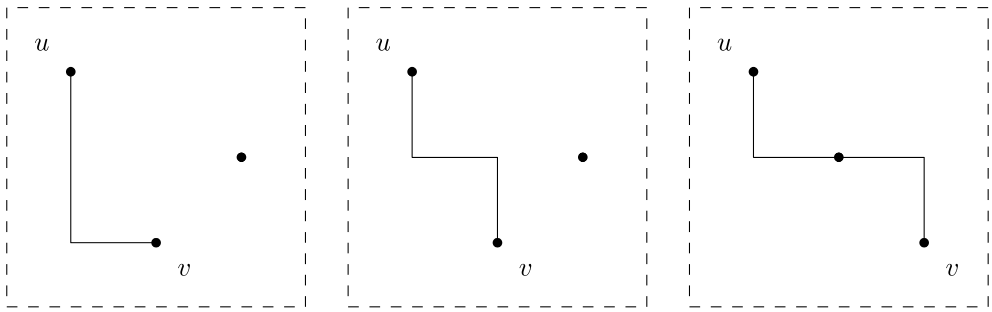

Figure 1.1: Examples of edges in planar orthogeodesic embeddings.

Katz et al. [17] have proven the NP-completeness of deciding, whether an orthogeodesic point set embedding exists by showing the equivalence of this problem and the decision problem, whether an embedding exists where every edge has at most one bend.

As introduced in [10], an orthogeodesic embedding is said to be **L-shaped** if every edge has at most one bend. Note that every edge in an L-shaped embedding consists of at most two line segments. If each two vertices in an L-shaped embedding have distinct x- and y-coordinates, then every edge has exactly one bend. We remark that in this case every edge looks like the capital letter "L". The leftmost example in Figure 1.1 illustrates an L-shaped embedded edge.

Given a finite set of points $P \subset \mathbb{R}^2$ where each two points have distinct x- and y-coordinates, we ask whether there exists a planar or nonplanar, L-shaped or orthogeodesic embedding of G in P. According to the definition of orthogeodesic embeddings, only the ordering of the points affect the existence of an embedding but not the actual coordinates. Without loss of generality, we can assume that

$$x_1 < x_2 < \ldots < x_n$$

and

$$y_{\sigma_1} < y_{\sigma_2} < \ldots < y_{\sigma_n}$$

hold for a certain permutation σ. This observation admits an equivalence relation on the subsets of $\mathbb{R}^2$ with n elements and distinct coordinates. The equivalence class of P is said to be a **general point set**. As we only consider general point sets in this thesis, we do not write the prefix "general" in later chapters anymore.

When deciding, whether a graph admits an embedding in a general point set P, we can also consider the equivalent point set $P_\pi = \{(1, \pi_1), \ldots, (n, \pi_n)\}$ instead, with π being a permutation. We remark that the inverse permutation $\sigma = \pi^{-1}$ fulfills $\pi_{\sigma_1} < \ldots < \pi_{\sigma_n}$, since $\pi_{\sigma_i} = i$ holds for every $1 \leq i \leq n$.

Since every general point set can be represented by $\{(1, \pi_1), \ldots, (n, \pi_n)\}$ with, π being a permutation, the set of general point sets is isomorphic to the **symmetric group**, that is, the set of permutations on $\{1, \ldots, n\}$. Furthermore, the number of general point sets is $n!$, because there are exactly $n! = n \cdot (n-1) \cdots 2 \cdot 1$ permutations on n elements.

A more general question is the following one: Given a graph G, what is the minimum natural number $m(G)$, such that G can be embedded in every general point set of size $m(G)$ such that the embedding fulfills certain requirements, like being planar or nonplanar, L-shaped or orthogeodesic?

An even more general question: What is the minimum natural number $m(n)$, such that any n-vertex graph can be embedded in every general point set of size $m(n)$ with certain properties? In this thesis we will also consider restrictions to certain subclasses of graphs, such as outerplanar graphs, trees, and caterpillars.

Definition 1. *We define*

- $f_{LTk}(n)$ *as the minimum natural number, such that any n-vertex k-tree admits a planar L-shaped embedding in every general point set of size* $f_{LTk}(n)$*,*

- $f_{NTk}(n)$ *as the minimum natural number, such that any n-vertex k-tree admits a nonplanar L-shaped embedding in every general point set of size* $f_{NTk}(n)$*, and,*

- $f_{OTk}(n)$ *as the minimum natural number, such that any* n*-vertex* k*-tree admits a planar orthogeodesic embedding in every general point set of size* $f_{OTk}(n)$.

Recall that in this context nonplanar means not-necessarily-planar. Note that $f_{LTk} \leq f_{LTk+1}$, $f_{NTk} \leq f_{NTk+1}$, and $f_{OTk} \leq f_{OTk+1}$ hold, because every k-tree is a $k+1$-tree by definition. Analogously, we define f_{LCk}, f_{NCk}, and f_{OCk} for k-caterpillars.

Giacomo et al. have investigated all of these functions and gave linear upper bounds for all of them, except for f_{LT3} and f_{LT4}, where they only obtained quadratic upper bounds. Table 1.1 summarizes the upper bounds provided by Giacomo et al. [10]. In their conclusion, they asked for a subquadratic upper bound on f_{LT3} and also for embeddings of other subclasses of graphs since they have only considered trees and caterpillars.

	Planar L-Shaped	Nonplanar L-Shaped	Planar Orthogeodesic
3-Cat.	n	n	n
3-Tree	$n^2 - 2n + 2$	n	n
4-Cat.	$3n - 2$	$n + 1$	$\lfloor 1.5n \rfloor$
4-Tree	$n^2 - 2n + 2$	$4n - 3$	$4n$

Table 1.1: Upper bounds given by Giacomo et al. [10].

In this thesis we will tackle both these questions. In Chapter 2, we consider orthogeodesic point set embeddings of outerplanar graphs and state that planar graphs do not admit orthogeodesic point set embeddings in general. In Chapter 3 and Chapter 5, we give an improvement of all not-yet-optimal upper bounds on trees and caterpillars, respectively, provided by Giacomo et al. Table 1.2 summarizes the best upper bounds currently known.

	Planar L-Shaped	Nonplanar L-Shaped	Planar Orthogeodesic
3-Cat.	n [10]	n [10]	n [10]
3-Tree	$0.334n^{1.585} + \mathcal{O}(n)$ [Cor. 4]	n [10]	n [10]
4-Cat.	$1.334n + \mathcal{O}(1)$ [Th. 20]	n [Th. 21]	$1.334n + \mathcal{O}(1)$ [Th. 18]
4-Tree	$0.339n^{1.585} + \mathcal{O}(n)$ [Th. 9]	$2.334n + \mathcal{O}(1)$ [Th. 7]	$1.5n + \mathcal{O}(1)$ [Th. 6]

Table 1.2: Best upper bounds currently known.

In Chapter 3.3.2, we define the saturation-property σ for trees and prove that every tree T admits a planar L-shaped embedding in every general point set of size $|V(T)| \cdot 2^{\sigma(T)}$. In Chapter 3.3.3, we state an analogous result for point sets: For every general point set P there exists a number $f(P)$

depending on the structure of P such that every $f(P)$-vertex tree admits a planar L-shaped embedding in P.

Another question we consider: What is the smallest natural number $m(n)$, such that every n-vertex graph admits a planar L-shaped embedding in at least half of all general point sets of size $m(n)$? By means of probability theory, we state the following equivalent formulation of this question: What is the smallest natural number $m(n)$, such that every n-vertex graph admits a planar L-shaped embedding in a point set P with probability at least $\frac{1}{2}$ if P is chosen uniformly at random among all point sets of size $m(n)$? In Chapter 4, we investigate this problem but restrict ourselves to trees since the problem turns out to be very involved even with this restriction to trees.

Definition 2. *We define $f_{LTk}^{1/2}(n)$ as the minimal natural number such that any n-vertex k-tree admits a planar L-shaped embedding in at least half of all general point sets of size $f_{LTk}^{1/2}(n)$.*

As stated above, $f_{LTk}^{1/2}(n)$ is also the smallest natural number such that any n-vertex k-tree admits a planar L-shaped embedding with probability at least $\frac{1}{2}$ in a point set that is chosen uniformly at random. In Chapter 4, we state that a certain class of trees can be embedded easily and then give a proof for $f_{LT3}^{1/2} \in \mathcal{O}(n \log n (\log \log n)^2)$. We also prove that $f_{LT4}^{1/2} \in \mathcal{O}(n^{\gamma_0 + \varepsilon})$ holds for every $\varepsilon > 0$, where $\gamma_0 = 1.331 \cdots$ is a real constant.

Chapter 2

Outerplanar Graphs

In this chapter, we analyze all types of embeddings of outerplanar graphs in point sets, that is, orthogeodesic and L-shaped, planar and nonplanar embeddings. As we make use of the two point sets $P = \{(1,1),(2,2),\ldots,(n,n)\}$ and $P' = \{(1,n),(2,n-1),\ldots,(n,1)\}$ very often in this thesis, we denote these two point sets as the **diagonal point sets** of size n. Note that any sub point set of a diagonal point set is a diagonal point set as well.

Theorem 1. *There exist outerplanar graphs that do not admit an orthogeodesic embedding in any diagonal point set.*

Proof. Assume that for every outerplanar graph G there exists a natural number $n \in \mathbb{N}$, such that G admits an orthogeodesic embedding in a diagonal point set of size n. Consider the outerplanar 6-vertex graph G that is depicted in Figure 2.1 and the diagonal point set P of size n with decreasing y-coordinate as the x-coordinate increases.

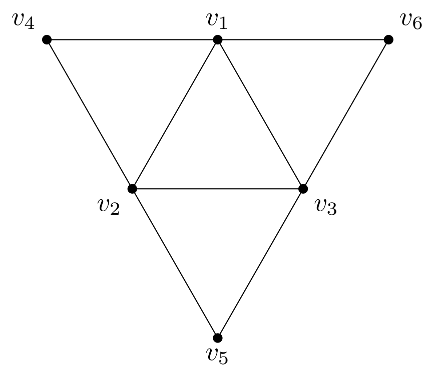

Figure 2.1: An outerplanar graph that does not admit an orthogeodesic embedding in any diagonal point set.

Without loss of generality, we can assume that the three points v_1, v_2, v_3 and the edge $\{v_1, v_3\}$ are drawn as exemplified in Figure 2.2. Note that the edge $\{v_1, v_3\}$ can have arbitrary bends since we consider orthogeodesic embeddings.

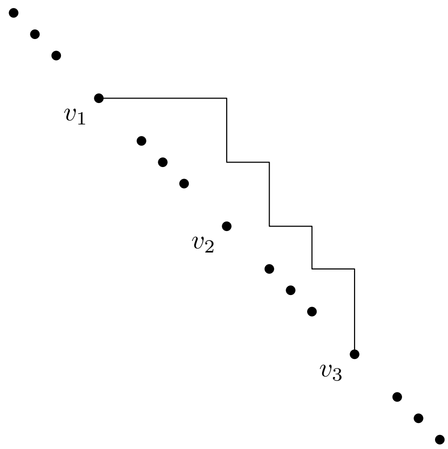

Figure 2.2: Placement of v_1, v_2, and v_3.

As we consider a diagonal point set, the only remaining points, where v_6 can be placed, are in the dashed and the dotted areas, as depicted in Figure 2.3. The vertices v_1 and v_3 are both connected to v_2 and v_6, and as v_2 is placed as illustrated and connected to v_1 and v_3, the vertex v_6 can not be placed in one of the dashed areas. Without loss of generality, v_6 is placed in the upper dotted area. As v_3 is connected to v_2, v_3 can not be connected to v_6, and therefore, G does not admit an orthogeodesic embedding in P. The arguments for the diagonal point set of size n with increasing y-coordinate as the x-coordinate increases are analogous. Hence, this gives a contradiction to our assumption. $\square$

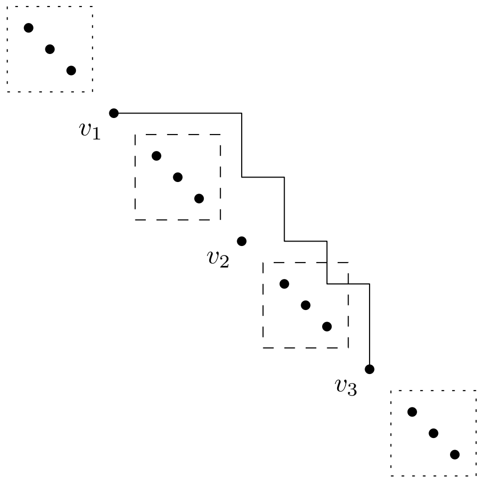

Figure 2.3: Placement of v_6.

We further remark that L-shaped embeddings are orthogeodesic by definition, and therefore, the illustrated graph does not admit an L-shaped embedding in diagonal point sets either. As planarity did not matter in the proof, neither planar nor nonplanar embeddings are admitted.

Lemma 1. *There exist planar graphs with maximum degree 3 that do not admit an orthogeodesic embedding in any diagonal point set.*

Proof. Assume that for every planar graph G there exists an $n \in \mathbb{N}$, such that G admits an orthogeodesic embedding in every point set of size n. Consider the planar 5-vertex graph G that is depicted in Figure 2.4 and a diagonal point set of size n.

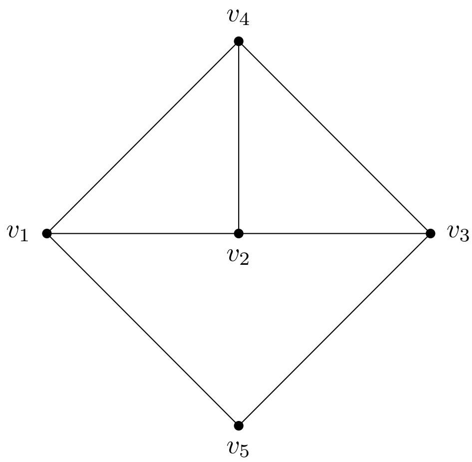

Figure 2.4: A planar graph with maximum degree 3 that does not admit an orthogeodesic embedding in any diagonal point set.

The topmost embedded vertex and bottommost embedded vertex each have at most two connections as we consider a diagonal point set. Therefore, at least two vertices have degree 2 or less. That is a contradiction. □

According to Kuratowski's theorem [19,20], every nonplanar graph contains K_5 or $K_{3,3}$ as a minor. A graph G is said to be a **minor** of a graph H if G can be obtained from H by a sequence of edge contractions, edge deletions, and vertex deletions (in any order), where the **contraction** of the edge $e = (u, v)$ denotes the identification of the two vertices u and v [5,19]. Analogously to the proof of Lemma 1, one can show that K_5 does not admit an orthogeodesic embedding in any point set. Also, the graph $K_{3,3}$ does not admit an orthogeodesic embedding in any diagonal point set, even though it might admit an embedding in certain point sets, as illustrated in Figure 2.5.

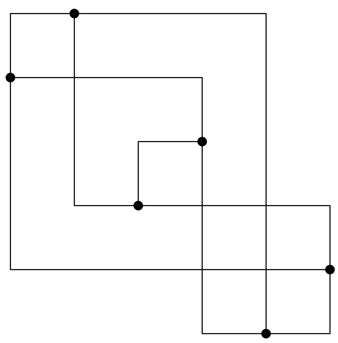

Figure 2.5: An orthogeodesic embedding of $K_{3,3}$.

Due to these observations we only consider outerplanar graphs with maximum degree 3 or less and trees with maximum degree 4 or less from now on.

2.1 Outerplanar Graphs with Maximum Degree 3 or Less

It is obvious, that every graph with maximum degree 2 or less is a collection of paths, cycles, and isolated vertices. Hence, we can embed such a graph by embedding each path, cycle, and isolated vertex separately. We observe that every n-vertex path admits a planar L-shaped embedding in every point set of size n. Such an embedding can be constructed by placing the vertices from left to right and by connecting them in a way, such that the right point of every edge is connected vertically, as exemplified in Figure 2.6.

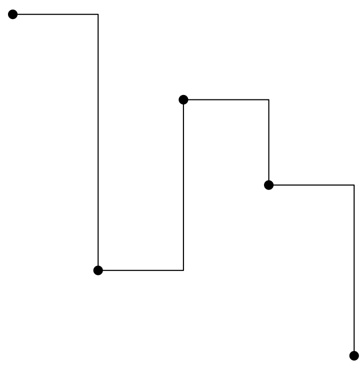

Figure 2.6: Embedding a path.

Lemma 2. *Every n-vertex cycle admits a planar L-shaped embedding in every point set of size n.*

Proof. Let a be the topmost point and b be the bottommost point of a point set P of size n. Obviously a and b are distinct for $n \geq 2$. The following three cases can occur:

- In case a is the rightmost point, we embed an $n-1$-vertex path in the $n-1$ points $P \backslash \{a\}$ and close the cycle as illustrated in Figure 2.7.

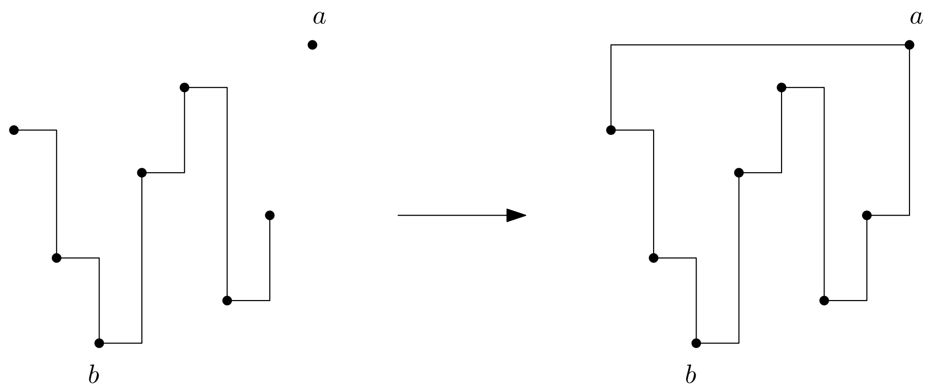

Figure 2.7: Embedding a cycle. Case 1.

- In case b is the rightmost point, we embed in an analogous manner.
- In case another point $c \in P \backslash \{a, b\}$ is the rightmost point, consider the embedding of an $n-2$-vertex path in $P \backslash \{a, b\}$. If c is connected from below, c is also connected from below when embedding an $n-1$-vertex path in $P \backslash \{a\}$ as stated above. Closing the cycle by connecting both path endings to a yields an embedding, as exemplified in Figure 2.8.

Continue analogously if c is connected from above, and use b to close the cycle.

□

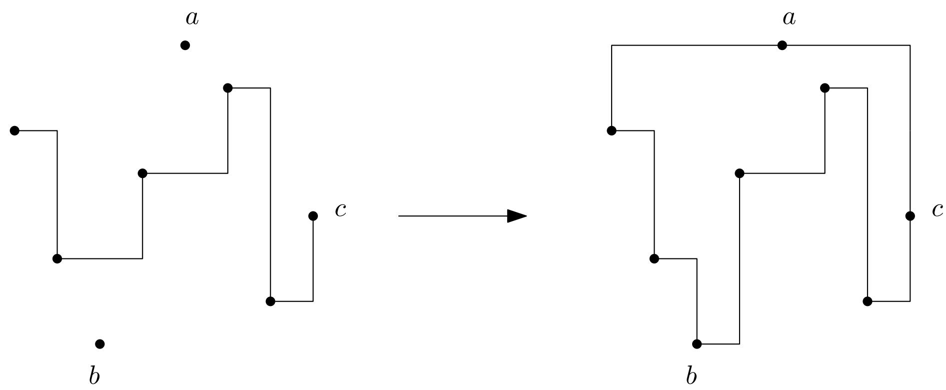

Figure 2.8: Embedding a cycle. Case 3.

Corollary 1. *Every n-vertex graph with maximum degree 2 or less admits a planar L-shaped embedding in every point set of size n.*

Due to this Corollary we will not consider graphs with maximum degree 2 or less anymore in this thesis as the embedding of those graphs is trivial.

2.1.1 Planar L-Shaped Embeddings

Theorem 2. *There exist outerplanar graphs with maximum degree 3 or greater that do not admit a planar L-shaped embedding in any diagonal point set.*

Proof. Assume that for every outerplanar graph G with maximum degree 3 there exists an $n \in \mathbb{N}$, such that G admits an L-shaped planar embedding in every point set of size n. Consider the outerplanar graph G with 4 vertices that is depicted in Figure 2.9 and a diagonal point set of size n.

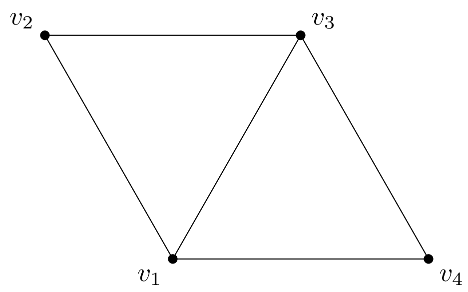

Figure 2.9: An outerplanar graph that does not admit a planar L-shaped embedding in any diagonal point set.

The three points v_1, v_2, v_3 and the edges are drawn as depicted in Figure 2.10. We write v_a, v_b, v_c with $\{a, b, c\} = \{1, 2, 3\}$ as any order of v_1, v_2, v_3 is possible.

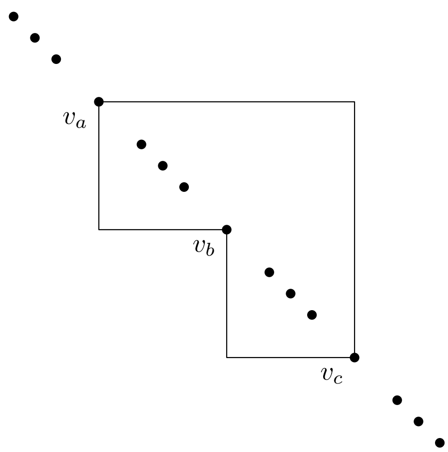

Figure 2.10: Placement of v_1, v_2, and v_3.

Recall that we consider a diagonal point set, and therefore, the only remaining points, where v_4 can be placed, are in the dashed and the dotted areas, as depicted in Figure 2.11. The vertex v_4 can not be placed in the upper dotted area, because it could only be connected to v_a then, whereas it must have two neighbors. The same holds for the lower dotted area. The vertex v_4 can also not be placed in a dashed area, because it could only be connected to v_b then. Therefore, the graph G does not admit a planar L-shaped embedding in the point set P. That is a contradiction to our assumption. □

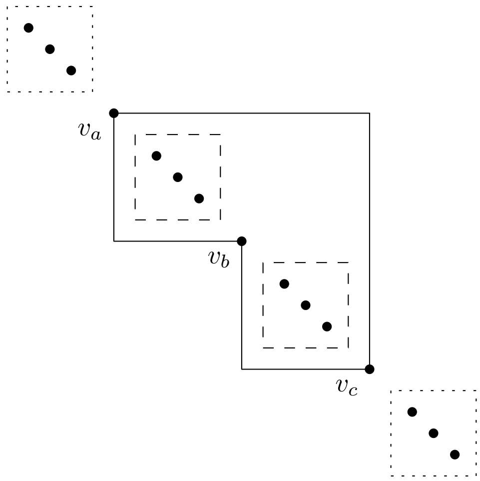

Figure 2.11: Placement of v_4.

2.1.2 Nonplanar L-Shaped Embeddings

Consider the following fundamental combinatorial result by Erdős and Szekeres [8]:

Lemma 3 (Erdős and Szekeres [8])**.** *Any point set of size* $n^2 + 1$ *contains a diagonal point set of size* $n + 1$*.*

Recall from the definition of general point sets (in Chapter 1) that we consider the equivalence classes of point sets. We remark that $n^2 + 1$ is the best possible bound since point sets of size n^2 exist that do not contain a diagonal point set of size $n + 1$. For example, the sequence $\pi_1, \pi_2, \ldots, \pi_{n^2}$ with $\pi_i := n(2\lfloor \frac{i}{n} \rfloor + 1) - i$ yields a point set of size n^2 that has the desired properties. Figure 2.12 shows the point set P_π that corresponds to this sequence for $n = 4$.

Figure 2.12: A point set of size 16 that does not contain a diagonal point set of size 5.

We will also make use of results from graph theory to cope with some proofs in this thesis. The following property of nontrivial outerplanar graphs will help us to prove our main result, where a graph is said to be **nontrivial** if it has at least two vertices. Note that there exists a unique **single-vertex-graph** and also a unique **empty-graph**, that is, a graph without vertices.

Lemma 4 ([5]). *Every nontrivial outerplanar graph contains at least two vertices of degree 2 or less.*

To put the proof of the following theorem as brief as possible, we will make use of the following two considerations, which will also be used for later proofs:

1. We will only consider connected graphs, where a graph is said to be **connected** if there exists a path between every pair of vertices. This definition gives an equivalence relation on an undirected graph, where the equivalence classes are said to be the **connected components**. This consideration is valid, because each connected component can be embedded separately and the mapping $f(n) = (n-1)^2 + 1$ is superadditive on $\mathbb{N}$, that is, $f(a) + f(b) \leq f(a+b)$ holds. This inequality holds since
$$f(a+b) - f(a) - f(b) = 2(ab-1) \geq 0$$
holds for every pair of natural numbers $a, b \in \mathbb{N}$.

2. We will only consider point sets that are large enough to contain a diagonal point set of a certain size. The reader should keep in mind

that this consideration might not lead to the best upper bound, and hence, further improvements might be possible.

In the following proof we will also make use of bridges, where a **bridge** is an edge whose deletion increases the number of connected components of a graph [19]. Recall that the deletion of a bridge does not yield new bridges, and therefore, the number of bridges decreases whenever a bridge is deleted.

Theorem 3. *Every outerplanar n-vertex graph with maximum degree 3 admits a nonplanar L-shaped embedding in every point set of size* $(n-1)^2+1$.

Proof. According to Lemma 3, any point set of size $(n-1)^2+1$ contains a diagonal point set of size n. Therefore, we only consider embeddings on diagonal point sets of size n.

We give a proof by induction on the number of bridges for the following statement: Let $G=(V,E)$ be a connected outerplanar n-vertex graph with maximum degree 3 or less, and with two distinct vertices $w_1, w_2 \in V$ of degree 2 or less. Recall that such vertices exist according to Lemma 4. Then G admits a nonplanar L-shaped embedding in every diagonal point set of size n, such that

1. w_1 allows a connection from above,
2. w_2 allows connections from below and from the right side, and
3. any other vertex $v \in V \backslash \{w_1, w_2\}$ with degree 2 or less allows a connection from above or from the left side.

Base case: Let $G=(V,E)$ be a nontrivial bridgeless connected outerplanar n-vertex graph with maximum degree 3 or less, and let P be a diagonal point set of size n. In G all vertices have degree 2 or 3 as the graph is bridgeless. Note that a bridgeless connected outerplanar graph with maximum degree 2 is a cycle. According to Chapter 2.1, such a graph admits a planar L-shaped embedding in P. Furthermore, this embedding also fulfills the desired properties.

By definition of outerplanarity, all vertices belong to the unbounded face, and therefore, all vertices lie on a polygon that is also a subset of the edges by definition. The polygon is simple as the graph is bridgeless, connected, and each vertex has degree 2 or 3. Hence, for this proof we can assume that the vertices are labeled clockwise in such a way, that v_1 and v_k are two vertices with degree 2, as exemplified in Figure 2.13.

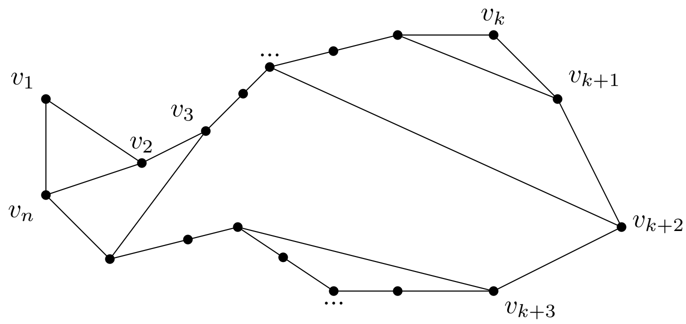

Figure 2.13: A bridgeless connected outerplanar graph.

We first embed the vertices and the polygon as sketched in Figure 2.14.

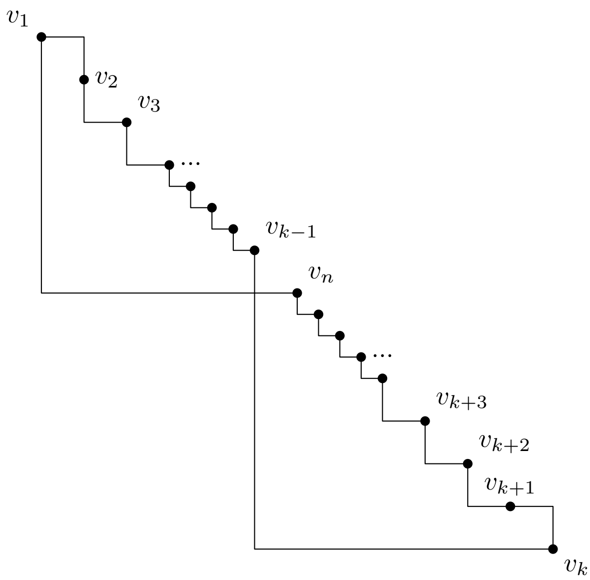

Figure 2.14: Embedding of the vertices and the outer polygon.

Using this embedding each vertex allows another connection from above or from the right side (or both). Hence, we are able to embed the remaining edges as exemplified in Figure 2.15. Note that v_1 and v_k do not have further connections, because they have degree 2.

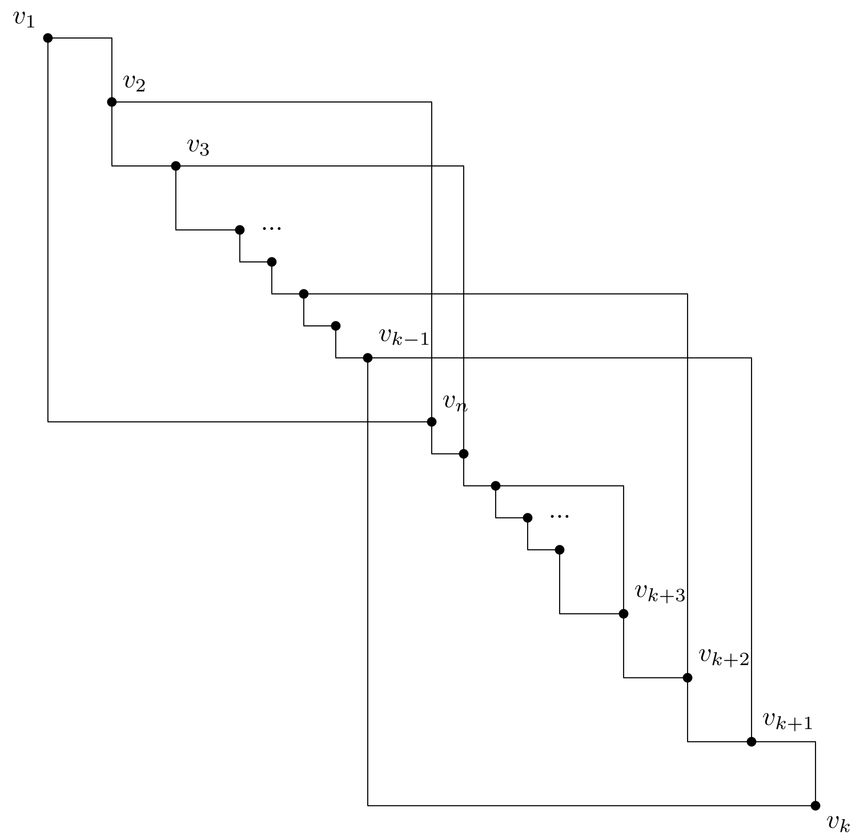

Figure 2.15: Embedding of the remaining edges.

This embedding fulfills the desired properties by construction.

Inductive step: Let $G = (V, E)$ be an n-vertex outerplanar graph, let $w_1, w_2 \in V$ be two distinct vertices of degree 2 or less, let $e \in E$ be a bridge, and let P be a diagonal point set of size n. The deletion of $e = \{v_1, v_2\}$ yields two connected components $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ where $n = |V| = |V_1| + |V_2|$. Without loss of generality, $v_1 \in V_1$, $v_2 \in V_2$, and $w_2 \in V_2$ hold.

- Case 1: $w_1 \in V_1$ and $w_2 \in V_2$.
  - Case 1a: $w_1 \neq v_1$ and $w_2 \neq v_2$. As we deleted the edge e, the vertex v_1 has degree 2 or less in G_1, and v_2 has degree 2 or less in G_2. Hence, by the induction hypothesis, G_1 can be embedded in the topmost $|V_1|$ points, such that v_1 is the rightmost and w_1 the topmost vertex in the embedding of G_1. Also, G_2 can be embedded in the bottommost $|V_2|$ points, such that v_2 is the topmost and

w_2 the rightmost vertex. The edge e can be drawn as sketched in Figure 2.16 and by construction, all desired properties are fulfilled.

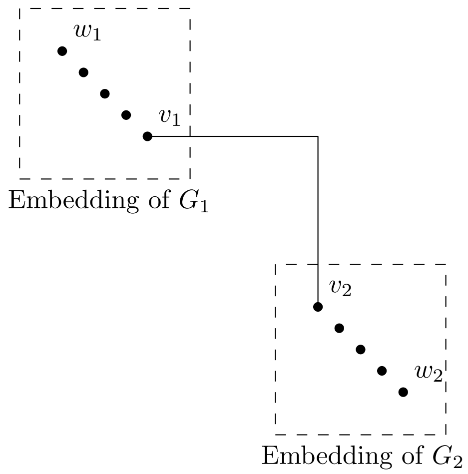

Figure 2.16: Embedding of a graph with bridges. Case 1a.

– Case 1b: $w_1 = v_1$ and $w_2 \neq v_2$. In this case $v_1 = w_1$ has degree at most 1 in G_1. According to Lemma 4, there exists another vertex u in G_1 with degree at most 2. Hence, G_1 can be embedded in the topmost $|V_1|$ points, such that v_1 is the topmost and u the rightmost vertex. If v_1 is connected from the right side, we can continue with the mirrored embedding of G_1. We can embed G_2 analogously to Case 1a. As v_1 has degree 1 in G_1, v_1 can be connected to v_2 as outlined in Figure 2.17. All required properties are fulfilled by this construction.

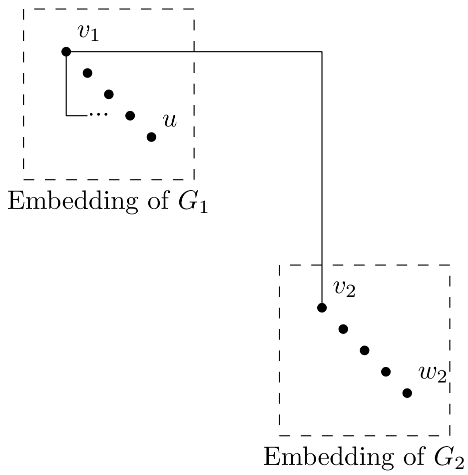

Figure 2.17: Embedding of a graph with bridges. Case 1b.

- Case 1c: $w_1 \neq v_1$ and $w_2 = v_2$. Analogously to Case 1b.
- Case 1d: $w_1 = v_1$ and $w_2 = v_2$. Analogously to Case 1b and 1c. Note that the embeddings of both connected components might need to be flipped to draw the edge e.

- Case 2: $w_1, w_2 \in V_2$. By the induction hypothesis, G_2 can be embedded in the bottommost $|V_2|$ points, such that v_2 is the topmost and w_2 is the rightmost point. Furthermore, w_1 allows either a connection from left or from below by the induction hypothesis. If w_1 does not allow a connection from above, mirroring the embedding of G_2 does the trick. According to Lemma 4, there exists another vertex u in G_1 with degree at most 2. Hence, G_1 can be embedded in the topmost $|V_1|$ points, such that u is the topmost and v_1 is the rightmost vertex. We can draw the edge e as sketched in Figure 2.18 and by construction, all required properties are fulfilled.

□

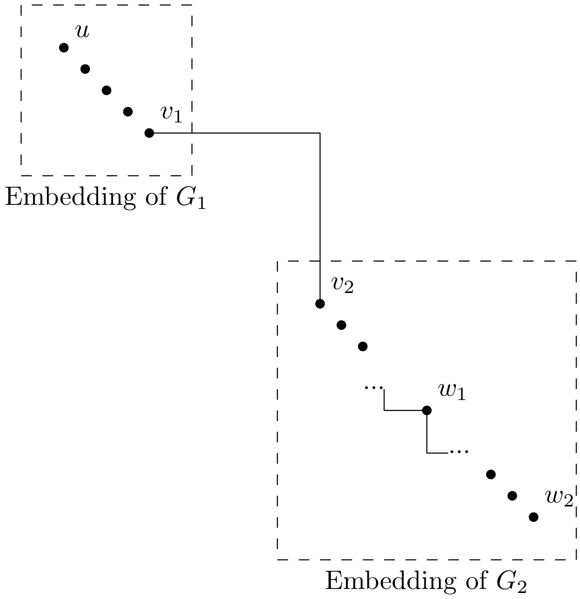

Figure 2.18: Embedding of a graph with bridges. Case 2.

We remark that a proof by induction like this one yields a recursive embedding algorithm.

2.1.3 Planar Orthogeodesic Embeddings

Theorem 4. *Every outerplanar n-vertex graph with maximum degree 3 admits a planar orthogeodesic embedding in every point set of size* $4(n-1)^2+1$.

Proof. Analogously to the proof of the previous theorem, we give a proof by induction on the number of bridges, that any connected outerplanar n-vertex graph $G = (V, E)$ with maximum degree 3 or less, with two distinct vertices $w_1, w_2 \in V$ of degree 2 or less, admits a planar orthogeodesic embedding in every diagonal point set of size n, such that

1. w_1 allows a connection from above,

2. w_2 allows connections from below and from the right side, and

3. any other vertex $v \in V \backslash \{w_1, w_2\}$ with degree 2 or less allows a connection from above or from the left side.

The inductive step will be the same as in the previous proof, because planarity was not violated by any construction in any case. Therefore, we only need to show the induction basis to prove this theorem.

Base case: Let $G = (V, E)$ be a nontrivial bridgeless connected outerplanar n-vertex graph with maximum degree 3 or less, and let P be a diagonal point set of size n. Recall from the previous proof, that all vertices in G have degree 2 or 3 as the graph is bridgeless. We can assume that the vertices are labeled clockwise such that v_1 and v_k are two vertices with degree 2, as exemplified in Figure 2.13.

First, we embed the vertices and the polygon as exemplified in Figure 2.19. The vertices $v_1, \ldots, v_k$ are placed in ascending order and $v_{k+1}, \ldots, v_n$ are placed in descending order, such that for any edge from v_{k-i} to v_{k+j} in the graph the vertex v_{k+j} is placed between v_{k-i} and v_{k-i+1}. We remark that every vertex has degree at most 3, and therefore, this construction is valid so far.

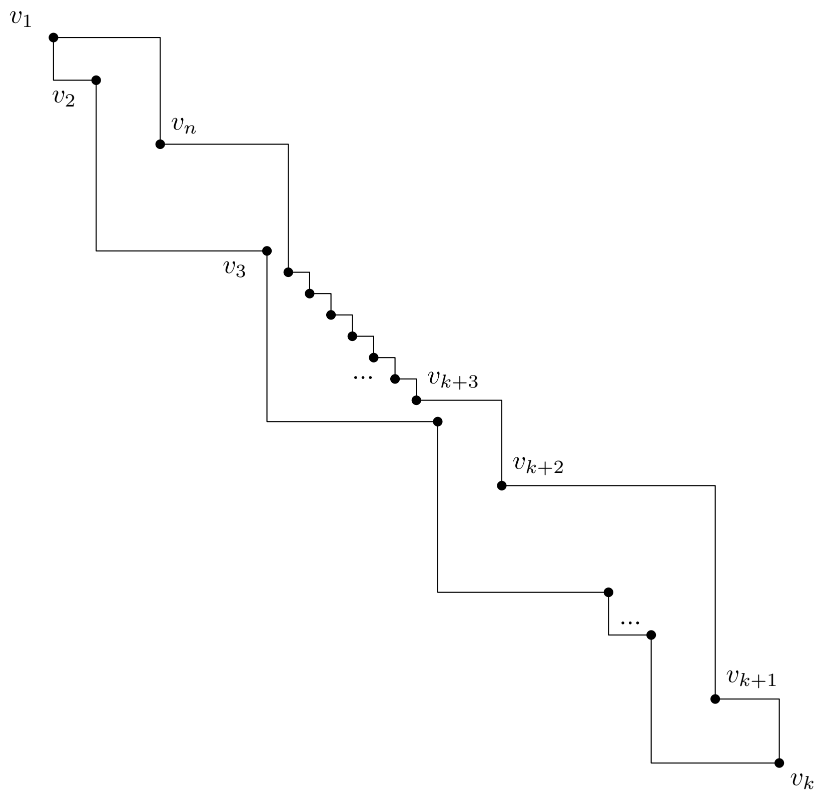

Figure 2.19: Embedding of the vertices and the outer polygon.

Note that v_1 and v_k do not have further connections, as they only have degree 2. Using this placement of the vertices, we are able to embed the remaining edges as illustrated in Figure 2.20 - except for the edges that are drawn as non-L-shaped connections in this figure, but we will soon argue that these edges can be drawn in a valid way. Moreover, note that also the third requirement might not be fulfilled by this construction.

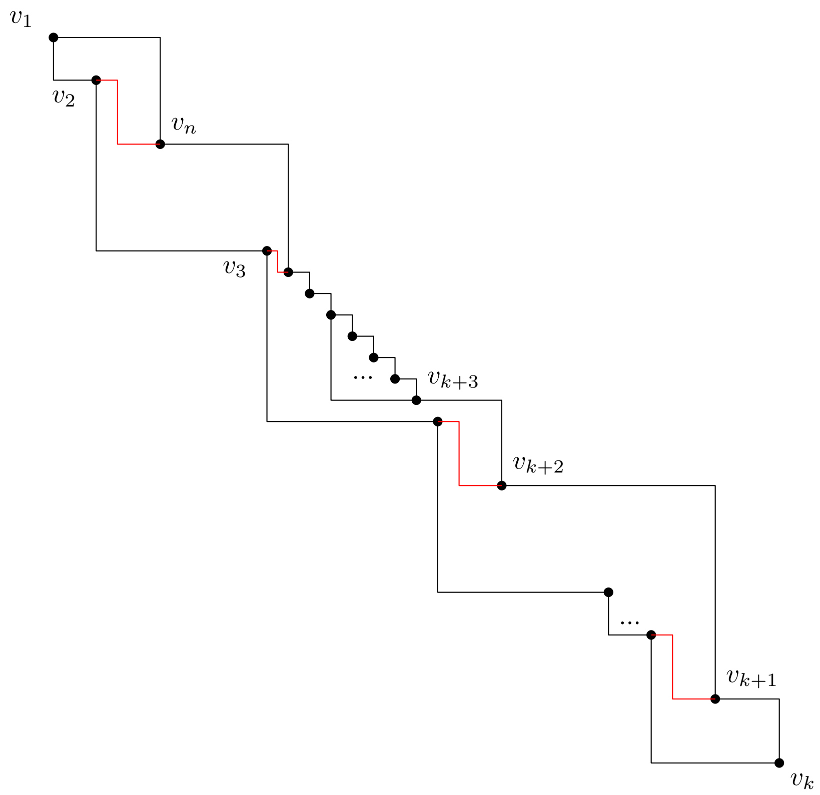

Figure 2.20: Embedding of the remaining edges. Non-L-shaped connections are colored red.

The solution to both problems: we can assume that between each two points of our current embedding a further point is present, as there are still $n-1$ additional points in our diagonal point set by construction. Note that $((n+(n-1))-1)^2+1 = 4(n-1)^2+1$ holds. Therefore, the non-L-shaped drawn edges as illustrated in the previous figure are drawn in a valid way. Furthermore, we can make use of those additional points to modify our embedding by re-drawing every L-shaped edge connected to a degree 2 vertex as depicted in Figure 2.21 and Figure 2.22. Recall that those bends are valid by definition of orthogeodesic embeddings, even though none of the vertices are embedded in an additional point.

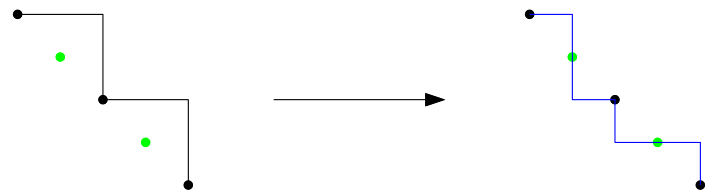

Figure 2.21: Modification of the edges connected to degree 2 vertices. The additional points are colored green.

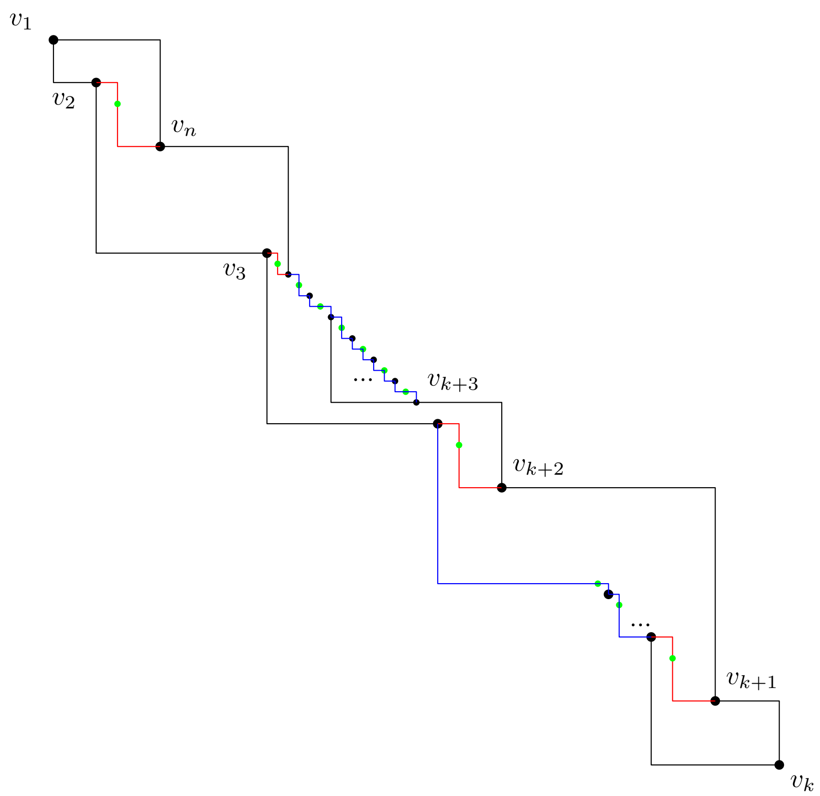

Figure 2.22: Final Embedding. The additional points are colored green and the replaced edges are colored blue.

The orthogeodesic embedding we constructed is planar and fulfills all desired properties. □

In contrast to our first results, the last two theorems state that every outerplanar graph with maximum degree 3 admits a planar orthogeodesic embedding. Also, a nonplanar L-shaped embedding is admitted in any point set if the point set is large enough. We remark that it might be possible to improve the multiplicative constant for both upper bounds - there might even exist sub-quadratic bounds.

2.2 Approximation of Outerplanar Graphs

Consider an arbitrary connected outerplanar n-vertex graph $G = (V, E)$ with maximum degree greater or equal to 4. As already stated earlier, G might not admit any embedding in any point set. So what else can we do with G? In some practical applications it makes sense to replace the graph G with a similar graph G'. For example, we could choose a graph G' that contains G as a minor. In fact, we can always construct an outerplanar graph G' that contains our original graph G as minor and has maximum degree 3 or less. That is, because every vertex v with degree $d \geq 4$ can be replaced by the vertices $v'_1, \ldots, v'_{d-2}$ as exemplified in Figure 2.23.

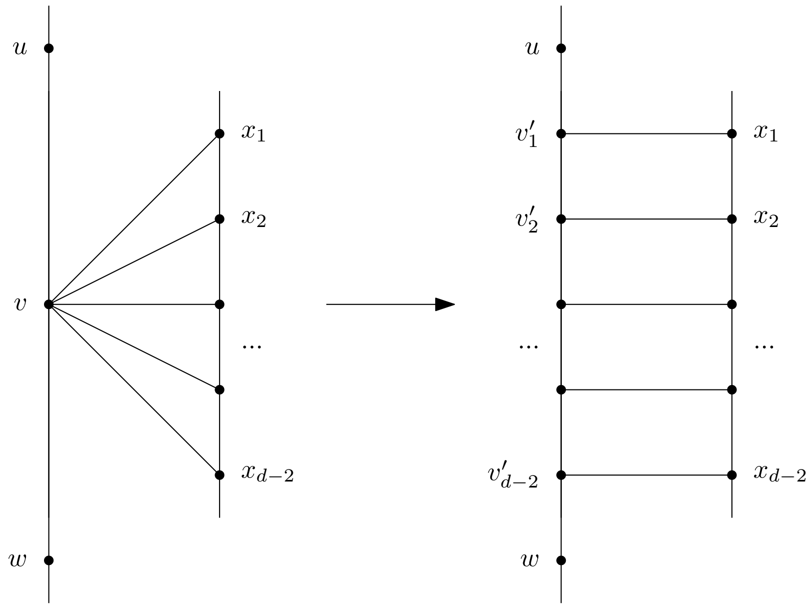

Figure 2.23: Construction of a graph G' with maximum degree 3 or less that contains G as minor.

Note that according to this construction, we can contract the edges that are connecting the vertices $v'_1, \ldots, v'_k$ to retrieve the original vertex v, and therefore, G' contains G as minor. Our new graph $G' = (V', E')$ has exactly

$$|V'| = \sum_{v \in V} \max\{\deg_G(v) - 2, 1\}$$

vertices. Since every additional edge in G can only increase the number of vertices in G', $|V'|$ is maximal if G is a maximal outerplanar graph. Hence, we can assume that $d_0 = d_1 = 0$ and $d_2 \leq \frac{|V|}{2}$ hold, where d_i denotes the number of vertices in G of degree i. As every outerplanar graph has at most $2|V| - 3$ edges [5],

$$|V'| \leq \sum_{v \in V} (\deg_G(v) - 2) + d_2 \leq 2|E| - 2|V| + \frac{|V|}{2} \leq 2.5|V| - 6$$

holds. Due to our earlier results, G' admits a planar orthogeodesic embedding in every point set of size $4(2.5n - 7)^2 + 1 = 25n^2 + \mathcal{O}(n)$, and a nonplanar L-shaped embedding in every point set of size $(2.5n - 7)^2 + 1 = 6.25n^2 + \mathcal{O}(n)$. We conjecture that the multiplicative constant for the quadratic term is not yet optimal in either of these two upper bounds.

Chapter 3

Trees

In this chapter, we discuss embeddings of trees in point sets. Giacomo et al. [10] have already given upper bounds for all cases, that is, the planar orthogeodesic case, the nonplanar L-shaped case, and the planar L-shaped case for 3-trees and 4-trees. In particular, they have stated that any n-vertex 3-tree can be embedded in any set of n points in the nonplanar L-shaped case and in the orthogeodesic case.

3.1 Planar Orthogeodesic Embeddings of 4-Trees

In this subchapter, we give an improvement of the bound $f_{OT4}(n) \leq 4n$ provided by Giacomo et al. [10].

Theorem 5.

$$f_{OT4}(n) \leq 2n.$$

Proof. We give a proof by induction on the number of vertices, that every n-vertex 4-tree $T = (V, E)$ with a fixed vertex $v \in V$ of degree 3 or less admits a planar orthogeodesic embedding in every point set P of size $2n$, that allows v to be connected to any point below P.

Base case: The embedding of a single vertex is trivial as it can be placed arbitrarily.

Inductive step: As v has degree 3 or less, the given tree can be considered as a 4-ary tree with root v. Hence, v has at most 3 subtrees with a, b, and c vertices, respectively, with $a + b + c = n - 1$ and $a, b, c \leq n - 1$ being non-negative integers. Given $2n$ points, we can (in this order)

1. cut off the bottommost point p,

2. cut off the $2a$ leftmost points and embed the first subtree,
3. cut off the $2b$ rightmost points and embed the second subtree,
4. cut off the $2c$ topmost points and embed the third subtree, and
5. use the remaining point to embed v

as illustrated in Figure 3.1. Recall that an edge might contain points of P, in which no vertices are embedded, as exemplified by the point q in the figure. Since we have cut off the bottommost point p, the embedded vertex v can be connected from below arbitrarily. □

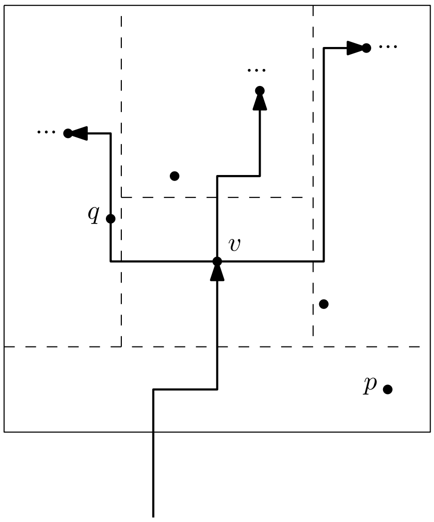

Figure 3.1: Embedding a subtree with root of degree 3 or less. Cutting lines are drawn dashed.

This proof by induction yields an embedding algorithm as follows: given a 4-tree, we pick an arbitrary leaf as starting vertex and embed the tree recursively as stated in the inductive step. Note that vertices of degree 1 or 2 can be embedded without using any additional points. Hence, we will give an amortized analysis of this algorithm. To achieve that, we will make use of the following two well-known results on trees. Chartrand and Zhang give a proof of the first one in their book Chromatic Graph Theory [5], whereas they left the second one as an exercise for the reader, since it results from the first one. Nevertheless, for completeness we prove both statements in this thesis, as their proofs are quite simple and short.

Lemma 5. *For every n-vertex tree the equation $\sum_{i=1}^{\infty} d_i(i-2) = -2$ holds, where d_i denotes the number of vertices of degree i.*

Proof. Let $T = (V, E)$ be an n-vertex tree. Then

$$\sum_{i=1}^{\infty} id_i = \sum_{v \in V} \deg(v) = 2|E| = 2n - 2 = \sum_{i=1}^{\infty} 2d_i - 2$$

holds, and therefore, $\sum_{i=1}^{\infty}(i-2)d_i = -2$ holds. □

Corollary 2. *Let $k > 1$. Every n-vertex tree has at most $\frac{n-2}{k-1}$ vertices of degree k or greater.*

Proof. Let $T = (V, E)$ be an n-vertex tree and let $U \subseteq V$ be the set of vertices, that have degree k or greater. Then

$$2n - 2 = 2|E| = \sum_{v \in V} \deg(v) \geq n + (k-1)|U|$$

holds, or equally, $|U| \leq \frac{n-2}{k-1}$. □

Note that according to these two statements, every nontrivial tree has at least two leaves, and furthermore, the number of leaves is exactly $d_1 = \sum_{i=2}^{\infty} d_i(i-2)$, where d_i is defined as in Lemma 5.

Theorem 6.

$$f_{OT4}(n) \leq \frac{3}{2}n - 1.$$

Proof. We revise the proof of the previous theorem and give an amortized analysis of the corresponding algorithm. If the root v of a subtree has degree 0 or 1, that is, v has degree 1 or 2 in the original tree, we do not need to waste the bottommost point of the given point set to give an embedding. Instead, we can embed v as the bottommost point, and in case there is a subtree, this subtree can be embedded above v as depicted in Figure 3.2. It is obvious, that the statement holds whenever v has degree 0. If v has degree 1, it only has one connection from above according to this construction, and therefore, v can be connected to any point below.

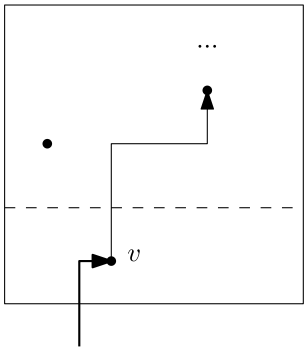

Figure 3.2: Embedding a subtree with root of degree 1. The cutting line is drawn dashed.

As we are embedding recursively, any vertex v - except for the starting vertex - has its degree reduced by one in the subtree we are looking at. According to the previous corollary, every 4-tree has at most $\frac{n-2}{2}$ inner vertices of degree 3 or 4, and therefore, the number of points wasted by the embedding algorithm is at most $\frac{n-2}{2}$. □

The multiplicative constant $\frac{3}{2}$ might be improved by embedding multiple vertices in each recursive call. Furthermore, one might improve the additional term -1 by enlarging the induction basis, but as we conjecture that the multiplicative constant $\frac{3}{2}$ is not tight at all, we did not waste too much time on that. Nevertheless, we will make use of this idea later on, to improve some other bounds.

3.2 Nonplanar L-Shaped Embeddings of 4-Trees

In this subchapter, we give an improvement of the bound $f_{NT4}(n) \leq 4n - 3$ provided by Giacomo et al. [10] by revising their proof. We first give a definition of ring-partitions and head to the idea of the improvement afterwards.

Given a point set P of size m, one can cut off the topmost, the bottommost, the leftmost, and the rightmost point. Recall that the leftmost point can also be the topmost point and so on. This cut-off process can be iterated until no more points are left. We call this partition the **ring-partition** of P. Each ring consists of either two, three or four points - except for the innermost ring that can also consist of one single point. Furthermore, there are at least $\lceil \frac{m}{4} \rceil$ and at most $\lceil \frac{m}{2} \rceil$ rings.

In the proof given by Giacomo et al. [10], a new ring in the ring partition is used for every vertex in the tree. Since every ring can contain up to 4 vertices, up to $3n$ points might be wasted using this strategy when embedding an n-vertex 4-tree. We give an improvement of the multiplicative constant 4 by reusing unused points of previous rings.

Theorem 7.

$$f_{NT4}(n) \leq \frac{7}{3}n + \mathcal{O}(1).$$

We remark that the notation $f(n) \leq g(n) + \mathcal{O}(h(n))$ is used as an abbreviation for $f \leq \hat{f}$ for some function $\hat{f}$ with $|\hat{f}(n) - g(n)| = \mathcal{O}(h(n))$.

Proof. Let T be a 4-tree with $n \geq 2$ vertices and P be a point set. Consider the following embedding algorithm: Create an empty list that holds the unused points and create a queue with the rings of the ring-partition of P.

We start with picking an arbitrary leaf u, root the tree at u, and place u on an arbitrary point in the innermost ring. In addition, all unused points of this ring are added to the list of unused points. Then we perform the following procedure recursively on the children of u: Without loss of generality, the current vertex v is connected to its parent from the right side.

- Case 1: $\deg(v) = 1$: Stop. No further points are required.
- Case 2: $\deg(v) = 2$: If the list of unused points is not empty, remove and use a point from the list as exemplified in Figure 3.3. Otherwise, place the vertex in a new ring and add the other points of this ring to the list.

  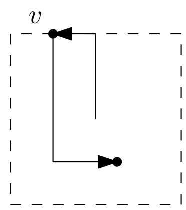

  Figure 3.3: Case 2 - recycle an unused point.

  Note that in this case points are only added to the list of unused points if the list is empty. Recall that we consider nonplanar embeddings, and therefore, crossings are allowed.

- Case 3: $\deg(v) = 3$: As there is a topmost and a bottommost point in every ring, and as v is connected from the right side, we can connect as sketched in Figure 3.4.

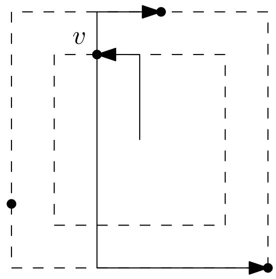

Figure 3.4: Case 3.

Note that in this case at most 2 points are added to the list of unused points.

- Case 4: $\deg(v) = 4$:
  - Case 4a: The next ring contains 4 points: Hence, there exist distinct left-, top-, and bottommost points. As v is connected from the right side, we can connect as illustrated in Figure 3.5. Furthermore, we add the rightmost point to the list of unused points.

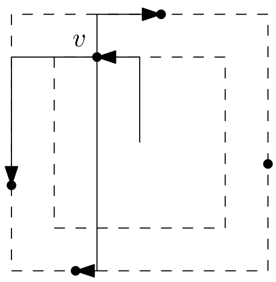

Figure 3.5: Case 4a.

  - Case 4b: The next ring contains at most 3 points: As the left-, bottom- and topmost point might not be distinct, we might need to use one more ring. When using two rings, there certainly exist three distinct points that are on the left side of v, below v, and above v, respectively. We can continue analogously to Case 4a, but in this case, up to 4 points might be added to the list of unused points.

Now we give an amortized analysis of this recursive algorithm. It is obvious, that we only need to consider Case 3 and Case 4, since Case 2 can only increase the number of unused points up to a constant value of 3. According to Corollary 2, an upper bound on the number of unused points - except for a constant offset - can be calculated as

$$\begin{aligned} \text{maximize } & 1x_3 + 4x_4 \\ \text{subject to } & \sum_{i=1}^{4} x_i = n \\ & \sum_{i=1}^{4} (i-2)x_i = -2 \\ & x_i \in \mathbb{N}_0 \qquad\qquad , 1 \leq i \leq 4, \end{aligned}$$

where x_i denotes the number of points of degree i. We can further consider the relaxation of this problem, whereas in the relaxation we have $x_i \in \mathbb{R}$ and $x_i \geq 0$ for every i. Consider a feasible solution x of the relaxation with $x_2 = \varepsilon > 0$. We can construct another feasible solution x' that gives a greater objective function value than x as

$$x'_1 := x_1 + \frac{\varepsilon}{2}, \quad x'_2 := 0, \quad x'_3 := x_3 + \frac{\varepsilon}{2}, \quad x'_4 := x_4.$$

Therefore, any feasible solution with $x_2 > 0$ can not be optimal. We can rewrite the problem to

$$\begin{aligned} \text{maximize } & 1x_3 + 4x_4 \\ \text{subject to } & x_1 + x_3 + x_4 = n \\ & x_1 = 2 + x_3 + 2x_4 \\ & x_1, x_3, x_4 \geq 0. \end{aligned}$$

By using the restriction $x_1 = 2 + x_3 + 2x_4$, we can eliminate the variable x_1 and write the problem as

$$\begin{aligned} \text{maximize } & 1x_3 + 4x_4 \\ \text{subject to } & 2x_3 + 3x_4 = n - 2 \\ & 2 + x_3 + 2x_4 \geq 0 \\ & x_3, x_4 \geq 0. \end{aligned}$$

Now we can drop the second restriction $2 + x_3 + 2x_4 \geq 0$, as it is always fulfilled since $x_3, x_4 \geq 0$. Furthermore, we can eliminate the variable x_3 by using the remaining equation.

$$\begin{aligned} \text{maximize } & \frac{1}{2}n + \frac{5}{2}x_4 - 1 \\ \text{subject to } & 3x_4 \leq n - 2 \\ & x_4 \geq 0. \end{aligned}$$

The optimal value of this problem is $\frac{1}{2}n + \frac{5}{2}\frac{n-2}{3}$, and therefore, the number of unused points is at most $\frac{4}{3}n + \mathcal{O}(1)$.

□

Since we conjecture that the multiplicative constant $\frac{7}{3}$ can be further improved, we did not focus on the additive constant. As stated in the previous subchapter, the multiplicative constant might be improved by embedding more than one vertex in each step.

3.3 Planar L-Shaped Embeddings of k-Trees

In this subchapter, we will give a sub-quadratic upper bound on f_{LT4} and hence, also on f_{LT3}. As Giacomo et al. [10] have already investigated diagonal point sets to prove a quadratic upper bound on f_{LT4}, we will need another approach to achieve our goal. We will define a function $f^{\uparrow}_{LTk}$ such that $f_{LTk} \leq f^{\uparrow}_{LTk}$ holds, and then give a sub-quadratic upper bound on this new function. We remark that every upper bound on $f^{\uparrow}_{LTk}$ is also an upper bound on f_{LTk} by definition.

Definition 3. *Let $T = (V, E)$ be a k-tree and let $v \in V$ be a vertex with degree less than k. We define $f^{\uparrow}_{LTk}(T; v)$ as the minimal natural number, such that for any point set P with $|P| \geq f^{\uparrow}_{LTk}(T; v)$ and any point $q \in \mathbb{R}^2$ with $\min_{p \in P} y(p) > y(q)$ there exists a planar L-shaped embedding of T in P, where v can be connected to q such that q is connected from above. We further define*

$$f^{\uparrow}_{LTk}(n) := \max_{\substack{T = (V,E)\ k\text{-tree} \\ |V| = n \\ \deg_T(v) < k}} f^{\uparrow}_{LTk}(T; v)$$

for $n \in \mathbb{N}$ and $f^{\uparrow}_{LTk}(0) := 0$.

Note that $f_{LTk}(n) \leq f^{\uparrow}_{LTk}(n)$ holds, because of the more restrictive definition.

Lemma 6. *Let $n \in \mathbb{N}$. Then*

$$f^{\uparrow}_{LT4}(n) \leq \max_{\substack{a,b,c \in \mathbb{N}_0 \\ a+b+c+1=n \\ a \geq b \geq c}} 1 + f^{\uparrow}_{LT4}(a) + 2f^{\uparrow}_{LT4}(b) + 2f^{\uparrow}_{LT4}(c).$$

Proof. We give a proof by induction on n.

Base case: $n = 1$: As $f^{\uparrow}_{LT4}(0) = 0$, $f^{\uparrow}_{LT4}(1) = 1$, and because $a = b = c = 0$ is the only possible setting for $a + b + c + 1 = 1$, the inequality holds.

Inductive step: Let $T = (V, E)$ be an n-vertex 4-tree and let $v \in V$ be a vertex with degree less than 4. The removal of v leaves three subtrees T_1, T_2, and T_3, where T_i can be the empty tree. Let a, b, and c denote the number of vertices in the subtree T_1, T_2, and T_3, respectively. Furthermore, let P be a point set with $|P| \geq 1 + f^{\uparrow}_{LT4}(a) + 2f^{\uparrow}_{LT4}(b) + 2f^{\uparrow}_{LT4}(c)$ and let q be a point below P, that is, $y(q) < \min_{p \in P} y(p)$. We can partition $P = A \uplus (B_1 \uplus C \uplus B_2)$ with $|A| \geq f^{\uparrow}_{LT4}(a)$, $|B_1| = |B_2| = f^{\uparrow}_{LT4}(b)$, and $|C| = 1 + 2f^{\uparrow}_{LT4}(c)$ such that

$$\min_{a \in A} y(a) > \max_{p \in B_1 \uplus C \uplus B_2} y(p), \quad \max_{b \in B_1} x(b) < \min_{c \in C} x(c), \quad \max_{c \in C} x(c) < \min_{b \in B_2} x(b)$$

hold, as depicted in Figure 3.6. The vertical line through the point q partitions C into two parts C_1 and C_2 with

$$\max_{c \in C_1} x(c) < x(q) < \min_{c \in C_2} x(c),$$

where one part must contain at least $f^{\uparrow}_{LT4}(c) + 1$ points. Without loss of generality the left part C_1 does, because otherwise, we could continue mirrored. Note that $C_1 = C$ and $C_2 = \emptyset$ also gives a valid partition of the set C.

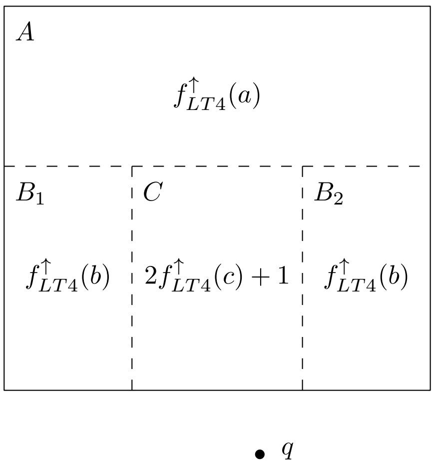

Figure 3.6: Visualization of the partitioning of P.

We embed v as the topmost point of C_1. By the induction hypothesis, each of the 3 subtrees can be embedded recursively in A, B_1, and $C_1 \setminus \{v\}$, respectively, as exemplified in Figure 3.7. Note that we can not embed a subtree recursively in B_2 since v is already connected from the right side. $\square$

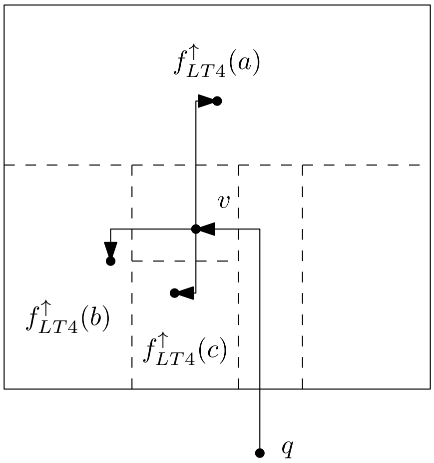

Figure 3.7: Visualization of the embedding.

For this subchapter we define the integer sequence $(f_n)_{n\in\mathbb{N}_0}$ as

$$f_0 := 0,\ f_1 := 1,\ f_k := \max_{\substack{a,b,c\in\mathbb{N}_0\\ a+b+c+1=n\\ a\geq b\geq c}} 1 + f_a + 2f_b + 2f_c.$$

According to the previous lemma, we have $f_{LT4}(n) \leq f^{\uparrow}_{LT4}(n) \leq f_n$.

Now we state some well-known analytical lemmas which will be used to prove the next theorem.

Lemma 7 (Mean Value Theorem [30])**.** *Let f be a real-valued continuous function on the closed interval $[a, b]$, which is differentiable on the open interval (a, b). Then there exists a point $\xi \in (a, b)$ such that*

$$f(b) - f(a) = f'(\xi)(b - a).$$

Lemma 8 (Jensen's Inequality [15,28,31])**.** *Let $X \subset \mathbb{R}^d$ be a convex set and let f be a real-valued convex function on X. For any $x_1, \dots, x_k \in X$ and $\lambda_1, \dots, \lambda_k \in [0, 1]$ with $\sum_{i=1}^{k} \lambda_i = 1$ the following inequality holds:*

$$f(\sum_{i=1}^{k} \lambda_i x_i) \leq \sum_{i=1}^{k} \lambda_i f(x_i).$$

For a finite point set $S = \{s_1, \dots, s_k\} \subset \mathbb{R}^d$ we define the **convex hull** of S as

$$\operatorname{conv} S := \left\{ x \in \mathbb{R}^d \;\middle|\; \exists \lambda_1, \dots, \lambda_k \geq 0\colon \sum_{i=1}^{k} \lambda_i = 1,\ \sum_{i=1}^{k} \lambda_i s_i = x \right\}.$$

Furthermore, let $C = \operatorname{conv} S$ be the convex hull of a finite set $S \subset \mathbb{R}^2$. We denote the minimal set S' with $C = \operatorname{conv} S'$ as the **corners** of C. Note that there exists a unique minimal set S' with $C = \operatorname{conv} S'$, and therefore, the set of corners is well-defined [2,12].

Lemma 9 (Maximum Principle [28])**.** *Let $S \subset \mathbb{R}^d$ be an finite set, let $C = \operatorname{conv} S$, and let f be a real-valued convex function on C. Then*

$$\max_{x \in C} f(x) = \max_{x \in S} f(x).$$

Proof. Let $x \in C$ and $S = \{s_1, \ldots, s_k\}$. Since $C = \operatorname{conv} S$, there exists $\lambda \in [0,1]^k$ such that $\sum_{i=1}^k \lambda_i = 1$ and $x = \sum_{i=1}^k \lambda_i s_i$ hold. According to Jensen's Inequality, the following inequality holds:

$$f(x) = f(\sum_{i=1}^k \lambda_i s_i) \le \sum_{i=1}^k \lambda_i f(s_i) \le \sum_{i=1}^k \lambda_i \max_{j \in \{1,\ldots,k\}} f(s_j) = \max_{j \in \{1,\ldots,k\}} f(s_j)$$

$\square$

Now we state a lemma on convex sets to keep the proof of the following theorem a bit shorter.

Lemma 10. *Let $d \in \mathbb{N}$, let $M > 0$, let*

$$C = \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d \;\middle|\; x_1 \ge \ldots \ge x_d \ge 0,\ \sum_{i=1}^d x_i = M \right\},$$

and let e_i denote the i-th unit vector in $\mathbb{R}^d$. Let $s_i = \frac{M}{i}(e_1 + \ldots + e_i)$ for $i \in \{1, \ldots, d\}$. Then $S = \{s_1, \ldots, s_d\}$ is the set of corners of C.

Proof. Let $x = (x_1, \ldots, x_d) \in C$. With $y_d := x_d$ and $y_i := x_i - x_{i+1} \ge 0$ for $i \in \{1, \ldots, d-1\}$ we have

$$\sum_{j=i}^d y_j = \left(\sum_{j=i}^{d-1} (x_j - x_{j+1}) \right) + x_d = x_i$$

for any $1 \le i \le d$. Furthermore, let $\lambda_i := \frac{i}{M} y_i \ge 0$ for $i \in \{1, \ldots, d\}$. Then we can write

$$\sum_{i=1}^d \lambda_i \cdot s_i = \sum_{i=1}^d y_i \left(\sum_{j=1}^i e_j \right) = \sum_{j=1}^d \left(\sum_{i=j}^d y_i \right) e_j = \sum_{j=1}^d x_j e_j = x.$$

We also observe that

$$\sum_{i=1}^{d} \lambda_i = \frac{1}{M} \sum_{i=1}^{d} i y_i = \frac{1}{M} \left(x_1 + \sum_{i=2}^{d} x_i \underbrace{(i-(i-1))}_{=1} \right) = \frac{M}{M} = 1$$

holds, and therefore, $C = \operatorname{conv} S$. Furthermore, as all elements in S are linearly independent, S is the set of corners of C. $\square$

Theorem 8.

$$f_{LT4}(n) \leq f^{\uparrow}_{LT4}(n) \leq f_n \leq c_1((n+1)^{\log_2 3} - 1) \leq c_1 n^{\log_2 3} + \mathcal{O}(n),$$

with $c_1 = \frac{6}{5^{\log_2 3} - 1} = 0.5076\cdots$.

Proof. Let $g(x) := c_1((x+1)^{\log_2 3} - 1)$ be a real-valued function on the non-negative real numbers $[0, \infty)$. We give a proof by induction on n, that $f_n \leq g(n)$ holds for $n \in \mathbb{N}_0$.

Base case:

n	f_n	$g(n)$
0	0	$0.000\cdots$
1	1	$1.015\cdots$
2	2	$2.388\cdots$
3	4	$4.061\cdots$
4	6	$6.000\cdots$
5	7	$8.180\cdots$

Inductive step: Let $n \geq 6$. According to the definition of f_n and the induction hypothesis,

$$f_n = \max_{\substack{a,b,c \in \mathbb{N}_0 \\ a+b+c+1=n \\ a \geq b \geq c}} 1 + f_a + 2f_b + 2f_c \leq \max_{\substack{a,b,c \in \mathbb{N}_0 \\ a+b+c+1=n \\ a \geq b \geq c}} 1 + g(a) + 2g(b) + 2g(c)$$

holds, and therefore, by relaxation,

$$f_n \leq \max_{\substack{a,b,c \in \mathbb{R} \\ a+b+c+1=n \\ a \geq b \geq c \geq 0}} 1 + g(a) + 2g(b) + 2g(c).$$

As g is a convex function on $[0, \infty)$, the functions

$$G_1(a,b,c) := g(a), \quad G_2(a,b,c) := g(b), \quad G_3(a,b,c) := g(c)$$

are also convex on $[0, \infty)^3$. Furthermore, as the sum of convex functions is a convex function as well [28], the function $G := 1 + G_1 + 2 \cdot G_2 + 2 \cdot G_3$ is convex on the convex set

$$C := \{(a, b, c) \in \mathbb{R}^3 \mid a + b + c + 1 = n,\ a \geq b \geq c \geq 0\}.$$

Hence, according to the Lemma 9 and Lemma 10, we have

$$f_n \leq \max_{(a,b,c) \in S} 1 + g(a) + 2g(b) + 2g(c),$$

where $S = \{(n-1, 0, 0), (\frac{n-1}{2}, \frac{n-1}{2}, 0), (\frac{n-1}{3}, \frac{n-1}{3}, \frac{n-1}{3})\}$ is the set of corners of C. Now we show that the value of $1 + g(a) + 2g(b) + 2g(c)$ is smaller or equal to $g(n)$ in each of these four corners:

- According to the Mean Value Theorem, $g(n) - g(n-1) = g'(\xi)$ holds for some $\xi \in (n-1, n)$. Since $g'(x) = c_1 \cdot \log_2 3 \cdot (x+1)^{\log_2 3 - 1}$ and $\log_2 3 > 1$ hold, g' is increasing on $[1, \infty)$. Thus, $g'(x) \geq g'(2) = 1.188\cdots$ holds for any $x \geq 1$ and we have

  $$1 + g(n-1) + 2g(0) + 2g(0) = 1 + g(n-1) \leq g(n).$$

- $1 + g(\frac{n-1}{2}) + 2g(\frac{n-1}{2}) + 0 = 1 + 3g(\frac{n-1}{2}) \leq g(n)$ holds, since $1 \leq 2c_1$ and $1 + 3g(\frac{n-1}{2}) \leq c_1(2 + 3(\frac{n+1}{2})^{\log_2 3} - 3) = c_1(\frac{3}{3}(n+1)^{\log_2 3} - 1) = g(n)$.

- Consider the function $h(x) := g(x) - (1 + 5g(\frac{x-1}{3}))$. We can write

  $$\begin{aligned} h'(x) &= \gamma((x+1)^{\alpha-1} - \beta(x+2)^{\alpha-1}) \\ &= \gamma((x+1)^{\alpha-1} - (x+2)^{\alpha-1} + (1-\beta)(x+2)^{\alpha-1}), \end{aligned}$$

  with $\alpha = \log_2 3 = 1.584\cdots$, $\beta = \frac{5}{3^{\log_2 3}} = 0.876\cdots$, and a positive constant γ. According to the Mean Value Theorem, we have

  $$h'(x) = \gamma(-(\alpha-1)\xi^{\alpha-2} + (1-\beta)(x+2)^{\alpha-1})$$

  with some $\xi \in (x+1, x+2)$. Since $1 < \alpha < 2$ holds, the function $\phi(x) = x^{\alpha-2} = \frac{1}{x^{2-\alpha}}$ is monotonically decreasing on $(0, \infty)$ and $\psi(x) = x^{\alpha-1}$ is monotonically increasing on $(0, \infty)$. Thus, we have

  $$\begin{aligned} h'(x) &\geq \gamma((1-\alpha)(x+2)^{\alpha-2} + (1-\beta)(x+2)^{\alpha-1}) \\ &= \gamma(1 - \alpha + (1-\beta)(x+2))(x+2)^{\alpha-2}. \end{aligned}$$

  For every $x \geq 6$ we have

  $$1 - \alpha + (x+2)(1-\beta) \geq 1 - \alpha + 8(1-\beta) = 0.403\cdots,$$

and therefore, $h'(x) \geq 0$ holds on $[6, \infty)$. Furthermore, since $h(6) = 0.108 \cdots > 0$ holds, we have $h(x) \geq 0$ on $[6, \infty)$. Hence,

$$1 + g(\frac{n-1}{3}) + 2g(\frac{n-1}{3}) + 2g(\frac{n-1}{3}) = 1 + 5g(\frac{n-1}{3}) \leq g(n)$$

holds for $n \geq 6$.

Altogether we have $f_n \leq g(n)$. $\square$

Note that $\log_2 3 = 1.5849 \cdots$ holds, and therefore, this is a sub-quadratic upper bound. The multiplicative constant $c_1 = 0.50767 \cdots$ of this bound might be improved slightly by using a better function g. But using this approach, the multiplicative constant can not become smaller than the constant $c_2 := \frac{6.5}{5^{\log_2 3}} = 0.50707 \cdots$, because

$$f_{5 \cdot 2^k - 1} \geq 3 f_{5 \cdot 2^{k-1} - 1} + 1 \geq \ldots \geq 3^k f_4 + \sum_{i=0}^{k-1} 3^i \geq 6 \cdot 3^k + \frac{3^k - 1}{3 - 1} \geq \frac{13 \cdot 3^k - 1}{2},$$

and therefore,

$$f_n \geq 6.5 \cdot 3^{\log_2 \frac{n+1}{5}} = c_2 (n+1)^{\log_2 3} \geq c_2 n^{\log_2 3}$$

must hold for certain values of n. Figure 3.8 gives an illustration. One would have to improve the upper bound on $f^{\uparrow}_{LT4}$ first to achieve a further improvement by using this approach.

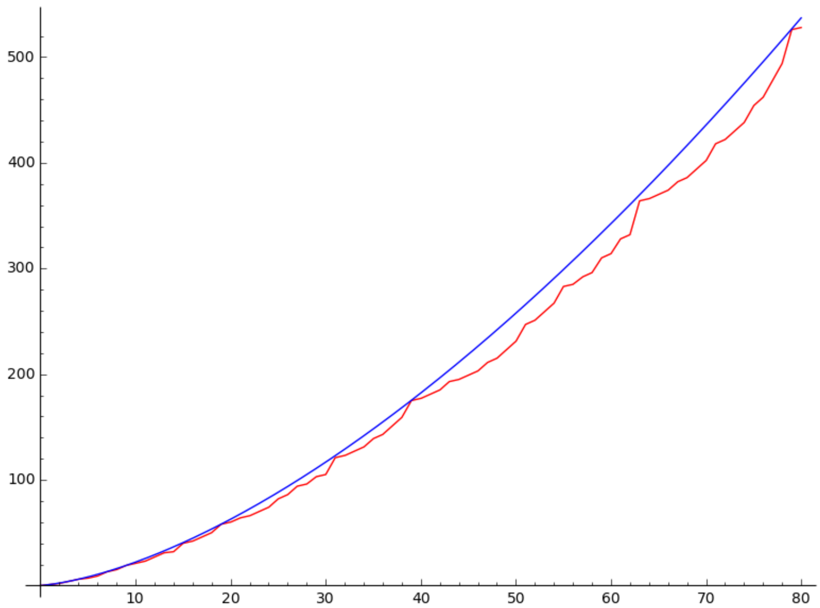

Figure 3.8: A plot of the sequence $(f_n)_{n \in \mathbb{N}_0}$ (red) and the function g from the proof of Theorem 8 (blue).

Now we will apply this approach to 3-trees. Since we only need to consider two variables in the proof for 3-trees, we can even achieve $\frac{1}{2}$ as multiplicative constant.

Corollary 3.

$$f_{LT3}(n) \leq 0.5n^{\log_2 3} + \mathcal{O}(n).$$

Proof. Let $g(x) = \frac{1}{2}((x+1)^{\log_2 3} - 1)$. Analogously to the proof above, we only need to consider the convex set

$$C := \{(a,b) \in \mathbb{R}^2 \mid a+b+1 = n,\ a \geq b \geq 0\}$$

with the corners $S = \{(n-1, 0), (\frac{n-1}{2}, \frac{n-1}{2})\}$. We show that the value of $1 + g(a) + 2g(b)$ is smaller or equal to $g(n)$ in those corners:

- As in the previous proof, we have

  $$1 + g(n-1) + 2g(0) = 1 + g(n-1) \leq g(n)$$

  according to the Mean Value Theorem.

- $1 + g(\frac{n-1}{2}) + 2g(\frac{n-1}{2}) = 1 + 3g(\frac{n-1}{2}) = g(n)$, since

$$1 + 3g(\frac{n-1}{2}) = \frac{2 + 3(\frac{n+1}{2})^{\log_2 3} - 3}{2} = \frac{3\frac{1}{3}(n+1)^{\log_2 3} - 1}{2} = g(n).$$

□

3.3.1 Jordan's Separator Lemma

To give a further improvement of the upper bound on f_{LTk}, we make use of graph theory, in particular a basic result on trees. A vertex v in an n-vertex tree is called $\frac{1}{2}$-**separator**, if the removal of v leaves connected components, where the size of every component is smaller or equal to $\frac{n}{2}$ [6,11]. The following statement was shown by Camille Jordan in 1869 [6,16]:

Lemma 11 (Jordan's Separator Lemma [16])**.** *There exists a $\frac{1}{2}$-separator in every tree.*

We will make use of this lemma to improve the upper bound on f_{LTk}. Recall that in the previous proofs we have stated upper bounds on $f^{\uparrow}_{LTk}$, but not directly on f_{LTk}. The following statement on convex sets will also be a useful tool when doing so.

Lemma 12. *Let $d \geq 3$, let $M > 0$, let S be the set of corners of the convex set*

$$C = \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d \;\middle|\; 0 \leq x_1, \ldots, x_d \leq \frac{M}{2},\ \sum_{i=1}^{d} x_i = M \right\},$$

and let e_i denote the i-th unit vector in $\mathbb{R}^d$. Then for every $s \in S$ there exist i and j with $i \neq j$ such that $s = \frac{M}{2}(e_i + e_j)$ holds.

Proof. Assume for a contraction that a corner $x = (x_1, \ldots, x_d) \in S$ exists that does not fulfill this property. Without loss of generality, $x_1, x_2, x_3 > 0$ and $x_1, x_2 < \frac{M}{2}$ hold. We can choose an $\varepsilon > 0$ such that $\varepsilon \leq x_1, x_2 \leq \frac{M}{2} - \varepsilon$ holds. Let $u := (x_1 + \varepsilon, x_2 - \varepsilon, x_3, \ldots, x_d)$ and $v := (x_1 - \varepsilon, x_2 + \varepsilon, x_3, \ldots, x_d)$. As $u, v \in C \backslash \{x\}$ and $x = \frac{u+v}{2}$ hold, this is a contraction. $\square$

Theorem 9.

$$f_{LT4}(n) \leq c_3 n^{\log_2 3} + \mathcal{O}(n)$$

holds with $c_3 = \frac{2}{3} \cdot \frac{6}{5^{\log_2 3} - 1} = 0.3384 \cdots$.

Proof. Let $T = (V, E)$ be an n-vertex tree and let g be defined as in the proof of Theorem 8. According to Jordan's Lemma, there exists a $\frac{1}{2}$-separator v. We denote the sizes of the connected components left after the removal of v as a, b, c, and d, respectively. We remark that $a + b + c + d + 1 = n$ and $0 \leq a, b, c, d \leq \frac{n}{2}$ hold. Let P be a point set of size $1 + f^{\uparrow}_{LT4}(a) + f^{\uparrow}_{LT4}(b) + f^{\uparrow}_{LT4}(c) + f^{\uparrow}_{LT4}(d)$. We can pick a point q of P and partition $P \backslash \{q\}$ into four parts A, B, C, and D of size $f^{\uparrow}_{LT4}(a)$, $f^{\uparrow}_{LT4}(b)$, $f^{\uparrow}_{LT4}(c)$, and $f^{\uparrow}_{LT4}(d)$, respectively, such that

$$\max_{a \in A} x(a) < \min_{p \in C \uplus \{q\} \uplus D} x(p), \qquad \max_{p \in C \uplus \{q\} \uplus D} x(p) < \min_{b \in B} x(b),$$

$$\min_{c \in C} y(c) > y(q) > \max_{d \in D} y(d).$$

According to Theorem 8, we can embed each connected component separately as illustrated in Figure 3.9, and thus, T can be embedded in P.

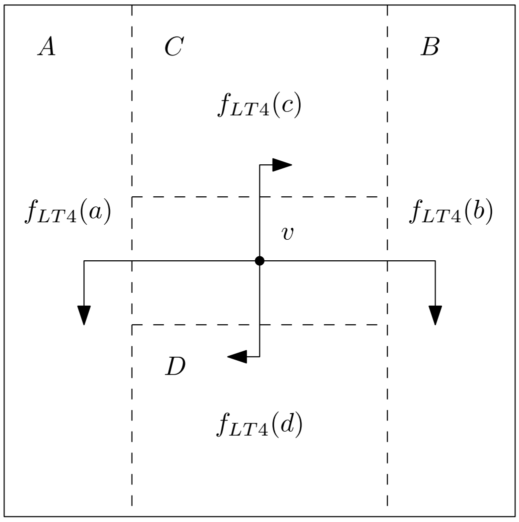

Figure 3.9: Partition of P.

By relaxation, we can write

$$\begin{aligned}
&\max_{\substack{a,b,c,d\in\mathbb{N}_0\\ 0\le a,b,c,d\le \frac{n}{2}\\ a+b+c+d+1=n}} 1+f^{\uparrow}_{LT4}(a)+f^{\uparrow}_{LT4}(b)+f^{\uparrow}_{LT4}(c)+f^{\uparrow}_{LT4}(d)\\
\le\ &\max_{\substack{a,b,c,d\in\mathbb{N}_0\\ 0\le a,b,c,d\le \frac{n}{2}\\ a+b+c+d+1=n}} 1+g(a)+g(b)+g(c)+g(d)\\
\le\ &\max_{\substack{a,b,c,d\in\mathbb{R}\\ 0\le a,b,c,d\le \frac{n}{2}\\ a+b+c+d+1=n}} 1+g(a)+g(b)+g(c)+g(d),
\end{aligned}$$

where g is defined as in the proof of Theorem 8. Recall that g is a monotonically increasing function in the nonnegative real numbers. According to the Maximum Principle, we have

$$\begin{aligned}
&\max_{\substack{a,b,c,d\in\mathbb{R}\\ 0\le a,b,c,d\le \frac{n}{2}\\ a+b+c+d+1=n}} 1+g(a)+g(b)+g(c)+g(d)\\
\le\ &\max_{\substack{a,b,c,d\in\mathbb{R}\\ 0\le a,b,c,d\le \frac{n}{2}\\ a+b+c+d=n}} 1+g(a)+g(b)+g(c)+g(d)\\
=\ &\max_{(a,b,c,d)\in S} 1+g(a)+g(b)+g(c)+g(d),
\end{aligned}$$

where S is the set of corners of the convex set

$$\{(a,b,c,d)\in\mathbb{R}^4 \mid 0\le a,b,c,d\le \frac{n}{2},\ a+b+c+d=n\}.$$

According to the previous lemma, for every corner $s \in S$ there exist $i \neq j$ such that $s = \frac{n}{2}(e_i + e_j)$ holds, where e_i denotes the i-th unit vector. Thus, we have

$$\max_{(a,b,c,d)\in S} 1 + g(a) + g(b) + g(c) + g(d) = 1 + 2g(\frac{n}{2}) = \frac{2}{3}g(n) + \mathcal{O}(n).$$

$\square$

Note that the convexity on the function g was crucial for this proof. Analogously to this theorem, we will prove the following statement for 3-trees, where the multiplicative constant for $f^{\uparrow}_{LT3}$ will slightly be improved.

Corollary 4.

$$f_{LT3}(n) \leq \frac{1}{3}n^{\log_2 3} + \mathcal{O}(n).$$

We remark that by improving the upper bound on $f^{\uparrow}_{LTk}$ and by using this approach, one can further improve the multiplicative constant of the upper bound on f_{LTk} - at least as long as the upper bound on $f^{\uparrow}_{LTk}$ is super-linear.

3.3.2 The Saturation of a Tree

Consider the following sequence of rooted k-ary trees $(T^{\star}_{k,h})_{h\in\mathbb{N}_0}$, where $T^{\star}_{k,0}$ has only one single vertex, and $T^{\star}_{k,h+1}$ has a root r that has k subtrees that are all equal to $T^{\star}_{k,h}$. By definition, the tree $T^{\star}_{k,h}$ has exactly $\frac{k^{h+1}-1}{k-1}$ vertices and height h, where the **height** of a rooted tree is the number of edges on the longest path from the root to a leaf. Furthermore, all leaves are at the same height and all inner vertices have exactly k children. We denote the trees $T^{\star}_{k,h}$ as **perfect k-ary trees**. For $k = 2$ the trees $T^{\star}_{2,h}$ are called **perfect binary trees** and for $k = 3$ the trees $T^{\star}_{3,h}$ are said to be **perfect ternary trees**. [7,27]

We will now introduce the saturation-property of unrooted trees, which is similar to the height-property of rooted trees. Later on we will use this definition to show, that trees with low saturation can be embedded much easier in small point sets than trees with high saturation. As a consequence, given a certain point set, trees with saturation up to a certain value can be proven to allow an embedding. Unfortunately, since perfect binary trees have very high saturation, we can not improve our $\mathcal{O}(n^{\log_2 3})$ bound in general.

Consider an n-vertex tree $T = (V, E)$. By rooting T at a certain vertex r, we can determine the function $\sigma_{T,r} : V \to \mathbb{N}$ recursively, where

$$\sigma_{T,r}(v) = \max\{0, \sigma_{T,r}(u_1), \sigma_{T,r}(u_2) + 1, \ldots, \sigma_{T,r}(u_k) + 1\}$$

holds for every vertex v with children $u_1, \ldots, u_k$ and $\sigma_{T,r}(u_1) \geq \ldots \geq \sigma_{T,r}(u_k)$. Note that this function is well defined. We define the **saturation** of T as

$$\sigma(T) := \min_{r \in V} \sigma_{T,r}(r) = \min_{r \in V} \max_{v \in V} \sigma_{T,r}(v).$$

The equality of these two expressions follows directly from the definition of the function $\sigma_{T,r}$. Figure 3.10 gives an example.

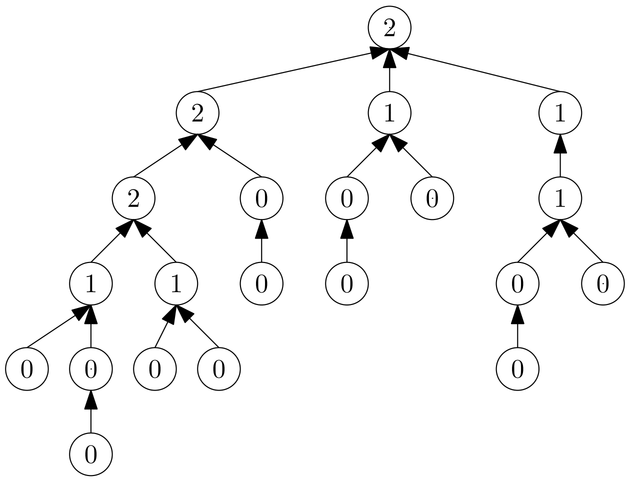

Figure 3.10: Example of a tree and the corresponding function $\sigma_{T,r}$.

Lemma 13.

1. *Any tree T with $\sigma(T) \geq h$ contains the perfect binary tree $T^\star_{2,h}$ as a minor.*

2. *Any tree T contains at least $2^{\sigma(T)+1} - 1$ vertices.*

3. *Any n-vertex tree T has a saturation of at most $\sigma(T) \leq \log_2(n+1)$.*

Proof. Recall that for any $k < h$ the perfect binary tree $T^\star_{2,h}$ contains $T^\star_{2,k}$ as a minor by definition. Hence, we give a proof by induction on n, that every n-vertex tree with saturation $\sigma(T) = h$ contains $T^\star_{2,h}$ as a minor. The other two statements follow directly from the first one.

Base case: $n = 1$: The one-vertex-tree has saturation 0 by definition, and since $T^\star_{2,0}$ is the one-vertex-tree, the statement follows directly.

Inductive step: Let T be an n-vertex tree. We can root the tree T at a vertex r, such that $\sigma(T) = \sigma_{T,r}(r) = h$ holds. Furthermore, let $u_1, \ldots, u_k$ be the children of r with $\sigma_{T,r}(u_1) \geq \ldots \geq \sigma_{T,r}(u_k)$.

According to the definition of $\sigma_{T,r}$, either $\sigma_{T,r}(u_1) = h$ or $\sigma_{T,r}(u_1) = \sigma_{T,r}(u_2) = h - 1$ holds. In the first case, the subtree containing u_1 contains $T^\star_{2,h}$ as a minor by the induction hypothesis, and therefore, so does T. In the second case, the subtrees containing u_1 and u_2 each contain $T^\star_{2,h-1}$ as a minor by the induction hypothesis, and therefore, T contains $T^\star_{2,h}$ as a minor. □

Recall from Chapter 1 that k-trees must not be confused with k-ary trees.

Theorem 10. *Every 4-tree $T = (V, E)$ admits a planar L-shaped embedding in every point set of size $|V| \cdot 2^{\sigma(T)}$.*

Proof. Let $\sigma_{T,r}$ such that $\sigma(T) = \sigma_{T,r}(r)$ holds. We describe an algorithm that proceeds similar as stated in the proof of Theorem 9.

Analogously to the proof of Theorem 9, we partition the given point set as sketched in Figure 3.6. For this proof we denote the upper part in a partition as non-wasting-area and the others as wasting-areas. To embed the tree, we place the root r as stated in Theorem 9 and continue embedding the subtrees recursively. When embedding a (sub-)tree with root v recursively, we do not embed its subtree with the maximum number of vertices in the non-wasting-area as one might assume, but its subtree with the maximum saturation. By definition, in all other subtrees the saturation is less than the saturation of v. Hence, there are at most $\sigma(T)$ recursive calls in which a wasting-area is given as parameter, and therefore, for every vertex $v \in V$ at most $2^{\sigma(T)}$ points are needed. Thus, every tree $T = (V, E)$ admits an embedding in every point set P of size $|P| \geq |V| \cdot 2^{\sigma(T)}$. □

Note that this theorem also gives a quadratic upper bound for the general case, since $2^{\sigma(T)} \leq 2^{\log_2(|V|+1)} \leq 2^{\log_2(2|V|)} = 2|V|$ holds.

3.3.3 The Orthogonal Convex Hull of a Point Set

Consider an arbitrary point set P. A point $p \in P$ is said to be an **inner point** of P, if there exist points $p_1, p_2, p_3, p_4 \in P \backslash \{p\}$, such that

$$x(p_1), x(p_2) \leq x(p) \leq x(p_3), x(p_4), \quad y(p_1), y(p_3) \leq y(p) \leq y(p_2), y(p_4)$$

hold. Otherwise p is said to be an **outer point** of P. We define the **orthogeodesic convex hull** of P as the set of outer points of P. Figure 3.11 gives an example of this definition. Note that any point within the areas with dashed borders is an inner point by definition. For more information on orthogonal convex hulls we refer to [21–25].

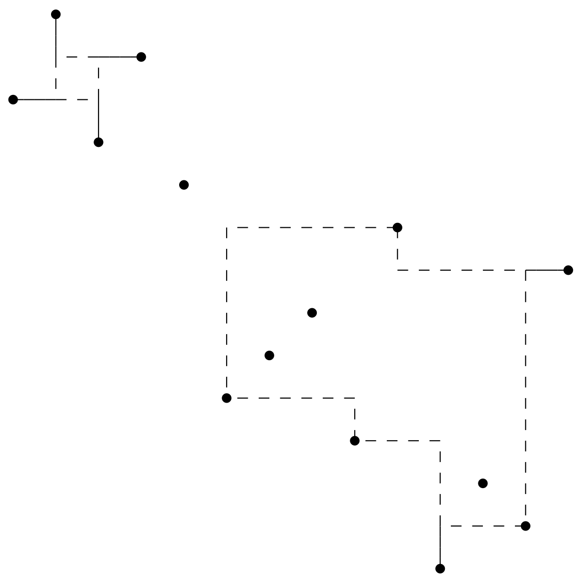

Figure 3.11: Example of an orthogeodesic convex hull of a point set.

As we can also compute the orthogeodesic convex hull for the inner points of P, we can inductively define the **orthogeodesic onion peeling** of P as the resulting point set partition. Since any point set of size 4 or greater has at least 4 outer points, there are at most $\lceil\frac{|P|}{4}\rceil$ layers. Furthermore, each layer contains at least 4 points, except for the innermost one, that can also contain just one single point.

Now we state an important result by Giacomo et al. [10]. They used this statement in combination with Lemma 3 to provide a quadratic upper bound on f_{LT4} in their paper.

Lemma 14 ([10])**.** *Every n-vertex 4-tree admits a planar L-shaped embedding in every diagonal point set of size n.*

By using this lemma in combination with orthogeodesic onion peelings, we can prove the following theorem:

Theorem 11. *Let P be a point set and let further $k_1, \ldots, k_l$ denote the number of points in the layers of the orthogeodesic onion peeling of P. Then P contains a diagonal point set of size*

$$n := \max\left\{2l-1, \left\lceil\frac{k_1}{4}\right\rceil, \ldots, \left\lceil\frac{k_l}{4}\right\rceil\right\}.$$

As a consequence, every n-vertex 4-tree admits a planar L-shaped embedding in the point set P.

Proof. We give a proof in three parts:

1. We give a proof by induction on l, that in every point set with l layers in its orthogeodesic onion peeling there exists a diagonal point set of size $2l-1$.

   Base case: $l = 1$: Every single point is a trivial diagonal point set of size 1.

   Inductive step: Consider a point set P and its l-layer orthogeodesic onion peeling. By definition, the orthogeodesic convex hull of P is exactly the first layer, and the union of the layers 2 up to l gives the inner points of P. Hence, by the induction hypothesis, the inner points of P contain a diagonal point set of length $2l-3$. Without loss of generality, we can assume that the y-coordinate in this diagonal point set be increasing as the x-coordinate increases. Let a denote the bottommost and b denote the topmost point of this diagonal point set. As a is an inner point, there exists a point $a' \in P$, that is bottom left of a. Analogously, there exists a point $b' \in P$, that is top right of b. Therefore, there exists a diagonal point set of size $2l-1$, as exemplified in Figure 3.12.

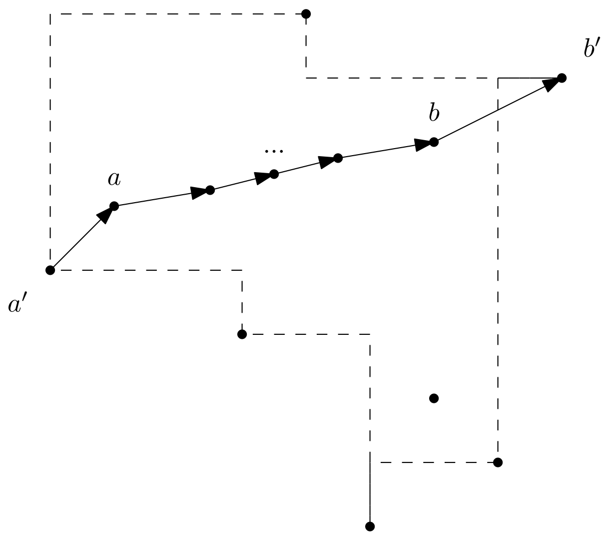

Figure 3.12: Diagonal point set of size $2l-1$.

2. Now we give a proof, that in every point set with $k \geq 4$ points in one of the onion layers there exists a diagonal point set of size $\lceil \frac{k}{4} \rceil$. Without loss of generality, we can assume that k points be in the orthogeodesic

convex hull. Furthermore, as we only consider points in the convex hull for this proof, we can assume that there are no inner points. Therefore, we can assume that the given point set is of size k and each point lies on the convex hull.

Recall that there is a unique leftmost point p_L, a rightmost point p_R, a topmost point p_T, and a bottommost point p_B in every point set, whereas these points do not need to be distinct as already stated in Chapter 3.2. All the other points have to lie in at least one of the rectangles induced by the points

- p_T and p_L,
- p_T and p_R,
- p_B and p_L, and
- p_B and p_R,

respectively. Figure 3.13 gives an illustration. Note a point might lie in multiple rectangles, as exemplified by the point q in the figure.

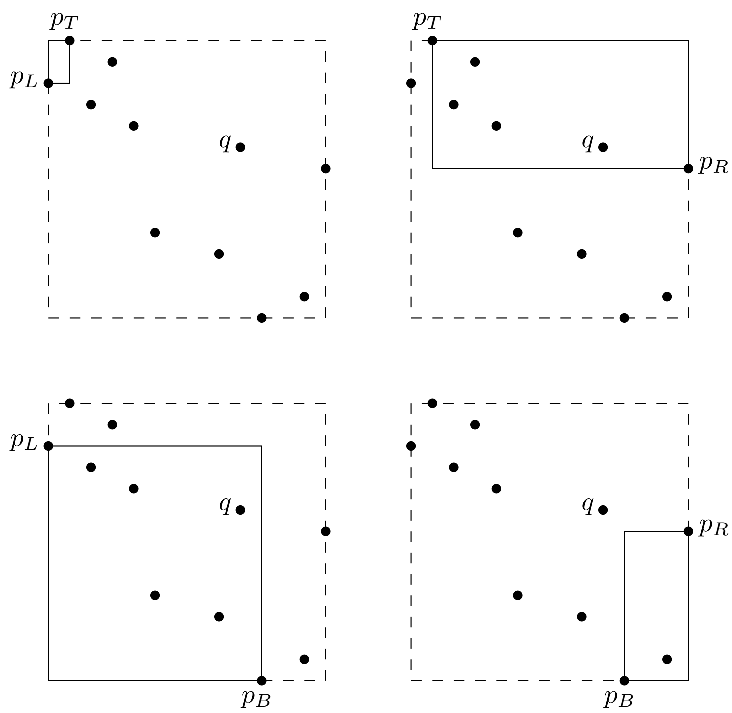

Figure 3.13: The four induced rectangles.

Without loss of generality, at least $\lceil \frac{k}{4} \rceil$ points of the orthogeodesic convex hull lie in the rectangle induced by p_L and p_B. These points give the desired diagonal point set of size $\lceil \frac{k}{4} \rceil$ as sketched in Figure 3.14, because they must have decreasing y-coordinate as the x-coordinate is increasing.

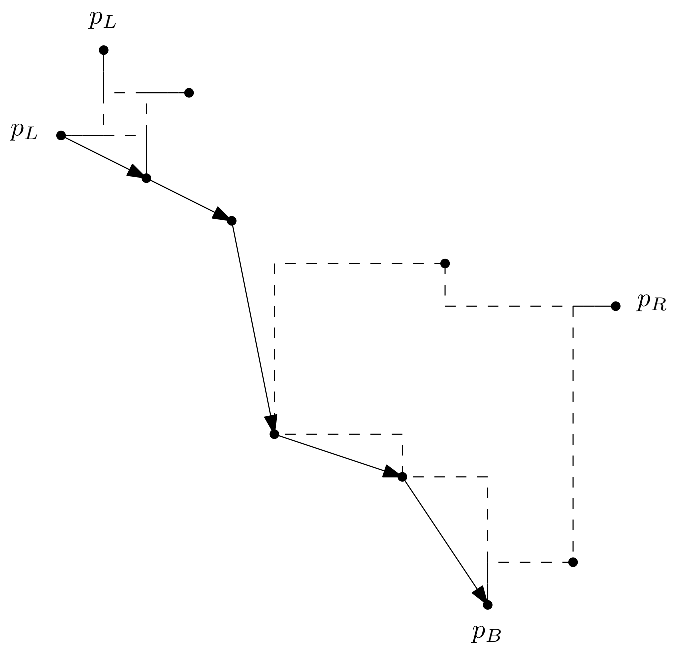

Figure 3.14: Diagonal point set of size $\lceil \frac{k}{4} \rceil$.

3. According to the previous lemma, every n-vertex 4-tree admits a planar L-shaped embedding in P.

□

As a consequence of this theorem, any point set P of size $2n^2 + 4n$ or greater has at least $\lceil \frac{n}{2} \rceil + 1$ layers or a layer with at least $4n$ points in its orthogeodesic onion peeling. Thus, we can use this result to prove a quadratic upper bound on f_{LTk} without making use of Lemma 3. According to the remark to that lemma, there exist point sets of size $(n-1)^2$ that do not contain a diagonal point set of size n, and therefore, we can not give a sub-quadratic upper bound for the general case using this approach. To give further improvements for certain trees, one could try to combine this approach with the concept of the saturation of trees.

Chapter 4

Probabilistic Approaches for Trees

Recall that every perfect k-ary tree of height greater than 1 (defined in Chapter 3.3.2) is a $k+1$-tree (defined in Chapter 1) by definition. In the beginning of this chapter, we consider the embedding of perfect binary trees and perfect ternary trees which are 3-trees and 4-trees, respectively. By making use of probability theory, we are able to provide a quasilinear upper bound on $m(n)$, where $m(n)$ denotes the smallest natural number such that every n-vertex perfect 4-ary tree admits a planar L-shaped embedding in at least half of all point sets of size $m(n)$. We prove an analogous statement for perfect 4-ary trees by using the same idea.

Later on, we apply this idea in combination with Jordan's Separator Lemma (Lemma 11) to handle arbitrary 3-trees and arbitrary 4-trees. We provide a quasilinear upper bound on $f_{LT3}^{1/2}$ and prove that $f_{LT4}^{1/2} \in \mathcal{O}(n^{\gamma_0+\varepsilon})$ holds for every $\varepsilon > 0$ where $\gamma_0 = 1.331\cdots$ is a real constant. Unfortunately we have not been able to provide a quasilinear upper bound on $f_{LT4}^{1/2}$.

4.1 Perfect Binary Trees

First, we state basic results on randomly chosen point sets, which we will use throughout the whole chapter. Note that in this thesis we only consider the (discrete) uniform distribution on the set of point sets of fixed size. Afterwards, we will state some results on perfect binary trees on randomly chosen point sets. As these ideas also work for ternary trees, we will state analogous results for perfect ternary trees in Chapter 4.2.

Recall from the Introduction (Chapter 1) that each point set of size n is isomorphic to a permutation of size n, and that the set of point sets of size n is isomorphic to the symmetric group. Therefore, exactly $n!$ different point sets of size n exist. We consider the probability space $(\Omega_n, \mathcal{P}(\Omega_n), \mathbb{P}_n)$ for every natural number n where

1. the sample space Ω_n is the set of all point sets of size n,
2. $\mathcal{P}(\Omega_n)$ is the power set of Ω_n, that is, $\mathcal{P}(\Omega_n) = \{\Omega' \mid \Omega' \subseteq \Omega_n\}$, and
3. $\mathbb{P}_n$ is the probability mass function of the discrete uniform distribution, that is, $\mathbb{P}_n(\{P\}) = \frac{1}{|\Omega_n|} = \frac{1}{n!}$ holds for every $P \in \Omega_n$.

Note that the symbols P, $\mathbb{P}$, and $\mathcal{P}$ must not be confused. We will also write $\mathbb{P}$ for $\mathbb{P}_n$ as the n is given implicitly by the point sets. Recall from probability theory that

$$\mathbb{P}(\Omega') = \mathbb{P}\left(\bigcup_{P\in\Omega'} \{P\}\right) = \sum_{P\in\Omega'} \mathbb{P}\left(\{P\}\right) = \frac{|\Omega'|}{|\Omega_n|}$$

holds for every $\Omega' \subseteq \Omega_n$.

From now on, let the symbol $\sim$ denote the equivalence relation on the set of point as introduced in Chapter 1.

Lemma 15. *Let $m, n \in \mathbb{N}$ with $m \leq n$ and let $P' \in \Omega_m$ be a fixed point set. Then*

$$\mathbb{P}\left(\{\ \{(i, y_i)\}_{i=1}^n \in \Omega_n \mid \{(i, y_i)\}_{i=1}^m \sim P'\ \}\right) = \frac{1}{m!}.$$

Proof. Let $P' = \{(i, y_i')\}_{i=1}^m \in \Omega_m$ and let

$$\Omega' := \{\ \{(i, y_i)\}_{i=1}^n \in \Omega_n \mid \{(i, y_i)\}_{i=1}^m \sim P'\ \}.$$

Recall that Ω_n is isomorphic to the symmetric group S_n. So, how many elements does the set Ω' contain? Equivalently, how many permutations $\pi \in S_n$ exist such that $\pi_i < \pi_j \Leftrightarrow y_i' < y_j'$ holds for $1 \leq i < j \leq m$? There are $\binom{n}{m}$ possibilities to partition the set $\{1, \ldots, n\}$ into two sets A and B of size m and $n - m$, respectively. Obviously, there is only one unique ordering $a_1, \ldots, a_m$ of the elements in A fulfilling $a_i < a_j \Leftrightarrow y_i' < y_j'$ for $1 \leq i < j \leq m$. Also, observe that the ordering of the elements in B does not affect the desired property. As there are $(m - n)!$ possibilities to order the elements in B, there are exactly

$$\binom{n}{m}(m-n)! = \frac{n!}{m!}$$

elements in Ω'. Since there are $n!$ elements in Ω_n, $\mathbb{P}_n(\Omega') = \frac{1}{m!}$ holds. $\square$

Recall that for a function $f : X \to Y$ and a subset $A \subseteq Y$, the set $\{x \in X \mid f(x) \in A\}$ is exactly the definition of $f^{-1}(A)$, that is, the preimage of A. Hence, we can rewrite the statement of the previous lemma as follows:

Corollary 5. *Let $f_m : \Omega_n \to \Omega_m$ be the measurable function with*

$$f_m(\{(i, y_i)\}_{i=1}^n) \sim \{(i, y_i)\}_{i=1}^m.$$

Then $\mathbb{P}_n(f_m^{-1}(P')) = \frac{1}{m!} = \mathbb{P}_m(P')$ holds for every $P' \in \Omega_m$.

In terms of measure theory, we can also write $f_m(\mathbb{P}_n) = \mathbb{P}_m$. Moreover, since

$$\mathbb{P}_n(f_m^{-1}(P')) = \mathbb{P}_n(\{P \in \Omega \mid f_m(P) = P'\}) = \frac{1}{m!}$$

holds for a fixed $P' \in \Omega_m$, the probability that $f_m(P)$ is equal to P' is $\frac{1}{m!}$ if the point set P is chosen uniformly at random. Therefore, $f_m(P)$ is also uniformly distributed on Ω_m if P is chosen uniformly at random on Ω_n.

Lemma 16. *Let n, k, h, and l be natural numbers with $n = k + l + h$. If P is chosen uniformly at random among all point sets of size n, and if $P = (A \uplus B) \uplus C$ is the (unique) partition of P with $|A| = k$, $|B| = l$, $|C| = h$, and*

$$\min_{p \in A \uplus B} y(p) > \max_{c \in C} y(c), \quad \max_{a \in A} x(a) < \min_{b \in B} x(b)$$

as sketched in Figure 4.1, then there exists a point $c_0 \in C$ with

$$x(c_0) < \min_{b \in B} x(b)$$

with probability at least

$$1 - \left(1 - \frac{|A|}{|P|}\right)^{|C|}.$$

We denote such a point $c_0 \in C$ (if it exists) as **candidate point for** B **with respect to** A.

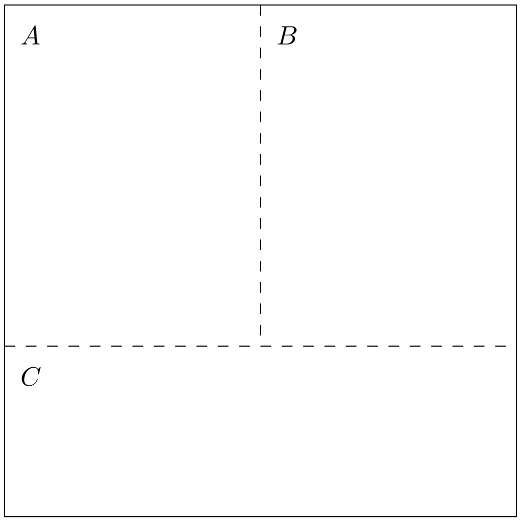

Figure 4.1: Partition of the point set.

Proof. Recall from Chapter 1 that every point set $P \in \Omega_n$ can be represented by an equivalent point set $P_\pi = \{(i, \pi_i)\}_{i=1}^n$ with π being a permutation.

Let $P = \{(i, \pi_i)\}_{i=1}^n \in \Omega_n$ be a fixed point set with π being a permutation. Consider the (well defined) sets A, B, and C that partition P as in the statement of this lemma. We can order the elements in $C = \{c_1 \ldots, c_h\}$ such that $y(c_i)$ is decreasing as i increases. This ordering is well-defined, and thus, we can decide, whether a certain c_i is a candidate point for B with respect to A.

Now, we define $E_i \subseteq \Omega$ such that $P \in E_i$ holds if and only if c_i is a candidate point in P. We remark that c_i depends on the set P as stated above. By this definition, we can write

$$\mathbb{P}(\{P \in \Omega_n \mid \exists i\colon c_i \text{ is a candidate point}\}) = \mathbb{P}(\cup_{i=1}^h E_i),$$

and since $\mathbb{P}$ is a probability measure, we have

$$\mathbb{P}(\cup_{i=1}^h E_i) = 1 - \mathbb{P}(\cap_{i=1}^h \overline{E_i}) = 1 - \prod_{i=1}^h \mathbb{P}(\overline{E_i} | \cap_{j=1}^{i-1} \overline{E_j})$$

according to the multiplication rule of probability [14,29]. Note that we make use of conditional probability measures here. Recall that the conditional probability of $Y \in \Omega_n$ given $X \in \Omega_n$ with $\mathbb{P}(X) > 0$ is defined as

$$\mathbb{P}(Y|X) = \frac{\mathbb{P}(X \cap Y)}{\mathbb{P}(X)}.$$

Consider the subset $P' := (A \uplus B) \uplus \{c_1, \ldots, c_k\}$. By definition $P' \in \Omega_m$ holds with $m = |A| + |B| + k$. Moreover, we can write $P' \sim \tilde{f}_m(P)$ where $\tilde{f}_m : \Omega_n \to \Omega_m$ is the measurable function with

$$\tilde{f}_m(\{(x_i, i)\}_{i=1}^n) \sim \{(x_i, i)\}_{i=n-m+1}^n .$$

Analogously to the previous corollary, one can show that $\tilde{f}_m(\mathbb{P}_n) = \mathbb{P}_m$ holds, and therefore, P' is also uniformly distributed among all point sets of size m if P is chosen uniformly at random among all point sets of size n.

So what is $\mathbb{P}(\overline{E_1})$? Let $m = |A| + |B| + 1$. Consider the points set $P' = \tilde{f}_m(P) \in \Omega_m$ where P is the original point set of size n. We can write $P' = \{(i, \pi'_i)\}_{i=1}^m$ with π' being a permutation of the set $\{1, \ldots, m\}$. By definition of $\tilde{f}_m$, $P' \sim (A \uplus B) \uplus \{c_1\}$. Hence, consider the partition $P' = (A' \uplus B') \uplus \{q\}$ such that q is the bottommost point, $|A'| = k$, $|B'| = l$, and

$$\max_{a \in A} x(a) < \min_{b \in B} x(b).$$

By definition, $P \in E_1$ holds if and only if $x(q) < \min_{b \in B} x(b)$. On one hand, $x(q)$ can not be greater than $|A| + 1$, because otherwise, there would exist a point $b \in B$ with $x(b) < x(q)$. On the other hand, if $x(q)$ is less or equal to $|A| + 1$, all the points in B have to be placed to the right of q since $\max_{a \in A} x(a) < \min_{b \in B} x(b)$. Thus, we have

$$\mathbb{P}(E_1) = \sum_{x=1}^{|A|+1} \mathbb{P}(\{P \in \Omega_m \mid (x, 1) \in P\}).$$

Furthermore, since $\mathbb{P}(\{P' \in \Omega_m \mid (x, y) \in P'\}) = \frac{1}{m}$ holds for every fixed $x, y \in \{1, \ldots, m\}$, we have

$$\mathbb{P}(\overline{E_1}) = 1 - \frac{|A| + 1}{|A| + |B| + 1}.$$

The evaluation of $\mathbb{P}(\overline{E_i} | \cap_{j=1}^{i-1} \overline{E_j})$ is a little bit trickier for $i > 1$ as the points $c_1, \ldots, c_{i-1}$ are already placed with $\min_{b \in B} x(b) < x(c_1), \ldots, x(c_{i-1})$, but when considering these points we can write

$$\mathbb{P}(\overline{E_i} | \cap_{j=1}^{i-1} \overline{E_j}) = \frac{|B| + (i-1)}{|A| + |B| + (i-1) + 1} = 1 - \frac{|A| + 1}{|A| + |B| + i}$$

analogously to $\mathbb{P}(\overline{E_1})$. All in all,

$$\mathbb{P}(\cup_{i=1}^h E_i) = 1 - \prod_{i=1}^h \left(1 - \frac{|A| + 1}{|A| + |B| + i}\right) \geq 1 - \left(1 - \frac{|A|}{|A| + |B| + |C|}\right)^{|C|}.$$

□

Theorem 12. *Let $h \in \mathbb{N}_0$, let $n = 2^{h+1}$, and let $T^\star_{2,h}$ be the perfect binary tree with $n-1$ vertices. Pick a point set P uniformly at random among all point sets of size $2n\log_2^2 n$. Then $T^\star_{2,h}$ admits a planar L-shaped embedding in P with probability at least $\frac{1}{2}$.*

Proof. Note that $T^\star_{2,0}$ always admits an embedding in a single point, because $T^\star_{2,0}$ is a single vertex. Let $\alpha := 2\log_2 n = 2h$. Let P be a point set chosen uniformly at random among all point sets of size $2n\log_2^2 n = \alpha n \log_2 n$. Consider the following recursive algorithm: We partition $P = (A \uplus B) \uplus C$ with $|A| = |C| = \alpha\frac{n}{2} = h \cdot n$ and $|B| = 2\alpha\frac{n}{2}\log_2\frac{n}{2}$ as stated in Lemma 16. Note that this partition is valid since $n\log_2 n = n + n\log_2\frac{n}{2}$. According to that Lemma 16, there exists a candidate point in C for B with respect to A with probability at least

$$1 - \left(1 - \frac{\alpha\frac{n}{2}}{\alpha n \log_2 n}\right)^{\alpha\frac{n}{2}} = 1 - \left(1 - \frac{1}{2\log_2 n}\right)^{\alpha\frac{n}{2}}.$$

If no candidate point exists we stop. Otherwise, we embed the root r of the tree as the candidate point, then split B into a left half B_L and a right half B_R, and continue embedding both subtrees recursively in the point sets B_L and B_R, respectively. We remark that the subtrees of the root have $\frac{n}{2}-1$ vertices each, and B_L and B_R have $\alpha\frac{n}{2}\log_2\frac{n}{2}$ points each. Therefore, we can continue recursively until a leaf is reached. If the algorithm does not fail at any time, we can connect the embedded vertices as depicted in Figure 4.2.

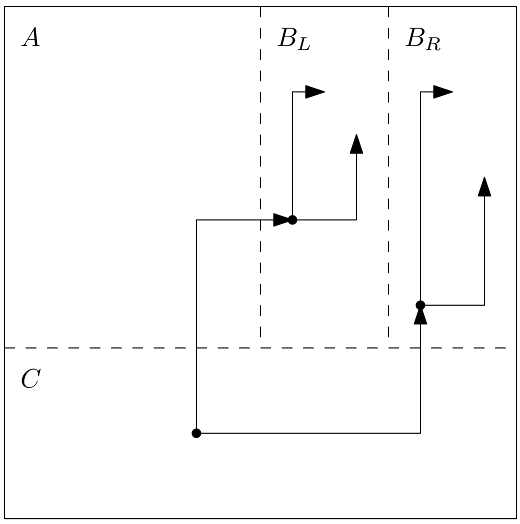

Figure 4.2: Recursive embedding.

Recall that the embedding of leaves is trivial. Thus, we only need to analyze the recursive embedding of non-trivial subtrees. As any non-trivial subtree has at least 3 vertices, $\log_2 m \geq 2$ holds.

We are now going to prove that

$$1 - \left(1 - \frac{1}{2\log_2 m}\right)^{\alpha \frac{m}{2}} \geq 1 - \frac{1}{n}$$

holds, and therefore, when considering a non-trivial subtree of size $m-1$, a candidate point exists with probability at least $1 - \frac{1}{n}$.

Since $\log_2 m > 0$ holds, we have

$$1 - \left(1 - \frac{1}{2\log_2 m}\right)^{\alpha \frac{m}{2}} = 1 - \left(\left(1 - \frac{1}{2\log_2 m}\right)^{2\log_2 m}\right)^{\alpha \frac{m}{4\log_2 m}}.$$

Now consider the function $\phi(x) := (1 - \frac{1}{x})^x$. We have

$$\phi'(x) = \left(e^{x\ln(1-\frac{1}{x})}\right)' = \phi(x)\left(x\ln(1 - \frac{1}{x})\right)' = \phi(x)\left(\ln(1 - \frac{1}{x}) + \frac{1}{x-1}\right),$$

with $e = 2.718\cdots$ being Euler's number. According to the Mean Value Theorem, for every $y \in (0,1)$ there exists $\nu \in (0,y)$, such that we can write

$$\ln(1-y) = \ln(1-y) - \ln(1-0) = (y-0)\frac{1}{\nu - 1} \geq y\frac{1}{y-1} = \frac{1}{1 - \frac{1}{y}}.$$

Hence, we have $\ln(1 - \frac{1}{x}) + \frac{1}{x-1} \geq \frac{1}{1-x} + \frac{1}{x-1} = 0$, and therefore, $\phi'(x) \geq 0$ holds for $x \geq 1$. As a consequence, ϕ is monotonically increasing on $[1, \infty)$. Furthermore, $\phi(x) \leq \frac{1}{e}$ holds on $[1, \infty)$, because $\lim_{x\to\infty} \phi(x) = \frac{1}{e}$. Thus, we have

$$1 - \left(\underbrace{\left(1 - \frac{1}{2\log_2 m}\right)^{2\log_2 m}}_{\leq \frac{1}{e}}\right)^{\alpha \frac{m}{4\log_2 m}} \geq 1 - \left(\frac{1}{e}\right)^{\alpha \frac{m}{4\log_2 m}}.$$

Now consider the function $\psi(x) := \frac{x}{\ln x}$. The only root of $\psi'(x) = \frac{\ln x - 1}{\ln^2 x}$ on the interval $(1, \infty)$ is in $x = e$, and therefore, $\psi(x) \geq e$ holds on $(1, \infty)$. Thus, we have

$$\frac{m}{4\log_2 m} = \frac{\ln 2}{4}\psi(m) \geq \frac{e\ln 2}{4} \geq \frac{\ln 2}{2}$$

for $m > 1$, and therefore

$$1 - \left(\frac{1}{e}\right)^{\alpha \frac{m}{4\log_2 m}} \geq 1 - \left(\frac{1}{e}\right)^{\frac{\alpha \ln 2}{2}} = 1 - \frac{1}{2^{\frac{\alpha}{2}}} = 1 - \frac{1}{n}.$$

Note that using this estimation the probability for the existence of a candidate point does not depend on the size of the subtree, but on the size of the original tree during the whole algorithm. All in all, the algorithm does not fail with probability at least

$$\left(1 - \frac{1}{n}\right)^{\frac{n}{2}-1} \geq \left(1 - \frac{1}{n}\right)^{\frac{n}{2}} \geq \left(\frac{1}{4}\right)^{\frac{1}{2}} = \frac{1}{2},$$

since $T^\star_{2,h}$ has exactly $n - 1 - \frac{n}{2} = \frac{n}{2} - 1$ inner vertices and $\phi(x) \geq \frac{1}{4}$ holds on $[2, \infty)$. $\square$

Note that some of the constants in the proof above can be improved easily, but an $o(n \log^2 n)$ bound on the size of the point set can not be achieved. Furthermore, one could revise the proof such that the heights of the vertices are taken into consideration. This might give a slightly better bound on the probability for success, as not all inner vertices have the same height.

We will now give an improvement of this upper bound by using another simple idea.

Theorem 13. *Let $\varepsilon \in (0, 1)$, let $h \in \mathbb{N}_0$, let $n = 2^{h+1}$, and let $T^\star_{2,h}$ be the perfect binary tree with $n - 1$ vertices. Pick a point set P uniformly at random among all point sets of size $\left\lceil \frac{C}{\varepsilon^2} n \log_2^{1+\varepsilon} n + 4n - 4 \right\rceil$ where $C = \frac{80}{e^3(\ln 2)^2} = 8.290\cdots$ is a real constant. Then $T^\star_{2,h}$ admits a planar L-shaped embedding in P with probability at least $\frac{1}{2}$.*

Proof. Let $\alpha := \frac{C \log_2 n}{\varepsilon^2} = \frac{80 \ln n}{\varepsilon^2 e^3 (\ln 2)^3}$. According to the Mean Value Theorem, for any $x \geq 1$ there exists $\xi \in (x, x + 1)$ such that

$$(x + 1)^\varepsilon - x^\varepsilon = \varepsilon \xi^{\varepsilon - 1} \geq \varepsilon \xi^{-1} \geq \varepsilon (x + 1)^{-1} \geq \varepsilon (2x)^{-1}$$

holds. As a consequence,

$$\begin{aligned}
\lceil \alpha(2m)\log_2{}^{\varepsilon}(2m) + 8m - 4\rceil &\geq \alpha 2m\log_2{}^{\varepsilon}(2m) + 8m - 4 \\
&= \alpha 2m(\log_2 m + 1)^{\varepsilon} + 8m - 4 \\
&\geq \alpha 2m\left(\log_2{}^{\varepsilon} m + \frac{\frac{\varepsilon}{2}}{\log_2 m}\right) + 8m - 4 \\
&= \alpha 2m\log_2{}^{\varepsilon} m + \alpha 2\frac{\frac{\varepsilon}{2}m}{\log_2 m} + 8m - 4 \\
&= 2\left(\alpha m\log_2{}^{\varepsilon} m + 4m - 4\right) + 2\left(\frac{\alpha\frac{\varepsilon}{2}m}{\log_2 m}\right) + 4 \\
&\geq 2\left\lceil \alpha m\log_2{}^{\varepsilon} m + 4m - 4\right\rceil + 2\left\lceil \frac{\alpha\frac{\varepsilon}{2}m}{\log_2 m}\right\rceil
\end{aligned}$$

holds for every $m = 2^k$ with $k \in \mathbb{N}$. According to this observation, we can modify the algorithm from the proof of Theorem 12: Given a perfect binary tree with $m-1$ vertices with $\log_2 m \geq 2$ and a point set P' of size $\lceil \alpha m\log_2{}^{\varepsilon} m + 4m - 4\rceil$ that is chosen uniformly at random, we can partition a subset $P'' \subseteq P'$ into $P'' = (A \uplus B) \uplus C$ such that $|A| = |C| = \left\lceil \frac{\alpha\frac{\varepsilon}{2}\frac{m}{2}}{\log_2 \frac{m}{2}}\right\rceil$ and $|B| = 2\left\lceil \alpha\frac{m}{2}\log_2{}^{\varepsilon}\frac{m}{2} + 4\frac{m}{2} - 4\right\rceil$ as stated in Lemma 16. According to that lemma, there exists a candidate point in C for B with respect to A with probability at least

$$\begin{aligned}
1 - \left(1 - \frac{|A|}{|P''|}\right)^{|C|} &\geq 1 - \left(1 - \frac{\frac{\alpha\frac{\varepsilon}{2}\frac{m}{2}}{\log_2\frac{m}{2}}}{\alpha m\log_2{}^{\varepsilon} m + 4m - 4 + 1}\right)^{\frac{\alpha\frac{\varepsilon}{2}\frac{m}{2}}{\log_2\frac{m}{2}}} \\
&\geq 1 - \left(1 - \frac{\frac{\frac{\alpha\varepsilon}{4}m}{\log_2 m}}{5\alpha m\log_2 m}\right)^{\frac{\frac{\alpha\varepsilon}{4}m}{\log_2 m}} \\
&\geq 1 - \left(1 - \frac{1}{\frac{20}{\varepsilon}\log_2{}^2 m}\right)^{\frac{\alpha\varepsilon m}{4\log_2 m}}.
\end{aligned}$$

Analogously to the proof of Theorem 12, we can write

$$1 - \left(1 - \frac{1}{\frac{20}{\varepsilon}\log_2{}^2 m}\right)^{\frac{\alpha\varepsilon m}{4\log_2 m}} \geq 1 - \left(\frac{1}{e}\right)^{\frac{\alpha\varepsilon^2 m}{80\log_2{}^3 m}}.$$

Consider the function $\psi_3(x) := \frac{x}{\ln^3 x}$. In an analogous manner to the proof of Theorem 12, we can show that $\psi_3(x) \geq \psi_3(e^3) = e^3$ holds for $x \in (1, \infty)$.

By definition of α, we have

$$\frac{\alpha\varepsilon^2 m}{80{\log_2}^3 m} \geq \frac{\alpha\varepsilon^2(\ln 2)^3 e^3}{80} = \ln n,$$

where n is the number of vertices in the original tree, and therefore,

$$1 - \left(\frac{1}{e}\right)^{\frac{\alpha\varepsilon^2 m}{80{\log_2}^3 m}} \geq 1 - \left(\frac{1}{e}\right)^{\ln n} = 1 - \frac{1}{n}.$$

Note that this bound only depends on the number of vertices in the original tree. Hence, if we apply this algorithm recursively as in the proof of Theorem 12, the algorithm does not fail with probability at least $\frac{1}{2}$. $\square$

The multiplicative constant in this bound might be slightly improved by analyzing the behavior of $(1 - \frac{1}{x})^x$.

Corollary 6. *Let $n = 2^{h+1} > 4$ with $h \in \mathbb{N}$, and let $T^\star_{2,h}$ be the perfect binary tree with $n - 1$ vertices. Pick a point set P uniformly at random among all point sets of size $\lceil Cn \log_2 n(\log_2 \log_2 n)^2 + 4n - 4\rceil$ where $C = \frac{2 \cdot 80}{e^3(\ln 3)^2} = 16.580\cdots$ is a real constant. Then $T^\star_{2,h}$ admits a planar L-shaped embedding in P with probability at least $\frac{1}{2}$.*

Proof. Let $\varepsilon := \frac{1}{\log_2 \log_2 n} \in (0, 1)$. We can write

$$\frac{1}{\varepsilon^2} n \log_2^{1+\varepsilon} n = \frac{1}{\varepsilon^2} n(\log_2 n)2^{\varepsilon \log_2 \log_2 n} = 2(\log_2 \log_2 n)^2 n \log_2 n,$$

and, according to the previous theorem, the statement holds. $\square$

4.2 Perfect Ternary Trees

First, we give a statement on randomly chosen point sets that is a generalization of Lemma 16. By using this lemma and the same ideas as in Chapter 4.1, we state analogous results for perfect ternary trees.

Lemma 17. *Pick a point set P uniformly at random among all point sets of size n. Let $P = (B_1 \uplus A \uplus B_2) \uplus C$ be the (unique) partition of P such that*

$$\min_{p \in B_1 \uplus A \uplus B_2} y(p) > \max_{c \in C} y(c), \quad \max_{b \in B_1} x(b) < \min_{a \in A} x(b), \quad \max_{a \in A} x(a) < \min_{b \in B_2} x(b)$$

hold, as depicted in Figure 4.3. Then there exists a point $c_0 \in C$ with

$$\max_{b \in B_1} x(b) < x(c_0) < \min_{b \in B_2} x(b)$$

with probability at least $1 - \left(1 - \frac{|A|}{|P|}\right)^{|C|}$.

We denote such a point $c_0 \in C$ as **candidate point for B_1 and B_2 with respect to A**.

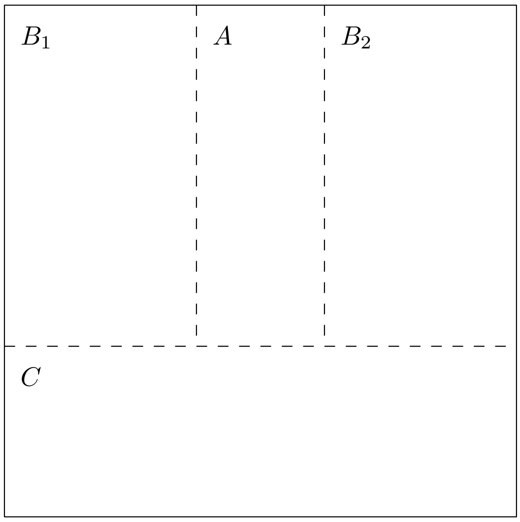

Figure 4.3: Partition of the point set.

Proof. Analogous to the proof of Lemma 16. □

Theorem 14. *Let $h \in \mathbb{N}_0$, let $n = 3^{h+1}$, and let $T^\star_{3,h}$ be the perfect ternary tree with $\frac{n-1}{2}$ vertices. Pick a point set P uniformly at random among all point sets of size $2n \log_3^2 n$. Then $T^\star_{3,h}$ admits a planar L-shaped embedding in P with probability at least $\frac{1}{2}$.*

Proof. We give a proof analogous to the proof of Theorem 12. Note that $T^\star_{3,0}$ admits an embedding in every point as it is a single vertex. Let $\alpha := 2\log_3 n$. Let P be a point set chosen uniformly at random among all point sets of size $\alpha n \log_3 n$. Consider the following recursive algorithm. We partition $P = (B_1 \uplus A \uplus B_2) \uplus C$ such that $|A| = |C| = \alpha\frac{n}{2}$, $|B_1| = 2\alpha\frac{n}{3}\log_3\frac{n}{3}$ and $|B_2| = \alpha\frac{n}{3}\log_3\frac{n}{3}$ holds as stated in Lemma 17. According to that lemma, there exists a candidate point in C for B_1 and B_2 with respect to A with probability at least

$$1 - \left(1 - \frac{\alpha\frac{n}{2}}{\alpha n \log_3 n}\right)^{\alpha\frac{n}{2}} = 1 - \left(1 - \frac{1}{2\log_3 n}\right)^{\alpha\frac{n}{2}}.$$

Note that this partition is valid, because 2 divides α and 3 divides n. If no candidate point exists we stop. Otherwise, we embed the root r of the tree as the candidate point, split B_2 into a left half B_L and a right half B_R, and continue embedding three subtrees recursively in the point sets B_1, B_L, and

B_R, respectively. Remark that the subtrees of the root have $\frac{\frac{n}{3}-1}{2}$ vertices each, and B_1, B_L, and B_R have $\alpha\frac{n}{3}\log_3\frac{n}{3}$ points each. Therefore, we can continue recursively until a leaf is reached. If the algorithm does not fail at any time, we can construct an embedding as depicted in Figure 4.4. Note the rotation by 90 degrees when embedding in B_L.

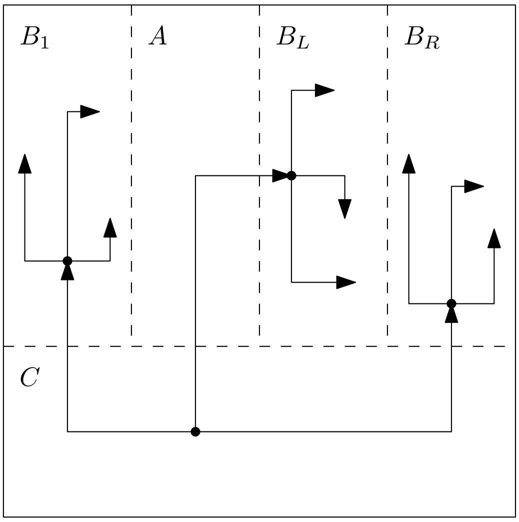

Figure 4.4: Recursive embedding.

Recall that the embedding of leaves is trivial. Thus, we only need to analyze the recursive embedding of non-trivial subtrees. As any non-trivial subtree has at least 3 vertices, $\log_3 m \geq 2$ holds.

Since $\log_3 m > 0$ holds, we have

$$1-\left(1-\frac{1}{2\log_3 m}\right)^{\alpha\frac{m}{2}} = 1-\left(\left(1-\frac{1}{2\log_3 m}\right)^{2\log_3 m}\right)^{\alpha\frac{m}{4\log_3 m}}.$$

According to the proof of Theorem 12, the function $\phi(x) := (1-\frac{1}{x})^x$ is monotonically increasing on $[1,\infty)$ and $\lim_{x\to\infty}\phi(x) = \frac{1}{e}$. Hence, we can write

$$1-\left(\underbrace{\left(1-\frac{1}{2\log_3 m}\right)^{2\log_3 m}}_{\leq\frac{1}{e}}\right)^{\alpha\frac{m}{4\log_3 m}} \geq 1-\left(\frac{1}{e}\right)^{\alpha\frac{m}{4\log_3 m}}.$$

According to the proof of Theorem 12, the function $\psi(x) := \frac{x}{\ln x}$ fulfills $\psi(x) \geq e$ on $(1,\infty)$.

Since $\frac{e}{4} = 0.679\cdots > \frac{1}{2}$,

$$\frac{m}{4\log_3 m} = \frac{\ln 3}{4}\psi(m) \geq \frac{e\ln 3}{4} \geq \frac{\ln 3}{2}$$

holds for $m > 1$, and therefore

$$1 - \left(\frac{1}{e}\right)^{\alpha\frac{m}{4\log_3 m}} \geq 1 - \left(\frac{1}{e}\right)^{\frac{\alpha\ln 3}{2}} \geq 1 - \frac{1}{3^{\frac{\alpha}{2}}} \geq 1 - \frac{1}{n}.$$

Remark that by using this estimation, the probability for the existence of a candidate point does not depend on the size of the subtree, but on the size of the original tree during the whole algorithm. All in all, the algorithm does not fail with probability at least

$$\left(1 - \frac{1}{n}\right)^{\frac{n-3}{6}} \geq \left(1 - \frac{1}{n}\right)^{\frac{n}{6}} \geq \left(\frac{1}{4}\right)^{\frac{1}{6}} = 0.793\cdots \geq \frac{1}{2},$$

since $T^{\star}_{3,h}$ has exactly $\frac{n-1}{2} - \frac{n}{3} = \frac{n-3}{6}$ inner vertices and $\phi(x) \geq \frac{1}{4}$ holds on $[2,\infty)$. □

Theorem 15. *Let $\varepsilon \in (0,1)$, let $h \in \mathbb{N}_0$, let $n = 3^{h+1}$, and let $T^{\star}_{3,h}$ be the perfect ternary tree with $\frac{n-1}{2}$ vertices. Pick a point set P uniformly at random among all point sets of size $\left\lceil \frac{C}{\varepsilon^2} n \log_3^{1+\varepsilon} n + 2.5n - 2.5 \right\rceil$ where $C = \frac{56}{e^3(\ln 3)^2} = 2.310\cdots$ is a real constant. Then $T^{\star}_{3,h}$ admits a planar L-shaped embedding in P with probability at least $\frac{1}{2}$.*

Proof. Let $\alpha := \frac{C\log_3 n}{\varepsilon^2} = \frac{56\ln n}{\varepsilon^2 e^3(\ln 3)^3}$. Analogously to the proof of Theorem 13,

$$\begin{aligned}
\left\lceil \alpha(3m)\log_3{}^{\varepsilon}(3m) + \frac{15}{2}m - \frac{5}{2} \right\rceil &\geq \alpha 3m\log_3{}^{\varepsilon}(3m) + \frac{15}{2}m - \frac{5}{2} \\
&= \alpha 3m(\log_3 m + 1)^{\varepsilon} + \frac{15}{2}m - \frac{5}{2} \\
&\geq \alpha 3m\left(\log_3{}^{\varepsilon} m + \frac{\frac{\varepsilon}{2}}{\log_3 m}\right) + \frac{15}{2}m - \frac{5}{2} \\
&= \alpha 3m\log_3{}^{\varepsilon} m + \alpha 3\frac{\frac{\varepsilon}{2}m}{\log_3 m} + \frac{15}{2}m - \frac{5}{2} \\
&= 3\left(\alpha m\log_3{}^{\varepsilon} m + \frac{5}{2}m - \frac{5}{2}\right) + 2\frac{\alpha\frac{3\varepsilon}{4}m}{\log_3 m} + 5 \\
&\geq 3\left\lceil \alpha m\log_3{}^{\varepsilon} m + \frac{5}{2}m - \frac{5}{2} \right\rceil + 2\left\lceil \frac{\alpha\frac{3\varepsilon}{4}m}{\log_3 m} \right\rceil
\end{aligned}$$

holds for every $m = 3^k$ with $k \in \mathbb{N}$. According to this observation, we can modify the algorithm from the proof of Theorem 14: Given a perfect ternary tree with $\frac{m-1}{2}$ vertices and $\log_3 m \geq 2$, and a uniformly at random chosen point set P' of size $\lceil \alpha m {\log_3}^{\varepsilon} m + \frac{5}{2}m - \frac{5}{2} \rceil$, we can partition a subset $P'' \subseteq P'$ into $P'' = (B_1 \uplus A \uplus B_2) \uplus C$ such that $|A| = |C| = \left\lceil \frac{\alpha \frac{3\varepsilon}{4} \frac{m}{3}}{\log_3 \frac{m}{3}} \right\rceil$, $|B_1| = \lceil \alpha \frac{m}{3} {\log_3}^{\varepsilon} \frac{m}{3} + \frac{5}{2}\frac{m}{3} - \frac{5}{2} \rceil$, and $|B_2| = 2 \lceil \alpha \frac{m}{3} {\log_3}^{\varepsilon} \frac{m}{3} + \frac{5}{2}\frac{m}{3} - \frac{5}{2} \rceil$ as stated in Lemma 17. According to that lemma, there exists a candidate point in C for B_1 and B_2 with respect to A with probability at least

$$\begin{aligned}
1 - \left(1 - \frac{|A|}{|P''|}\right)^{|C|} &\geq 1 - \left(1 - \frac{\frac{\alpha \frac{3\varepsilon}{4} \frac{m}{3}}{\log_3 \frac{m}{3}}}{\alpha m {\log_3}^{\varepsilon} m + 2.5m - 2.5 + 1}\right)^{\frac{\alpha \frac{3\varepsilon}{4} \frac{m}{3}}{\log_3 \frac{m}{3}}} \\
&\geq 1 - \left(1 - \frac{\frac{\frac{\alpha\varepsilon}{4} m}{\log_3 m}}{3.5 \alpha m \log_3 m}\right)^{\frac{\frac{\alpha\varepsilon}{4} m}{\log_3 m}} \\
&\geq 1 - \left(1 - \frac{1}{\frac{14}{\varepsilon} {\log_3}^2 m}\right)^{\frac{\alpha\varepsilon m}{4 \log_3 m}} \\
&\geq 1 - \left(\frac{1}{e}\right)^{\frac{\alpha\varepsilon^2 m}{56 {\log_3}^3 m}}.
\end{aligned}$$

According to the proof of Theorem 13,

$$\frac{\alpha\varepsilon^2 m}{56 {\log_3}^3 m} \geq \frac{\alpha\varepsilon^2 (\ln 3)^3 e^3}{56} = \ln n$$

holds where n is the number of vertices in the original tree, and therefore,

$$1 - \left(\frac{1}{e}\right)^{\frac{\alpha\varepsilon^2 m}{56 {\log_3}^3 m}} \geq 1 - \left(\frac{1}{e}\right)^{\ln n} = 1 - \frac{1}{n}$$

holds. Note that this bound only depends on the number of vertices in the original tree. Hence, if we apply this algorithm recursively as in the proof of Theorem 14, the algorithm does not fail with probability at least $\frac{1}{2}$. $\square$

Corollary 7. *Let $n = 3^{h+1} > 9$ with $h \in \mathbb{N}$, and let $T^{\star}_{3,h}$ be the perfect ternary tree with $\frac{n-1}{2}$ vertices. Pick a point set P uniformly at random among all point sets of size $\lceil Cn \log_3 n (\log_3 \log_3 n)^2 + 2.5n - 2.5 \rceil$ where $C = \frac{3 \cdot 56}{e^3 (\ln 3)^2} = 6.930 \cdots$ is a real constant. Then $T^{\star}_{3,h}$ admits a planar L-shaped embedding in P with probability at least $\frac{1}{2}$.*

Proof. Let $\varepsilon := \frac{1}{\log_3 \log_3 n} \in (0,1)$. We can write

$$\frac{1}{\varepsilon^2} n \log_3^{1+\varepsilon} n = \frac{1}{\varepsilon^2} n (\log_3 n) 3^{\varepsilon \log_3 \log_3 n} = 3(\log_3 \log_3 n)^2 n \log_3 n,$$

and, according to the previous theorem, the statement holds. □

4.3 Arbitrary Trees

Every approach introduced in Chapter 4.1 and Chapter 4.2 works for perfect binary trees and perfect ternary trees only. We could modify the corresponding algorithms in a way, such that trees which are similar to a perfect tree can also be processed, but in general, we need a new idea to make things work.

Recall from Jordan's Lemma (Lemma 11) that every tree admits a $\frac{1}{2}$-separator. We will combine this statement with our algorithms from Chapter 4.1 and Chapter 4.2 to handle arbitrary trees. Furthermore, since 3-trees can be processed with the same algorithms as 4-trees, we will state an algorithm in Chapter 4.3.1 that works for both 3-trees and 4-trees, and highlight the parts of the algorithm that are only needed for 4-trees. In Chapter 4.3.2, we give an analysis of the 3-trees case. Later on, we investigate the 4-trees case in Chapter 4.3.3.

4.3.1 The Algorithm

We are now going to state a recursive algorithm for finding an embedding of the vertices. Recall that in all the recursive algorithms we stated earlier this chapter, we started by placing the root as a candidate point in the first recursive layer. In the second layer, we placed the children of the root, and so on. Figure 4.5 sketches this idea.

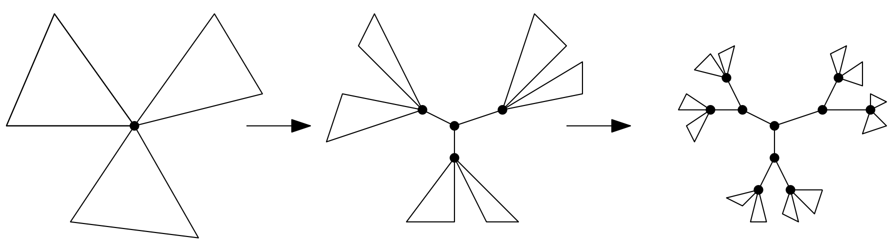

Figure 4.5: Idea of the previous algorithms.

In the following algorithm, we are going to proceed in a slightly different way. Unlike in the previous algorithms, where we embedded the roots the subtrees as candidate points, we now embed the half-separators of the subtrees as the candidate points in each recursive step. Hence, we start by placing the half-separator of the tree in the first layer. In the second layer, we determine the half-separators of the given subtrees and embed them as the candidate points. Figure 4.6 gives an illustration.

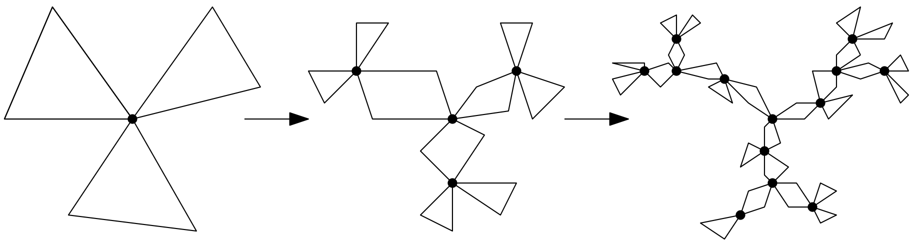

Figure 4.6: Idea of the following algorithm.

Note that we describe an algorithm for finding an embedding of the vertices. If an embedding of the vertices can be found by using this algorithm, the embedding of the edges can be obtained directly as depicted by the arrows in the figures. We remark that these arrows highlight connectability-requirements.

For the following algorithm let α be a constant natural number and let $f : \mathbb{N}_0 \to \mathbb{N}_0$ be a function that fulfills $f(0) = 0$ and

$$f(n) \geq \max_{\substack{a_1,\ldots,a_5,b,c\in\mathbb{N}_0 \\ a_1,\ldots,a_5\leq\frac{n}{2},\ b\leq\frac{n}{3},\ c\leq\frac{n}{8} \\ a_1+\ldots+a_5+b+c=n}} 4\alpha n + f(a_1) + \ldots + f(a_5) + f(b) + f(c)$$

for every $n \in \mathbb{N}$. When considering the 3-trees only, we only require f to fulfill

$$f(n) \geq \max_{\substack{n_1,\ldots,n_5\in\mathbb{N}_0 \\ n_1,\ldots,n_5\leq\frac{n}{2} \\ n_1+\ldots+n_5=n}} 4\alpha n + f(n_1) + \ldots + f(n_5)$$

for every $n \in \mathbb{N}$.

Now consider the following recursive algorithm: Given a nontrivial n-vertex tree $T = (V, E)$ with maximum degree at most 4, two distinct vertices $v_L, v_B \in V$ of degree at most 3, and a point set P of size $f(n)$, we want to embed T in P, such that v_L allows a connection from the left side and v_B allows a connection from below.

In the 3-tree case, all given vertices have degree at most 3 and v_L, v_B have degree at most 2.

According to Jordan's Lemma, there exists a vertex $v \in V$ such that each of the four induced subtrees T_1, T_2, T_3, and T_4 has at most $\frac{|V|}{2}$ vertices. Without loss of generality, we can assume that $v_B \neq v$ holds. Otherwise, we could continue mirrored. Let further n_i be the size of the subtree T_i for any $1 \leq i \leq 4$. We remark that T_i can be the empty tree, and thus, n_i can be 0.

The following cases can occur:

- Case 1: $v_L = v$: Note that v has at most three subtrees since it has degree at most 3. Without loss of generality, $v_B \in T_1$ holds and T_4 is the empty tree, as sketched in Figure 4.7.

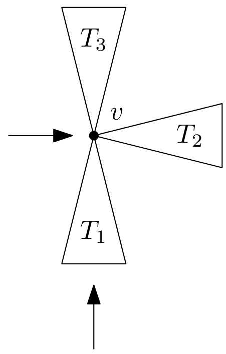

Figure 4.7: The tree in Case 1.

Analogously to the proofs in the previous subchapter, we can partition a subset $P' \subseteq P$ into

$$P' = C \uplus ((B_1 \uplus B_2) \uplus A \uplus B_3)$$

as exemplified in Figure 4.8, such that $|A| = |C| = \alpha n$ and $|B_i| = f(n_i)$.

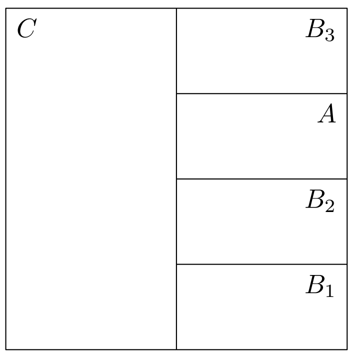

Figure 4.8: The point set partition in Case 1.

If a candidate point $c_0 \in C$ exists for $(B_1 \uplus B_2)$ and B_3 with respect to A, we can embed v in c_0, embed the subtrees T_1, T_2, and T_3 recursively in B_1, B_2, and B_3, respectively, and connect the vertices as sketched in Figure 4.9. Otherwise we stop.

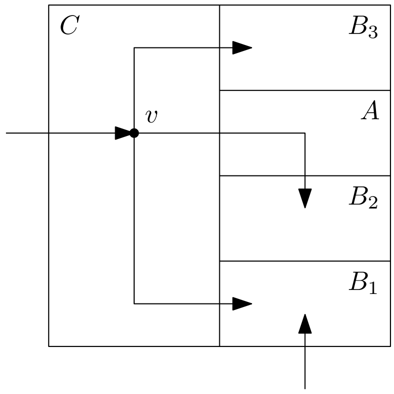

Figure 4.9: The embedding in Case 1.

The subtree T_1 needs to be embedded in a way, such that v_B is connectable from below and the root of T_1, that is, the vertex that is connected to v in the original tree, is connectable from the left side. Also, note that the subtrees T_2 and T_3 that are embedded in B_2 and B_3, respectively, only have one connectability-requirement. This is not a problem since we can choose an arbitrary leaf in those trees and require it to be connectable from a certain direction without loss of generality. Furthermore, when embedding T_2 in B_2 the root vertex is required to be connectable from above. In that case, we can continue with the mirrored or the rotated point set.

We remark that each arrow in Figure 4.9 visualizes a connectability-requirement, that is, a vertex that needs to be connectable from a certain direction. Recall that we choose some of these requirements arbitrarily, and therefore, not every arrow represents an edge.

Note that according to this construction, the number of points needed to embed T is at most

$$f(n_1) + f(n_2) + f(n_3) + 2\alpha n.$$

- Case 2: $v_L \in T_i$ and $v_B \in T_j$ with $i \neq j$: Without loss of generality, $v_B \in T_1$, $v_L \in T_2$, $n_1 \geq n_2$, and $n_3 \geq n_4$.

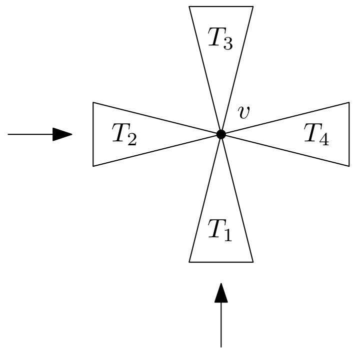

Figure 4.10: The tree in Case 2.

– Case 2a: T_4 is one of the smallest subtrees, that is, $n_4 \leq n_1, n_2, n_3$. As exemplified in Figure 4.11, we partition a subset $P' \subseteq P$ into

$$P' = ((B_1 \uplus B_2) \uplus A \uplus B_3) \uplus C \uplus D$$

such that $|A| = |C| = \alpha n$, $|B_i| = f(n_i)$, and $|D| = 2f(n_4)$.

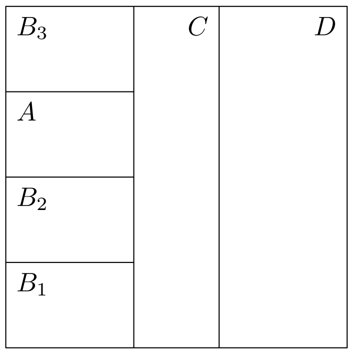

Figure 4.11: The point set partition in Case 2a.

If a candidate point $c_0 \in C$ exists for $(B_1 \uplus B_2)$ and B_3 with respect to A, we can use c_0 to partition $D = D_1 \uplus D_2$ such that

$$\min_{d \in D_1} y(d) > y(c_0) > \max_{d \in D_2} y(d).$$

Without loss of generality, $|D_2| \geq f(n_4)$ holds. Now we can embed T_4 in D_2 and continue analogously to Case 1 with the recursive embedding as sketched in Figure 4.12. We remark that if v has degree 3 or less, T_4 is the empty tree and we do not need such a part D. This observation will be crucial to provide a quasilinear bound for arbitrary 3-trees.

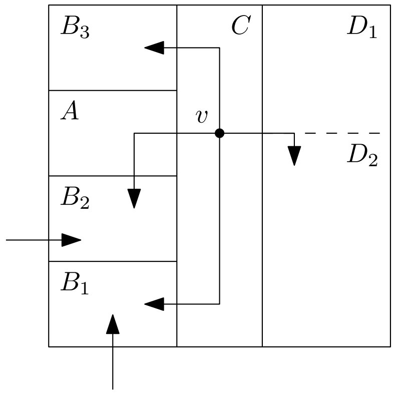

Figure 4.12: The embedding in Case 2a.

– Case 2b: T_2 is the smallest subtree, that is, $n_2 < n_1, n_3, n_4$. We proceed analogously to Case 2a, whereas we embed T_2 in D_2 and T_4 in B_2 as illustrated in Figure 4.13.

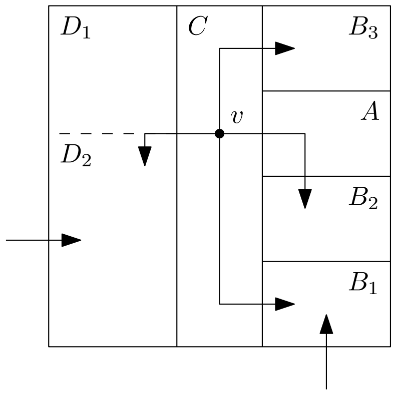

Figure 4.13: The embedding in Case 2b.

Note that according to the constructions in Case 2a and Case 2b, the number of points needed to embed T is at most

$$f(n_1) + f(n_2) + f(n_3) + 2f(n_4) + 2\alpha n,$$

where $n_1 \geq n_2 \geq n_3 \geq n_4$ are the numbers of vertices in the subtrees of v in T. We remark that $n_1, n_2 \leq \frac{n}{2}$, $n_3 \leq \frac{n}{3}$, and $n_4 \leq \frac{n}{4}$ hold.

When considering 3-trees only, we can assume that $n_4 = 0$ holds, and therefore, at most

$$f(n_1) + f(n_2) + f(n_3) + 2\alpha n,$$

points are needed to embed T.

- Case 3: $v_L, v_B \in T_i$. Without loss of generality, $v_L, v_B \in T_1$ holds and $n_2 \geq n_3 \geq n_4$. Since T is a tree, there exists a unique path P_L from v to v_L and a unique path P_B from v to v_B.
  - Case 3a: $v_B \in P_L$ or $v_L \in P_B$. Without loss of generality, $v_B \in P_L$. Let T'_1, T'_2, T'_3, and T'_4 be the subtrees of v_B in T_1, where $v_L \in T'_1$ and where v is either adjacent to a vertex in T'_3 or adjacent to v_B. Figure 4.14 gives an illustration. Let n'_i be the size of the subtree T'_i for any $1 \leq i \leq 4$.

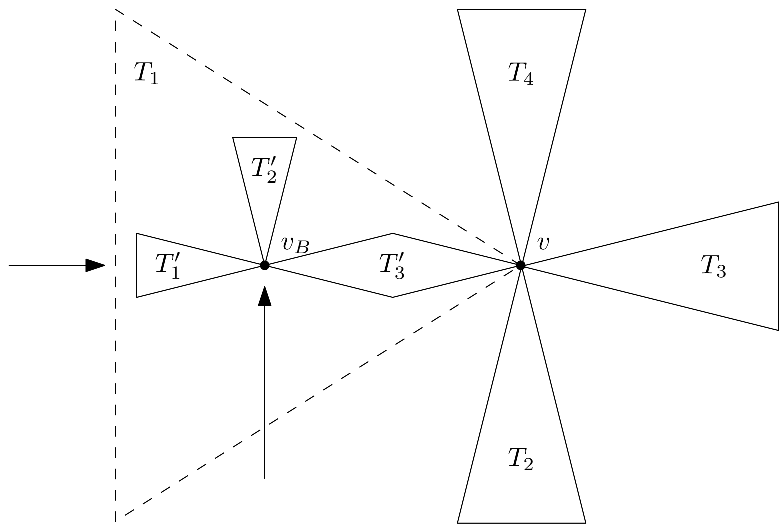

Figure 4.14: The tree in Case 3a.

As exemplified in Figure 4.15, we can partition a subset $P' \subseteq P$ into

$$P' = ((B_1 \uplus B_2) \uplus A \uplus B_3) \uplus C \uplus D$$

such that $|A| = |C| = \alpha n$, $|B_2| = f(n_2)$, $|B_3| = f(n_3)$, and $|D| = 2f(n_4)$. Furthermore, we partition

$$B_1 = (B'_1 \uplus A' \uplus (B'_2 \uplus B'_3)) \uplus C'$$

such that $|A'| = |C'| = \alpha n$, $|B'_i| = f(n'_i)$.

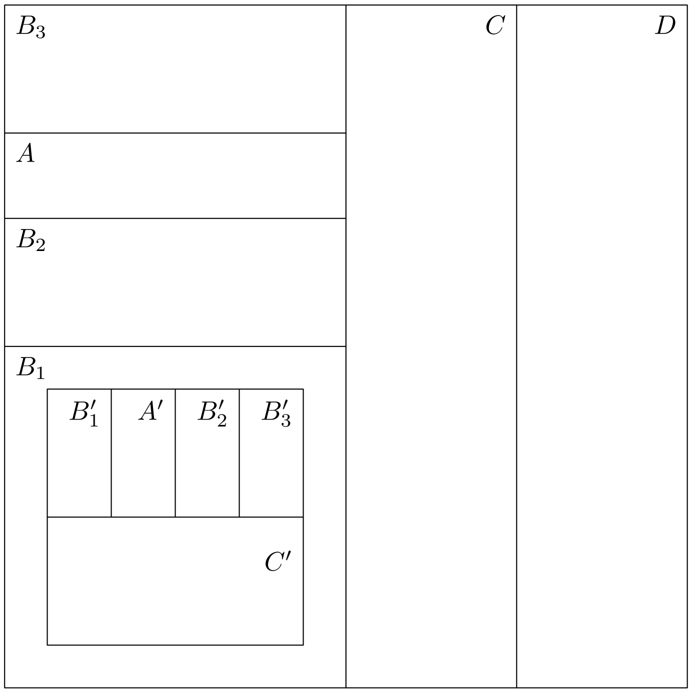

Figure 4.15: The point set partition in Case 3a.

If a candidate point $c_0 \in C$ exists for $(B_1 \uplus B_2)$ and B_3 with respect to A, and if a candidate point $c'_0 \in C'$ exists for B'_1 and $(B'_2 \uplus B'_3)$ with respect to A', we can embed v in c_0 and v_B in c'_0. As in Case 2, we can split $D = D_1 \uplus D_2$, and $|D_2| \geq f(n_4)$ holds without loss of generality. We continue with the recursive embedding of T_2 in B_2, T_3 in B_3, T_4 in D_2, and T'_i in B'_i as illustrated in Figure 4.16. In the case that v and v_B are adjacent, that is, if B'_3 is the empty tree, the vertices v and v_B can be connected directly.

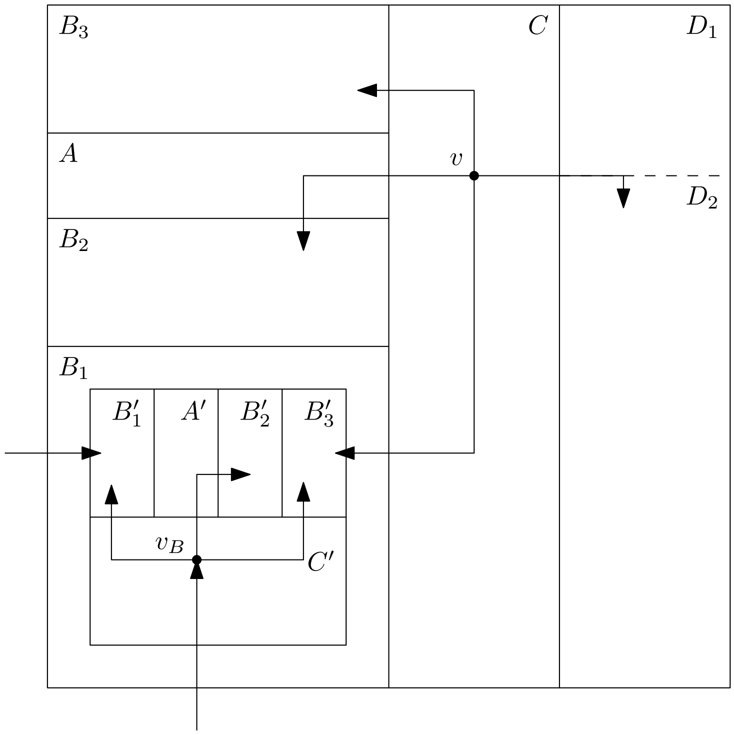

Figure 4.16: The embedding in Case 3a.

Note that according to this construction, the number of points needed to embed T is at most

$$f(n_1') + f(n_2') + f(n_3') + f(n_2) + f(n_3) + 2f(n_4) + 4\alpha n.$$

Since $\frac{n}{2} \geq n_1 \geq n_2 \geq n_3 \geq n_4$ holds, we have $n_2, n_3 \leq \frac{n}{2}$, and $n_4 \leq \frac{n}{3}$, and furthermore, $n_i' \leq \frac{n}{2}$.
When considering 3-trees only, we can assume that $n_4 = 0$ and $n_3' = 0$ hold, and therefore, at most

$$f(n_1') + f(n_2') + f(n_2) + f(n_3) + 4\alpha n$$

points are needed to embed to T.

– Case 3b: $v_B \notin P_L$ and $v_L \notin P_B$. There exists a unique vertex w in T_1 with subtrees T_1', T_2', T_3', and T_4', where $v_L \in T_1'$, $v_B \in T_2'$, and where v is either adjacent to a vertex in T_3' or adjacent to w. Figure 4.17 gives an illustration. Let n_i' be the size of the subtree T_i' for any $1 \leq i \leq 4$.

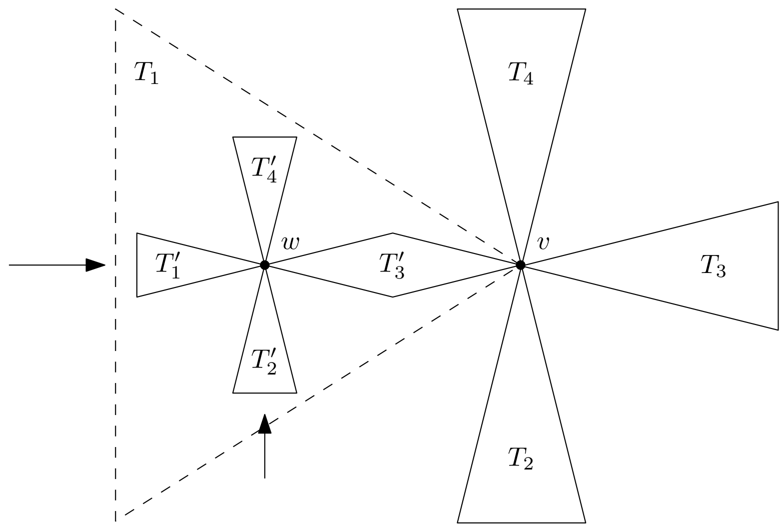

Figure 4.17: The tree in Case 3b.

* Case 3b1: The subtree T_4' is one of the smallest subtrees, that is, $n_4' \leq n_1', n_2', n_3'$. We proceed analogously to Case 3a, whereas we partition

$$B_1 = D' \uplus C' \uplus (B_1' \uplus A' \uplus (B_2' \uplus B_3'))$$

such that $|A'| = |C'| = \alpha n$, $|B_i'| = f(n_i')$, and $|D'| = 2f(n_4')$. If a candidate point $c_0' \in C'$ exists, we can split $D' = D_1' \uplus D_2'$ such that

$$\max_{d' \in D_1'} x(d') < x(c_0') < \min_{d' \in D_2'} x(d')$$

holds. Without loss of generality, $|D_2'| \geq f(n_4')$, and therefore, we can embed T_4' in D_2' as illustrated in Figure 4.18. In the case that v and w are adjacent, that is, if T_3' is the empty tree, the vertices v and w can be connected directly.

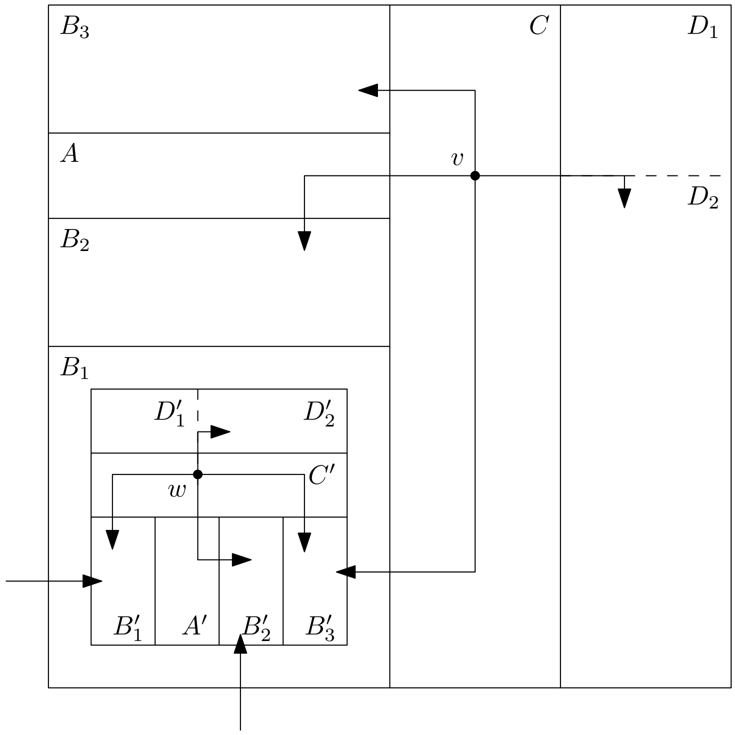

Figure 4.18: The embedding in Case 3b1.

- Case 3b2: The subtree T'_3 is one of the smallest subtrees, that is, $n'_3 \leq n'_1, n'_2$ and $n'_3 < n'_4$. We proceed analogously to Case 3b1, whereas we partition B_1 such that $|B'_1| = f(n'_4)$, $|B'_2| = f(n'_1)$, $|B'_3| = f(n'_2)$ and $|D'| = 2f(n'_3)$ hold, and embed T'_4 in B'_1, T'_1 in B'_2, T'_2 in B'_3, and T'_4 in D'_2 as illustrated in Figure 4.19.

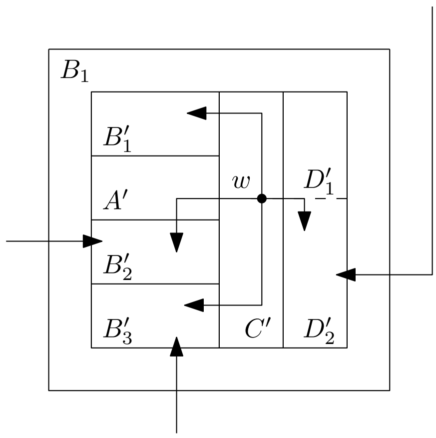

Figure 4.19: The embedding in Case 3b2.

* Case 3b3: The subtree T'_2 is one of the smallest subtrees, that is, $n'_2 \leq n'_1$ and $n'_2 < n'_3, n'_4$. We proceed analogously to Case 3b1, whereas we partition B_1 such that $|B'_1| = f(n'_3)$, $|B'_2| = f(n'_4)$, $|B'_3| = f(n'_1)$ and $|D'| = 2f(n'_2)$ hold, and embed T'_3 in B'_1, T'_4 in B'_2, T'_1 in B'_3, and T'_2 in D'_2 as illustrated in Figure 4.20.

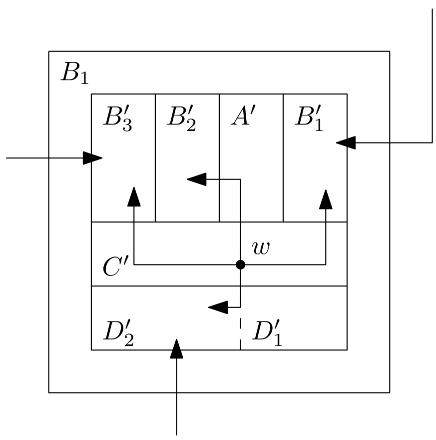

Figure 4.20: The embedding in Case 3b3.

* Case 3b4: The subtree T'_1 is the smallest subtree, that is, $n'_1 < n'_2, n'_3, n'_4$. We proceed analogously to Case 3b1, whereas we partition B_1 such that $|B'_1| = f(n'_2)$, $|B'_2| = f(n'_3)$, $|B'_3| = f(n'_4)$ and $|D'| = 2f(n'_1)$ hold, and embed T'_2 in B'_1, T'_3 in B'_2, T'_4 in B'_3, and T'_1 in D'_2 as illustrated in Figure 4.21.

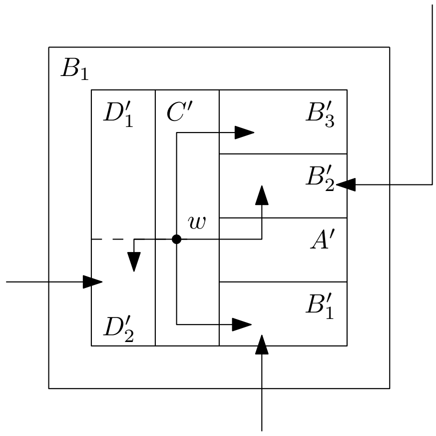

Figure 4.21: The embedding in Case 3b4.

Note that according to the constructions in Case 3b1, Case 3b2, Case 3b3, and Case 3b4, the number of points needed to embed T is at most

$$f(n'_1) + f(n'_2) + f(n'_3) + 2f(n'_4) + f(n_2) + f(n_3) + 2f(n_4) + 4\alpha n,$$

where $n'_1 \geq n'_2 \geq n'_3 \geq n'_4$ are the numbers of vertices in the subtrees of w in T_1. Since $\frac{n}{2} \geq n_2 \geq n_3 \geq n_4$ hold, we also have $n_1, n_2, n_3 \leq \frac{n}{2}$, and $n_4 \leq \frac{n}{3}$. Furthermore, we have $n'_1 \leq \frac{n}{2}$, $n'_2 \leq \frac{n}{4}$, $n'_3 \leq \frac{n}{6}$ and $n'_4 \leq \frac{n}{8}$.
When considering 3-trees only, we can assume that $n_4 = 0$ and $n'_4 = 0$ hold, and therefore, at most

$$f(n'_1) + f(n'_2) + f(n_2) + f(n_3) + 4\alpha n$$

points are needed to embed to T.

4.3.2 Analysis of the 3-Trees Case

Consider the algorithm stated in Chapter 4.3.1. As the analysis of that algorithm is a bit tough when considering 4-trees, we give an analysis for the 3-trees version first.

Theorem 16.

$$f_{LT3}^{1/2}(n) \leq 8n \log_2^2 n + \mathcal{O}(n \log_2 n).$$

Proof. Let $T = (V, E)$ be an n-vertex 3-tree. Let $\alpha := \lceil 2 \log_2 n \rceil$. Consider the function $f(x) = (4\alpha + 5)(x + x \log_2 x)$ on the nonnegative real numbers.

Note that $\lim_{x\to 0^+} x\ln x = 0$ holds, and therefore, we can write $f(0) = 0$. We observe that

$$f(x) = (4\alpha + 5)x + 2f(\frac{x}{2})$$

holds for any $x \geq 0$ and that f is a convex function. Analogously to the proof of Theorem 9,

$$f(m) \geq \max_{\substack{m_1,\ldots,m_5\in\mathbb{N}_0\\ m_1,\ldots,m_5\leq\frac{m}{2}\\ m_1+\ldots+m_5=m}} (4\alpha+5)m + f(m_1) + \ldots + f(m_5)$$

holds for every $m \in \mathbb{N}$. The function $g\colon \mathbb{N}_0 \to \mathbb{N}_0$ with $g(x) := \lceil f(x) \rceil$ satisfies

$$g(m) \geq \max_{\substack{m_1,\ldots,m_5\in\mathbb{N}_0\\ m_1,\ldots,m_5\leq\frac{m}{2}\\ m_1+\ldots+m_5=m}} 4\alpha m + g(m_1) + \ldots + g(m_5)$$

for every $m \geq 1$, and therefore, g fulfills all properties that are required by the algorithm. Hence, we only need to bound the probability of failure for the algorithm to prove this theorem. Analogously to the proof of Theorem 12, we can show that a candidate point exists with probability at least

$$1 - \left(1 - \frac{\alpha m}{4\alpha m \log_2 m + m}\right)^{\alpha m} \geq 1 - \left(1 - \frac{1}{5\log_2 m}\right)^{\alpha m} \geq 1 - \left(\frac{1}{e}\right)^{\frac{\alpha m}{5\log_2 m}},$$

and since $\frac{x}{5\log_2 x} \geq \frac{e\ln 2}{5} \geq \frac{\ln 2}{2}$ holds for any $x > 1$,

$$1 - \left(\frac{1}{e}\right)^{\alpha\frac{m}{5\log_2 m}} \geq 1 - \left(\frac{1}{2}\right)^{\frac{\alpha}{2}} = 1 - \frac{1}{n}.$$

As stated in the proof Theorem 12, the probability of the algorithm not failing is at least $\frac{1}{2}$. □

Analogously to the proof of Theorem 13 and the Corollary corresponding to that theorem, we can also show that $f_{LT3}^{1/2}(n) = \mathcal{O}(n\log_2 n(\log_2\log_2 n)^2)$ holds. Thus, we did not work out the multiplicative constant properly in the proof above.

4.3.3 Analysis of the 4-Trees Case

We state two lemmas that will help us obtain a sub-quadratic bound on the function f_{LT4}.

Lemma 18. *Let S be the minimal set of corners of the convex set*

$$C = \left\{ (a_1, \ldots, a_5, b, c) \in \mathbb{R}^7 \;\middle|\; \begin{array}{l} a_1, \ldots, a_5, b, c \geq 0, \\ a_1, \ldots, a_5 \leq \frac{M}{2},\; b \leq \frac{M}{3},\; c \leq \frac{M}{8}, \\ a_1 + \ldots + a_5 + b + c = M \end{array} \right\},$$

and let e_i denote the i-th unit vector in $\mathbb{R}^7$. Then every $s \in S$ can either be written as

- $s = M(\frac{1}{2}e_i + \frac{1}{2}e_j)$ *with $i, j \in \{1, \ldots 5\}$ and $i \neq j$,*
- $s = M(\frac{1}{2}e_i + \frac{1}{6}e_j + \frac{1}{3}e_6)$ *with $i, j \in \{1, \ldots 5\}$ and $i \neq j$,*
- $s = M(\frac{1}{2}e_i + \frac{3}{8}e_j + \frac{1}{8}e_7)$ *with $i, j \in \{1, \ldots 5\}$ and $i \neq j$, or*
- $s = M(\frac{1}{2}e_i + \frac{1}{24}e_j + \frac{1}{3}e_6 + \frac{1}{8}e_7)$ *with $i, j \in \{1, \ldots 5\}$ and $i \neq j$.*

Proof. Without loss of generality, $M = 1$. Let $s = (a_1, \ldots, a_5, b, c) \in S$ be a corner of C. Without loss of generality, $a_1 \geq \ldots \geq a_5$.

- $a_2 \neq 0$ must hold, because $a_1 + b + c \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{8} = \frac{23}{24} < 1$.
- $a_3 = 0$ holds. Assume to the contrary that $a_3 > 0$. Then $a_2 < \frac{1}{2}$ must hold, because otherwise $a_1 + a_2 + a_3 > 1$ would give a contraction. We can define $\varepsilon := \min\{a_3, \frac{1}{2} - a_2\} > 0$ and write $s = \frac{1}{2}u + \frac{1}{2}v$ with

  $$u = (a_1, a_2 + \varepsilon, a_3 - \varepsilon, a_4, \ldots), v = (a_1, a_2 - \varepsilon, a_3 + \varepsilon, a_4, \ldots) \in C \backslash \{s\}.$$

  This is a contradiction since s is a corner of C.
- $a_1 = \frac{1}{2}$ holds. Assume to the contrary that $a_1 < \frac{1}{2}$ holds. Since $\varepsilon := \min\{\frac{1}{2} - a_1, a_2\} > 0$, we can show that $a_1 = \frac{1}{2}$ holds analogously to the previous point.

As a consequence, $a_1 = \frac{1}{2}$, $a_2 > 0$, $a_3 = a_4 = a_5 = 0$ must hold. Furthermore, since $a_3 + b + c = \frac{1}{2}$ must hold, we can write $a_3 = \frac{1}{2} - b - c$. As $0 \leq b + c \leq \frac{11}{24}$ holds, $0 \leq a_3 \leq \frac{1}{2}$ is always fulfilled, and therefore, we only need to consider the corners of the set

$$C' := \left\{ (b, c) \in \mathbb{R}^2 \;\middle|\; 0 \leq b \leq \frac{1}{3},\; 0 \leq c \leq \frac{1}{8} \right\}.$$

It is obvious that $\{(0,0), (\frac{1}{3}, 0), (0, \frac{1}{8}), (\frac{1}{3}, \frac{1}{8})\}$ is the minimal set of corners of C'. The statement follows directly since $a_3 = \frac{1}{2} - b - c$. $\square$

Lemma 19. *Let $a, b, c, d \geq 0$ with $a+b+c+d = 1$, let $\varepsilon \geq 0$, and let $\gamma \geq 1$ with $a^\gamma + b^\gamma + 2c^\gamma + 2d^\gamma + \varepsilon \leq 1$. Furthermore, let f_γ be the function on the nonnegative real numbers with $f_\gamma(x) = x^\gamma$. Then*

$$f_\gamma(ax) + f_\gamma(bx) + 2f_\gamma(cx) + 2f_\gamma(dx) + \varepsilon f_\gamma(x) \leq f_\gamma(x)$$

holds for any $x \geq 0$.

Proof.

$$f_\gamma(ax)+f_\gamma(bx)+2f_\gamma(cx)+2f_\gamma(dx)+\varepsilon f_\gamma(x) = \underbrace{(a^\gamma + b^\gamma + 2c^\gamma + 2d^\gamma + \varepsilon)}_{\leq 1} f_\gamma(x).$$

□

From now on, let $\gamma_0 := 1.3319\cdots$ be the unique solution of the equation

$$\left(\frac{1}{2}\right)^\gamma + \left(\frac{1}{24}\right)^\gamma + 2\left(\frac{1}{3}\right)^\gamma + 2\left(\frac{1}{8}\right)^\gamma = 1.$$

Corollary 8. *Let $\gamma > \gamma_0$ and let $\delta_\gamma = \frac{1}{24^{\gamma_0}} - \frac{1}{24^\gamma}$. Then the function f_γ - defined as in the previous lemma - fulfills*

1. $f_\gamma(x) \geq f_\gamma(\frac{1}{2}x) + f_\gamma(\frac{1}{2}x) + \delta_\gamma x$,
2. $f_\gamma(x) \geq f_\gamma(\frac{1}{2}x) + f_\gamma(\frac{1}{6}x) + 2f_\gamma(\frac{1}{3}x) + \delta_\gamma x$,
3. $f_\gamma(x) \geq f_\gamma(\frac{1}{2}x) + f_\gamma(\frac{3}{8}x) + 2f_\gamma(\frac{1}{8}x) + \delta_\gamma x$, *and*
4. $f_\gamma(x) \geq f_\gamma(\frac{1}{2}x) + f_\gamma(\frac{1}{24}x) + 2f_\gamma(\frac{1}{3}x) + 2f_\gamma(\frac{1}{8}x) + \delta_\gamma x$

for any $x \geq 1$.

Proof. As the function $\phi_c(x) = \frac{1}{c^x}$ is strictly monotonically decreasing on the positive real numbers, we have $\phi_c(\gamma) + \delta_\gamma \leq \phi_c(\gamma_0)$ for any $c \in \{\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{24}\}$. Furthermore, since $\gamma > \gamma_0 > 1.3 > 1$ holds,

1. $\left(\frac{1}{2}\right)^\gamma + \left(\frac{1}{2}\right)^\gamma + \delta_\gamma \leq \left(\frac{1}{2}\right)^{\gamma_0} + \left(\frac{1}{2}\right)^{\gamma_0} \leq \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^1 = 1$,
2. $\left(\frac{1}{2}\right)^\gamma + \left(\frac{1}{6}\right)^\gamma + 2\left(\frac{1}{3}\right)^\gamma + \delta_\gamma \leq \left(\frac{1}{2}\right)^{1.3} + \left(\frac{1}{6}\right)^{1.3} + 2\left(\frac{1}{3}\right)^{1.3} = 0.98\cdots \leq 1$,
3. $\left(\frac{1}{2}\right)^\gamma + \left(\frac{3}{8}\right)^\gamma + 2\left(\frac{1}{8}\right)^\gamma + \delta_\gamma \leq \left(\frac{1}{2}\right)^{1.3} + \left(\frac{3}{8}\right)^{1.3} + 2\left(\frac{1}{8}\right)^{1.3} = 0.81\cdots \leq 1$, and
4. $\left(\frac{1}{2}\right)^\gamma + \left(\frac{1}{24}\right)^\gamma + 2\left(\frac{1}{3}\right)^\gamma + 2\left(\frac{1}{8}\right)^\gamma + \delta_\gamma \leq \left(\frac{1}{2}\right)^{\gamma_0} + \left(\frac{1}{24}\right)^{\gamma_0} + 2\left(\frac{1}{3}\right)^{\gamma_0} + 2\left(\frac{1}{8}\right)^{\gamma_0} = 1$.

By the previous lemma and since $f_\gamma(x) \geq x$ holds for $x \geq 1$, our statement follows directly. □

Theorem 17. *Let $\varepsilon > 0$.*

$$f_{LT4}^{1/2}(n) = \mathcal{O}(n^{\gamma_0+\varepsilon}).$$

Proof. Let $T = (V, E)$ be an n-vertex 4-tree, let $\alpha := \lceil \log_2 n \rceil$, let $\gamma := \gamma_0 + \frac{\varepsilon}{2}$, and let f_γ be defined as in Lemma 19. Furthermore, let δ_γ be defined as in the previous corollary. Consider the function f on the nonnegative real numbers with $f(x) := \frac{4\alpha+7}{\delta_\gamma} f_\gamma(x)$. We observe that f fulfills

1. $f(x) \geq f(\frac{1}{2}x) + f(\frac{1}{2}x) + (4\alpha + 7)x$,
2. $f(x) \geq f(\frac{1}{2}x) + f(\frac{1}{6}x) + 2f(\frac{1}{3}x) + (4\alpha + 7)x$,
3. $f(x) \geq f(\frac{1}{2}x) + f(\frac{3}{8}x) + 2f(\frac{1}{8}x) + (4\alpha + 7)x$, and
4. $f(x) \geq f(\frac{1}{2}x) + f(\frac{1}{24}x) + 2f(\frac{1}{3}x) + 2f(\frac{1}{8}x) + (4\alpha + 7)x$

for any $x \geq 1$.

According to Lemma 18 and the Maximum Principle (Lemma 9), we have

$$f(n) \geq \max_{\substack{a_1,\ldots,a_5,b,c \in \mathbb{N}_0 \\ a_1,\ldots,a_5 \leq \frac{n}{2},\ b \leq \frac{n}{3},\ c \leq \frac{n}{8} \\ a_1+\ldots+a_5+b+c=n}} (4\alpha + 7)n + f(a_1) + \ldots + f(a_5) + f(b) + f(c).$$

As we consider $n \geq 1$, the function $g \colon \mathbb{N}_0 \to \mathbb{N}_0$ with $g(x) := \lceil f(x) \rceil$ fulfills

$$g(n) \geq \max_{\substack{a_1,\ldots,a_5,b,c \in \mathbb{N}_0 \\ a_1,\ldots,a_5 \leq \frac{n}{2},\ b \leq \frac{n}{3},\ c \leq \frac{n}{8} \\ a_1+\ldots+a_5+b+c=n}} 4\alpha n + g(a_1) + \ldots + g(a_5) + g(b) + g(c),$$

and therefore, g fulfills all properties that are required by the algorithm stated in Chapter 4.3.1. Analogously to the proof of Theorem 16, that is, the 3-tree version of this theorem, we can show that the algorithm does not fail with probability at least $\frac{1}{2}$. Thus, we have $f, g \in \mathcal{O}(n^\gamma \log_2 n) \subset \mathcal{O}(n^{\gamma_0+\varepsilon})$. $\square$

Chapter 5

Caterpillars

In this chapter, we consider embeddings of caterpillars. Giacomo et al. have already shown that any n-vertex 3-caterpillar admits a planar L-shaped embedding in any point set of size n [10]. Hence, we only consider 4-caterpillars in this thesis.

Let $C = (V, E)$ be a 4-caterpillar. Recall from Chapter 1 that every caterpillar has a spine, that is, a path of inner vertices. Let $v_1, \ldots, v_k$ be the vertices along the spine. Consider the sequence of degrees $a_1, \ldots, a_k$ along the spine with $a_i = \deg(v_i) \in \{2, 3, 4\}$ for any $1 \leq i \leq k$. We denote this sequence as the **sequence** of the caterpillar C and use the notation $(a_1, \ldots, a_k)$. Also, we make use of the following notations:

- If C admits an embedding in a point set P, then we say that $(a_1, \ldots, a_k)$ admits an embedding in P.
- If C admits an embedding in a point set P in a way, such that a leaf v, that is adjacent to v_k, is embedded as the rightmost point and v is connected vertically, then we say that $(a_1, \ldots, a_{\vec{k}})$ admits an embedding in P.
- If C admits an embedding in a point set P in a way, such that a leaf v, that is adjacent to v_1, is embedded as the leftmost point and v is connected horizontally, then we say that $({}^{\rightarrow}a_1, \ldots, a_k)$ admits an embedding in P.
- If C admits an embedding in a point set P in a way, such that
  - a leaf u, that is adjacent to v_1, is embedded as the leftmost point,
  - a leaf v, that is adjacent to v_k, is embedded as the rightmost point,
  - u is connected horizontally, and

– v is connected vertically,

then we say that $({}^{\rightarrow}a_1, \ldots, a_k^{\rightarrow})$ admits an embedding in P.

The following fundamental lemma on caterpillars will be used several times in this chapter. Note that the lemma holds for every type of embeddings, that is, L-shaped or orthogeodesic, and planar or nonplanar embeddings.

Lemma 20. *Let $(a_1, \ldots, a_k)$ be the sequence of a caterpillar and let further $l \in \{1, \ldots, k-1\}$.*

1. *If $(a_1, \ldots, a_l^{\rightarrow})$ and $({}^{\rightarrow}a_{l+1}, \ldots, a_k)$ both admit an embedding in any point set of size $m_1 + 1$ and $m_2 + 1$, respectively, then $(a_1, \ldots, a_k)$ also admits an embedding in any point set of size $m_1 + m_2$.*

2. *If $({}^{\rightarrow}a_1, \ldots, a_l^{\rightarrow})$ and $({}^{\rightarrow}a_{l+1}, \ldots, a_k)$ both admit an embedding in any point set of size $m_1 + 1$ and $m_2 + 1$, respectively, then $({}^{\rightarrow}a_1, \ldots, a_k)$ also admits an embedding in any point set of size $m_1 + m_2$.*

3. *If $(a_1, \ldots, a_l^{\rightarrow})$ and $({}^{\rightarrow}a_{l+1}, \ldots, a_k^{\rightarrow})$ both admit an embedding in any point set of size $m_1 + 1$ and $m_2 + 1$, respectively, then $(a_1, \ldots, a_k^{\rightarrow})$ also admits an embedding in any point set of size $m_1 + m_2$.*

4. *If $({}^{\rightarrow}a_1, \ldots, a_l^{\rightarrow})$ and $({}^{\rightarrow}a_{l+1}, \ldots, a_k^{\rightarrow})$ both admit an embedding in any point set of size $m_1 + 1$ and $m_2 + 1$, respectively, then $({}^{\rightarrow}a_1, \ldots, a_k^{\rightarrow})$ also admits an embedding in any point set of size $m_1 + m_2$.*

Proof. We only give a proof of statement 4 as the other statements can be proven analogously. Let $P = \{p_1, \ldots, p_m\}$ be a point set of size $m = m_1 + m_2$ with $p_i = (i, y_i)$ for any $1 \leq i \leq m$. Let $P_1 := \{p_1, \ldots, p_{m_1+1}\}$, let C_1 be the caterpillar corresponding to the sequence $(a_1, \ldots, a_l)$, and let $v_1, \ldots, v_l$ be the inner vertices of C_1 with $\deg v_i = a_i$. By our assumption, we can embed $({}^{\rightarrow}a_1, \ldots, a_l^{\rightarrow})$ in P_1. Let $p \in P_1$ be the point where v_l is embedded.

Now let $P_2 := \{p, p_{m_1+1}, \ldots, p_m\}$. By our assumption, we can embed $({}^{\rightarrow}a_{l+1}, \ldots, a_k^{\rightarrow})$ in P_2. By construction, we can remove the point p_{m_1+1} from the first embedding and merge both embeddings as exemplified in Figure 5.1. This gives an embedding of $({}^{\rightarrow}a_1, \ldots, a_k^{\rightarrow})$. □

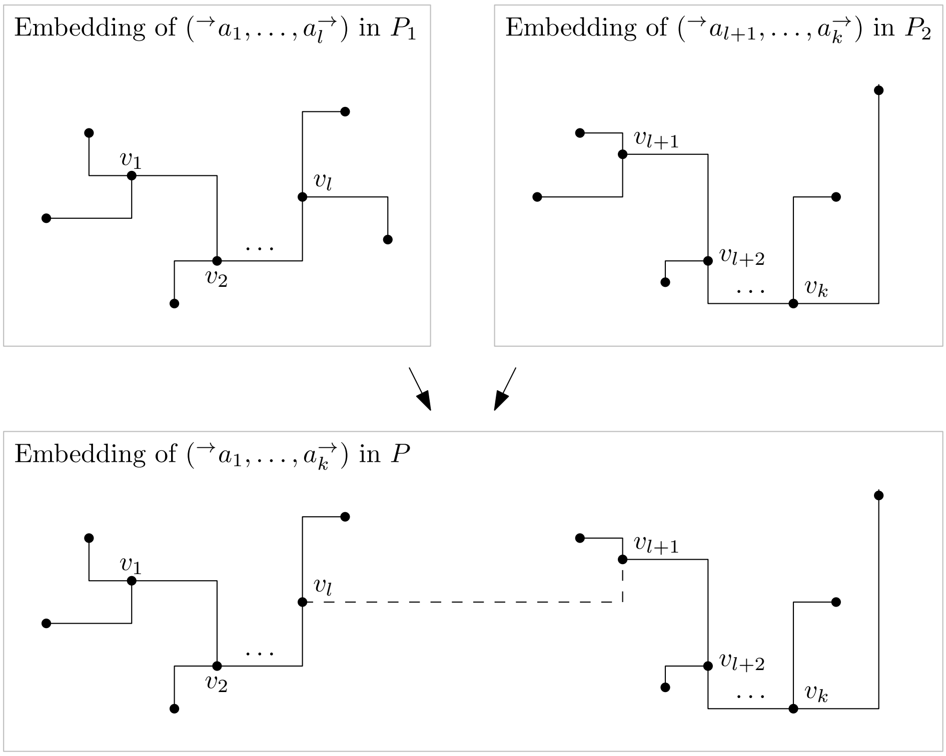

Figure 5.1: Construction in statement 4.

We denote this divide-and-conquer-strategy as stated in the lemma above as a **splitting** of a caterpillar. Note that we can apply this splitting-idea recursively to embed a caterpillar in a point set. The parts produced by a recursive splitting are called **pieces** of the caterpillar.

5.1 Planar Orthogeodesic Embeddings

We give an improvement of the $1.5n + \mathcal{O}(1)$ bound on $f_{OC4}(n)$ given by Giacomo et al. [10].

Theorem 18.

$$f_{OC4} \leq \frac{4}{3}n + \mathcal{O}(1).$$

Proof. Let $(a_1, \ldots, a_k)$ be the sequence of a caterpillar C with at least 3 vertices, and let P be a point set. Because of the more restrictive definition, any embedding of $(^{\rightarrow}a_1, \ldots, a_k^{\rightarrow})$ in P is also an embedding of C. According to Lemma 20, we can split the caterpillar into pieces of length 1 and embed each of them separately. Hence, we only need to consider the embeddings of $(^{\rightarrow}2^{\rightarrow})$, $(^{\rightarrow}3^{\rightarrow})$, and $(^{\rightarrow}4^{\rightarrow})$. The following statements hold:

1. $(^{\rightarrow}2^{\rightarrow})$ admits a planar orthogeodesic embedding in any point set P of size 3 as illustrated in Figure 5.2. Note that the embedding of this caterpillar piece is trivial, because it is a path.

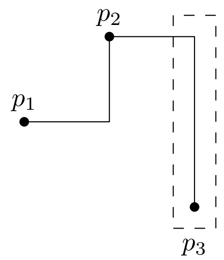

Figure 5.2: Planar orthogeodesic embedding of $(^{\rightarrow}2^{\rightarrow})$ in P.

   As no points are left unused, we need only 1 point per vertex to embed the caterpillar piece $(^{\rightarrow}2^{\rightarrow})$.

2. Now we prove that $(^{\rightarrow}3^{\rightarrow})$ admits a planar orthogeodesic embedding in any point set P of size 4. Let $P = \{p_1, \ldots, p_4\}$ with $x(p_1) < \ldots < x(p_4)$. Without loss of generality, $y(p_3) > y(p_1)$. There are two cases:

   - Case 1: $y(p_2) > y(p_1)$: We can embed v_1 in p_3 and draw the edges as illustrated in Figure 5.3.

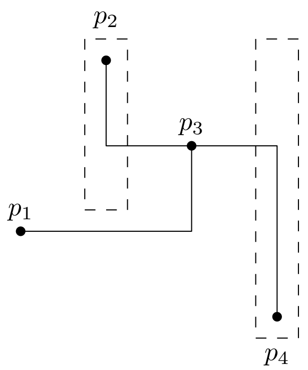

Figure 5.3: Planar orthogeodesic embedding of $(^{\rightarrow}3^{\rightarrow})$ in P - Case 1.

   - Case 2: $y(p_2) < y(p_1)$: We can embed v_1 in p_3 and draw the edges as illustrated in Figure 5.4.

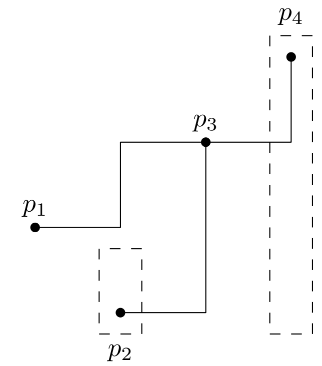

Figure 5.4: Planar orthogeodesic embedding of $(^{\rightarrow}3^{\rightarrow})$ in P - Case 2.

As no points are left unused in any case, we need only 1 point per vertex to embed the caterpillar piece $(^{\rightarrow}3^{\rightarrow})$.

3. Now we prove that $(^{\rightarrow}4^{\rightarrow})$ admits a planar orthogeodesic embedding in any point set P of size 6. Let $P = \{p_1, \ldots, p_6\}$ with $x(p_1) < \ldots < x(p_6)$. We can partition $P = \{p_1\} \uplus (P^+ \uplus P^-) \uplus \{p_6\}$ such that

$$\min_{p \in P^+} y(p) > y(p_1) > \max_{p \in P^-} y(p).$$

Without loss of generality, $|P^+| \geq |P^-|$ holds.

- Case 1: $|P^+| \geq 3$: Let $P^+ = \{p_L, p_T, p_C\}$ with $x(p_L) < x(p_T)$, $x(p_L) < x(p_C)$ and $y(p_T) > y(p_C)$. We can embed C as illustrated in Figure 5.5.

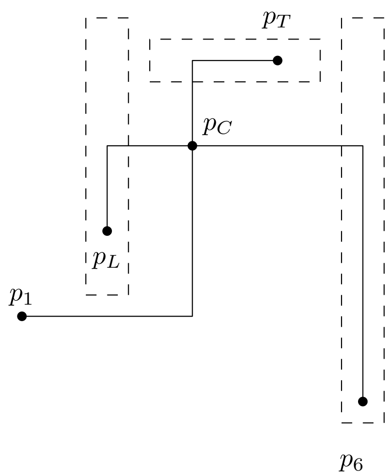

Figure 5.5: Planar orthogeodesic embedding of $(^{\rightarrow}4^{\rightarrow})$ in P - Case 1.

- Case 2: $|P^+| = |P^-| = 2$: Without loss of generality, $p_2 \in P^-$. Let $P^+ = \{p_T, p_C\}$ with $y(p_T) > y(p_C)$. Since $x(p_2) < x(p_T)$, (p_C) holds, we can embed C as illustrated in Figure 5.6.

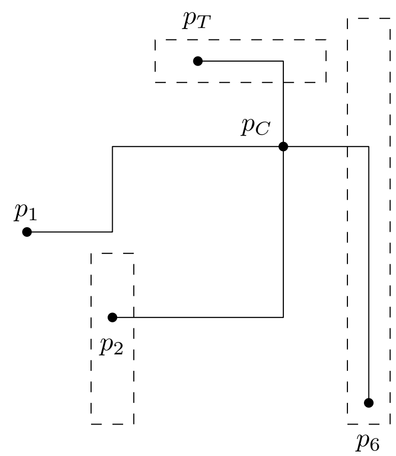

Figure 5.6: Planar orthogeodesic embedding of $(^{\rightarrow}4^{\rightarrow})$ in P - Case 2.

We remark that the leftmost point, the rightmost point, and the vertices embedded in those points appear in multiple embeddings, and therefore, we do not consider them for the points-per-vertex-rate. Since at most 1 point is left unused in Case 1 and Case 2, we only need $\frac{6-2}{5-2} = \frac{4}{3}$ points per vertex to embed the caterpillar piece $(^{\rightarrow}4^{\rightarrow})$.

All in all, at most $\frac{4}{3}$ points per vertex are needed to embed any 4-caterpillar. □

We conjecture that the multiplicative constant $\frac{4}{3}$ can be further improved by using embedding techniques analogous to those stated in the following subchapter.

5.2 Planar L-Shaped Embeddings

We give an improvement of the $3n+\mathcal{O}(1)$ bound on $f_{LC4}(n)$ given by Giacomo et al. [10].

Theorem 19.

$$f_{LC4} \leq \frac{5}{3}n + \mathcal{O}(1).$$

Proof. We give a proof analogous to the proof of Theorem 18.

1. $(^{\rightarrow}2^{\rightarrow})$ admits a planar L-shaped embedding in any point set P of size 3 as illustrated in item 1 of the proof of Theorem 18. Only 1 point per vertex is needed to embed such a caterpillar piece.

2. Now we prove that $(^{\rightarrow}3^{\rightarrow})$ admits a planar L-shaped embedding in any point set P of size 5. Let $P = \{p_1, \ldots, p_5\}$ with $x(p_1) < \ldots < x(p_5)$. We can partition $P = \{p_1\} \uplus (P^+ \uplus P^-) \uplus \{p_5\}$ such that

$$\min_{p \in P^+} y(p) > y(p_1) > \max_{p \in P^-} y(p).$$

Without loss of generality, $|P^+| \geq 2$ holds. We can embed $(^{\rightarrow}3^{\rightarrow})$ as stated in Case 1 of item 2 in the proof of Theorem 18. Only $\frac{5-2}{4-2} = \frac{3}{2}$ points per vertex are needed to embed such a caterpillar piece.

3. Now we prove that $(^{\rightarrow}4^{\rightarrow})$ admits a planar L-shaped embedding in any point set P of size 7. Let $P = \{p_1, \ldots, p_7\}$ with $x(p_1) < \ldots < x(p_7)$. We can partition $P = \{p_1\} \uplus (P^+ \uplus P^-) \uplus \{p_7\}$ such that

$$\min_{p \in P^+} y(p) > y(p_1) > \max_{p \in P^-} y(p).$$

Without loss of generality, $|P^+| \geq 3$ holds. We can embed $(^{\rightarrow}4^{\rightarrow})$ as stated in Case 1 of item 3 in the proof of Theorem 18. Only $\frac{7-2}{5-2} = \frac{5}{3}$ points per vertex are needed to embed such a caterpillar piece.

All in all, at most $\frac{5}{3}$ points per vertex are needed to embed any 4-caterpillar. $\square$

Recall from the proof of Theorem 18 that the reason for the "-2" in the numerator and the denominator are the leftmost point, the rightmost point, and the vertices embedded in the points, which are embedded multiple times.

An approach to achieve a better bound, is to split the caterpillar into larger pieces. The drawback of this approach is that the analysis of larger pieces becomes more difficult as there are many more cases that can occur. Therefore, we will give a computer-assisted proof.

5.2.1 A Computer-Assisted Proof

The following statements can be validated by a computer in a few minutes of CPU time. When doing these calculations, we have observed that most of the cases can be handled easily, and therefore, we conjecture that a reasonable short proof can be done by a smart case distinction.

Lemma 21 (Computer-Assisted)**.** *The following statements hold:*

1. $({}^{\rightarrow}3, 2^{\rightarrow})$ *admits a planar L-shaped embedding in any point set P of size 5. Only $\frac{5-2}{5-2} = 1$ point per vertex is needed to embed such a caterpillar piece.*

2. $({}^{\rightarrow}3, 3^{\rightarrow})$ *admits a planar L-shaped embedding in any point set P of size 6. Only $\frac{6-2}{6-2} = 1$ point per vertex is needed to embed such a caterpillar piece.*

3. $({}^{\rightarrow}3, 4^{\rightarrow})$ *admits a planar L-shaped embedding in any point set P of size 9. Only $\frac{9-2}{7-2} = \frac{7}{5}$ points per vertex are needed to embed such a caterpillar piece.*

4. $({}^{\rightarrow}4, 2^{\rightarrow})$ *admits a planar L-shaped embedding in any point set P of size 7. Only $\frac{7-2}{6-2} = \frac{5}{4}$ points per vertex are needed to embed such a caterpillar piece.*

5. $({}^{\rightarrow}4, 3^{\rightarrow})$ *admits a planar L-shaped embedding in any point set P of size 8. Only $\frac{8-2}{7-2} = \frac{6}{5}$ points per vertex are needed to embed such a caterpillar piece.*

6. $({}^{\rightarrow}4, 4^{\rightarrow})$ *admits a planar L-shaped embedding in any point set P of size 10. Only $\frac{10-2}{8-2} = \frac{4}{3}$ points per vertex are needed to embed such a caterpillar piece.*

It is obvious that the embedding of $({}^{\rightarrow}3, 2^{\rightarrow})$ can be done analogously to the proof of Theorem 18, but since the embedding of $({}^{\rightarrow}4, 4^{\rightarrow})$ is pretty tough, we did not write a proof for any of the statements above. We remark that $f_{LC4}(n) \leq \frac{8}{5}n + \mathcal{O}(1)$ follows directly from the lemma above, but we will not give a proof for this statement as we give a much stronger result in Theorem 20.

Lemma 22. *The following statements hold:*

1. $({}^{\rightarrow}3, 4, 2^{\rightarrow})$ *admits a planar L-shaped embedding in any point set P of size 10. Only $\frac{10-2}{8-2} = \frac{4}{3}$ points per vertex are needed to embed such a caterpillar piece.*

2. $({}^{\rightarrow}3, 4, 3^{\rightarrow})$ *admits a planar L-shaped embedding in any point set P of size 11. Only $\frac{11-2}{9-2} = \frac{9}{7} = 1.285\cdots$ points per vertex are needed to embed such a caterpillar piece.*

3. $(^{\rightarrow}3, 4, 4^{\rightarrow})$ *admits a planar L-shaped embedding in any point set P of size 13. Only $\frac{13-2}{10-2} = \frac{11}{8} = 1.375$ points per vertex are needed to embed such a caterpillar piece.*

Proof. According to Lemma 20, we can embed $(^{\rightarrow}3, 4, a^{\rightarrow})$ by embedding the pieces $(^{\rightarrow}3^{\rightarrow})$ and $(^{\rightarrow}4, a^{\rightarrow})$ separately.

1. By embedding $(^{\rightarrow}3^{\rightarrow})$ and $(^{\rightarrow}4, 2^{\rightarrow})$ separately, we can achieve a points-per-vertex rate of at most
$$\frac{3+5}{2+4} = \frac{8}{6} = \frac{4}{3}.$$
2. By embedding $(^{\rightarrow}3^{\rightarrow})$ and $(^{\rightarrow}4, 3^{\rightarrow})$ separately, we can achieve a points-per-vertex rate of at most
$$\frac{3+6}{2+5} = \frac{9}{7}.$$
3. By embedding $(^{\rightarrow}3^{\rightarrow})$ and $(^{\rightarrow}4, 4^{\rightarrow})$ separately, we can achieve a points-per-vertex rate of at most
$$\frac{3+8}{2+6} = \frac{11}{8}.$$

□

We remark that $f_{LC4}(n) \leq \frac{11}{8}n + \mathcal{O}(1)$ follows directly from the lemma above, but we will not give a proof for this statement as we give a much stronger result in Theorem 20.

Lemma 23. *The following statements hold:*

1. $(^{\rightarrow}3, 4, 4, 2^{\rightarrow})$ *admits a planar L-shaped embedding in any point set P of size 14. Only $\frac{14-2}{11-2} = \frac{4}{3}$ points per vertex are needed to embed such a caterpillar piece.*
2. $(^{\rightarrow}3, 4, 4, 3^{\rightarrow})$ *admits a planar L-shaped embedding in any point set P of size 15. Only $\frac{15-2}{12-2} = \frac{13}{10}$ points per vertex are needed to embed such a caterpillar piece.*
3. $(^{\rightarrow}3, 4, 4, 4^{\rightarrow})$ *admits a planar L-shaped embedding in any point set P of size 17. Only $\frac{17-2}{13-2} = \frac{15}{11} = 1.363\cdots$ points per vertex are needed to embed such a caterpillar piece.*

Proof. According to Lemma 20, we can embed $({}^{\rightarrow}3,4,4,a^{\rightarrow})$ by embedding the pieces $({}^{\rightarrow}3,4^{\rightarrow})$ and $({}^{\rightarrow}4,a^{\rightarrow})$ separately.

1. By embedding $({}^{\rightarrow}3,4^{\rightarrow})$ and $({}^{\rightarrow}4,2^{\rightarrow})$ separately, we can achieve a points-per-vertex rate of at most
$$\frac{7+5}{5+4} = \frac{12}{9} = \frac{4}{3}.$$

2. By embedding $({}^{\rightarrow}3,4^{\rightarrow})$ and $({}^{\rightarrow}4,3^{\rightarrow})$ separately, we can achieve a points-per-vertex rate of at most
$$\frac{7+6}{5+5} = \frac{13}{10}.$$

3. By embedding $({}^{\rightarrow}3,4^{\rightarrow})$ and $({}^{\rightarrow}4,4^{\rightarrow})$ separately, we can achieve a points-per-vertex rate of at most
$$\frac{7+8}{5+6} = \frac{15}{11}.$$

□

We remark that $f_{LC4}(n) \leq \frac{15}{11}n + \mathcal{O}(1)$ follows directly from the lemma above, but we will not give a proof for this statement as we give a much stronger result in Theorem 20. Recall that the the multiplicative constants $\frac{3}{2}$, $\frac{7}{5}$, $\frac{11}{8}$, and $\frac{15}{11}$ were achieved by applying the same idea on caterpillar pieces of length 1, 2, 3, and 4, respectively. One can observe from this integer sequence, that caterpillar pieces of length m seem to lead to a points-per-vertex rate of at most $\frac{4m-1}{3m-1}$. In the following theorem we will give a proof for this observation.

Theorem 20. *Let $\varepsilon > 0$ be fixed.*
$$f_{LC4} \leq \left(\frac{4}{3} + \varepsilon\right) n + \mathcal{O}(1).$$

Proof. Let $M \geq 6$ be an even natural number such that $\frac{4}{3} + \varepsilon > \frac{4M-1}{3M-1}$ holds.

Consider the caterpillar sequence $({}^{\rightarrow}a_1,\ldots,a_k^{\rightarrow})$. If $k \leq M$ holds, only a constant (with respect to n) number of points is needed to embed this caterpillar sequence. We remark that this constant is increasing as ε decreases. Otherwise the following cases can occur:

- Case 1: $a_1 \neq 3$ or $a_2 \neq 4$ (or both). According to Lemma 21, we can embed $({}^{\rightarrow}a_1, a_2{}^{\rightarrow})$ with a points-per-vertex rate of at most $\frac{4}{3}$ and continue by embedding $({}^{\rightarrow}a_3, \ldots, a_k{}^{\rightarrow})$.

- Case 2: $a_1 = 3$, $a_2 = a_3 = \ldots = a_{l-1} = 4$, and $a_l \neq 4$ for $3 \leq l \leq M$ odd. Recall that $({}^{\rightarrow}3, 4, 3^{\rightarrow})$ can be embedded with a points-per-vertex rate of $\frac{9}{7}$ and that $({}^{\rightarrow}3, 4, 2^{\rightarrow})$ can be embedded with a points-per-vertex rate of $\frac{4}{3}$. Hence, let $l \geq 5$ and let $h = l - 5$. We can embed $({}^{\rightarrow}a_1, a_2, a_3{}^{\rightarrow})$, $({}^{\rightarrow}a_4, a_5{}^{\rightarrow})$, $({}^{\rightarrow}a_6, a_7{}^{\rightarrow})$, …, $({}^{\rightarrow}a_{l-1}, a_l{}^{\rightarrow})$ separately with a points-per-vertex rate of

  $$\frac{11 + 4h + 6}{8 + 3h + 5} < \frac{4}{3}$$

  if $a_l = 2$, and a points-per-vertex rate of

  $$\frac{11 + 4h + 5}{8 + 3h + 4} = \frac{4}{3}$$

  if $a_l = 3$, and then continue by embedding $({}^{\rightarrow}a_{l+1}, \ldots, a_k{}^{\rightarrow})$.

- Case 3: $a_1 = 3$, $a_2 = a_3 = \ldots = a_{l-1} = 4$, and $a_l \neq 4$ for $4 \leq l \leq M$ even. Recall that $({}^{\rightarrow}3, 4, 4, 3^{\rightarrow})$ can be embedded with a points-per-vertex rate of $\frac{13}{10}$ and that $({}^{\rightarrow}3, 4, 4, 2^{\rightarrow})$ can be embedded with a points-per-vertex rate of $\frac{4}{3}$. Hence, let $l \geq 6$ and let $h = l - 6$. We can embed $({}^{\rightarrow}a_1, a_2, a_3, a_4{}^{\rightarrow})$, $({}^{\rightarrow}a_5, a_6{}^{\rightarrow})$, $({}^{\rightarrow}a_7, a_8{}^{\rightarrow})$, …, $({}^{\rightarrow}a_{l-1}, a_l{}^{\rightarrow})$ separately with a points-per-vertex rate of

  $$\frac{15 + 4h + 6}{11 + 3h + 5} < \frac{4}{3}$$

  if $a_l = 2$, and a points-per-vertex rate of

  $$\frac{15 + 4h + 5}{11 + 3h + 4} = \frac{4}{3}$$

  if $a_l = 3$, and then continue by embedding $({}^{\rightarrow}a_{l+1}, \ldots, a_k{}^{\rightarrow})$.

- Case 4: $a_1 = 3$ and $a_2 = \ldots = a_M = 4$. In this case, we can embed the caterpillar pieces $({}^{\rightarrow}a_1, a_2{}^{\rightarrow})$, …, $({}^{\rightarrow}a_{M-1}, a_M{}^{\rightarrow})$ separately with a points-per-vertex rate of $\frac{7+4(M-2)}{5+3(M-2)} = \frac{4M-1}{3M-1}$ and continue by embedding $({}^{\rightarrow}a_{M+1}, \ldots, a_k{}^{\rightarrow})$.

$\square$

5.3 Nonplanar L-Shaped Embeddings

We give an improvement of the $n+1$ bound on f_{NC4} given by Giacomo et al. [10].

Lemma 24 (Computer Assisted)**.** *The following statements hold:*

1. $(^{\rightarrow}2^{\rightarrow})$ *admits an L-shaped embedding in any point set P of size 3. Only* 1 *point per vertex is needed to embed such a caterpillar piece.*

2. $(^{\rightarrow}3^{\rightarrow})$ *admits an L-shaped embedding in any point set P of size 4. Only* 1 *point per vertex is needed to embed such a caterpillar piece.*

3. $(^{\rightarrow}2, 4^{\rightarrow})$ *admits an L-shaped embedding in any point set P of size 6. Only* 1 *point per vertex is needed to embed such a caterpillar piece.*

4. $(^{\rightarrow}3, 4^{\rightarrow})$ *admits an L-shaped embedding in any point set P of size 7. Only* 1 *point per vertex is needed to embed such a caterpillar piece.*

5. $(^{\rightarrow}4, 4^{\rightarrow})$ *admits an L-shaped embedding in any point set P of size 8. Only* 1 *point per vertex is needed to embed such a caterpillar piece.*

Lemma 25.

$(4^{\rightarrow})$ *admits an L-shaped embedding in any point set P of size 5. Only* 1 *point per vertex is needed to embed such a caterpillar piece.*

Proof. Observe that the y-coordinate of the rightmost point does not affect the result. Figure 5.7 states 12 cases of the $24 = 4!$ possible cases - the other 12 cases are vertical mirrors of the ones in the figure. □

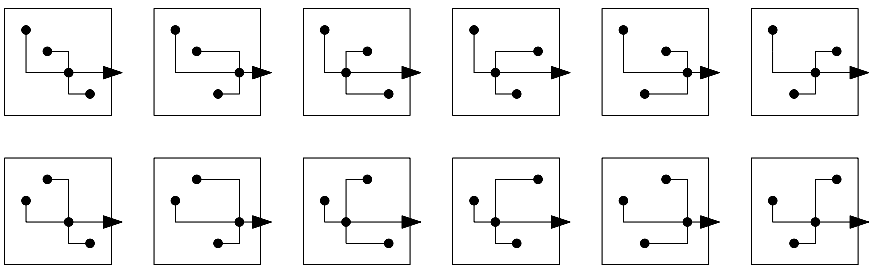

Figure 5.7: Enumeration of all cases.

Now, we combine the statements of these two lemmas and give a proof of the following theorem:

Theorem 21.

$$f_{NC4}(n) = n.$$

Proof. To embed $(a_1, \ldots, a_k{}^{\rightarrow})$ recursively we can continue as follows:

- If $a_k = 2$ or $a_k = 3$ holds, we embed $(a_1, \ldots, a_{k-1}{}^{\rightarrow})$ and $({}^{\rightarrow}a_k^{\rightarrow})$ separately.
- If $a_k = 4$ and $k \geq 2$ hold, we embed $(a_1, \ldots, a_{k-2}{}^{\rightarrow})$ and $({}^{\rightarrow}a_{k-1}, 4^{\rightarrow})$ separately.
- If $a_k = 4$ and $k = 1$ hold, we embed $(4^{\rightarrow})$.

$\square$

Chapter 6

Conclusion and Outlook

In Chapter 2, we provide (planar) graphs that do not admit a point set embedding in diagonal point sets, and thus, we restrict ourselves to outerplanar graphs. We prove that an outerplanar graph might not admit a planar L-shaped embedding in a given point set, and further provide a quadratic upper bound on the number of points needed in the nonplanar L-shaped case and in the planar orthogeodesic case. As we have made use of diagonal point sets in the proofs of these statements, we ask whether sub-quadratic bounds exist for these cases.

In Chapter 3, we give improvements of the upper bounds on the number of points needed to embed trees, which were provided by Giacomo et al. [10]. In particular, we provide an $\mathcal{O}(n^{\log_2 3})$ bound on f_{LT4}. This also gives an upper bound on f_{LT3}, because every 3-tree is a 4-tree by definition. Moreover, for f_{LT3} we give a slightly improved multiplicative constant.

Since we have made use of several ideas in this thesis, we ask for further embedding techniques that allow improvements. In particular, we ask whether $f_{LT3} \in o(n^{\log_2 3})$ and $f_{LT4} \in o(n^{\log_2 3})$. We believe that both statements hold and that the bound for the 3-trees case is easier to provide. Thus, one should first investigate whether $f_{LT3} \in o(n^{\log_2 3})$. If a proof of this statement were to be found, it might be generalized to 4-trees.

Also, it would be interesting to know, if the saturation-property, introduced in in Chapter 3.3.2, can be combined with the result based on the structure of the orthogonal convex hull, which was stated in Chapter 3.3.3. Moreover, we conjecture that these two ideas can be generalized - at least for the 3-trees case.

In Chapter 4 we state some probabilistic results. The question arises, whether the ideas from Chapter 4 can also be applied to achieve deterministic results. Moreover, we leave it as an open question, whether $f^{1/2}_{LT4}$ is bounded by a quasilinear function.

In Chapter 5, we give some improvements on the number of points needed to embed 4-caterpillars. For some lemmas we came up with computer-assisted proofs, but we conjecture that a reasonable short proof can also be done for those lemmas.

Moreover, we conjecture that none of the multiplicative constants greater than 1 stated in Chapter 5 are optimal yet, and thus, one might work out those constants more precisely. A more exhaustive computer-assisted calculation might lead to better multiplicative constants.

The reader might have noticed that in this thesis we have only considered upper bounds on the number of points when embedding trees. Giacomo et al. [10] asked, whether a non-trivial lower bound exists for any type of embeddings. We tried to tackle this question but it seems to be very involved. On one hand, it is hard to prove that an n-vertex tree can not be embedded in a point set of size n if n is large, since there are $n!$ possible vertex-mappings. On the other hand, if n is very small the embedding seems to be easy. In particular, every tree on 6 vertices or less is a caterpillar, and therefore, such graphs are easy to embed. That is, because every non-caterpillar tree contains the 7-vertex tree depicted in Figure 6.1 as a minor.

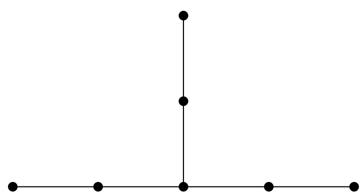

Figure 6.1: A tree that is not a caterpillar.

Because of the more restrictive definition, the planar L-shaped case is the most difficult one. Moreover, we conjecture that, in general, 4-trees are more difficult to embed than 3-trees. It would be interesting to know, if there exists an n-vertex tree T with maximum degree 4 or less and a point set P of size n with T not admitting a planar L-shaped embedding in P. We conjecture that every n-vertex 4-tree with $n \leq 9$ admits a planar L-shaped embedding in any point set of size n. To answer this question, we did some calculations, but no counterexample could be found so far.

Moreover, we believe the 13-vertex 4-tree depicted in Figure 6.2 to be one of the "hardest" 4-trees on 13 vertices. Hence, we ask whether this 4-tree admits a planar L-shaped embedding in every point set of size 13. Note that there are $13! \approx 6.2 \cdot 10^9$ possible vertex mappings that might need to be checked for each of the 13! point sets.

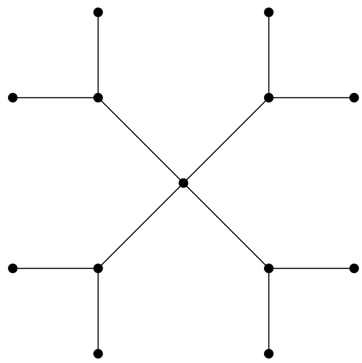

Figure 6.2: A 13-vertex 4-tree.

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