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Hamilton circles in cubic planar Cayley graphs

MASTER'S THESIS

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Statutory Declaration

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Abstract

A popular conjecture, which goes back to a problem raised by Lovász, states that every finite connected Cayley graph with at least three vertices contains a Hamilton cycle. Although this statement can not be claimed valid for all infinite graphs, it probably is for some classes of them.

This thesis deals with the planar cubic Cayley graphs classified by their connectivity and number of ends.

We provide a characterization of the Hamiltonian planar cubic Cayley graphs of connectivity one or two. It turns out that there is a class of two-connected planar cubic Cayley graphs with infinitely many ends which are nonhamiltonian. These graphs might provide counterexamples to a problem presented by Georgakopoulos.

The finite or one-ended planar cubic Cayley graphs are the well-known regular spherical, Euclidean or hyperbolic tessellations or the rotation subgroups of their symmetry groups. All of them are shown to be Hamiltonian.

We prove that all two-ended three-connected cubic planar Cayley graphs with two generators have a Hamilton circle. Applying the twist-amalgamation or the twist-squeeze-amalgamation introduced by Mohar and Georgakopoulos respectively to finite or one-ended graphs, graphs with two or infinitely many ends are obtained. We show that all graphs which are constructed as a twist-amalgamation are Hamiltonian. Some special classes of graphs constructed as twist-squeeze-amalgamation are proved to be Hamiltonian.

Keywords Hamilton circle, Cayley graph, infinite graph, twist-squeeze-amalgamation, twist-amalgamation, Lovász problem, tessellation, symmetry group.

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Kurzfassung

Eine berühmte Vermutung, die auf ein von Lovász aufgeworfenes Problem zurückgeht, besagt, dass jeder endliche, zusammenhängende Cayley-Graph mit mindestens drei Knoten einen hamiltonschen Kreis besitzt. Obwohl diese Aussage nicht für alle unendlichen Graphen zutrifft, dürfte sie für einige Klassen von unendlichen Graphen gelten.

Diese Masterarbeit behandelt planare kubische Cayley-Graphen getrennt nach ihrer Zusammenhangszahl und Anzahl der Enden.

Wir geben eine Charakterisierung der hamiltonschen kubischen planaren Cayley-Graphen mit Zusammenhangszahl Eins oder Zwei an. Es stellt sich heraus, dass eine Klasse von zwei-zusammenhängenden planaren kubischen Cayley-Graphen mit unendlich vielen Enden existiert, die nicht hamiltonsch sind. Diese Graphen könnten Gegenbeispiele zu einem von Georgakopoulos vorgelegten Problem liefern.

Die planaren kubischen Cayley-Graphen mit höchstens einem Ende sind die wohlbekannten sphärischen, euklidschen oder hyperbolischen Parkettierungen oder die Drehungs-Untergruppen ihrer Symmetriegruppen. Für alle von ihnen wird gezeigt, dass sie nicht hamiltonsch sind.

Wir beweisen, dass alle zweiendigen, dreizusammenhängenden kubischen planaren Cayley-Graphen mit zwei Erzeugern einen hamiltonschen Kreis haben. Indem man die von Mohar bzw. Georgakopoulos eingeführte Twist-Amalgamation oder Twist-Squeeze-Amalgamation auf endliche Graphen oder Graphen mit einem Ende anwendet, erhält man Graphen mit zwei oder unendlich vielen Enden. Wir zeigen, dass alle Graphen, die als Twist-Amalgamation gebildet werden, hamiltonsch sind. Für einige spezielle Klassen von Graphen, die als Twist-Squeeze-Amalgamation konstruiert werden, wird nachgewiesen, dass sie hamiltonsch sind.

Stichwörter Hamilton-Kreis, Cayley-Graph, Unendlicher Graph, Twist-Amalgamation, Twist-Squeeze-Amalgamation, Lovász-Problem, Parkettierung, Symmetriegruppe.

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This thesis is dedicated to my family.

Contents

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List of Symbols and Abbrevations

Symbol	Explanation	Definition
$\emptyset$	The empty set.	
$\mathbb{N}$	Set of positive integers, $\mathbb{N} = \{1, 2, 3, \ldots\}$.	
$\mathbb{N}_0$	Set of non-negative integers, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.	
$A \cong B$	A is *isomorphic* to B.	
$A \leq B$	A is a *subgroup* of B.	
$\kappa(G)$	The *connectivity* of G.	def. 2.5.1.
$g(G)$	The *girth* of G.	def. 2.5.1.
$\Gamma(G, S)$	The *Cayley color digraph* of a group G with connection set S.	def. 2.3.1.
$\mathrm{Cay}(G, S)$	The *Cayley graph* of a group G with connection set S.	def. 2.3.1.
$\mathrm{Aut}(G)$	The *automorphism group* of G.	
$\Omega(G)$	The set of ends of G.	def. 2.5.4.
$\overline{X}$	The closure of X.	def. 2.5.7.
$\lvert G \rvert$	The *topological end space* or the number of vertices of G of a graph G.	def. 2.5.6.
$\mathcal{C}(G)$	The *(topological) cycle space.*	def. 2.5.9.
$G^{\underline{n}}$	The n-th *twist-graph.*	def. 2.6.3.
$G^{\underline{\infty}}$	The *twist-amalgamation* of G.	def. 2.6.3.
$G^{\overline{n}}$	The n-th *twist-squeeze-graph.*	def. 2.6.4.
$G^{\overline{\infty}}$	The *twist-squeeze-amalgamation* of G.	def. 2.6.4.
$\{p, q\}$	A *regular polyhedron* or *tessellation* with q p-gons meeting at each vertex (Schläfli symbol), see [CM72, chapter 4].	def. 4.1.1.
$[p, q]$	The *symmetry group* of $\{p, q\}$.	def. 4.1.2.
$[p, q]^+$	The *rotation subgroup.*	def. 4.1.3.

CHAPTER 1

Introduction

Cayley graphs are a very common method to display the structure of a group. There are different variants including colored or uncolored, directed or undirected versions of such graphs. In the present thesis, we study the uncolored, undirected Cayley graphs. More precisely, the cubic planar Cayley graphs are considered. *Cubic* means that the graphs are regular of degree 3.

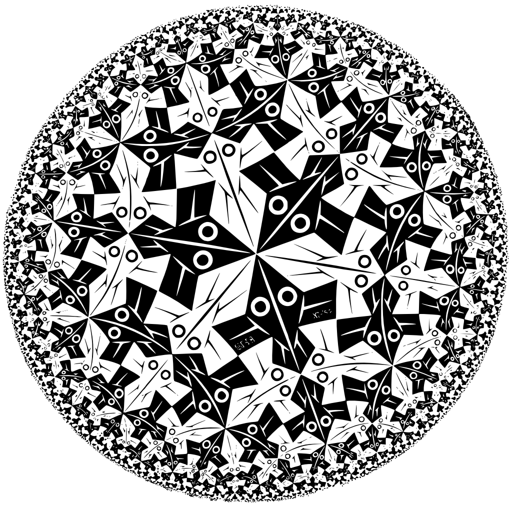
The cubic planar Cayley graphs have been characterized by Georgakopoulos [Geo11b, Geo11a]. There are finite graphs (graphs with finitely many vertices) and infinite graphs among them. The infinite graphs can be classified by their number of ends. Ends are equivalence classes of rays in the graph and can be imagined as directions where the graph extends to infinity. There are infinite graphs with one, two or infinitely many ends.

In finite graphs, a *Hamilton cycle* is a closed walk that visits every vertex exactly once. The notion of a *Hamilton circle* is the generalization of a Hamilton cycle to infinite graphs.

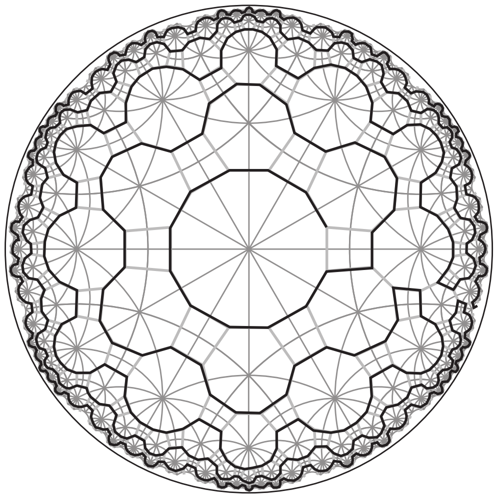
1.1. Motivation

The studied finite or one-ended graphs are regular tessellations of the sphere or the Euclidean or hyperbolic plane or Cayley graphs of rotation subgroups (see chapter 4 for details). Many of the famous artistic woodcuts and lithographs by M.C. Escher (1898–1972) include repeating patterns in the Euclidean or hyperbolic plane, which are based on regular or semi-regular tessellations. See [EBL82] for M.C. Escher's artistic work.

When generating Escher-like repeated patterns using a computer, it is desired to find a path through the underlying Cayley graph that visits every vertex exactly one time. If this path is a ray, it forms a one-way infinite *Hamilton path*. If the spanning path is a double-ray, it forms a two-way-infinite *Hamilton circle*. Dunham [Dun09, dSJ03] uses Hamilton paths in infinite Cayley graphs to render patterns originally designed by M.C. Escher. Figure 1.1 shows M.C. Escher's *Circle limit I* [EBL82, p. 319] and a Hamilton path in the corresponding Cayley graph [dSJ03, p. 453]. This path has been used in Dunham's construction to draw the Circle limit I on a computer.



(a) Circle limit I by M.C. Escher



(b) A Hamilton path in the corresponding graph.

Figure 1.1.: Dunham's construction of repeating patterns.

A very popular conjecture, which goes back to a problem introduced in 1969 by Lovász, is that every finite connected Cayley graph with at least 3 vertices contains a Hamilton cycle. Many special classes of Cayley graphs, for example graphs of groups of prime-power order or Abelian groups, are known to be Hamiltonian and hence support the Lovász-conjecture.

The conjecture can not be simply transferred to infinite graphs, since there are many nonhamiltonian infinite Cayley graphs, for instance the graph of the free group on two generators, depicted in figure 2.3. However, it seems probable that it is also true for some classes of infinite Cayley graphs. Georgakopoulos conjectures that every finitely generated 3-connected planar Cayley graph is Hamiltonian (conjecture 2.7.6).

1.2. Overview

In chapter 2 we provide the definitions and background concerning Cayley graphs and Hamilton circles. *Cayley color digraphs* and *Cayley graphs* are discussed in sections 2.3 and 2.4. Sabidussi's theorem 2.4.3 offers a characterization of Cayley graphs. Since groups are mostly defined in this thesis and in the related work by *group presentations*, we briefly introduce free groups and presentations in section 2.1. *Products, extensions* and *amalgamations* make it possible to construct overgroups of given groups (section 2.2). These operations allow to characterize multi-ended groups and decompose them in finitely many steps into finite or one-ended groups (theorem 2.6.2). In a similar way, many of the multi-ended cubic planar Cayley graphs can be constructed as *twist-amalgamation* or as *twist-squeeze-amalgamation* of a finite or one-ended cubic planar Cayley graph (section 2.6). We cover the specifics of infinite graphs and related concepts such *ends, topological end space, circles* in section 2.5. The classes of Cayley graphs which are already known to have Hamilton circles are described in section 2.7.

The planar cubic Cayley graphs of connectivity 1 are nonhamiltonian (chapter 3). We give a characterization of the Hamiltonian cubic Cayley graphs of connectivity 2 (theorem 3.0.6). Of the eight different types of 2-connected graphs, only one contains nonhamiltonian graphs.

The finite or one-ended graphs are considered in chapter 4. All of them are tessellations or Cayley graphs of rotation subgroups (section 4.1). We prove that every finite or one-ended planar cubic Cayley graph is Hamiltonian. Moreover, the studied finite or one-ended graphs are used to obtain two-ended or infinitely-ended graphs by twist-amalgamation or twist-squeeze-amalgamation.

The multi-ended graphs of connectivity 3 are treated in chapter 5. In section 5.1 we show, that all 2-ended 3-connected cubic planar Cayley graphs with 2 generators have a Hamilton circle. We prove that all graphs obtained by twist-amalgamation are Hamiltonian (section 5.2). The twist-squeeze-amalgamation is more difficult to handle. Some classes of graphs constructed as twist-squeeze-amalgamation are proved to be Hamiltonian. We use both finite and one-ended graphs for this amalgamation operation. However, the transitions applied in the corresponding proofs are rather complicated and can not simply be transferred to all types of twist-squeeze-amalgamations. Therefore we can not offer a final solution for all of the considered graphs.

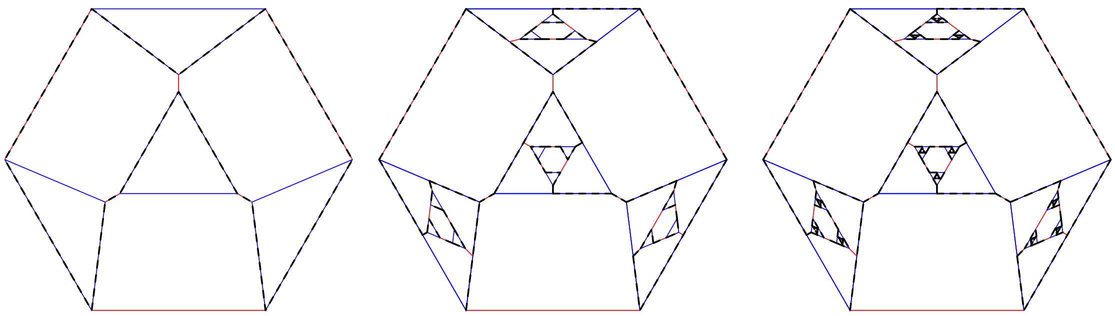
Appendix B contains a list of graphs that are discussed in this master's thesis.

1.3. Basic ideas

The finite or one-ended cubic planar Cayley graphs are (spherical, Euclidean or hyperbolic) tessellations or graphs of rotation subgroups. For these graphs we can rely on many facts already known. Thus, we are able to prove that all of them are Hamiltonian.

Among the multi-ended graphs, two classes are of special interest: those which can be expressed as twist-amalgamation or as twist-squeeze-amalgamation of a finite or one-ended graph.

To obtain the twist-squeeze-amalgamation of a cubic graph H, a copy of H is embedded inside every monochromatic face of H and the copies are glued along the cycle bounding the face. The glueing is done in a special way: One of the sides of the cycle is rotated so that the edges incident with that cycle on either side do not have any common endvertex. The embedding and glueing operations are repeated recursively for the new monochromatic faces (see figure 1.2 and figure 2.5). The rotation assures that the resulting graphs are cubic.

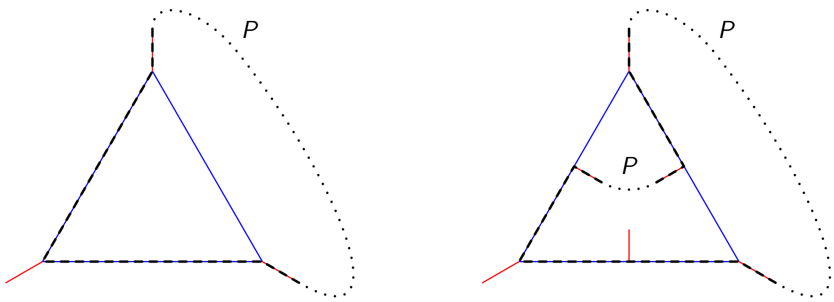


(a) A finite cubic Cayley graph H. (b) First twist-graph. (c) Twist-amalgamation.

Figure 1.2.: A Hamilton circle in the twist-amalgamation of a finite Cayley graph H.

Starting with a Hamilton circle D in the finite or one-ended graph H (dashed line in figure 1.2(a)), we try to find a transition that transforms D into a Hamilton circle of the first twist-graph which embeds a copy of H in every monochromatic cycle of H (figure 1.2(b)). If this is done carefully enough, repeated application of this transition will lead to a Hamilton circle in the twist-amalgamation of H (figure 1.2(c)).

When the monochromatic cycles of H are triangles, the desired transition is easy to find. Consider a monochromatic triangle C of H, as outlined in figure 1.3(a) (the Hamilton circle is black and dashed or dotted in this figure). We embed H inside C and glue the graphs along C as described before. Figure 1.3(b) shows a Hamilton circle D' of the newly created graph. D' uses the same type of edges in the same order inside and outside of C (the path P appears twice in the figure). This transition has to be repeated for all monochromatic triangles of H. An example of this construction is presented in figure 1.2. See section 5.2 for details.



(a) The Hamilton circle D. (b) The Hamilton circle D' after embedding H in the blue triangle.

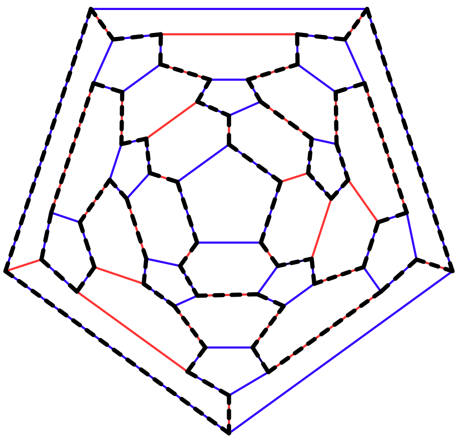
Figure 1.3.: The transition if the monochromatic cycles are triangles.

In other cases, it is slightly more difficult to find suitable transitions. If H has a monochromatic cycle C of size ≥ 4, there may be two (finite or infinite) disjoint paths P and Q connecting different pairs of vertices of C. A solution for some configurations containing two such paths is depicted in figure 5.4.

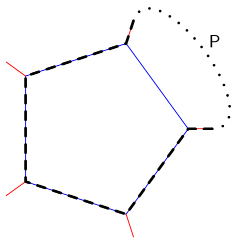
However, often it is not enough to take an arbitrary Hamilton circle D of H and find a suitable transition for the monochromatic cycles where the embedded graphs are glued. Instead we need appropriate Hamilton circles in the base graph H and compatible pairs of monochromatic cycles which are merged. This results in different types of monochromatic cycles that are incident with the Hamilton circle. For each type we need a correct transition.

It is not very difficult to prove that repeated application of the aforementioned transitions leads to a connected infinite subgraph $\overline{D}$ of the twist-amalgamation of H that meets all vertices exactly twice. Furthermore, every vertex of $\overline{D}$ has degree 2. Nevertheless, it is not obvious that $\overline{D}$ is a circle, since it is not clear that every end has degree 2. Indeed, the literature [Geo09] shows that the hardest part about the quest for Hamilton circles is by far guaranteeing injectivity at the ends.

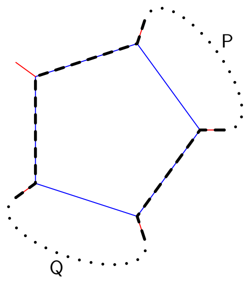
Consider for example $G = \mathrm{Cay}\left\langle a, b \mid b^2, a^{10}, (a^2b)^3 \right\rangle$, which is the twist-amalgamation of $H = \mathrm{Cay}\left\langle a, b \mid b^2, a^5, (ab)^3 \right\rangle$. H is shown in figure 1.4(a) with its Hamilton cycle D dashed.



(a) $H = \mathrm{Cay}\left\langle a, b \mid b^2, a^5, (ab)^3 \right\rangle$.



(b) 5-cycle type 1.

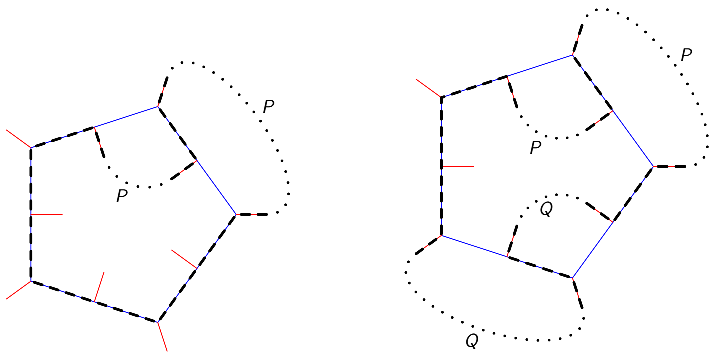


(c) 5-cycle type 2.

Figure 1.4.: A Hamilton cycle in $H = \mathrm{Cay}\left\langle a, b \mid b^2, a^5, (ab)^3 \right\rangle$.

Note that H contains two different types of 5-cycles with respect to D. When we try to find a method to transform D into a Hamilton cycle of the first twist-graph of H, the approach shown in figure 1.5 might be the first solution. Every 5-cycle of type 1 (figure 1.4(b)) is treated analogously to the triangles in our aforesaid example: The new Hamilton cycle D' uses the same type of edges inside and outside of the 5-cycle (figure 1.5(a)). The 5-cycles of type 2 (figure 1.5(a)) are handled according to figure 1.5(a). These transitions have to be applied to all 5-cycles of H.

Repeated use of the described procedure leads to some problems in the limit graph. Let C be a 5-cycle of type 2 in H. After applying the transition, there are ten vertices incident to C. Let p_1, p_2, q_1, q_2 be vertices on C, so that p_1Pp_2 and q_1Qq_2 are subpaths of the Hamilton cycle. Taking into account the chosen Hamilton cycle D of H, it follows from the construction that there exists a (type 2) 5-cycle C_1 of the first twist-graph of H that



(a) Type 1. (b) Type 2.

Figure 1.5.: A transition for the 5-cycles of H.

is incident with both P and Q. After applying the transition to C_1 (and all other 5-cycles of the first twist-graph of H), there are two subpaths $p_1P'p_2$ and $q_1Q'q_2$ of the Hamilton cycle, so that a (type 2) 5-cycle C_2 of the second twist-graph of H is incident with both P' and Q'. After k steps of this construction, we obtain a Hamilton circle D_k of the k-th twist-graph of H and two subpaths $p_1P^{(k)}p_2$ and $q_1Q^{(k)}q_2$ of D_k. Both paths $P^{(k)}$ and $Q^{(k)}$ are incident with a (type 2) 5- cycle C_k. In the limit graph, the considered loops $P^{(k)}$ and $Q^{(k)}$ converge to a common end ω. Hence, D_∞ contains an end with degree ≥ 4.

The result of our first approach is not a Hamilton circle in the twist-amalgamation of H. Fortunately, there is a possibility to repair it. In general, we can try

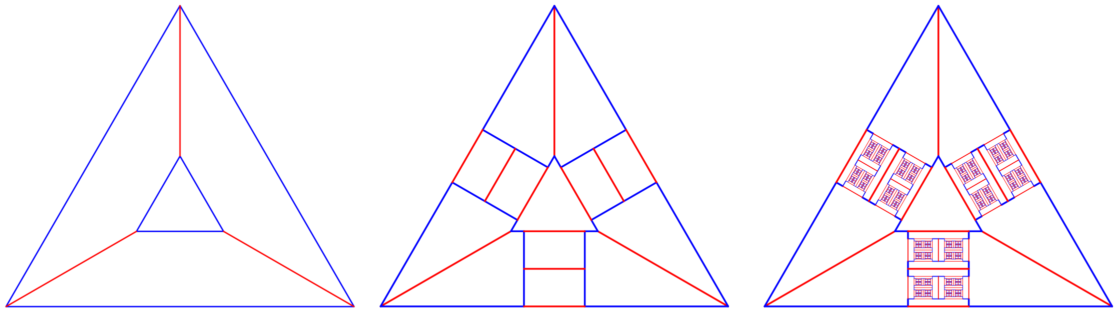
- to use *different Hamilton cycles* in the base graph H, or
- to *modify the transitions*.

In this example, the second method—modification of the second transition—is successful. The solution can be found in figure 5.6. When considering the twist-amalgamation of $H = \mathrm{Cay}\left\langle a, b \mid b^2, a^4, (ab)^4 \right\rangle$ the same problem (ends with degree ≥ 4) occurs. In this case, using a different Hamilton circle provides a solution (see figure 5.5).

The twist-squeeze-amalgamation is constructed similarly to the twist-amalgamation. This time, a copy of H is embedded inside every alternately-colored face of H and the copies are glued along the cycle bounding the face. However, this glueing operation includes subdivision of edges as displayed in figure 1.6.

The first approach to find a Hamilton circle in the twist-squeeze-amalgamation is similar to the above construction in twist-amalgamations. Starting with a Hamilton circle in the basic graph H we distinguish different types of alternately-colored cycles C in H depending on how the Hamilton circle runs through C. For each type we find a suitable transition which allows us to preserve the Hamilton circle when we embed a copy of H inside C and glue the graphs as required.

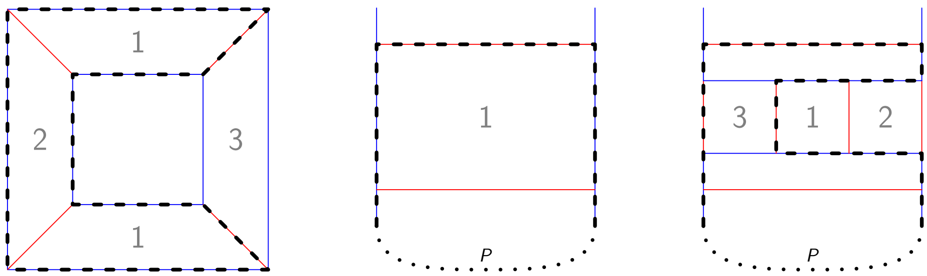
As an example, consider the cubic Cayley graph H in figure 1.7, which has three different types of alternately-colored cycles with respect to the dashed Hamilton cycle shown.



(a) A finite cubic Cayley graph H. (b) First twist-squeeze-graph. (c) Twist-squeeze-amalgamation.

Figure 1.6.: The twist-squeeze-amalgamation of a finite Cayley graph H.

Figure 1.7(b) depicts a cycle of type 1 before applying the transition and figure 1.7(c) the same cycle after applying the transition.



(a) A finite cubic Cayley graph H. (b) Cycle C before transition. (c) Cycle C after transition.

Figure 1.7.: A possible transition for twist-squeeze-graphs.

Together with suitable transitions for the other types this leads to a Hamilton circle in the twist-squeeze-amalgamation of H. As before, the most difficult task is to prove that every end has degree 2 in the resulting subgraph.

For the graph which is shown in figure 1.6 and some other twist-squeeze-amalgamations we use a slightly different method to obtain a Hamilton circle. Starting with an alternately-colored 4-cycle, in which a copy of H is embedded, we choose a path through the graph and save this configuration as a module M_1. We embed these two modules into the alternately-colored 4-cycles of M_1 to obtain a new module M_2 and so forth. Glueing three copies of the module M_k along the alternately-colored 4-cycles of H leads to a Hamilton circle in the k-th twist-squeeze-graph. As the limit graph we get a Hamilton circle in the twist-squeeze-amalgamation of H. The construction is depicted in figure 5.11. Hamilton circles in twist-squeeze-amalgamation are discussed in detail in section 5.3.

The method of composing Hamilton cirlces in the basis graphs into a Hamilton cirlce in the twist-amalgamation was proposed by Bojan Mohar (private communication), and we are grateful to him for this idea.

CHAPTER 2

Definitions and Facts

In 1878, Arthur Cayley was the first to use colored digraphs (which he called *diagrams*) to graphically represent finite groups [Cay78]. Today, *Cayley graphs* are a widely used tool in geometric, algebraic and combinatorial group theory. The studied groups are in many cases defined by *presentations* using generators and relations. For an overview of the basic concepts, see [MKS66, chapter 1] or [LS77].

2.1. Free groups and group presentations

Definition 2.1.1. Let F be a group and X a subset of F. F is called *free with basis* X or *free on* X if for every group G and every function $\phi : X \to G$ there is a unique homomorphism $\psi : F \to G$ extending ϕ such that the diagram in figure 2.1 commutes (ι is the embedding of X into F).

$$\begin{tikzcd} X \arrow[r, "\iota"] \arrow[dr, "\phi"'] & F \arrow[d, "\psi"] \\ & G \end{tikzcd}$$

Figure 2.1.: Definition of free groups.

Definition 2.1.2. Let F be a group with identity element e and $X \subseteq F$. A sequence $w = (v_1^{\alpha_1}, v_2^{\alpha_2}, v_3^{\alpha_3}, \dots)$ with $v_i \in X$ and $\alpha_i \in \{-1, 0, 1\}$ for all i is called a *word in* X. The constant sequence $(e, e, e, \dots)$ is the *empty word*, denoted by ϵ.

If for some index n, $v_i = e$ for all $i > n$, then w is considered as a *final word.* The smallest index, that satisfies this condition is called the *length of* w denoted by $|w|$. The empty word is set to have length 0.

For a nonempty final word w in X of length $n > 0$ the *spelling of* w is an alternative notation of the form

$$w = v_1^{\alpha_1} v_2^{\alpha_2} v_3^{\alpha_3} \dots v_n^{\alpha_n}.$$

A final word w is *reduced*, if it satisfies one of the following two conditions:

- $|w| \leq 1$,
- $|w| = n$, $w = v_1^{\alpha_1} v_2^{\alpha_2} v_3^{\alpha_3} \dots v_n^{\alpha_n}$ and $v_i^{\alpha_i} \neq v_{i+1}^{-\alpha_{i+1}}$ and for all $1 \leq i \leq n-1$ it holds that $\alpha_i \in \{-1, 1\}$.

The *inverse* word of $w = v_1^{\alpha_1} v_2^{\alpha_2} \dots v_n^{\alpha_n}$ is $w^{-1} = v_n^{-\alpha_n} \dots v_2^{-\alpha_2} v_1^{-\alpha_1}$

Theorem 2.1.3. Let X be a set. Then a free group F with basis X exists.

For a proof, see [Rot94, Theorem 11.1]. The free group F on X is constructed as follows. The elements of F are defined by the reduced words in X. The product in F is the concatenation of words, the identity element is the empty word and the inverse elements are the inverse words.

Theorem 2.1.4. Every group G is a quotient of a free group.

Proof. Let $X = \{x_g \mid g \in G\}$. Then $\phi : X \to G,\ x_g \mapsto g$ is a bijection. According to theorem 2.1.3 a free group F with basis X exists. Let $\psi : F \to G$ be a homomorphism extending ϕ. Because ψ is surjective, $G \cong F/\ker\psi$ by the first isomorphism theorem. □

By finding generators of $\ker\psi$, the *presentation of* G is obtained.

Definition 2.1.5. Let X be a set and F a free group with basis X. Furthermore, let W be a family of words in X and R the normal subgroup of F which is generated by W. If $G \cong F/R$, then the ordered pair $\langle X \mid W \rangle$ is called a *presentation of* G. X is called the set of *generators* and W the set of *relations.*

Corollary 2.1.6 (Theorem 2.1.4). Every group G has a presentation.

Remarks.

- The relations W are sometimes written as equations rather than words (for example $w = 1$ instead of w or $u = v$ instead of uv^{-1}).
- Every finite group G has a finite presentation.

Example 2.1.7. The free group F_2 on two generators has the presentation $\langle a, b \mid \emptyset \rangle$.

Example 2.1.8. For $n \in \mathbb{N}$ the cyclic group Z_n has the presentation $\langle a \mid a^n \rangle$.

Example 2.1.9. The dihedral group D_5 has the presentation $\langle a, b \mid a^5, b^2, abab \rangle$.

2.2. Products and extensions

Let $G = \langle X_G \,|\, W_G \rangle$ and $H = \langle X_H \,|\, W_H \rangle$ be groups. Then the *direct product*

$$G \times H = \{(g, h) \mid g \in G, h \in H\}$$

with multiplication

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2)$$

has the presentation

$$G \times H = \langle X_G \cup X_H \,|\, W_G \cup W_H \cup W \rangle,$$

where $W = \left\{ aba^{-1}b^{-1} \mid a \in G, b \in H \right\}$.

Definition 2.2.1. Consider two groups $G = \langle X_G \,|\, W_G \rangle$ and $H = \langle X_H \,|\, W_H \rangle$. Then the group

$$G * H = \langle X_G \cup X_H \,|\, W_G \cup W_H \rangle$$

is called the *free product* of G and H.

Remarks.

- Both G and H are subgroups of $G * H$.
- If G and H are non-trivial groups, then $|G * H| = \infty$.

Another way to construct groups is by *amalgamating* a common subgroup F in G and H (Definition 2.2.2) or embed a group G in a new group H, such that two isomorphic copies of subgroups A, B of G are conjugate in H (Definition 2.2.3). See [Ser77] or the translation [Ser80] for details about these constructions.

Definition 2.2.2. Let F be a group and $G = \langle X_G \,|\, W_G \rangle$ and $H = \langle X_H \,|\, W_H \rangle$ groups containing an isomorphic copy of F. $\iota_G : F \hookrightarrow G$ and $\iota_H : F \hookrightarrow H$ are the monomorphisms embedding F into G and H respectively. Let

$$W_F = \left\{ \iota_G(a) \iota_H(a)^{-1} \mid a \in F \right\}.$$

Then the group

$$G *_F H = \langle X_G \cup X_H \,|\, W_G \cup W_H \cup W_F \rangle$$

is called the *free product of G and H with amalgamated subgroup F*.

Remark. The free product $G * H$ is a special case of the free product with amalgamation $G *_F H$ with trivial group F.

Another construction was given by Higman, Neumann and Neumann [HNN49].

Definition 2.2.3. Let $G = \langle X \mid W \rangle$ be a group with two subgroups A and B isomorphic by a mapping $\phi : A \to B$. Choose

$$U = \{t^{-1}at\phi(a)^{-1} \mid a \in A\}.$$

Then the group

$$G*_\phi = \langle X \cup \{t\} \mid W \cup u \rangle$$

is called the *HNN-extension of G relative to ϕ*. t is called the *stable letter* and A, B the *associated subgroups*.

2.3. Cayley color digraphs

The term *Cayley graph* is not always defined in the same way. It is used for directed or ordinary graphs, for labelled or colored or none of both. In the present work *Cayley color digraph* stands for a colored digraph, *Cayley digraph* for an uncolored digraph and *Cayley graph* for the underlying ordinary graph.

Definition 2.3.1. Let G be a group with identity element e. Let S be a finite subset of the elements of G, such that $e \notin S$. The *Cayley color digraph*—also known as *Cayley color diagram*—$\Gamma(G, S)$ of G with *connection set* S is defined as the arc-colored, directed graph, such that

- for each $s \in S$ there is a unique color c_s,
- $\Gamma(G, S)$ has $|G|$ vertices which are identified with the elements of G,
- there is an c_s-colored arc from g to h if and only if $h = gs$ for some $s \in S$.

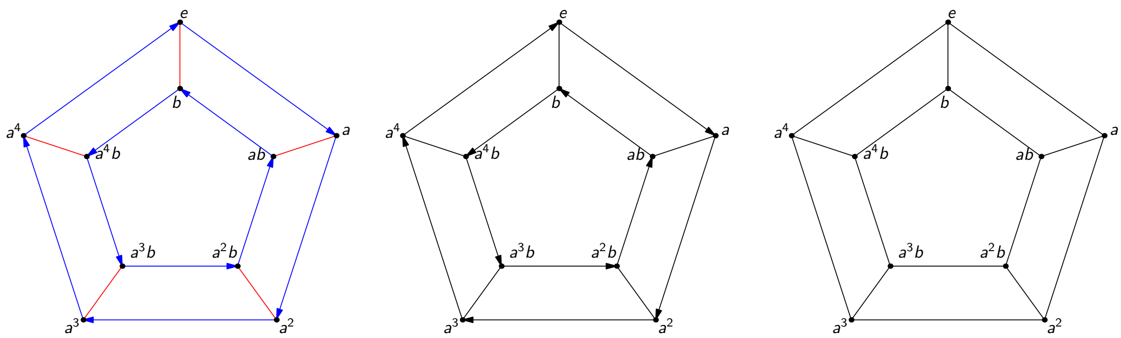
Remark (Bidirected-arc convention). If two vertices are joined by a pair of arcs (one arc in each direction), the pair of arcs can be replaced by an undirected edge. This is the case, if some $s \in S$ is an involution, i.e. $s = s^{-1}$.

Remarks.

- Depending on the chosen set S of generators, there may be different Cayley color digraphs derived from the same group G.
- If $\langle S \mid W \rangle$ is a presentation of G, the Cayley color digraph $\Gamma(G, S)$ can be written as $\Gamma \langle S \mid W \rangle$.
- The underlying uncolored digraph of a Cayley color digraph is called *Cayley digraph.*

Example 2.3.2 (continued from example 2.1.9). The Cayley color digraph, Cayley digraph and Cayley graph of the dihedral group D_5 are shown in figure 2.2.

Example 2.3.3 (continued from example 2.1.7). The Cayley color digraph of the free group F_2 on two generators a (blue) and b (red) is shown in figure 2.3. Because it is an infinite graph, the figure is limited to words of a maximum length of 7.



(a) Cayley color digraph of D_5 (b) Cayley digraph of D_5. (c) Cayley graph of D_5.

Figure 2.2.: Graphs of the dihedral group D_5.

Proposition 2.3.4. Let G be a finite group and S a subset of G satisfying $e \notin S$. Then $\Gamma(G, S)$ is weakly connected if and only if S generates G.

PROOF. Assume that G is generated by S. For an arbitrarily chosen $g \in G$ let

$$g = s_1^{m_1} s_2^{m_2} s_3^{m_3} \cdots s_k^{m_k},\ k \in \mathbb{N} \text{ and for all } 1 \leq i \leq k:\ m_i \in \{-1, 1\},\ s_i \in S.$$

be its expression by generators in S.

Then there is a sequence

$$a_1 = (e = v_0, v_1), a_2 = (v_1, v_2), a_3 = (v_2, v_3), \ldots, a_n = (v_{n-1}, g = v_n),$$

such that for all $1 \leq i \leq k$

- a_i is an c_{s_i}-colored arc of $\Gamma(G, S)$ if and only if $m_i = 1$,
- $\overline{a_i} = (v_i, v_{i-1})$ is an c_{s_i}-colored arc of $\Gamma(G, S)$ if and only if $m_i = -1$.

Hence, there is a path from e to g in the undirected version of $\Gamma(G, S)$. Since this condition holds for arbitrarily chosen $g \in G$, it follows that $\Gamma(G, S)$ is weakly connected.

To prove the converse direction, assume that $\Gamma(G, S)$ is weakly connected. Then for every $g \in G$ there is a path from e to g in the undirected version of $\Gamma(G, S)$. Thus, a sequence

$$a_1 = (e = v_0, v_1), a_2 = (v_1, v_2), a_3 = (v_2, v_3), \ldots, a_n = (v_{n-1}, g = v_n)$$

as before exists. From this sequence the equation

$$g = s_1^{m_1} s_2^{m_2} s_3^{m_3} \cdots s_k^{m_k},\ k \in \mathbb{N} \text{ and for all } 1 \leq i \leq k:\ m_i \in \{-1, 1\},\ s_i \in S$$

is obtained. Since this works for every $g \in G$, it follows that S generates G.

This condition means that $\Gamma(G, S)$ is weakly connected. $\square$

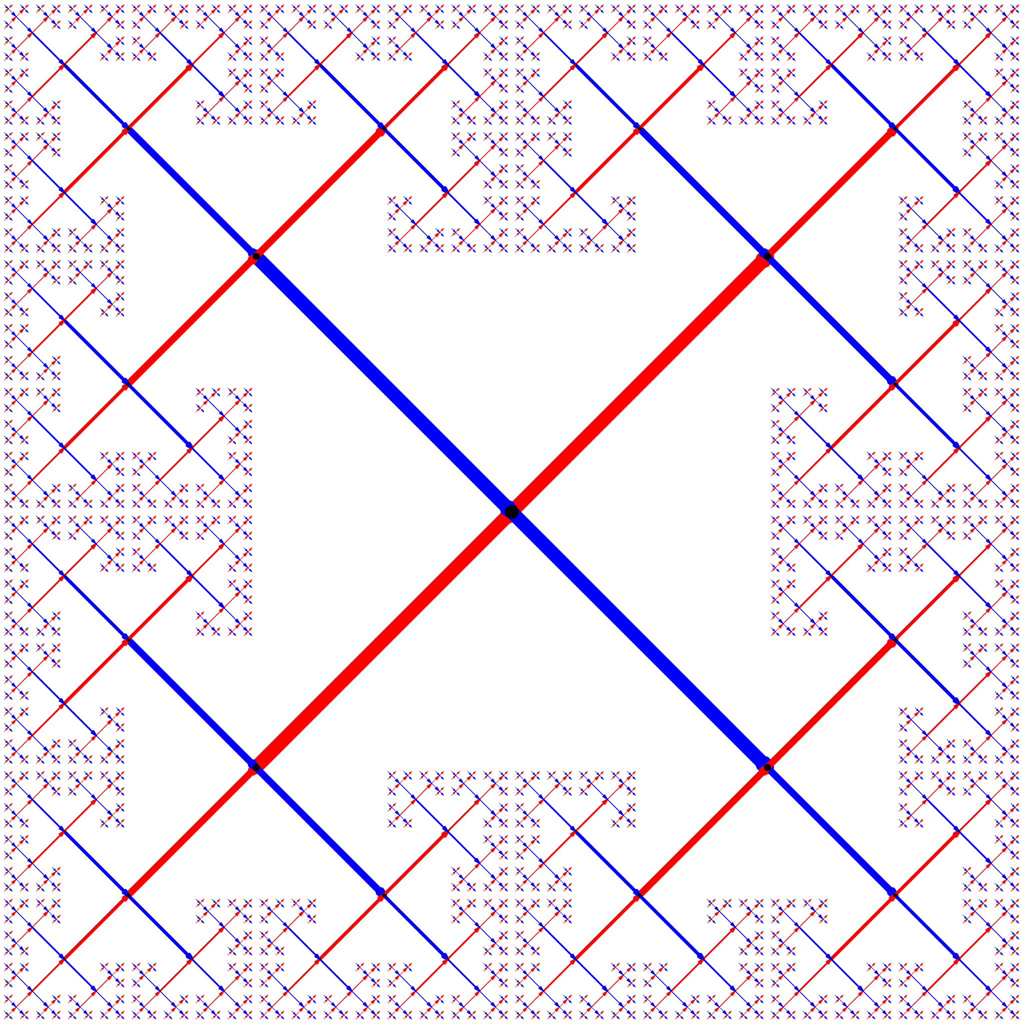


Figure 2.3.: Cayley color digraph of the free group F_2 on two generators.

An automorphism of a graph H is a permutation $\sigma : V(H) \to V(H)$ that preserves the structure of H. This is, for a colored directed graph directed adjacency with respect to the labels. For a Cayley color digraph $\Gamma(G, S)$, this implies that a permutation

$$\sigma : V(\Gamma(G, S)) \to V(\Gamma(G, S))$$

is an automorphism, if and only if for all $g, h \in G, s \in S$,

$$gs = h \Leftrightarrow \sigma(g)s = \sigma(h)$$

holds.

In other words, σ is a Cayley color digraph automorphism, iff for all $g \in G, s \in S$,

$$\sigma(gs) = \sigma(g)s \tag{2.1}$$

holds. Equivalently, σ is a Cayley color digraph automorphism, if and only if for all $g \in G, s \in S$,

$$\sigma(gs^{-1}) = \sigma(g)s^{-1}. \tag{2.2}$$

The set of automorphisms of H together with composition, forms the *automorphism group* which is written as $\mathrm{Aut}(H)$.

If S is a generating set of G, the Cayley color digraph $\Gamma(G, S)$ fully describes the underlying group G, as the following theorem states.

Theorem 2.3.5. Let $\Gamma(G, S)$ be a Cayley color digraph on a group G with identity element e and generating set S. Then

$$\mathrm{Aut}(\Gamma(G, S)) \cong G$$

holds independent of the choice of S.

Proof (cf. [Whi01, Theorem 4-8]). Consider

$$\alpha : G \to \mathrm{Aut}(\Gamma(G, S));\ g \mapsto \sigma_g \tag{2.3}$$

with

$$\sigma_g : V(\Gamma(G, S)) \to V(\Gamma(G, S));\ h \mapsto gh.$$

Now σ_g is a bijection and therefore a permutation of $V(\Gamma(G, S))$. Furthermore,

$$\sigma_g(h_1 h_2) = g h_1 h_2 = \sigma_g(h_1) h_2$$

proofs that σ_g is an automorphism. This shows that α is well-defined.

α is a group homomorphism

$$\alpha(g_1 g_2)(h) = \sigma_{g_1 g_2}(h) = g_1 g_2 h = \sigma_{g_1}(g_2 h) = \sigma_{g_1}(\sigma_{g_2}(h)) = \alpha(g_1)\alpha(g_2)(h).$$

and injective because

$$\ker \alpha = \{e\}.$$

Choose an arbitrary $\sigma \in \mathrm{Aut}(\Gamma(G, S))$. Then $\sigma(e) = g$ for some $g \in G$.

σ is an automorphism, so for any $h \in G$ with

$$h = s_1^{m_1} s_2^{m_2} s_3^{m_3} \cdots s_k^{m_k},\ k \in \mathbb{N} \text{ and for all } 1 \le i \le k:\ m_i \in \{-1, 1\},\ s_i \in S.$$

repeated application of (2.1) and (2.2) leads to

$$\begin{aligned} \sigma(h) = \sigma(eh) &= \sigma(e s_1^{m_1} s_2^{m_2} s_3^{m_3} \cdots s_k^{m_k}) \\ &= \sigma(e) s_1^{m_1} s_2^{m_2} s_3^{m_3} \cdots s_k^{m_k} = \sigma(e) h \\ &= gh = \sigma_g(h) = \alpha(g)(h). \end{aligned}$$

Hence, α is surjective and therefore a bijection between G and $\mathrm{Aut}(\Gamma(G, S))$. $\square$

Corollary 2.3.6. Let G be a group with generating set S and let H be a group with generating set T. If $\Gamma(G, S) \cong \Gamma(H, T)$ then $G \cong H$.

Definition 2.3.7. Let G be a group and X a non-empty set. The group action

$$G \times X \to X;\ (g, x) \mapsto g \cdot x$$

is called

- *transitive*, if for any $x, y \in X$ there is a $g \in G$, such that $g \cdot x = y$,
- *regular*, if for any $x, y \in X$ there is exactly one $g \in G$, such that $g \cdot x = y$.

Definition 2.3.8. A graph H is called *vertex-transitive*, if for any $x, y \in V(H)$ a $\sigma \in \mathrm{Aut}(H)$ exists such that $\sigma(x) = y$.

A graph H is vertex transitive, if and only if $\mathrm{Aut}(H)$ acts transitively on $V(H)$.

Proposition 2.3.9. Let $\Gamma(G, S)$ be a Cayley color digraph on a group G with connection set S. Then G acts regularly on $V(\Gamma(G, S))$.

Proof. Choose $x, y \in V(\Gamma(G, S))$ arbitrarily. Define a group action

$$\circ :\ G \times V(\Gamma(G, S)) \to V(\Gamma(G, S));\ g \circ v = gv. \qquad \square$$

Now $u \circ x = y$ if and only if $u = yx^{-1}$. Hence, the group action $\circ$ is regular.

Corollary 2.3.10. Let $\Gamma(G, S)$ be a Cayley color digraph on a group G with connection set S. Then $\mathrm{Aut}(\Gamma(G, S))$ acts transitively on $V(\Gamma(G, S))$. If S is a generating set of G, then $\mathrm{Aut}(\Gamma(G, S))$ acts regularly on $V(\Gamma(G, S))$.

Proof. Define a group action

$$\star :\ \mathrm{Aut}(\Gamma(G, S)) \times V(\Gamma(G, S)) \to V(\Gamma(G, S));\ \sigma \star v = \sigma(v).$$

Choose $x, y \in V(\Gamma(G, S))$ arbitrarily and let $\alpha(yx^{-1}) = \sigma_{yx^{-1}}$ be defined as in (2.3). Note that α is not necessarily a bijection because S does not need to be a generating set of G. Still, $\alpha(yx^{-1}) = \sigma_{yx^{-1}}$ is an automorphism of $\Gamma(G, S)$. Moreover,

$$\sigma_{yx^{-1}} \star x = y$$

holds. Hence, $\star$ is a transitive group action.

If S generates G, then α is a bijection between G and $\mathrm{Aut}(\Gamma(G, S))$. Every automorphism $\sigma \in \mathrm{Aut}(\Gamma(G, S))$ with $\sigma(x) = y$ satisfies $\alpha^{-1}(\sigma) = yx^{-1}$. Therefore, $\alpha(yx^{-1})$ is the only automorphism with this property and the group action $\star$ is regular. $\square$

Corollary 2.3.11. Every Cayley color digraph is vertex transitive and therefore regular.

The converse of Corollary 2.3.11 is not true, so vertex transitivity is not suitable to characterize Cayley color digraphs. There are vertex-transitive graphs which are not the Cayley color digraph of any group G, even if proper labels are chosen, see Corollary 2.4.5.

A better criterion to characterize Cayley color digraphs has been found by Sabidussi [Sab64]. This leads to Theorem 2.3.13.

Definition 2.3.12. An arc-coloring of a digraph H is called *proper*, if for every vertex $u \in V(H)$ no pair of arcs $(u,v),(u,w), v \neq w$ is assigned the same color.

Theorem 2.3.13. A properly arc-colored digraph H is a Cayley color digraph if and only if a subgroup G of $\mathrm{Aut}(H)$ exists that acts regularly on $V(H)$.

Proof. Let H be a Cayley color digraph of a group G with connection set S. Let α be as defined in (2.3). Then

$$\overline{G} = \{\alpha(g) \mid g \in G\} = \{\sigma_g \mid g \in G\}$$

satisfies

$$G \cong \overline{G} \leq \mathrm{Aut}(H).$$

Furthermore, define the group action

$$\diamond : \ \overline{G} \times V(\Gamma(G,S)) \to V(\Gamma(G,S)); \ \sigma_g \diamond v = \sigma_g(v) = gv.$$

Now $\diamond$ is a regular group action of $\overline{G}$ on $V(H)$.

To prove the opposite direction, assume that $L \leq \mathrm{Aut}(H)$ acts regularly on $V(H)$. Then H is a vertex transitive graph and $|V(H)| = |L|$. Take arbitrarily a $v \in V(H)$ and identify it by the identity element $e \in L$. Now for each $w \in V(H)$ exists a unique automorphism $\sigma_w \in L$ such that $\sigma_w(e) = w$. Identify w by the automorphism σ_w. Denote the successor set of e by S. For each $s \in S$ identify the color of the arc (e,s) by c_s. Since H is vertex-transitive, all arcs of H are c_s-colored, where $s \in S$.

Now let $g \in V(H), h \in V(H), s \in S$. H has an c_s-colored arc from e to s. Since g, h, s are automorphisms satisfying $g(e) = g, \ g(s) = gs$, there is an c_s-colored arc from g to h if and only if $gs = h$.

So H is the Cayley color digraph $\Gamma(L,S)$. □

Corollary 2.3.14. A weakly connected properly arc-colored digraph H is a Cayley color digraph if and only if $\mathrm{Aut}(H)$ acts regularly on $V(H)$.

Proof. If $\mathrm{Aut}(H)$ acts regularly on $V(H)$, then H is a Cayley color digraph according to theorem 2.3.13.

In order to prove the converse direction, assume that H is a weakly connected Cayley color digraph of a group G with connection set S. Proposition 2.3.4 ensures that S generates G. Corollary 2.3.10 guarantees that $\mathrm{Aut}(H)$ acts regularly on $V(H)$. □

2.4. Cayley graphs

By deleting all labels and edge directions from the Cayley color digraph $\Gamma(G, S)$ and merging parallel edges, the (undirected) *Cayley graph* $\mathrm{Cay}(G, S)$ is obtained.

Definition 2.4.1. Let G be a group with identity element e. Let S be a finite subset of the elements of G, such that $e \notin S$. The *Cayley graph* $\mathrm{Cay}(G, S)$ is defined as the graph, such that

- $\mathrm{Cay}(G, S)$ has $|G|$ vertices which are identified with the elements of G,
- two vertices g and h are joined by an edge if and only if $h = gs$ for some $s \in S$ or $s^{-1} \in S$.

If $\langle S \mid W \rangle$ is a presentation of G, the Cayley graph $\mathrm{Cay}(G, S)$ can also be written as $\mathrm{Cay}\,\langle S \mid W \rangle$.

Remark. Although the Cayley graph $\mathrm{Cay}(G, S)$ is uncolored and undirected, its edges are sometimes referred to by the colors and directions in the corresponding Cayley color graph $\Gamma(G, S)$. For $s \in S$ the c_s-colored arcs are also called s-colored arcs. Subpaths of the Cayley graph can be defined by the sequence of edge-colors of the path and some vertices which lie on the path, if this definition is unambiguous.

Proposition 2.4.2. The Cayley graph $\mathrm{Cay}(G, S)$ shares some properties with the Cayley color digraph $\Gamma(G, S)$. More precisely,

- $\mathrm{Cay}(G, S)$ is connected if and only if S generates G,
- G acts regularly on $V(\mathrm{Cay}(G, S))$ by left multiplication,
- $\mathrm{Aut}(\mathrm{Cay}(G, S))$ acts transitively on $V(\mathrm{Cay}(G, S))$ and hence, $\mathrm{Cay}(G, S)$ is vertex-transitive and regular.

Moreover, the statement of Theorem 2.3.13 remains true for uncolored undirected Cayley graphs:

Theorem 2.4.3. A graph H is a Cayley graph if and only if a subgroup G of $\mathrm{Aut}(H)$ exists that acts regularly on $V(H)$.

Vertex transitivity is not an appropriate criterion to characterize Cayley graphs. An example for a vertex-transitive graph which is not a Cayley graph of any group G is the *Petersen graph* P, see figure 2.4.

Proposition 2.4.4. The Petersen graph P is not a Cayley graph of any group G.

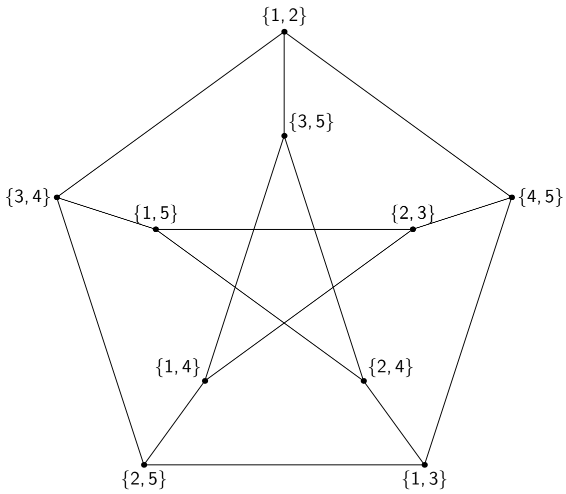


Figure 2.4.: The Petersen graph (Kneser graph $\mathrm{KG}_{5,2}$).

Proof. Assume that G is a subgroup of $\mathrm{Aut}(P)$ that acts regularly on $V(P)$.

$|G| = 10$, so G contains an element σ of order 5 (more precisely, G has to be the cyclic group Z_{10} or the dihedral group D_5). Since G acts regularly on $V(P)$, σ does not fix any vertex of P. Hence, σ is of the form

$$\sigma = (v_0\ v_1\ v_2\ v_3\ v_4)\ (w_0\ w_1\ w_2\ w_3\ w_4),$$

where v_i, $0 \le i \le 4$ and w_i, $0 \le i \le 4$ are the vertices of P.

Due to the connectivity of P there is an edge connecting v_i and w_j for some i, j. Without loss of generality let $\{v_0, w_0\} \in E(P)$. Therefore, $\{v_i, w_i\} \in E(P)$ for any $i \in \{0, 1, 2, 3, 4\}$.

As P is not bipartite, there is an edge connecting vertices in $\{v_0, v_1, v_2, v_3, v_4\}$ or in $\{w_0, w_1, w_2, w_3, w_4\}$. As a result, there are two cases.

- If $\{v_0, v_1\}$ is an edge of P, then $\{v_i, v_{i+1}\} \in E(P)$ for any $i \equiv 0, 1, 2, 3, 4 \mod 5$. P is regular of degree 3, so there are two more edges connecting w_0 and w_i for some i. If $i \in \{1, 4\}$, then P contains a cycle of length 4, which is a contradiction. Thus, $\{w_i, w_{i+2}\} \in E(P)$ for any $i \equiv 0, 1, 2, 3, 4 \mod 5$.
- If $\{v_0, v_2\}$ is an edge of P, then $\{v_i, v_{i+2}\} \in E(P)$ for any $i \equiv 0, 1, 2, 3, 4 \mod 5$. P is regular of degree 3, so there are two more edges connecting w_0 and w_i for some i. If $i \in \{2, 3\}$, then P contains a cycle of length 4, which is a contradiction. Thus, $\{w_i, w_{i+1}\} \in E(P)$ for any $i \equiv 0, 1, 2, 3, 4 \mod 5$.

Consider $\rho \in G$ such that $v_0\rho = w_0$. Necessarily, ρ is an involution because otherwise v_0 has degree 4. Moreover, $S = \{\sigma, \rho\}$ is the connection set of P. Hence, $v_i\rho = w_i$ for $i \in \{0, 1, 2, 3, 4\}$. Therefore, ρ is not an automorphism because not all edges are preserved. This is a contradiction. $\square$

Corollary 2.4.5. The Petersen graph P is not a Cayley color digraph of any group G.

Proposition 2.4.6. The Petersen graph P is vertex-transitive.

PROOF. P can be considered as the Kneser graph $\mathrm{KG}_{5,2}$ as introduced by Kneser [Kne55]. Therefore, the vertices of P will be identified by the two-element subsets of $\{1, 2, 3, 4, 5\}$, such that a pair of vertices is adjacent if and only if their labels are disjoint (see figure 2.4).

For any $\sigma \in S_5$ where S_5 denotes the symmetric group on the set $\{1, 2, 3, 4, 5\}$ define a function λ_σ on $V(P)$:

$$\lambda_\sigma : V(P) \to V(P); \ \ \{x, y\} \mapsto \{\sigma(x), \sigma(y)\}.$$

Given two vertices $\{a, b\}, \{c, d\}$ and a permutation $\sigma \in S_5$, the following statements are equivalent:

- $\{a, b\}$ and $\{c, d\}$ are adjacent,
- $\{a, b\}$ and $\{c, d\}$ are disjoint,
- $\{\sigma(a), \sigma(b)\}$ and $\{\sigma(c), \sigma(d)\}$ are disjoint,
- $\{\sigma(a), \sigma(b)\}$ and $\{\sigma(c), \sigma(d)\}$ are adjacent.

Hence, for each $\sigma \in S_5$ the map λ_σ is an edge-preserving permutation and therefore an automorphism of P.

Considering the injective group homomorphism

$$\kappa : S_5 \to \mathrm{Aut}(P); \ \ \sigma \mapsto \lambda_\sigma,$$

it is clear that there is a subgroup $H \leq \mathrm{Aut}(P)$, such that $S_5 \cong H$. In fact, even $S_5 \cong \mathrm{Aut}(P)$ holds, see [HS93, Theorem 4.6].

Let $\{a, b\}$ and $\{c, d\}$ be two arbitrarily selected, different vertices of P. If $\{a, b\}$ and $\{c, d\}$ are disjoint, choose $\pi = (a, c, b, d) \in S_5$ (in cycle notation). Otherwise suppose without loss of generality that $a = c$, $b \neq d$ and choose $\pi = (b, d) \in S_5$. Now $\kappa(\pi) \in \mathrm{Aut}(P)$ and $\kappa(\pi)(\{x, y\}) = \{c, d\}$. Hence, P is vertex-transitive. □

2.5. Finite and infinite Graphs

Definition 2.5.1. Let $G = (V, E)$ be a graph.

- An edge xy, such that $G \setminus \{x, y\}$ is disconnected, is called a *hinge.*
- An edge, such that the removal of this edge separates G is called a *bridge.*
- G is *k-connected* if for every subset $X \subseteq V$ with $|X| < k$, $G \setminus X$ is connected. The largest $k \in \mathbb{N}$, such that G is k-connected is called the *connectivity number* or *connectivity* $\kappa(G)$.
- The *girth* $g(G)$ is the length of a shortest circle contained in G. For acyclic graphs, the girth is defined to be infinity.

Chapter 8 of Diestel's book [Die05] provides a good introduction to the theory of infinite graphs.

Definition 2.5.2. A graph G is called *locally finite* if every vertex of G has a finite degree.

Remark. All graphs considered in the following chapters are finite or locally finite.

Definition 2.5.3. A one-way infinite path, that is a graph $G = (V, E)$ with

$$\begin{aligned} V &= \{x_0, x_1, x_2, \dots\} \\ E &= \{x_0x_1, x_1x_2, x_2x_3, \dots\}, \end{aligned}$$

such that $x_i \neq x_j$ if $i \neq j$ is a *ray*. A two-way infinite path, i.e. a graph $G = (V, E)$ with

$$\begin{aligned} V &= \{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\} \\ E &= \{x_{-2}x_{-1}, x_{-1}x_0, x_0x_1, x_1x_2, x_2x_3, \dots\}, \end{aligned}$$

such that $x_i \neq x_j$ if $i \neq j$ is a *double ray.*

Definition 2.5.4. Consider the following equivalence relation on the set of rays in G: Two rays R_1, R_2 in G are equivalent if, for every finite subset $S \subseteq V(G)$, both R_1 and R_2 have a subray in the same connected component of G. This means that two rays are equivalent if they can not be separated by a finite set of vertices. An equivalence class of rays in G under this relation is called an *end* of G.

The *set of ends* of G is denoted by $\Omega(G)$.

Definition 2.5.5. The *vertex-degree* of an end is the maximum number of disjoint rays (or arcs, see definition 2.5.7) in it and the *edge-degree* of an end is the maximum number of edge-disjoint rays (or arcs) in it. Ends with infinite vertex-degree are called *thick ends*, ends with finite vertex-degree are called *thin ends.*

Definition 2.5.6. Let $G = (V, E)$ be an (infinite) graph with ends $\Omega(G)$. The *topological end space* $|G|$ is constructed as follows. Every edge $e = uv$ is a homeomorphic image of the interval $[0, 1]$.

- As basic open sets around an inner point of an edge e choose the subsets of e that correspond to open subintervals of $[0, 1]$ by the homeomorphism.
- Let $(uv)_\epsilon$ be the subset of uv corresponding to the interval $[0, \epsilon)$ by the homeomorphism between $[0, 1]$ and uv. The basic open sets around the vertex u are the union of all $(uv)_\epsilon$, for all $0 < \epsilon < 1$ and all neighbour vertices v of u in G.
- For every end ω and every finite $S \subseteq V$, let $C(S, \omega)$ be the connected component of $G - S$ that contains a ray of ω. Moreover, let $\Omega(S, \omega)$ be the set of ends with a ray in $C(S, \omega)$ and $E_\epsilon(S, \omega)$ the set of inner points of edges from S to $C(S, \omega)$ with a distance less than ϵ to their endpoint (the metric is defined by the bijection between $[0, 1]$ and the edge).

  The basic open sets around the end ω are all sets of the form

  $$C(S, \omega) \cup \Omega(S, \omega) \cup E_\epsilon(S, \omega)$$

  for every finite $S \subseteq V$ and every $0 < \epsilon \leq 1$.

Definition 2.5.7.

- $\overline{X}$ is the *closure* of $X \subseteq |G|$,
- a *standard subspace* of $|G|$ is a subspace which fully contains every edge if an inner point of the edge is included,
- an *arc* in $|G|$ is the homeomorphic image of the interval $[0, 1]$, the images of 0 and 1 are called its *endpoints*,
- a *circle* in $|G|$ is the homeomorphic image of the unit circle $S^1 \subseteq \mathbb{R}^2$,
- a *circuit* is the edge set of a circle.

Remark. If D is the circuit of a circle C, the closure of $\bigcup D$ is C.

Lemma 2.5.8 ([Die05, Lemma 8.5.6]). Let G be a locally finite graph. A closed standard subspace of $|G|$ is a circle if and only if

- C is connected and
- every vertex of C has degree 2 and
- every end of C has (vertex-)degree 2.

Definition 2.5.9. Let G be a locally finite graph. A family of $(D_i)_{i\in I}$ of subsets of $E(G)$ is called *thin* if there is no edge which lies in D_i for infinitely many $i \in I$. The *sum* $\sum_{i\in I} D_i$ of a thin family is the set of all edges e_i which lie in D_i for an odd number of $i \in I$. The *(topological) cycle space* $\mathcal{C}(G)$ is the set containing all sums of thin families of circuits.

Lemma 2.5.10 ([Die05, Theorem 8.5.8]). Let G be a connected, locally finite graph. The cycle space $\mathcal{C}(G)$ contains exactly the subsets of $F \subseteq E(G)$ such that every finite cut of G contains an even number of edges of F. Moreover, every element of $\mathcal{C}(G)$ is a disjoint sum of circuits.

2.6. Multi-ended groups and amalgamations

Definition 2.6.1. The number of ends of a group G is the number of ends of the Cayley graph $\mathrm{Cay}(G, S)$ for an arbitrarily chosen generating set S.

Remarks ([Geo08, Theorems 13.5.5 and 13.5.7]).

- The number of ends of every finite group is 0.
- The number of ends of every finitely generated group is well-defined (the number of ends of a Cayley graph is independent of its generating set S).
- The number of ends of a finitely generated group is $0, 1, 2$ or ∞.

Stallings characterized groups with infinitely many ends as those having a nontrivial decomposition by HNN-extensions or free products with amalgamations (defined in section 2.2).

Theorem 2.6.2 ([Sta68], [Sta71]). A finitely generated group G has infinitely many ends if and only if either

- $G = K *_F H$ where F is finite with index ≥ 2 in K and H and one of these indices being ≥ 3
- $G = H*_\phi$ where the subgroups identified by ϕ have index ≥ 2 in H.

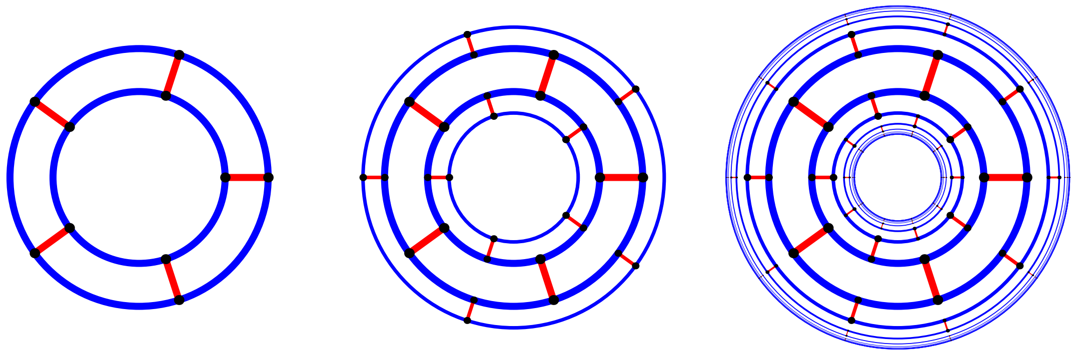
In his *accessibility theory* [Dun85], Dunwoody proved that every group with a finite presentation can be obtained from finite or one-ended groups using only finitely many steps of the operations in theorem 2.6.2. On the other hand, there are finitely generated groups which do not possess this property [Dun93].

Mohar [Moh06] constructed a similar amalgamation operation which he called *tree amalgamation*. He conjectured that, using the tree amalgamation, all planar Cayley graphs could be decomposed into finite or one-ended planar Cayley graphs, which would be a statement similar to theorem 2.6.2 for Cayley graphs instead of finitely generated groups.

To receive infinitely-ended (or two-ended) cubic planar Cayley graphs from one-ended cubic planar Cayley graphs, Mohar proposed the *twist-amalgamation*.

Definition 2.6.3. Let $G = \mathrm{Cay}\left\langle a, b \mid b^2, a^n, (ab)^m \right\rangle, n \geq 3, m \geq 2$. Then G is a cubic planar Cayley graph containing a-colored cycles of length n (see section 4.2).

For every a-colored cycle embed a copy of G in the face bounded by C and glue it along C as follows. Every edge of C is divided into two a-colored edges. The b-colored edges on C are alternately inwards and outwards. The embedded copy of G has the subdivision of C as a face boundary. See figure 2.5 for the case $n = 5, m = 2$.



(a) $G = \mathrm{Cay}\left\langle a, b \mid b^2, a^5, (ab)^2 \right\rangle$. (b) First twist-graph. (c) Twist-amalgamation.

Figure 2.5.: Twist-amalgamation of $G = \mathrm{Cay}\left\langle a, b \mid b^2, a^5, (ab)^2 \right\rangle$.

After performing one step of this operation, a cubic planar graph $G^{\underline{1}}$ (the first twist-graph), which has again a-colored cycles of length n is obtained. Another step of the described operation embedding copies of G in the a-colored cycles of $G^{\underline{1}}$ leads to $G^{\underline{2}}$. After n steps the *n-th twist-graph* $G^{\underline{n}}$ is produced. As n goes to infinity, the resulting graph $G^{\underline{\infty}}$ is called the *twist-amalgamation* of G.

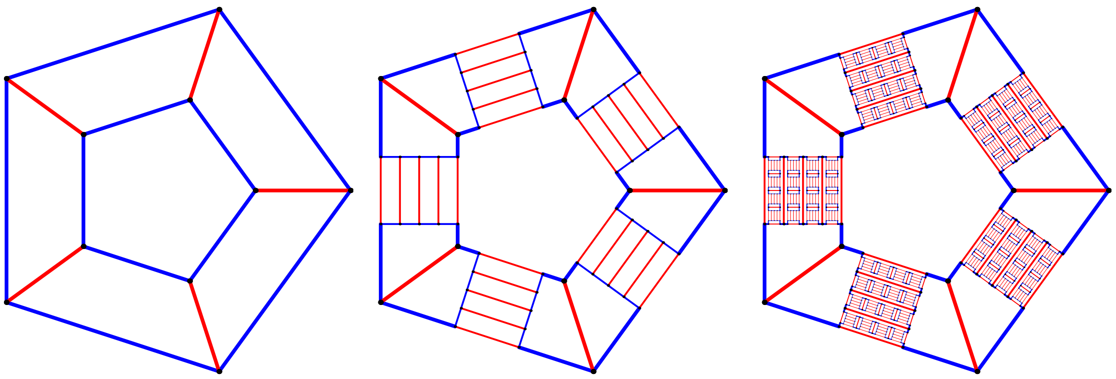
Remarks.

- $G^{\underline{0}} = G$.
- The twist-amalgamation of $G = \mathrm{Cay}\left\langle a, b \mid b^2, a^n, (aba^{-1}b)^m \right\rangle, n \geq 3, m \geq 1$ is defined analogously.
- The twist-amalgamation of a cubic planar Cayley graph is again a cubic planar Cayley graph.

However, not all cubic planar Cayley graphs can be obtained from finite or one-ended cubic planar Cayley graphs by *twist-amalgamation*. Georgakopoulos [Geo11a] suggested another amalgamation operation, which he calls *twist-squeeze-amalgamation*.

Definition 2.6.4. Let $G = \mathrm{Cay}\left\langle a, b \mid b^2, a^n, (ab)^m \right\rangle, n \geq 3, m \geq 2$. Then G is a cubic planar Cayley graph containing cycles of length $2m$ with alternating colors a and b.

For every such alternating cycle C embed a copy of G in the face bounded by C and glue it along C as follows. Every a-colored edge of C is divided into three edges: two a-colored edges and one b-colored edge in the middle. The embedded copy of G has the subdivision of C as a face boundary. See figure 2.6 for the case $n = 5, m = 2$.



(a) $G = \mathrm{Cay}\left\langle a, b \mid b^2, a^5, (ab)^2 \right\rangle$. (b) First twist-squeeze-graph. (c) Twist-squeeze-amalgamation.

Figure 2.6.: Twist-squeeze-amalgamation of $G = \mathrm{Cay}\left\langle a, b \mid b^2, a^5, (ab)^2 \right\rangle$.

After performing one step of this operation, a cubic planar graph $G^{\overline{1}}$ (the first twist-squeeze-graph), which has again alternately-colored cycles of length $2m$, is obtained. Another step of the described operation embedding copies of G in the alternately-colored cycles of $G^{\overline{1}}$ leads to $G^{\overline{2}}$. After n steps the *n-th twist-graph* $G^{\overline{n}}$ is produced. As n goes towards infinity, the resulting graph $G^{\overline{\infty}}$ is called the *twist-squeeze-amalgamation* of G.

Remarks.

- $G^{\overline{0}} = G$.

- The twist-squeeze-amalgamation of other types of cubic planar Cayley graphs is defined analogously.
- The twist-squeeze-amalgamation of a cubic planar Cayley graph is again a cubic planar Cayley graph.

2.7. Hamiltonicity of Cayley graphs

Definition 2.7.1. A *Hamilton cycle* in a finite graph G is a closed walk that visits every vertex exactly once. A graph G which contains a Hamilton cycle is called *Hamiltonian*.

This definition can be extended to infinite graphs using the topological end space.

Definition 2.7.2. A *Hamilton circle* is a circle in $|G|$ (see Definition 2.5.7) that contains every vertex of G. An infinite graph G is called *Hamiltonian* if $|G|$ contains a Hamilton circle.

Remark. Since a Hamilton circle is closed and contains every vertex it also contains every end of G.

In 1969, Lovász asked, whether every finite connected vertex-transitive graph contains a Hamiltonian path [Guy70, Problem 11]. In fact, there is no known example of a finite vertex-transitive graph without a Hamiltonian path. Moreover, only four finite connected vertex-transitive graphs with at least 3 vertices which do not have a Hamilton cycle are known, cf. [GR01, p. 45] and [Bab95, p. 25].

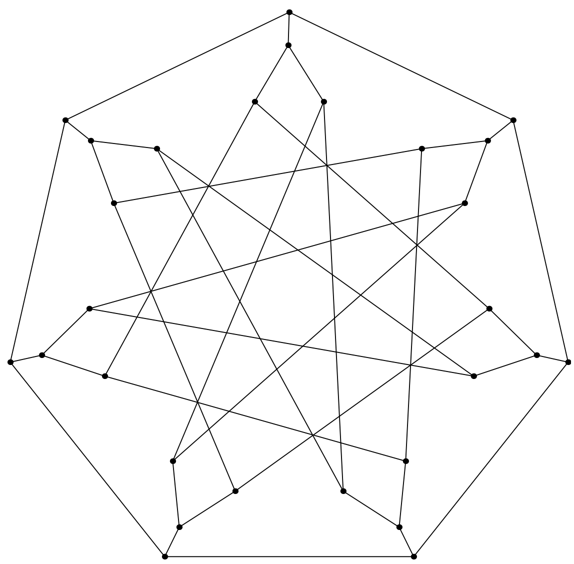


Figure 2.7.: The Coxeter graph.

- The Petersen graph (see figure 2.4) is vertex transitive (see proposition 2.4.6) and nonhamiltonian,
- the Coxeter graph (figure 2.7) is also nonhamiltonian (see [Tut60] for a proof) and vertex-transitive,
- the two graphs which are obtained from the Petersen graph and the Coxeter graph respectively by replacing each vertex by a triangle are nonhamiltonian and vertex-transitive, cf. [GR01, p. 45] and [Bab95, p. 25].

None of these graphs is a Cayley graph. This leads to conjecture 2.7.3 and the weaker conjecture 2.7.4, which are both widespread.

Conjecture 2.7.3. Every finite connected vertex-transitive graph with at least 3 vertices, except the four graphs mentioned above, contains a Hamilton cycle.

Conjecture 2.7.4. Every finite connected Cayley graph with at least 3 vertices contains a Hamilton cycle.

However, Babai did not agree and posed conjecture 2.7.5.

Conjecture 2.7.5 (Babai 1994). for some $c > 0$ there are infinitely many connected Cayley graphs without cycles of length $\geq (1-c)n$, where n is the number of vertices of the graph. Cf. [Bab95, p. 25]

Remarks.

- For Cayley digraphs, conjecture 2.7.4 is wrong. There are infinitely many counterexamples, namely
$$\Gamma\left(\mathrm{Sym}(n), \{(1,2),(1,2,3,\ldots,n)\}\right),$$
where $\mathrm{Sym}(n)$ is the symmetric group of order n. Those digraphs are nonhamiltonian if n is an even number with $n \geq 4$. For a proof, see [GR01, Corollary 3.8.2].
- For infinite graphs, conjecture 2.7.4 is also not correct. Obviously, every Cayley graph $\mathrm{Cay}(G,S)$ on a free group G of rank $r \geq 2$ with minimal generating set S is only 1-connected and therefore nonhamiltonian, see example 2.3.3.

Some special cases of conjecture 2.7.4 are solved:

- If G is a finite Abelian group, then all connected Cayley graphs $\mathrm{Cay}(G,S)$ are Hamiltonian, see [CQ81] and [CQ83],
- if $|G| = p^k$, where p is a prime and $|G| \neq 2$, then all connected Cayley graphs $\mathrm{Cay}(G,S)$ are Hamiltonian [Wit86],
- if $|G|$ can be expressed in one of the following possibilities (with p, q, r being distinct primes)
  - ⋆ kp with $1 \leq k \leq 31$ and $k \neq 24$,
  - ⋆ kpq with $1 \leq k \leq 5$,

  - $\star$ pqr,
  - $\star$ kp^2 with $1 \leq k \leq 4$,
  - $\star$ kp^3 with $1 \leq k \leq 2$,

  then all connected Cayley graphs $\mathrm{Cay}(G, S)$ are Hamiltonian [KMM$^+$10],

- if G is a finitely generated, infinite Abelian group, then every Cayley graph $\mathrm{Cay}(G, S)$ has a spanning double-ray and hence a Hamilton circle, see [Jun89] and [NW59].

As follows from the above, connected Cayley graphs on groups G with $3 \leq |G| \leq 71$ are Hamiltonian.

See the survey [CG96] and [KMM$^+$10] for more results on Hamiltonicity of special classes of Cayley graphs.

In [Geo09], Georgakopoulos asks if every connected 1-ended locally finite Cayley graph has a Hamilton circle.

Conjecture 2.7.6 (Georgakopoulos [Geo11a]). Every finitely generated 3-connected planar Cayley graph is Hamiltonian.

2.8. Cubic planar Cayley graphs

Motivated by the discovery of the amalgamation operations which are described in section 2.6, Georgakopoulos characterized the planar cubic Cayley graphs of connectivity two [Geo11b] and later all planar cubic Cayley graphs [Geo11a]. He used a rather complex distinction of cases considering connectivity, number of ends, number of generators, spin behavior, existence of one-colored or two-colored cycles, etc. Finally he got a classification containing 37 different types.

To distinguish the cases, some definitions are necessary concerning embeddings and spin behavior of edges and colors.

Definition 2.8.1. An *embedding σ of G* is, unless stated differently, a topological embedding of $|G|$ in the Euclidean plane $\mathbb{R}^2$, which is a drawing without crossing edges.

Definition 2.8.2. For a fixed embedding σ of a Cayley graph $\mathrm{Cay}(G, S) = (V, E)$ define the *spin* of $x \in V$ as the cyclic order of S, such that $s_2 \in S$ is a successor of $s_1 \in S$ if the edge $\{x, xs_2\}$ occurs immediatly after $\{x, xs_1\}$ when moving counter-clockwise around x.

Remark. If $\mathrm{Cay}(G, S) = (V, E)$ is cubic every vertex $x \in V$ has exactly 3 neighbours and there are only 2 possible spins.

Definition 2.8.3. Let $\mathrm{Cay}(G, S) = (V, E)$ be a cubic Cayley graph with a fixed embedding σ. An edge $e = xy \in E$ is called *spin-preserving*, if x and y have the same spin in σ. Otherwise it is called *spin-reversing*.

Definition 2.8.4. An embedding σ of $\mathrm{Cay}(G, S) = (V, E)$ is *consistent* if there are no two edges $u, v \in E$, such that u and v have the same color and u is spin-reversing and v is spin-preserving.

CHAPTER 3

Graphs of connectivity 1 or 2

A planar cubic Cayley graph G has connectivity 1, if and only if G has a bridge. Hence, $\kappa(G) = 1$ if and only if one of the generators of G does not satisfy a cyclic relation. Therefore, the only two types of 1-connected cubic planar Cayley graphs are

$$\mathrm{Cay}\left\langle a, b \mid b^2, a^n \right\rangle, n \in (\mathbb{N} \setminus \{1, 2\}) \cup \{\infty\},$$
$$\mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bc)^n \right\rangle, n \in \mathbb{N} \cup \{\infty\}.$$

Obviously, none of these is Hamiltonian since the removal of a single vertex disconnects the graph.

Proposition 3.0.5. Let G be a cubic, planar Cayley graph with $\kappa(G) = 1$. Then G is not Hamiltonian.

There are nine different types of cubic planar 2-connected Cayley graphs [Geo11b]. One type is a degenerate case, the other types are listed in table 3.1 and table 3.2. Surprisingly, not all of the graphs have a Hamilton circle. A class consisting of nonhamiltonian Cayley graphs exists.

Theorem 3.0.6. Let G be a cubic, planar Cayley graph with $\kappa(G) = 2$. Then G is Hamiltonian, unless $G = \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (b(cb)^n d)^m \right\rangle$, $m \geq 2, n \geq 3$.

Proof. The Hamiltonicity of the different types of cubic, planar Cayley graph with $\kappa(G) = 2$ is discussed in the propositions of this chapter. The nonhamiltonian graphs are considered in proposition 3.2.4. $\square$

Corollary 3.0.7. Let G be a cubic, planar, nonhamiltonian Cayley graph satisfying $\kappa(G) = 2$. Then G

- is generated by 3 involutions,
- has no 2-colored cycle,
- has no hinge and
- all generators of G preserve spin.

Proof. See the classification of the graphs in table 3.1 and table 3.2. $\square$

3.1. Graphs with two generators

The three different types of cubic planar Cayley graphs G with $\kappa(G) = 2$ which are generated by two elements are listed in table 3.1.

G has two generators a, b and b is an involution	a has infinite order and preserves spin	b reverses spin	$G = \text{Cay}\left\langle a, b \mid b^2, (aba^{-1}b^{-1})^n \right\rangle$, $n \geq 2$ G is Hamiltonian, see proposition 3.1.3.
		b preserves spin	$G = \text{Cay}\left\langle a, b \mid b^2, (ab)^n \right\rangle$, $n \geq 2$ G is Hamiltonian, see proposition 3.1.2.
	a has finite order	a reverses spin, b preserves spin	$G = \text{Cay}\left\langle a, b \mid b^2, a^4, (a^2b)^n \right\rangle$, $n \geq 2$ G is Hamiltonian, see proposition 3.1.1.

Table 3.1.: The cubic planar Cayley graphs of connectivity 2 with 2 generators.

Proposition 3.1.1. Every graph $G = \text{Cay}\left\langle a, b \mid b^2, a^4, (a^2b)^n \right\rangle$, $n \geq 2$ is Hamiltonian.

Proof. Let $v_0, v_1, v_2, \dots, v_{2n-1}$ be a cycle in G consisting of a-colored edges. Define two double-rays

$$P = \dots aba^{-1}baba^{-1}b\underbrace{a}_{v_0v_1}ba^{-1}b\dots = (aba^{-1}b)^{\infty}$$

$$Q = \dots a^{-2n+3}ba^{2n-3}b\underbrace{a^{-2n+3}}_{v_2v_3v_4\dots v_{2n-1}}ba^{2n-3}b\dots = (a^{2n-3}ba^{-2n+3}b)^{\infty}$$

by their edge-colors, such that P contains v_0v_1 as an a-edge arc and Q contains a subpath $v_2v_3v_4\dots v_{2n-1}$ consisting of a-colored edges.

Now it is easy to see that

- P and Q are disjoint,
- $V(P \cup Q) = V(G)$,
- $\overline{P}$ connects the two ends ω_1, ω_2 of G,
- $\overline{Q}$ connects ω_1 and ω_2,
- $\overline{P \cup Q}$ is a Hamilton circle in $|G|$. □

In figure 3.1 the Hamilton circle is black and dashed, a is blue and b is red.

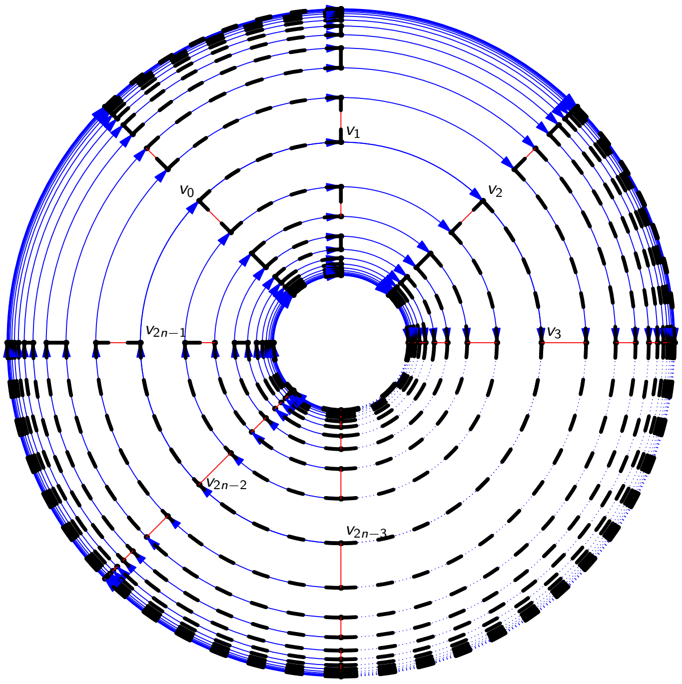


Figure 3.1.: A Hamilton circle in $G = \mathrm{Cay}\,\langle a, b \mid b^2, a^4, (a^2b)^n\rangle$, $n \geq 2$.

Proposition 3.1.2. Every graph $G = \mathrm{Cay}\,\langle a, b \mid b^2, (ab)^n\rangle$, $n \geq 2$ is Hamiltonian.

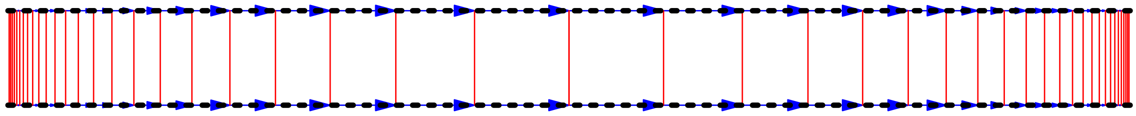


Figure 3.2.: A Hamilton circle in $G = \mathrm{Cay}\left\langle a, b \mid b^2, (ab)^2 \right\rangle$.

Proof. Depending on the choice of n, this graph is either a 2-way infinite ladder, shown in figure 3.2, or a graph with infinitely many ends, as shown in figure 3.3.

For $n = 2$, G is a 2-way infinite ladder and hence 2-ended. The ends are denoted ω_1 and ω_2. When all rungs (i.e. all b-labelled edges) are deleted, the remaining graph H is the disjoint union of two double-rays. Its closure $\overline{H} = H \cup \{\omega_1, \omega_2\}$ is a Hamilton circle in $|G|$, dashed in figure 3.2.

For $n \geq 3$, let H be again the union of all a-labelled edges. Then the closure $\overline{H}$ of H in $|G|$, that is the union of H with all ends of G, is again a Hamilton circle. In order to prove this claim, construct a suitable mapping from $\overline{H}$ to the unit circle, i.e. a homeomorphism from $\overline{H}$ to S^1. The infinitely-ended graph G consists of alternately a- and b-colored cycles of length $2n$ with every b-colored edge being a hinge. Take an arbitrary cycle C in G and label the $2n$ vertices of C

$$00, 01, 10, 11, 20, 21, 30, 31, \ldots, (n-1)0, (n-1)1$$

in clockwise order. Now consider a cycle B that shares an edge with C (called neighbour circle). Two adjacent vertices of B are already labelled $s0$ and $s1$, where s is a string consisting of the characters $0, 1, 2, \ldots, n-1$. Label the vertices of B

$$s0, s00, s01, s10, s11, s1$$

in clockwise order. Repeat this procedure for all neighbour circles of C, then also for their neighbour circles and so forth, to label all vertices of G. Now for every cycle except C consider the edge between $s01$ and $s10$ (where s is the result of the label construction before). By construction, this edge is a-labelled. Let D_s denote the double ray consisting only of a-edges which contains the edge $(s01, s10)$. Furthermore, let R_0 be the double ray consisting of a-edges containing the edge $(00, 01)$ and R_i the double ray consisting of a-edges containing the edge $((i-1)1, i0)$, where $1 \leq i \leq n-1$.

Map R_0 to one half of S^1, such that the points corresponding to 21 and 00 occur in clockwise order and split the other half into $2n-1$ parts, as outlined in figure 3.2 for $n = 3$. Leave every second part (starting with the first part) empty and map $R_1, R_2, \ldots, R_{n-1}$ to the remaining parts (in clockwise order), such that $(i-1)i$ and $i0$ occur in clockwise order. Split every empty part into three subparts and map $D_0, D_1, \ldots, D_{n-1}$ to the subparts in the middle (again in clockwise order). On D_i, $i01$ and $i10$ are in clockwise order.

Now repeat the following steps infinitely often for $0 \leq i \leq n-1$ and suitable strings s:

- In the middle third of the empty part between R_i and D_{is}, map D_{is0}, such that $s001$ and $s010$ are in clockwise order,

- in the middle third of the empty part between D_{js} and R_i, map D_{js1}, such that $j \equiv i-1 \mod n$, $0 \leq j \leq n-1$ and $s101$ and $s110$ are in clockwise order,
- in the middle third of the empty part between D_{s0} and D_s, map D_{s01}, such that $s0101$ and $s0110$ are in clockwise order,
- in the middle third of the empty part between D_s and D_{s1}, map D_{s10}, such that $s1001$ and $s1010$ are in clockwise order.

The construction of the mapping is outlined for $n = 3$ in figure 3.2. Each vertex of G is incident to exactly 2 edges of H and $\overline{H}$ forms a circle in $|G|$. So, $\overline{H}$ is a Hamilton circle of $|G|$. □

Proposition 3.1.3. Every graph $G = \mathrm{Cay}\left\langle a, b \mid b^2, (aba^{-1}b^{-1})^n \right\rangle$, $n \geq 1$ is Hamiltonian.

Proof. Such as the graphs considered in the last proposition, G consists of alternately a- and b-colored cycles of length $2n$ with every b-colored edge being a hinge. The only difference to $H = \mathrm{Cay}\left\langle a, b \mid b^2, (ab)^{2n} \right\rangle$ is the direction of some a-colored arcs. Every vertex of G can be represented as a reduced word of the form

$$x = a^{k_0} b a^{k_1} b a^{k_2} b \cdots a^{k_{n-1}} b a^{k_n}, \quad n \in \mathbb{N}_0,\ k_0, k_n \in \mathbb{Z}, \quad \forall\, 1 \leq i \leq n-1:\ k_i \in \mathbb{Z} \setminus \{0\}.$$

Now the function

$$\begin{aligned} &\phi : V(G) \to V(H); \\ &\quad \phi(x) = \phi(a^{k_0} b a^{k_1} b a^{k_2} b \cdots a^{k_{n-1}} b a^{k_n}) = a^{k_0} b a^{-k_1} b a^{k_2} b a^{-k_3} \cdots a^{(-1)^{n-1} k_{n-1}} b a^{(-1)^n k_n} \end{aligned}$$

is a graph-isomorphism between G and H. According to proposition 3.1.3, H is Hamiltonian. Therefore, G is also Hamiltonian. □

3.2. Graphs with three generators

The five non-degenerate types of cubic planar Cayley graphs G with two generators and $\kappa(G) = 2$ are listed in table 3.2.

Proposition 3.2.1. Every graph $G = \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bc)^2, (bcd)^m \right\rangle$, $m \geq 2$ is Hamiltonian.

Proof. The embedding of G is constructed step by step. Starting with the tetrahedral graph $G_0 = \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, bcd \right\rangle$ (see figure 4.1) in the first step, the d-colored arcs of G_0 are divided into m parts. Between each consecutive parts, a copy of G_0 is placed. In the resulting graph G_1, $m-1$ copies of G_0 are placed on all new d-colored arcs to receive G_2 and so forth. See [Geo11b, Theorem 4.6, case 1] for details of the construction.

In G_0, consider the path C_0 with color-sequence $bebe$, where e is the same color as d. The e-edges are the d-colored edges, which are replaced in the next step. Note that C_0 is uniquely determined by its edge-color sequence. C_0 is a Hamilton cycle in G_0 formed

G has three generators b, c, d				
G has a 2-colored cycle $(bc)^n = 1$			G has no 2-colored cycle	
G has no hinge		G has a hinge	G has no hinge	G has a hinge
b, c, d preserve spin	only c preserves spin	b, c, d reverse spin	b, c, d preserve spin	only b preserves spin
$G = \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bc)^2, (bcd)^m \right\rangle$, $m \geq 2$ G is Hamiltonian, see proposition 3.2.1.	$G = \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bc)^{2n}, (cbcd)^m \right\rangle$, $n, m \geq 2$ G is Hamiltonian, see proposition 3.2.2.	$G = \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bc)^n, (bd)^m \right\rangle$, $n, m \geq 2$ G is Hamiltonian, see proposition 3.2.3.	$G = \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (b(cb)^n d)^m \right\rangle$, $n, m \geq 2$ G is Hamiltonian iff $n = 2$, see proposition 3.2.4.	$G = \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bcbd)^n \right\rangle$, $n \geq 1$ G is Hamiltonian, see proposition 3.2.5.

Table 3.2.: The cubic planar Cayley graphs of connectivity 2 with 3 generators.

by all edges with colors b or d of G_0. Performing one step of the construction of G, every e-edge of C_0 is replaced by $d(bebd)^{m-1}$ to receive C_1. Following the path of C_1 with $e = d$ in G_1, this is again a Hamilton cycle consisting of all b-colored or d-colored edges of G_1. Replacing all e-edges in C_1 by $d(bebd)^{m-1}$ and considering $e = d$ leads to a Hamilton cycle C_2 of G_2 etc. In every step of this construction, C_n is a Hamilton cycle of G_n which consists of all edges with colors b or d.

Finally, the closure $\overline{C}$ of the circuit $C = \{e \in E(G) \,|\, e \text{ is } b\text{-colored or } d\text{-colored}\}$, is a subset of G that contains all vertices (and all ends since $\overline{C}$ is closed). In $\overline{C}$, every vertex has degree 2. From the structure of $\overline{C}$ it follows that any (standard subspace) neighbourhood of an arbitrary end is connected by exactly two edges to the remaining graph. Hence, all ends have vertex degree 2 and $\overline{C}$ is a Hamilton circle of G. □

Proposition 3.2.2. Every graph $G = \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bc)^{2n}, (cbcd)^m \right\rangle$, $n, m \geq 2$, is Hamiltonian.

Proof. The arguments in this proof are analogous to the proof of proposition 3.2.1. The construction of G starts with $G_0 = G = \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bc)^{2n}, cbcd \right\rangle$. Again, every d-edge is divided into m parts, placing a copy of G_0 between each two consecutive parts. See [Geo11b, Theorem 4.6, case 2] for a description of the construction. In G_0, consider the path C_0 with color-sequence $(be)^6$, where e is the same color as d. C_0 is a Hamilton cycle in G_0 formed by all b-edges or d-edges of G_0. In every step of the construction, every e-edge of C_{n-1} is replaced by $\left(d\,(be)^5\,b\right)^{m-1} d$ to obtain a Hamilton cycle C_n of G_n. Finally, the closure $\overline{C}$ of the circuit $C = \{e \in E(G) \,|\, e \text{ is } b\text{-colored or } d\text{-colored}\}$, is a Hamilton circle of G. □

Proposition 3.2.3. Every graph $G = \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bc)^n, (bd)^m \right\rangle$, $n, m \geq 2$, is Hamiltonian.

Proof. For $n = m = 2$, G is the 2-way-infinite ladder and hence Hamiltonian (see proposition 3.1.3). For any other choice of n and m, G has the structure of semi-regular trees merged at a common root, in which each vertex is replaced by a cycle. Two neighbour cycles share a b-colored edge. Hence, the structure of G is similar to the structure of the graphs in proposition 3.1.3. Let $C = \{e \in E(G) \mid e \text{ is } c\text{-colored of } d\text{-colored}\}$. Then the closure $\overline{C}$ is a Hamilton circle in G. The construction of a homeomorphism between $\overline{C}$ and S^1 is similar to the construction in the proof of proposition 3.1.3. □

Proposition 3.2.4. The graph $G = \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (b(cb)^n d)^m \right\rangle$, $m \geq 2$, $n \geq 2$, is Hamiltonian if and only if $n = 2$.

Proof. Assume that C is a Hamilton circle in $G = \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (b(cb)^n d)^m \right\rangle$, $m \geq 2$, $n \geq 3$. Using the notation of [Geo11b, Lemma 4.3], let D_0 be a double ray in G that is spanned by b and c and G_0 be the graph which results from G after contracting all edges that are not incident to D_0. Then G_0 consists of D_0 together with an infinite set $\mathcal{P}$ of pairwise disjoint d-colored paths of length 2. For every vertex $x \in V(D_0)$ there is a path $P \in \mathcal{P}$ joining x and $xb(cb)^n$.

Take an arbitrary $P \in \mathcal{P}$ and let $x, y \in V(G_0) \subseteq V(G)$ be the endvertices of P. Let $e_1, e_2 \in E(G)$ be the two d-edges incident to x and y. Since deleting e_1 and e_2 disconnects G, the Hamilton circle C necessarily contains those two edges. Moreover, x and y are joined by an arc of C which does not contain any edge of D_0.

For every $v \in V(D_0) \subseteq V(G)$, the vertices v and $vb(cb)^n$ are joined by an arc of C which does not contain any edge of D_0. Hence, C uses all d-edges of G incident to a vertex of G_0. Furthermore, C has to contain another edge of G_0 because every vertex has degree 2 in C. Consider two cases. In the first case, C contains a b-colored edge of G_0, in the second case, it does not.

case 1 If C contains a b-colored edge of G_0, it cannot contain any of the adjacent c-colored edges. Therefore, C must use the b-colored edge adjacent to those c-colored edges and so forth. Hence, in the first case, C has to contain all b-colored edges of G_0 but no other edges of G_0. Choose $x \in V(D_0)$ arbitrarily and let

$$x_1 = x,\ x_2 = xbc,\ x_3 = x(bc)^2$$

be vertices of D_0. Let R_1 be the arc of C connecting x_1 and an end of D_0 which uses the d-edge incident to x_1. Denote the endpoint of R_1 that is an end ω_1. Consider another arc R_2 of C connecting x_2 and ω_1 which uses the d-edge incident to x_2 and an arc R_3 of C connecting x_3 and ω_1 which uses the d-edge incident to x_3.

Then $R_1 \cap D_0$ consists of the b-edges from $x(bc)^{kn}$ to $x(bc)^{kn}b$, $R_2 \cap D_0$ consists of the b-edges from $x(bc)^{kn+1}$ to $x(bc)^{kn+1}b$ and $R_3 \cap D_0$ consists of the b-edges from $x(bc)^{kn+2}$ to $x(bc)^{kn+2}b$ for $k \in \mathbb{N}_0$. Since $n \geq 3$, the arcs R_1, R_2, R_3 are pairwise disjoint and have ω_1 as a common endpoint.

Hence, the vertex-degree of ω_1 in C is at least 3, which is a contradiction to the fact that C is a circle.

case 2 If C contains no b-colored edge of G_0, it has to contain a c-colored edge of G_0. Hence, C contains all c-colored edges of G_0 but no other edges of G_0. Analogously to case 1, choose $x \in V(D_0)$ arbitrarily and define x_1, x_2, x_3 and R_1, ω_1, R_2, R_3 in the same way. As before, the arcs R_1, R_2, R_3 are pairwise disjoint and have ω_1 as a common endpoint. Again, this is a contradiction to the fact that C is a circle.

Consider the special case $G = \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (b(cb)^2 d)^m \right\rangle$, $m \geq 2$. The edge set $D = \{e \in E(G) \mid e \text{ is } b\text{-colored or } d\text{-colored}\}$ is the circuit of a Hamilton circle.

To prove this claim, note that, using the notation of the first part of this proof, the edge set

$$D_0 = \{e \in E(G_0) \mid e \text{ is } b\text{-colored or } d\text{-colored}\}$$

forms two disjoint double-rays connecting the ends ω_1 and ω_2 of G_0 and containing all vertices of G_0. Hence $C_0 = \overline{D_0}$ is a Hamilton circle of G_0. Performing one step decontracting a double-ray D_i to transform G_{i-1} into G_i, two circles containing $D_0 \cup D_1 \cup \cdots \cup D_{i-1}$ and D_i respectively are merged along their two common d-edges to obtain a circle $C_i = \overline{D_i}$, where

$$D_i = \{e \in E(G_i) \mid e \text{ is } b\text{-colored or } d\text{-colored}\}.$$

In every step of this construction, C_i is a Hamilton circle.

$C = \overline{D}$ contains all vertices of G and (since it is closed) all ends of G. C is connected and every vertex has degree 2. From the structure of C it follows that any (standard subspace) neighbourhood of an arbitrary end is connected by exactly two edges to the remaining graph. Hence, all ends of C have vertex degree 2 and C is a Hamilton circle.□

Remark. Georgakopoulos presented the following problem [Geo09, problem 2]:

> Let G be a connected Cayley graph of a finitely generated group Γ. Prove that G has a Hamilton circle unless there is a $k \in \mathbb{N}$ such that Γ is the amalgamated product of more than k groups over a subgroup of order k.

The nonhamiltonian graphs in proposition 3.2.4 might provide counterexamples to this problem.

Proposition 3.2.5. Every graph $G = \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bcbd)^n \right\rangle$, $n \geq 1$, is Hamiltonian.

Proof. For $n = 1$, G is the 2-way-infinite ladder and hence Hamiltonian (see proposition 3.1.3). For any other choice of (n, m), G has the structure of regular trees merged to a common root, where every vertex is replaced by a cycle with edge-colors $(bcbd)^n$. Two neighbour cycles are merged along a b-colored edge and all cycles have a length of $4n$. Hence, G is isomorphic to $H = \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bc)^{2n}, (bd)^{2n} \right\rangle$, which is Hamiltonian according to proposition 3.2.3. □

Proposition 3.2.6. Every graph $G = \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bc)^n, cd \right\rangle$, $n \geq 1$, is Hamiltonian.

Proof. This is a degenerate case with two generators c and d being equal to each other. The resulting graph is a cycle with n double edges and n single edges and hence Hamiltonian. □

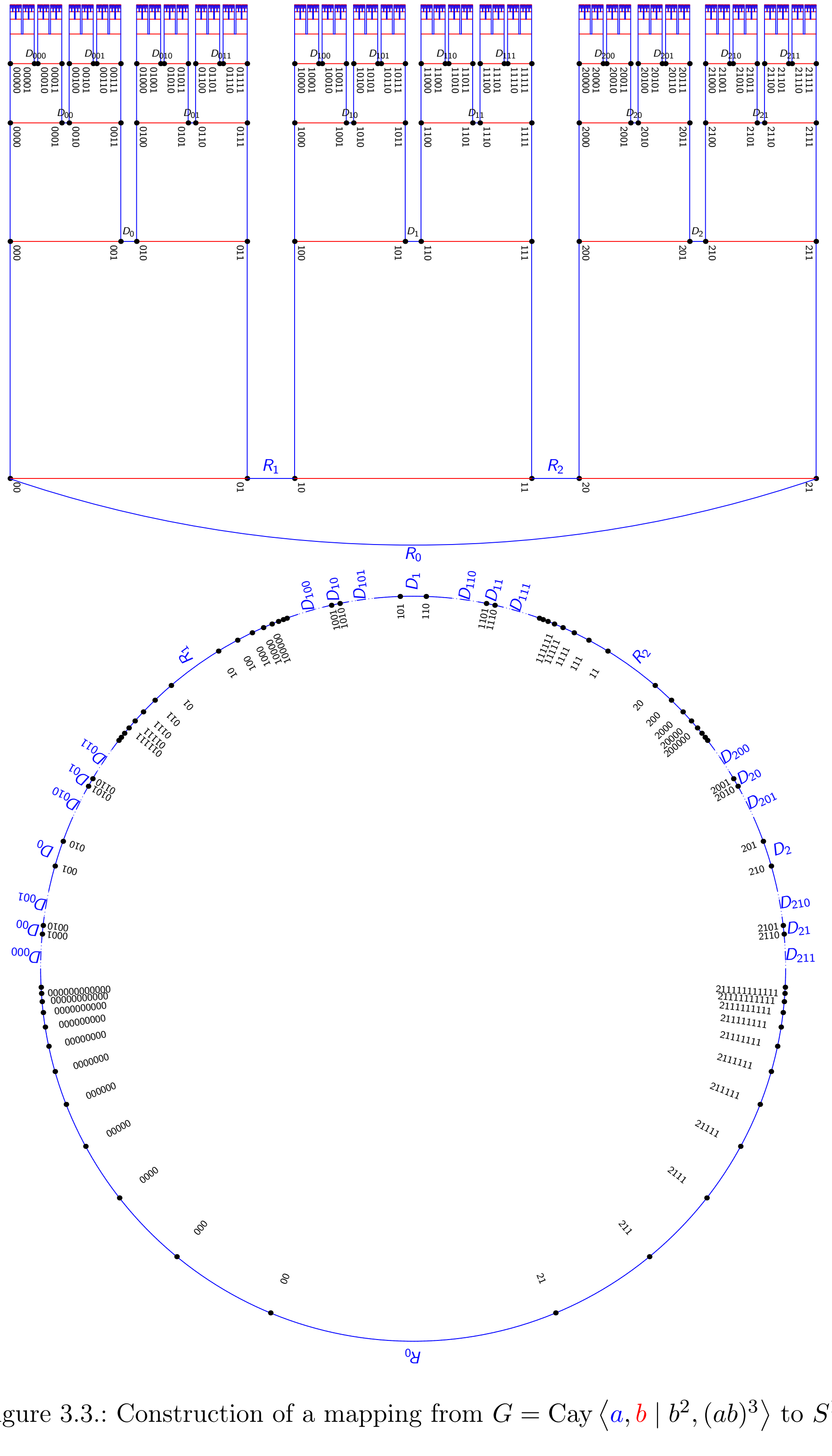


Figure 3.3.: Construction of a mapping from $G = \mathrm{Cay}\left\langle a, b \mid b^2, (ab)^3 \right\rangle$ to S^1.

CHAPTER 4

Finite and one-ended graphs

The finite and one-ended cubic planar Cayley graphs play a special role among the graphs considered in this thesis since they are used for twist-amalgamations and twist-squeeze-amalgamations to obtain new cubic planar Cayley graphs with more than one end.

For illustration and description of the finite planar cubic vertex-transitive graphs see [Zel77]. All of the finite or one-ended cubic planar Cayley graphs are 3-connected [Geo11a, theorem 7.1]. Furthermore, all of them are regular tessellations (of the sphere or the Euclidean plane or the hyperbolic plane) or Cayley graphs of rotation subgroups of the symmetry group of semi-regular tessellations, which is proved in this chapter.

Theorem 4.0.7. Let G be a finite or one-ended, cubic, planar Cayley graph. Then G is Hamiltonian.

Proof. There are 9 types of finite or one-ended, cubic, planar Cayley graphs. The graphs with two generators are listed in table 4.1, the graphs with three generators in table 4.3. The Hamiltonicity is proved in the propositions of the sections 4.2 and 4.3. $\square$

4.1. Tessellations and symmetry groups

See chapters 4 and 5 of [CM72] for an overview of tessellations and symmetry groups.

In the Euclidean space, a regular p-gon has an internal angle of $\phi = \pi\left(1 - \frac{2}{p}\right)$. Therefore, q regular p-gons can be placed around a common vertex, if

$$\begin{aligned} & & q\phi &= 2\pi \\ &\Leftrightarrow & q\pi\left(1-\frac{2}{p}\right) &= 2\pi \\ &\Leftrightarrow & qp - 2p - 2q &= 0 \\ &\Leftrightarrow & (q-2)(p-2) &= 4. \end{aligned}$$

For $(p, q) \in \mathbb{N}^2$, this equation has the solutions $(3, 6)$, $(4, 4)$ and $(6, 3)$. Hence, there are three regular tessellations of the Euclidean plane.

Definition 4.1.1. Let $p, q \in \mathbb{N}$, such that $p, q \geq 2$. Then $\{p, q\}$ denotes *regular tessellation* (of the sphere or the Euclidean plane or the hyperbolic plane) with q p-gons meeting at each vertex.

The regular Euclidean tessellations are $\{3, 6\}$, $\{4, 4\}$ and $\{6, 3\}$.

If $(q-2)(p-2) < 4$, a spherical geometry can be used to obtain a regular tessellation. In this case, the internal angle ϕ of a p-gon is greater than $\pi\left(1 - \frac{2}{p}\right)$. This results in the regular spherical tessellations

$$\{2, q\}, \{3, 3\}, \{3, 4\}, \{3, 5\}, \{4, 3\}, \{5, 3\}, \{q, 2\}$$

which correspond to the q-gonal hosedron (q great circles joining north- and south-pole), the five platonic solids and the q-gonal dihedron (q points on the equator joined to a circle).

If $(q-2)(p-2) < 4$, a hyperbolic geometry gives the regular hyperbolic tessellations $\{p, q\}$ with q p-gons meeting at each vertex. Here, the internal angle ϕ of a p-gon is less than $\pi\left(1 - \frac{2}{p}\right)$.

The symmetry group of $\{p, q\}$ is generated by three reflections R_1, R_2, R_3 in the side of a (spherical, Euclidean or hyperbolic) triangle with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$ and $\frac{\pi}{2}$.

Definition 4.1.2. Let $p, q \in \mathbb{N}$, such that $p, q \geq 2$. The *symmetry group* of $\{p, q\}$ is denoted by $[p, q]$.

A presentation of this symmetry group is

$$[p, q] = \left\langle R_1, R_2, R_3 \mid R_1^2, R_2^2, R_3^2, (R_1R_2)^p, (R_1R_3)^2, (R_2R_3)^q \right\rangle.$$

The Cayley graph of $[p, q]$ can be embedded as a semi-regular tessellation where a $2q$-gon, a $2p$-gon and a square meet at each vertex.

An important subgroup of $[p, q]$ is the *rotation subgroup* $[p, q]^+$.

Definition 4.1.3. Let $p, q \in \mathbb{N}$, such that $p, q \geq 2$. The subgroup of $[p, q]$ generated by the three rotations

$$R = R_1R_2,\ S = R_2R_3,\ T = R_1R_3.$$

is denoted by $[p, q]^+$.

However, the generating set $\{R, S, T\}$ is not minimal since two of the elements are sufficient to generate $[p, q]^+$. Hence, the presentations

$$[p, q]^+ = \left\langle S, T \mid S^q, T^2, (ST)^p \right\rangle = \left\langle R, S \mid R^p, S^q, (RS)^2 \right\rangle$$

are obtained. If $(p-2)(q-2) < 4$, the order of $[p, q]^+$ is

$$\frac{4pq}{4-(p-2)(q-2)},$$

otherwise the group is infinite. A possible embedding of the Cayley graph of $[p, q]^+$ is a semi-regular tessellation, where two $2p$-gons and one q-gon meet at each vertex.

These definitions can be generalized using the reflections R_1, R_2, R_3 in the side of a (not necessarily perpendicular) triangle with angles $\frac{\pi}{p}$, $\frac{\pi}{q}$ and $\frac{\pi}{r}$. If $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} > 1$, the underlying spherical geometry gives a finite symmetry group, if $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} \leq 1$, the Euclidean or hyperbolic geometry results in an infinite symmetry group. In all cases,

$$[p, q, r] = \left\langle R_1, R_2, R_3 \mid R_1^2, R_2^2, R_3^2, (R_1R_2)^p, (R_1R_3)^r, (R_2R_3)^q \right\rangle.$$

The rotation subgroup is

$$[p, q, r]^+ = \langle R, S, T \mid R^p, S^q, T^r \rangle.$$

Moreover, the Cayley graph of $[p, q, r]^+$ can be embedded as the semi-regular tessellation (of the sphere or the Euclidean plane or the hyperbolic plane) with one $2p$-gon, one $2q$-gon and one $2r$-gon meeting at each vertex.

For hyperbolic symmetry groups and their rotation subgroups, some results concerning Hamiltonicity are known, see [DJW95].

Lemma 4.1.4 (cf. [Rap59]). Let G be a connected, locally finite, one-ended graph which fulfills the following conditions.

1. There is a set $\mathcal{C}$ of pairwise-disjoint cycles containing every vertex just once and a set $\mathcal{R}$ of (not necessarily disjoint) 4-cycles containing every vertex.
2. Whenever two of the 4-cycles in $\mathcal{R}$ intersect, the intersection is only a single vertex and whenever one of the cycles in $\mathcal{C}$ intersects a 4-cycle in $\mathcal{R}$, the intersection is a single edge.
3. Every edge of G lies on one of the cycles in $\mathcal{C}$ or on one of the 4-cycles in $\mathcal{R}$ or both.

Then G is Hamiltonian.

A proof of this lemma can be found in [DJW95, theorem 3.1]. The Hamilton circle is constructed as the limit of a sequence of cycles. Every cycle is the boundary of a union U_i of faces in G. In every step, a polygon C_{i+1} of $\mathcal{C}$ is annexed to the present union U_i, such that a 4-cycle R_{i+1} of $\mathcal{R}$ connects C_{i+1} to some polygon C_j, $j \leq i$.

Lemma 4.1.5 (cf. [Geo10]). Every cubic, 1-ended planar graph G of girth $g(G) \geq 6$ is Hamiltonian.

A modified version of Georgakopoulos' proof is provided here.

Proof. Let x be an arbitrary face of G. By induction, construct a decomposition of the faces of G into disjoint sets $R_0 = \{x\}, R_1, R_2, \ldots$, such that for $i \geq 1$:

- The faces in R_i can be arranged in an order $f_1, f_2, \ldots, f_n$ such that f_j shares precisely one edge with f_k if $j \equiv k \pm 1 \mod n$ and f_j does not share an edge with any other face in R_i.
- The boundary of $\bigcup R_i$ consists of two disjoint cycles C_i and D_i. The interior U_i of $\bigcup R_i$ divides G into two parts. One of the two components of $G \setminus U_i$ is $F_i = \bigcup_{j<i} R_j$. The boundary of F_i is C_i.
- For any face f of R_i, $i \geq 1$, the number of faces in R_{i-1} that share an edge with f is exactly one or two.

Suppose that $R_0, R_1, \ldots, R_{i-1}$ are already defined and fulfill the above requirements. D_0 is the boundary of x.

Let R_i be the collection of all faces f of G such that f contains an edge of D_{i-1}, but f is not contained in R_{i-1}. The following steps are going to show that this collection meets the requirements for the induction.

Let $D_{i-1} = v_1 v_2 v_3 \ldots v_n v_1$ and for $1 \leq j \leq n$ let e_j be the edge such that e_j is incident to v_j and e_j is not contained in D_{i-1}. Choose a face f_1 of R_i, such that f_1 contains v_1 and for $2 \leq j \leq n$ let f_j be the face of R_i that shares the edge e_{j-1} with f_{j-1}.

Assume that there is a face f in R_i that shares three subsequent edges e_1, e_2, e_3 with faces in R_{i-1}. Suppose that e_1, e_2, e_3 occur in this order on D_{i-1}. Let g_j be the face in R_{i-1} incident with e_j for $1 \leq j \leq 3$. By the induction hypothesis, g_2 shares not more than two edges with R_{i-2}. Moreover, there is one edge between g_1 and g_2 and one edge between g_2 and g_3. Together with e_2 there are not more than 5 edges on the boundary of g_2. Hence, G contains a cycle of length ≤ 5. This is a contradiction.

Assume that f_j and $f_{j+1 \mod n}$ share two edges $e_1 = x_1 x_2$ and $e_2 = y_1 y_2$. W.l.o.g. let x_1, x_2, y_1, y_2 be the clockwise order of those vertices on the boundary of f_{i+1}. Then there are two cycles $C' = x_2 Q_1 y_1 P_1 x_2$ and $C'' = x_1 Q_2 y_2 P_2 x_1$ such that $P_1 \cup P_2$ is the boundary of f_j and $Q_1 \cup Q_2$ is the boundary of f_{i+1}. Either the side S_1 of C', which does not contain x_1, or the side S_2 of C'', which does not contain x_2 is a finite component of G. If S_1 is finite, consider the first twist-graph (see definition 2.6.3) of S_1 along C'. This leads to a finite, cubic, planar graph H of girth ≥ 6. Since every edge belongs to two faces and every face consists of at least six edges, $|F(H)| \leq 3|E(H)|$. Furthermore H is cubic, which means $|E(H)| = 6|V(H)|$. Together with Euler's formula $|V(H)| - |E(H)| + |F(H)| = 2$, this is a contradiction. If S_2 is finite, the twist-graph of S_2 along C'' gives the same contradiction.

Assume that for some $k \neq j : f_k = f_j$. Then there are two edges y_1y_2 and y_2y_3 in $f_k \cap D_{i-1}$. Suppose that y_1, y_2, y_3, y_4 lie on D_{i-1} in this order. Consider the cycle C, such that $C = y_2Py_3Qy_2$ and Q is a subpath of D_{i-1}, P is a subpath of the boundary of f_k. Since no face in R_i shares three subsequent edges e_1, e_2, e_3 with faces in R_{i-1} and the girth of G is ≥ 6, both sides of C are non-empty. One of the sides of C contains a finite component S of G. The twist-graph of S along C is a cubic, planar graph with girth ≥ 6. Again, this is a contradiction.

Assume that f_j and f_k share an edge $e = x_1x_2$, where $j \not\equiv i+1 \mod n$. Consider the edges y_1y_2 of $f_j \cap D_{i-1}$ and y_3y_4 of $f_k \cap D_{i-1}$. Let C be a cycle in G, such that $C = x_1P_1y_3Qy_2P_2x_1$ and Q is a subpath of D_{i-1}, P_2 is a subpath of the boundary of f_j, P_1 is a subpath of the boundary of f_k. Because of the same argument given above, this leads to a contradiction.

Hence, the boundary of $\bigcup R_i$ consists of two cycles C_i and D_i and all conditions of the induction are satisfied. This is also true for R_1, which provides the induction basis.

Therefore, the sequence $R_0, R_1, \ldots, R_{i-1}$ exists as claimed. This fact can be used to construct a double ray T spanning G.

Every vertex $v \in G$ lies on D_i for some $i \geq 0$. Call a vertex $v \in D_i$ *good* if v is adjacent to some vertex $w \in D_{i+1}$. Otherwise, v is adjacent to some vertex $w \in D_{i-1}$ and v is called *bad*.

Assume that v and w are adjacent vertices on D_i and both v and w are bad. Then a face f of R_i exists, such that $R_i \cap D_i = vw$. Since $E(f \cap D_{i-1}) \leq 2$, the boundary of f is a cycle of length ≤ 5. This is a contradiction. Hence, the neighbour vertices on D_i of a bad vertex are always good.

Start with a path T_0 spanning D_0. Note that all vertices of T_0 are good. Let p_i, q_i be the endvertices of T_i. Let p_i' and q_i' be the unique vertices of D_{i+1} that are adjacent to p_i and q_i respectively. Both p_i' and q_i' are bad. Consider the disjoint subpaths P_i and Q_i of D_{i+1}, such that

- $V(P_i) \cup V(Q_i) = V(D_{i+1})$,
- P_i has endvertices p_i' and q_i'',
- Q_i has endvertices q_i' and p_i'',
- $p_i'p_i''$ is an edge of D_{i+1},
- $q_i'q_i''$ is an edge of D_{i+1}.

Then

$$T_{i+1} = q_i''P_ip_i'p_iT_iq_iq_i'Q_ip_i''$$

is a path spanning $\bigcup_{j\leq i+1} D_j$ with good endvertices $p_{i+1} = p_i''$ and $q_{i+1} = q_i''$.

Finally, $D = \bigcup_{j \geq 0} P_j$ is a spanning double ray of G. □

Remark. Consider one side F of the Hamilton circle in the aforementioned proof. Then F is the union of faces $g_1, g_2, g_3, \ldots$, such that g_i and g_{i+1} intersect in exactly one edge and $g_i \cap g_j = \emptyset$ if $j \neq i \pm 1$. This fact is important for the construction of a Hamilton circle in the twist-amalgamation of G. See lemma 5.2.1 for details.

Corollary 4.1.6. Let $q \in \mathbb{N}$, with $q \geq 6$ and let G be the (graph of the) regular tessellation $\{q, 3\}$ of the Euclidean or hyperbolic plane. Then G is Hamiltonian.

Corollary 4.1.7. $G = \mathrm{Cay}([3,6]^+) = \mathrm{Cay}\left\langle S, T \mid S^6, T^2, (ST)^3 \right\rangle$ is Hamiltonian.

Proof. G can be embedded as the regular tessellation $\{6, 3\}$. □

Lemma 4.1.8. Let $p, q \in \mathbb{N}$, such that $(p-2)(q-2) > 4$. Then $G = \mathrm{Cay}([p,q])$ is Hamiltonian.

Proof. G can be embedded as the semi-regular tessellation of the hyperbolic plane, where a $2q$-gon, a $2p$-gon and a square meet at each vertex. G is connected, locally finite and one-ended. Let $\mathcal{R}$ be the collection of squares in G and $\mathcal{C}$ the collection of $2p$-gons in G. Then the requirements for lemma 4.1.4 are satisfied. Hence, G is Hamiltonian. □

Lemma 4.1.9 ([DJW95, theorem 8.3]). Let $q \in \mathbb{N}$, with $q \geq 7$ and let G be the (graph of the) regular tessellation $\{q, 3\}$ of the hyperbolic plane. Then G is Hamiltonian.

Lemma 4.1.10 ([DJW95, theorem 6.2]). Let $p, q \in \mathbb{N}$, with $(p-2)(q-2) > 4$ and $q \geq 4$. Then

$$G = \mathrm{Cay}([p,q]^+) = \mathrm{Cay}\left\langle S, T \mid S^q, T^2, (ST)^p \right\rangle$$

is Hamiltonian.

Dunham et. al start with G as the tessellation of the hyperbolic plane with two $2p$-gons and one q-gon at each vertex. After contracting all edges of q-gons the regular tessellation $\{p, q\}$ is obtained. In this tessellation, a set $P = \{P_1, P_2, P_3, \ldots\}$ of p-gons is determined, such that

- the union of the p-gons in P contains all vertices of the tessellation,
- P_i and P_{i+1} intersect in exactly one vertex for $i \geq 0$.

I is defined as the union of all q-gons of G together with the $2p$-gons corresponding to the p-gons in P. Then the boundary C of I is a Hamilton circle of G.

Remarks.

- One side of C contains all q-gons. This fact will be used in lemma 5.2.2, which is an important argument to prove the Hamiltonicity of the twist-amalgamation of G.
- Note that Dunham's construction only works if $q \geq 4$. Otherwise there is no such set P of polygons in the tessellation as required in his proof.

To get rid of this restriction, a construction for the case $q = 3$ is provided here.

Lemma 4.1.11. Let $p, q \in \mathbb{N}$, with $(p-2)(q-2) > 4$. Then

$$G = \mathrm{Cay}([p,q]^+) = \mathrm{Cay}\left\langle S, T \mid S^q, T^2, (ST)^p \right\rangle$$

is Hamiltonian.

Proof. If $q > 3$, apply lemma 4.1.10. Now consider $q = 3$.

Embed G as the tessellation with two $2p$-gons and one q-gon at each vertex. Contract all edges of q-gons to obtain the regular tessellation $G' = \{p, q\}$. According to lemma 4.1.5, G' is Hamiltonian.

When decontracting the triangles to transform G' into G, the Hamilton circle C' of G' can be transformed into a Hamilton circle C of G.

Consider an arbitrarily chosen triangle t of G. This triangle t is contracted to a vertex v_t of G', which lies on the Hamilton circle C'. Let $v_s \in G'$ and $v_u \in G'$ be the neighbours of v_t on C' and s and u the triangles of G corresponding to v_s and v_u. Let $s_1, s_2, s_3 \in V(G)$ be the vertices of s and $t_1, t_2, t_3 \in V(G)$ the vertices of t and $u_1, u_2, u_3 \in V(G)$ the vertices of u, such that t_1 is adjacent to s_1 and t_2 is adjacent to u_1.

When decontracting the vertex v_t to a triangle t, replace the subpath stu of C' in G' by the path $s_1t_1t_3t_2u_1$. The resulting subgraph is still a Hamilton circle.

Perform the same decontractions for all vertices of G' to obtain a Hamilton circle C of G. $\square$

Lemma 4.1.12 ([DJW95, theorem 3.2]). Let $p, q \in \mathbb{N}$, with $(p-2)(q-2) > 4$. The graph

$$G = \mathrm{Cay}([p,q]) = \mathrm{Cay}\left\langle R_1, R_2, R_3 \mid R_1^2, R_2^2, R_3^2, (R_1R_2)^p, (R_1R_3)^2, (R_2R_3)^q \right\rangle,$$

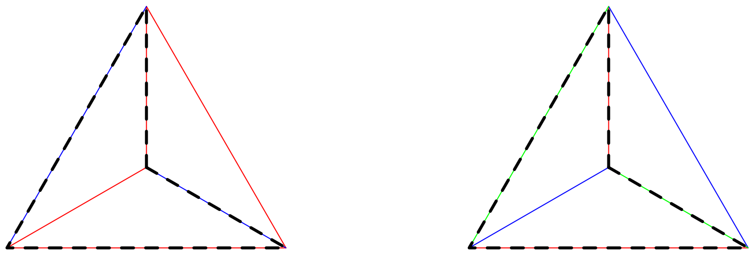
where $(p-2)(q-2) > 4$, is Hamiltonian.

Proof. G is a semi-regular tessellation of the hyperbolic plane with one square, one $2p$-gon and one $2q$-gon meeting at each vertex. Let $\mathcal{C}$ be the collection of $2p$-gons and $\mathcal{R}$ the collection of squares. Then the requirements of lemma 4.1.4 are satisfied and G is Hamiltonian. $\square$

Proposition 4.1.13. $G = \mathrm{Cay}([6,3]^+) = \mathrm{Cay}\left\langle S, T \mid S^3, T^2, (ST)^6 \right\rangle$ is Hamiltonian.

Proof. G can be embedded as the semi-regular tessellation of the Euclidean plane consisting of dodecagons and triangles. In each vertex, two dodecagons and one triangle intersect. By contracting the edges of all triangles, a new graph G' is obtained. G' is a regular tessellation of the plane with hexagons. According to lemma 4.1.7, G' has a Hamilton circle C'.

The decontraction of the triangles to transform G' into G is done in the same way as in the proof of lemma 4.1.11 to obtain a Hamilton C of G. $\square$



(a) $G = \mathrm{Cay}\left\langle a, b \mid b^2, a^2 b \right\rangle$.

(b) $G = \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, bcd \right\rangle$.

Figure 4.1.: The tetrahedral graph.

Proposition 4.1.14. $G = \mathrm{Cay}([4,4]^+) = \mathrm{Cay}\left\langle S, T \mid S^4, T^2, (ST)^8 \right\rangle$ is Hamiltonian.

Proof. G can be embedded as the semi-regular tessellation of the Euclidean plane consisting of squares and octagons (two octagons and one square meet at each vertex). If we contract the edges of all squares of G, the resulting graph G' is a regular tessellation of the plane with squares.

Let S be an arbitrary square of G' and $C_0 = \{S\}$. Define a sequence

$$C_i = \{T \,|\, T \text{ is a square of } G' \text{ and } \bigcup C_{i-1} \cap T \subseteq V(G')\}$$

and consider

$$\mathcal{C}' = \bigcup_{i \geq 0} C_i.$$

$\mathcal{C}'$ is a collection of squares in G'. Let $\mathcal{C}'$ be the collection of octagons of G corresponding to the squares of G' in $\mathcal{C}$ and $\mathcal{R}$ be the collection of all squares of G. The sets $\mathcal{C}$ and $\mathcal{R}$ satisfy the requirements of lemma 4.1.4. See figure 5.5(a) for an outline of the resulting Hamilton circle. $\square$

Remark. The aforementioned construction ensures that the Hamilton circle of G uses exactly two non-adjacent edges of every square of G. Therefore, the conditions of lemma 5.2.1 are fulfilled. This allows to prove that the twist-amalgamation of G is also Hamiltonian.

Corollary 4.1.15. The Cayley graphs

$$G_1 = \mathrm{Cay}([2,4,4]^+) = \mathrm{Cay}\left\langle R, S, T \mid R^2, S^4, T^4 \right\rangle,$$
$$G_2 = \mathrm{Cay}([4,4]) = \mathrm{Cay}\left\langle R_1, R_2, R_3 \mid R_1^2, R_2^2, R_3^2, (R_1R_2)^4, (R_1R_3)^2, (R_2R_3)^q \right\rangle$$

are Hamiltonian.

Proof. Consider $G = \mathrm{Cay}([4,4]^+) = \mathrm{Cay}\left\langle S, T \mid S^4, T^2, (ST)^8 \right\rangle$. Then $G \cong G_1 \cong G_2$, because all of these graphs can be embedded as semi-regular tessellations of the Euclidean plane where two octagons and one square meet at each vertex. G is Hamiltonian according to proposition 4.1.14. $\square$

G has two generators a, b	a reverses spin	b reverses spin	$G = \text{Cay}\left\langle a, b \mid b^2, (a^2ba^{-2}b)^m \right\rangle, m \geq 1$ G is Hamiltonian, see proposition 4.2.4.
		b preserves spin	$G = \text{Cay}\left\langle a, b \mid b^2, (a^2b)^m \right\rangle, m \geq 2,$ G is Hamiltonian, see proposition 4.2.3.
	a preserves spin	b reverses spin	$G = \text{Cay}\left\langle a, b \mid b^2, a^n, (ab)^m \right\rangle, n \geq 3, m \geq 2$ G is Hamiltonian, see proposition 4.2.2.
		b preserves spin	$G = \text{Cay}\left\langle a, b \mid b^2, a^n, (aba^{-1}b)^m \right\rangle, n \geq 3, m \geq 1$ G is Hamiltonian, see proposition 4.2.1.

Table 4.1.: The finite or one-ended cubic planar Cayley graphs with 2 generators.

4.2. Graphs with two generators

Proposition 4.2.1. Every graph $G = \text{Cay}\left\langle a, b \mid b^2, a^n, (ab)^m \right\rangle, n \geq 3, m \geq 2$, is Hamiltonian.

Proof. G is the Cayley graph of the rotation subgroup $[m, n]^+$ (see definition 4.1.3).

If $(n-2)(m-2) < 4$, the group $[m, n]^+$ is finite and has order

$$\frac{4nm}{4 - (n-2)(m-2)}.$$

All choices of (n, m) which result in finite groups are listed in table 4.2. Unless $m = 2$, the order of $[m, n]^+$ is an element of $\{12, 24, 60\}$. Cayley graphs on groups of these orders are Hamiltonian (see section 2.7). The case $(n, m) = (n, 2)$, $n \geq 3$ leads to the k-prism graph which contains $a^{n-1}ba^{n-1}b$ as a Hamilton cycle. All finite graphs of this type are shown in figure 4.2 with their Hamilton cycles boldly dashed.

If $(n-2)(m-2) = 4$, the group $[m, n]^+$ is infinite and G can be embedded as a semi-regular tessellation of the Euclidean plane. There are only three possible choices of (n, m):

n	m	$\lvert G\rvert$	Hamilton cycle
$n \geq 3$	2	$2n$	$(a^{n-1}b)^2$
3	3	12	$(a^2ba^{-2}b)^2$
3	4	24	$((a^2b)^2(a^{-2}b)^2)^2$
3	5	60	$((a^2b)^3(a^{-2}b)^3(a^2ba^{-2}b)^2)^2$
4	3	24	$(a^3ba^{-3}b)^3$
5	3	60	$a^4ba^{-1}ba^2b(a^{-1}bab)^3a^{-2}baba^{-4}ba^{-1}b(a^{-2}b)^3a^2ba^{-2}b(a^2b)^3ab$

Table 4.2.: The finite cases of $G = \text{Cay}\left\langle a, b \mid b^2, a^n, (ab)^m \right\rangle, n \geq 3, m \geq 2$.

- $(n, m) = (3, 6)$ leads to the regular tessellations of the Euclidean plane with hexagons. G is Hamiltonian according to corollary 4.1.7. Figure 5.7 depicts a part of the Hamilton circle.
- $(n, m) = (4, 4)$ gives a semi-regular tessellation with squares and octagons which is Hamiltonian according to proposition 4.1.14.
- $(n, m) = (6, 3)$ results in a semi-regular tessellation with triangles and dodecagons. G is Hamiltonian, see proposition 4.1.13.

If $(n-2)(m-2) > 4$, the group $[m, n]^+$ is infinite and G can be embedded as a semi-regular tessellation of the hyperbolic plane. The conditions of lemma 4.1.11 are satisfied and G is Hamiltonian. □

Proposition 4.2.2. Every graph $G = \text{Cay}\left\langle a, b \mid b^2, a^n, (aba^{-1}b)^m \right\rangle, n \geq 3, m \geq 1$, is Hamiltonian.

Proof. G can be embedded as a semi-regular tessellation of the sphere, Euclidean plane or hyperbolic plane, depending on the values of m, n. In each vertex, two $4m$-gons and one n-gon intersect. Hence, G is isomorphic to the Cayley graph of the group $[2m, n]^+$ and $G \cong \text{Cay}\left\langle a, b \mid b^2, a^n, (ab)^{2m} \right\rangle$. The Hamiltonicity of G follows from proposition 4.2.1. □

Proposition 4.2.3. Every graph $G = \text{Cay}\left\langle a, b \mid b^2, (a^2b)^m \right\rangle, m \geq 1$, is Hamiltonian.

Proof. G can be embedded as the regular tessellation $\{3m, 3\}$ (see definition 4.1.1).

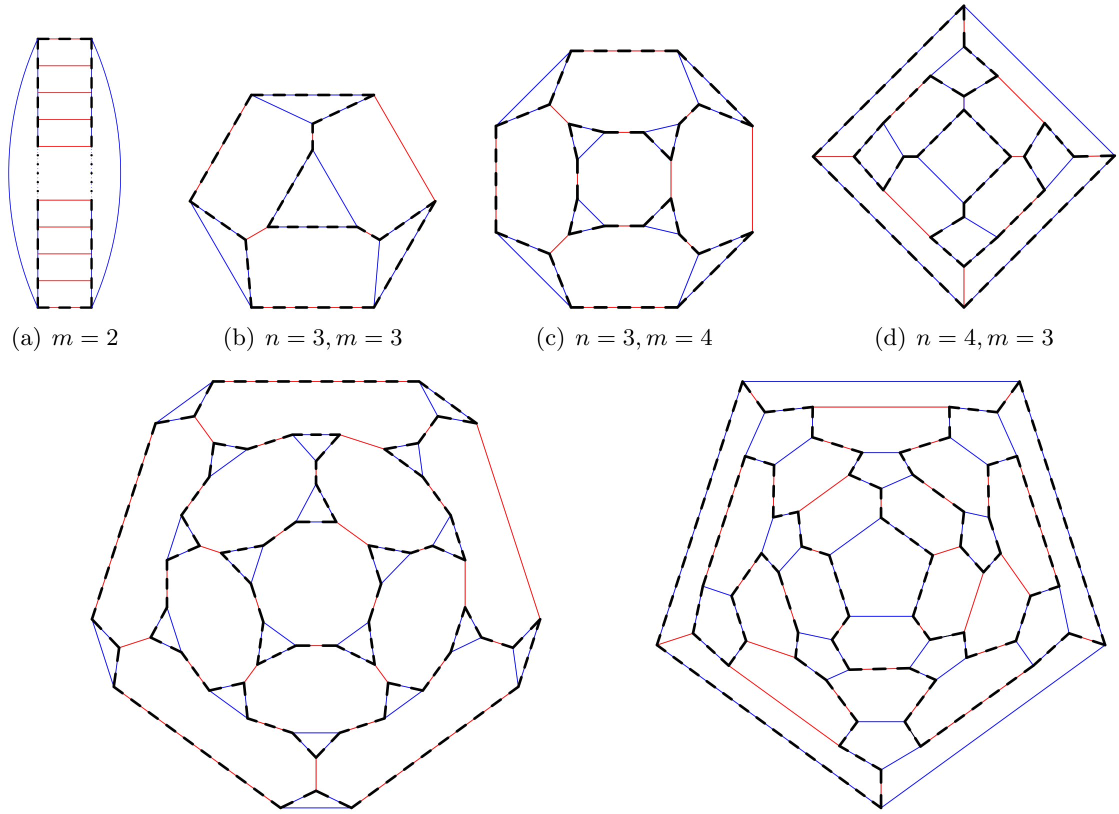
If $m = 1$, this is the tetrahedral graph (see figure 4.1) and therefore Hamiltonian.

The case $m = 2$ results in the hexagonal tiling of the Euclidean plane If $m \geq 3$, G tessellates the hyperbolic plane.

G is Hamiltonian according to corollary 4.1.6. □

Proposition 4.2.4. Every graph $G = \text{Cay}\left\langle a, b \mid b^2, (a^2ba^{-2}b)^m \right\rangle, m \geq 1$, is Hamiltonian.

Proof. G can be embedded as the regular tessellation $\{6m, 3\}$ and contains a Hamilton circle, see corollary 4.1.6. □



(a) $m = 2$ (b) $n = 3, m = 3$ (c) $n = 3, m = 4$ (d) $n = 4, m = 3$

(e) $n = 3, m = 5$ (f) $n = 5, m = 3$

Figure 4.2.: Finite cases of $G = \mathrm{Cay}\left\langle a, b \mid b^2, a^n, (ab)^m \right\rangle, n \geq 3, m \geq 2$.

4.3. Graphs with three generators

Proposition 4.3.1. Every graph $G = \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bcd)^n \right\rangle, n \geq 1$ is Hamiltonian.

Proof. G can be embedded as the regular tessellation $\{3n, 3\}$ and is isomorphic to $\mathrm{Cay}\left\langle a, b \mid b^2, (a^2b)^n \right\rangle$. Depending on the values of n, G is the tetrahedral graph (figure 4.1), or a tessellation of the Euclidean or hyperbolic plane. In all cases, G has a Hamilton circle, see proposition 4.2.3. □

Proposition 4.3.2. Every graph $G = \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (cbcdbd)^n \right\rangle, n \geq 1$ is Hamiltonian.

Proof. G can be embedded as the Euclidean or hyperbolic regular tessellation $\{6n, 3\}$. It is isomorphic to $\mathrm{Cay}\left\langle a, b \mid b^2, (a^2ba^{-2}b)^n \right\rangle$ and Hamiltonian, see proposition 4.2.4. □

Proposition 4.3.3. Every graph $G = \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bc)^n, (bdcd)^m \right\rangle$ with $n \geq 2$, $m \geq 1$ is Hamiltonian.

G has three generators b, c, d	all reverse spin	$G = \text{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bc)^n, (cd)^m, (db)^p \right\rangle, n, m, p \geq 2$ G is Hamiltonian, see proposition 4.3.4.
	d preserves spin	$G = \text{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bc)^n, (bdcd)^m \right\rangle, n \geq 2, m \geq 1$ G is Hamiltonian, see proposition 4.3.3.
	c, d preserve spin	$G = \text{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (cbcdbd)^n \right\rangle, n \geq 1$ G is Hamiltonian, see proposition 4.3.2.
	all preserve spin	$G = \text{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bcd)^n \right\rangle, n \geq 1$ G is Hamiltonian, see proposition 4.3.1.

Table 4.3.: The finite or one-ended cubic planar Cayley graphs with 3 generators.

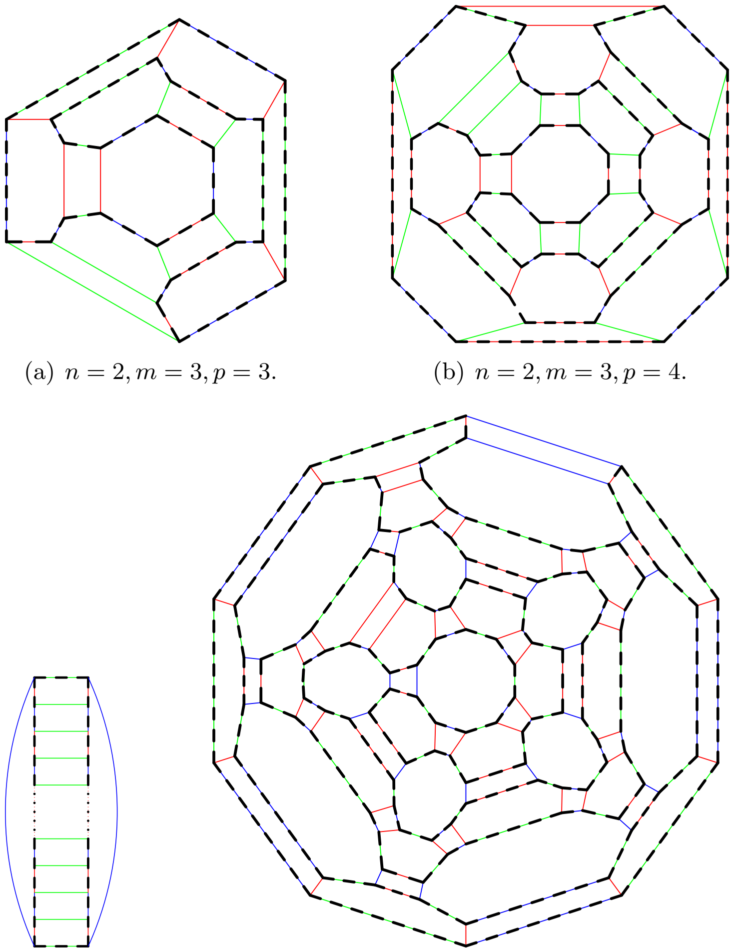
Proof. G can be embedded as a semi-regular tessellation with two $4m$-gons and one $2n$-gon meeting at each vertex. It is the Cayley graph of $[2m, 2n]^+$ and isomorphic to $\text{Cay}\left\langle a, b \mid b^2, a^{2n}, (ab)^{2m} \right\rangle, n \geq 3, m \geq 2$. According to proposition 4.2.1, G is Hamiltonian. $\square$

Proposition 4.3.4. Every graph $G = \text{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bc)^n, (cd)^m, (db)^p \right\rangle$, where $n, m, p \geq 2$ is Hamiltonian.

Proof. G is the Cayley graph of the hyperbolic rotation subgroup $[p, q, r]^+$. Hence, G can be embedded as the semi-regular tessellation with one $2p$-gon, one $2q$-gon and one $2r$-gone meeting at each vertex.

The involutions b, c, d can be exchanged without altering the graph. That is why the definition is symmetric in terms of m, n, p. Without loss of generality assume $m \leq n \leq p$.

If $\frac{1}{m} + \frac{1}{n} + \frac{1}{p} > 1$, a tessellation of the sphere and hence a finite graph is obtained. This



(a) $n = 2, m = 3, p = 3$.

(b) $n = 2, m = 3, p = 4$.

(c) $n = m = 2$.

(d) $n = 2, m = 3, p = 5$.

Figure 4.3.: Finite cases of $G = \text{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bc)^n, (cd)^m, (db)^p \right\rangle$.

leads to the (up to permutation of m, n, p) unique finite cases

$$(m, n, p) \in \{(2, 2, p \geq 2), (2, 3, 3), (2, 3, 4), (2, 3, 5)\},$$

see table 4.4. If $(m, n, p) = (2, 2, p \geq 2)$, G is the $2p$-prism graph which contains $a^{2p-1}ba^{2p-1}b$ as a Hamilton cycle. All other finite graphs of this type are also Hamiltonian, see figure 4.3. The Hamilton cycles are boldly dashed in this picture.

If $\frac{1}{m} + \frac{1}{n} + \frac{1}{p} = 1$, G is a semi-regular Euclidean tessellation. This is the case, if

$$(m, n, p) \in \{(2, 3, 6), (2, 4, 4), (3, 3, 3)\}.$$

- $(m, n, p) = (2, 3, 6)$ results in a semi-regular tessellation of the Euclidean plane with squares, hexagons and dodecagons. In this case,

$$G = \text{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bc)^2, (cd)^3, (db)^6 \right\rangle$$

n	m	p	$\|G\|$	Hamilton cycle
2	2	$p \geq 2$	$4p$	$(c(db)^{p-1}d)^2$
2	3	3	24	$(cdbd)^2bd(cd)^2bdcd(bd)^2cd$
2	3	4	48	$dc(dcdb)^3dbdc(db)^3dcd(cdbd)^3bdc(db)^3$
2	3	5	120	$cb(cbcd)^4cdcbcdc(bcdcbc)^4dcb(cd)^4cbc(bcdc)^4dcb(cdcbcd)^4$ $(cd)^2cb(cd)^4$

Table 4.4.: The finite cases of $G = \text{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bc)^n, (cd)^m, (db)^p \right\rangle$.

is the Cayley graph of the Euclidean symmetry group $[3, 6]$, which is Hamiltonian due to lemma 4.1.12.

- $(m, n, p) = (2, 4, 4)$ yields a semi-regular tessellation of the Euclidean plane with squares, hexagons and dodecagons.

$$G = \text{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bc)^2, (cd)^4, (db)^4 \right\rangle$$

is the Cayley graph of the Euclidean symmetry group $[4, 4]$, which is Hamiltonian due to corollary 4.1.15.

- $(m, n, p) = (3, 3, 3)$ results in the hexagonal tiling $\{6, 3\}$ of the Euclidean plane.

$$G = \text{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bc)^3, (cd)^3, (db)^3 \right\rangle$$

can be embedded as the regular tessellation $\{6, 3\}$, which is Hamiltonian according to corollary 4.1.6.

If $\frac{1}{m} + \frac{1}{n} + \frac{1}{p} < 1$, G is a semi-regular hyperbolic tessellation.

Consider $n = 2$ first. In this case, G is a tessellation with one square, one $2m$-gon and one $2p$-gon meeting in each vertex. Moreover,

$$\begin{aligned} \frac{1}{m} + \frac{1}{2} + \frac{1}{p} &< 1 \\ \Leftrightarrow mp - 2m - 2p &> 0 \\ \Leftrightarrow (p-2)(m-2) &> 4 \end{aligned}$$

Hence, G is isomorphic to the Cayley graph of the hyperbolic symmetry group $[m, p]$. According to lemma 4.1.12, G contains a Hamilton circle.

Now consider the case $n \geq 3$. The girth of G is greater or equal to 6. From lemma 4.1.5 the Hamiltonicity of G follows. $\square$

CHAPTER 5

Multi-ended graphs of connectivity 3

Georgakopoulos describes 18 different types of 3-connected multi-ended graphs. The six types which have two generators are two-ended graphs (section 5.1) or twist-amalgamations (section 5.2) or twist-squeeze-amalgamations (section 5.3).

The multi-ended graphs generated by three involutions are difficult to handle so we provide only some results for special cases which follow from the considered graphs with two generators.

5.1. Two-ended graphs of connectivity 3 with 2 generators

Among the cubic planar Cayley graphs of connectivity 3 with 2 generators, there are only two classes consisting completely of two-ended graphs, namely

- $G = \mathrm{Cay}\left\langle a, b \mid b^2, (a^2b)^2, (ab)^{2m} \right\rangle$, $m \geq 2$,
- $G = \mathrm{Cay}\left\langle a, b \mid b^2, a^2ba^{-2}b, (baba^{-1})^m \right\rangle$, $m \geq 2$.

All of those are Hamiltonian, which is proved in the two propositions of this chapter.

Theorem 5.1.1. Let G be a cubic planar Cayley graphs of connectivity 3 with 2 generators. Then G has a Hamilton circle.

Proposition 5.1.2. Every graph $G = \mathrm{Cay}\left\langle a, b \mid b^2, (a^2b)^2, (ab)^{2m} \right\rangle$, $m \geq 2$, is Hamiltonian.

Proof. Choose an arbitrary vertex $v \in V(G)$ and let

$$x = vb,\ y = va^{-2},\ z = yb.$$

For $m \geq 3$, define two double-rays P, Q

$$P = \dots (ba)^{2m-4}(baba^{-1})^2 \underbrace{b}_{vx} a(ba)^{2m-5}(baba^{-1})^2 \dots = \left((ba)^{2(m-2)}(baba^{-1})^2\right)^{\infty},$$

$$Q = \dots (ba)^{2m-4}(baba^{-1})^2 \underbrace{b}_{yz} a(ba)^{2m-5}(baba^{-1})^2 \dots = \left((ba)^{2(m-2)}(baba^{-1})^2\right)^{\infty},$$

by their arc-colors such that P contains vx as a b-edge and Q contains yz as a b-edge. For $m = 2$, let

$$P = \dots (baba^{-1})^2 \underbrace{b}_{vx} aba^{-1}baba^{-1}(baba^{-1})^2 \dots = \left((baba^{-1})^2\right)^{\infty},$$

$$Q = \dots (baba^{-1})^2 \underbrace{b}_{yz} aba^{-1}baba^{-1}(baba^{-1})^2 \dots = \left((baba^{-1})^2\right)^{\infty}.$$

Now it is easy to see that

- P and Q are disjoint,
- $V(P \cup Q) = V(G)$,
- $\overline{P}$ connects the two ends ω_1, ω_2 of G,
- $\overline{Q}$ connects ω_1 and ω_2,
- $\overline{P \cup Q}$ is a Hamilton circle in $|G|$.

In figure 5.1 the Hamilton circle is black and dashed, a is paleblue and b is red. □

Proposition 5.1.3. Every graph $G = \mathrm{Cay}\left\langle a, b \mid b^2, a^2ba^{-2}b, (baba^{-1})^m\right\rangle$, $m \geq 2$, is Hamiltonian.

Proof. Similar to the graphs considered in proposition 5.1.2, the faces of G are bounded by cycles consisting of 4 a-edges and 2 b-edges and the structure of the graphs is identical. The only difference between G and $H = \mathrm{Cay}\left\langle a, b \mid (a^2b)^2, (ab)^{2m}\right\rangle$ is the direction of some a-colored arcs. Every vertex of G can be represented as a reduced word of the form

$$x = a^{k_0}ba^{k_1}ba^{k_2}b \cdots a^{k_{n-1}}ba^{k_n}, \quad n \in \mathbb{N}_0,\ k_0, k_n \in \mathbb{Z}, \quad \forall\, 1 \leq i \leq n-1 :\ k_i \in \mathbb{Z} \setminus \{0\}.$$

The function

$$\begin{aligned} &\phi : V(G) \to V(H); \\ &\quad \phi(x) = \phi(a^{k_0}ba^{k_1}ba^{k_2}b \cdots a^{k_{n-1}}ba^{k_n}) = a^{k_0}ba^{-k_1}ba^{k_2}ba^{-k_3} \cdots a^{(-1)^{n-1}k_{n-1}}ba^{(-1)^n k_n} \end{aligned}$$

is a graph-isomorphism between G and H. According to proposition 5.1.2, H is Hamiltonian. Therefore, G is also Hamiltonian. □

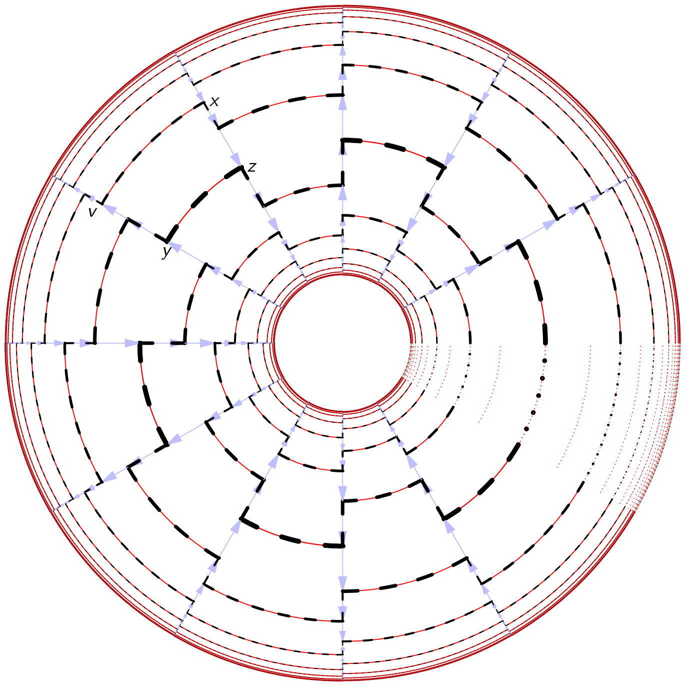


Figure 5.1.: A Hamilton circle in $G = \mathrm{Cay}\left\langle a, b \mid b^2, (a^2b)^2, (ab)^{2m}\right\rangle$, $m \geq 2$.

5.2. Twist-amalgamations

Definition 2.6.3 specifies how the application of the twist-amalgamation on finite or one-ended graphs yields multi-ended graphs. The twist-amalgamation of a graph

$$G = \mathrm{Cay}\left\langle a, b \mid b^2, a^n, (ab)^m\right\rangle,\ n \geq 3,\ m \geq 2$$

(see proposition 4.2.1) is

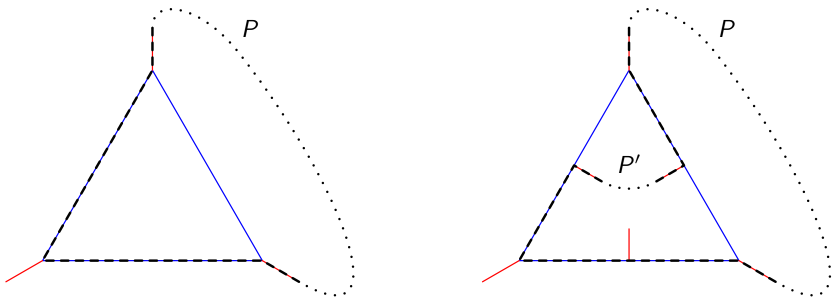
$$G^{\underline{\infty}} = \mathrm{Cay}\left\langle a, b \mid b^2, a^{2n}, (a^2b)^m\right\rangle,\ n \geq 3,\ m \geq 2.$$

In theorem 5.2.8 it will be proved that $G^{\underline{\infty}}$ is Hamiltonian for all $n \geq 3,\ m \geq 2$.

To describe the construction of a Hamilton circle in $G^{\underline{\infty}}$ consider the case $n = 3$ first. Start with a Hamilton circle C_0 in $G = G^{\underline{0}}$, which exists according to proposition 4.2.1. In every step, a copy of G is embedded inside every a-colored triangle of $G^{\underline{k}}$ to obtain $G^{\underline{k+1}}$. The Hamilton circle C_k of G_k can be transformed into a Hamilton circle C_{k+1} of G_{k+1} as explained below. By induction, G_k is Hamiltonian for every $k \geq 0$.

Obviously, C_k uses exactly two edges of every a-colored triangle in G_k. Hence, ba^2b or $ba^{-2}b$ is a subpath of C_k. W.l.o.g. let $C_k = ba^2bP$ (if necessary, reverse the orientation in C_k), where P is a subarc of C_k and a^2 are the edges of a triangle, see figure 5.2(a). Let $C_0 = ba^2bP'$.

Transform the Hamilton cycle $C_k = ba^2bP$ to $babP'ba^3bP$. This yields again a Hamilton circle as depicted in figure 5.2(b). Apply this transformation in every triangle of $G^{\underline{k}}$ to obtain a Hamilton circle C_{k+1} of $G^{\underline{k+1}}$.



(a) The Hamilton circle C_k.

(b) The Hamilton circle after embedding G in the blue triangle.

Figure 5.2.: The transition in the twist-graph construction for the case $n = 3$.

Now consider the limit graph C_∞ which is a subset of $G^{\underline{\infty}}$. It is clear that

- C_∞ visits all vertices of $G^{\underline{\infty}}$,
- C_∞ is connected (it meets every finite vertex cut),
- every vertex of C_∞ has degree 2.

There may be two different types of ends in C_∞. If G is a finite graph, all ends are obtained by repeated application of the twist operation. If G is one-ended, there are additional ends which correspond to the ends of copies of G that are used in the construction.

By lemma 2.5.8, in order to show that C_∞ is a Hamilton circle we need to check that every end ω of G^∞ has degree 2 in C_∞. If ω is one of the ends of copies of G, then this follows easily from the fact that C_0 is a Hamilton circle of G and therefore ω has degree 2 in C_0. For an end ω that is not of this kind, note that there is a basis of open neighbourhoods of ω, each of which is separated by the three b edges incident with a subdivided triangle as in figure 5.2(b) on one of its sides. Since C_∞ always uses 2 of the three edges from such a cut, it follows that ω has degree 2 in C_∞ as desired.

Lemma 5.2.1. Let $H = \mathrm{Cay}\left\langle a, b \mid b^2, a^n, (ab)^m \right\rangle,\ n \geq 3,\ m \geq 2$. Then H has a Hamilton circle D, such that for every a-colored cycle C of H, either

- D uses exactly $n-1$ edges of C, or
- D uses exactly $n-2$ edges of C and the two unused edges are non-adjacent.

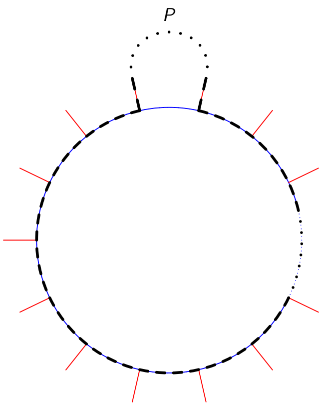
Proof. Consider the case $n = 3$ first. The a-colored cycles of H are triangles. Since H is a cubic graph and D is a Hamilton circle, D uses exactly two edges of every triangle C. For the rest of the proof, let $n \geq 4$.

- If $(n-2)(m-2) < 4$, H is a finite graph and the a-colored cycles are of length n. In figure 4.2, all finite graphs of this type are displayed with their Hamilton cycle dashed. It may be checked that the requirements are fulfilled.
- If $(n-2)(m-2) = 4$, embed H as a semi-regular tessellation of the Euclidean plane. The possible pairs of values of (n, m) are $(6, 3), (4, 4)$.

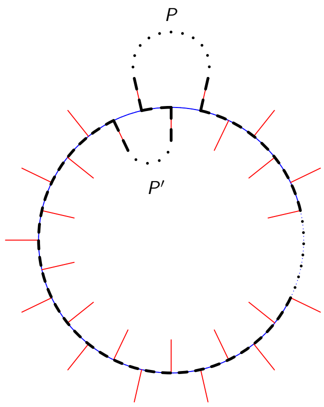
⋆ The first case $(n, m) = (6, 3)$ leads to a regular tessellation $\{6, 3\}$. According to lemma 4.1.5 this gives a Hamiltonian graph. The conditions of lemma 5.2.1 are satisfied (see the remark after the proof of lemma 4.1.5).

⋆ The case $(n, m) = (4, 4)$ results in a tessellation with squares and octagons. Proposition 4.1.14 shows that G has a Hamilton circle D. According to the remark after that proposition, D has the required form.

- If $(n-2)(m-2) > 4$, embed H as a semi-regular tessellation of the hyperbolic plane. Since $n \geq 4$, the construction of a Hamilton circle in G is outlined in lemma 4.1.10. Again, the resulting Hamilton circle meets the requirements. □



(a) The Hamilton circle C_k.



(b) The Hamilton circle after embedding H in the blue cycle.

Figure 5.3.: The transition in the twist-graph construction if C_k uses $n-1$ edges of C.

Lemma 5.2.2. Let

$$H = \mathrm{Cay}\left\langle a, b \mid b^2, a^n, (ab)^m \right\rangle, \ n \geq 3, \ m \geq 2, \ (n, m) \notin \{(4, 4), (5, 3), (6, 3)\}.$$

Then H has a Hamilton circle D, such that either

- D uses exactly $n-1$ edges of every a-colored cycle, or
- one side of D contains (the interior of) all a-colored cycles.

Proof. For $n = 3$ or $(n-2)(m-2) < 4$, the argument remains the same as in the proof of lemma 5.2.1.

The only remaining case is $(n-2)(m-2) > 4$ and $n \geq 4$, which results in a semi-regular tessellation of the hyperbolic plane. Lemma 4.1.10 ensures the existence of a Hamilton circle D, such that one side of D contains all a-colored cycles (see the remark after the proof of this lemma). □

Theorem 5.2.3. Let $H = \mathrm{Cay}\left\langle a, b \mid b^2, a^n, (ab)^m \right\rangle, \ n \geq 3, \ m \geq 2$ and $k \in \mathbb{N}$. Then the k-th twist-graph $H^{\underline{k}}$ is Hamiltonian.

Proof. In every step of the construction of $H^{\underline{k}}$, a copy of H is embedded inside every a-colored cycle of the graph. Throughout the following proof it is described how a Hamilton circle C_k in $H^{\underline{k}}$ can be transformed to yield a Hamilton circle C_{k+1} in $H^{\underline{k+1}}$ for $k \geq 0$.

Start with a Hamilton circle C_0 in $H = H^{\underline{0}}$. In every step, the Hamilton circle C_k satisfies the conditions of lemma 5.2.1 and if $(n, m) \notin \{(4, 4), (5, 3), (6, 3)\}$, the requirements of lemma 5.2.2 are also fulfilled. By induction it holds true that $H^{\underline{k}}$ has a Hamilton circle for every $k \geq 0$.

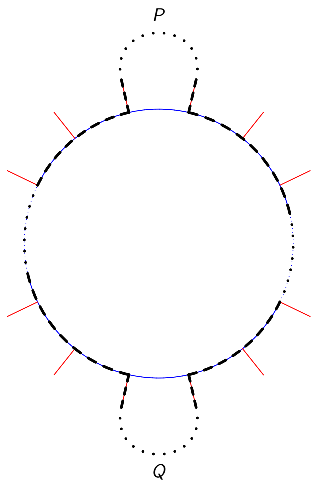
Consider an a-colored cycle C of $H^{\underline{k}}$. Then C_k uses either exactly $n - 1$ edges or exactly $n - 2$ edges of C.

- If C_k uses $n - 1$ edges of C, it can be expressed as $C_k = ba^{n-1}bP$, where P is a subarc of C_k and a^{n-1} is a subpath of both C and C_k (see figure 5.3(a)).

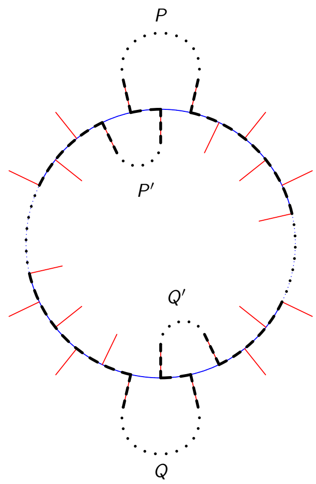
  Let D be a Hamilton cycle in H, such that $D = ba^{n-1}bP'$, where P' is a subarc of D. The existence of such a cycle D is trivial for $k = 0$ and follows from the subsequent construction for other values of k. Replace $C_k = ba^{n-1}bP$ by $ba^{2n-3}bP'ba^{-1}bP$ to obtain a Hamilton circle in the graph $H^{\underline{k}}$ with H embedded inside C (see figure 5.3(b)).

- If C_k uses $n-2$ edges of C, it can be expressed as $C_k = ba^ibQba^jbP$, where $i+j = n-2$ and P and Q are subarcs of C_k and a^i and a^j are subpaths of C (see figure 5.4(a)).

  Let D be a Hamilton cycle in H, such that $D = ba^ibQ'ba^jbP'$, where P' and Q' are subarcs of D and a^i and a^j are subpaths of the same a-cycle. The existence of such a cycle D is trivial for $k = 0$ and follows from the subsequent construction for other values of k. Replace $C_k = ba^ibQba^jbP$ by $ba^{2i-1}bQ'ba^{-1}bQba^{2j-1}bP'ba^{-1}bP$ to obtain a Hamilton circle in the graph $H^{\underline{k}}$ with H embedded inside C (see figure 5.4(b)).



(a) The Hamilton circle C_k.



(b) The Hamilton circle after embedding H inside C.

Figure 5.4.: The transition in the twist-graph construction if C_k uses $n - 2$ edges of C.

Repeat these transformations for all a-cycles of C_k to get a Hamilton circle C_{k+1} of $H^{\underline{k+1}}$.□

Corollary 5.2.4. Every graph

$$G = \mathrm{Cay}\left\langle a, b \mid b^2, a^{2n}, (a^2b)^m \right\rangle,\ n \geq 3,\ m \geq 2,\ (n, m) \notin \{(4,4), (5,3), (6,3)\}$$

is Hamiltonian.

Proof. G is the twist-amalgamation $H^{\underline{\infty}}$ of $H = \mathrm{Cay}\left\langle a, b \mid b^2, a^n, (ab)^m \right\rangle$. For every $k \geq 0$ the Hamilton circle C_k of $H^{\underline{k}}$ is constructed as in theorem 5.2.3. The most difficult part is to prove that all ends of C_∞ have degree 2.

Every end ω which is an end of a copy of H has degree 2, since D is a Hamilton circle of H and ω has degree 2 in D. In the rest of the proof we only consider ends which are obtained by repeated application of the twist operation.

If the Hamilton circle D uses exactly $n-1$ edges of every a-colored circle of H, all transitions are of the type shown in figure 5.3. In this case, P' is connected only by two edges to the a-colored cycle C (and hence to the rest of the graph). All faces incident with C are bounded by alternately-colored cycles. Since the alternately-colored circles remain unchanged, the separation of the subgraph of C_k which lies inside C is preserved in all further steps of the construction. Hence, C_∞ uses only two of the b-colored edges incident to C. Consider an arbitrarily chosen end ω of G (obtained by repeated application of the twist operation). There is a base of open neighbourhoods of ω each of which is separated by n b-colored edges incident with an a-colored circle. As a consequence of the facts mentioned above, C_∞ always uses two of the n edges from such a cut. Therefore ω has degree 2 in C_∞ as desired.

If D does not use exactly $n-1$ edges of every a-colored circle of H, there may be two different types of transitions, shown in figure 5.3 and figure 5.4. The first type of transitions is the same as before, so we consider the transition depicted in figure 5.4. The paths P' and Q' appear in the construction of C_{k+1}. P' and Q' are lying inside an a-colored circle C. It follows from lemma 5.2.2, that P' and Q' do not visit any common a-colored cycle. Therefore, P' and Q' are separated by alternately-colored cycles. As before, all faces incident with C are bounded by alternately-colored cycles and such cycles remain unchanged in all further steps. This makes it easy to separate P' from Q' and C (or to separate Q' from P' and C). C_∞ always uses two edges of the corresponding cuts, namely the edges connecting P' and C (or the edges connecting Q' and C). As a result, ω has degree 2 in C_∞.

C_∞ is a Hamilton circle of G, since

- C_∞ visits all vertices of $G^{\underline{\infty}}$,
- C_∞ is connected (it meets every finite vertex cut),
- every vertex of C_∞ has degree 2,
- every end of C_∞ has degree 2. □

Remark. For $(n, m) \in \{(4,4), (5,3), (6,3)\}$ and the Hamilton circles of chapter 4 the above construction does not yield a Hamilton circle C_∞.

In H there exists an a-colored n-cycle C, so that the considered Hamilton circle D of H uses $n-2$ edges of C. Let $D = ba^i bQ_0 ba^j bP_0$, where a^i and a^j are subpaths of the same a-cycle and $i+j = n-2$. Let p_1, p_2 be the endvertices of bP_0b and q_1, q_2 be the endvertices of bQ_0b. As a consequence, there is an n-cycle F_0, such that both P_0 and Q_0 contain at least one edge of F_0. Therefore, D uses exactly $n-2$ edges of F_0.

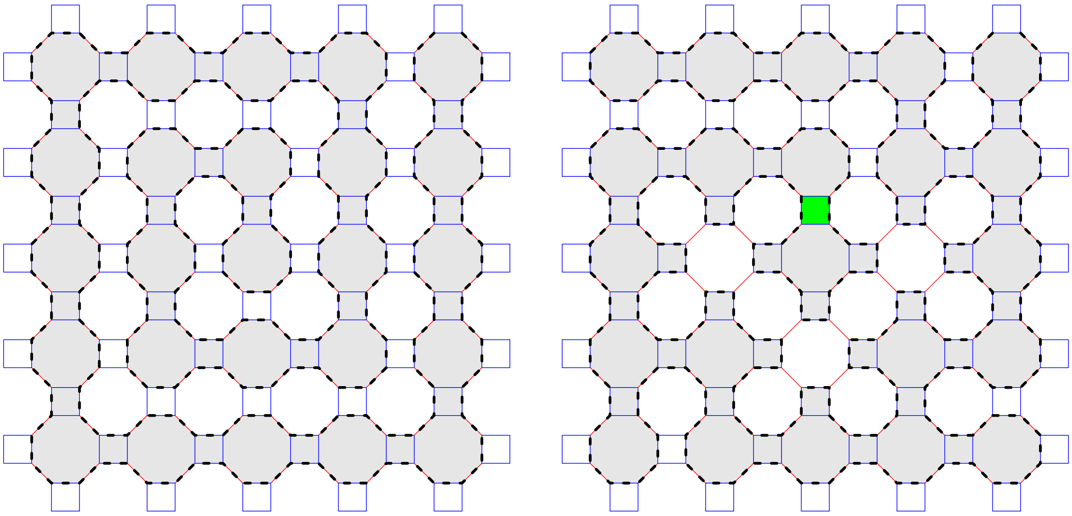
Applying one step of theorem 5.2.3 gives a Hamilton circle C_1 of $H^{\underline{1}}$. By construction, C_1 contains subpaths P_1 and Q_1, such that p_1, p_2 are the endvertices of bP_1b and q_1, q_2 are the endvertices of bQ_1b and both P_1 and Q_1 contain an edge of a common a-colored n-cycle F_1. Repeated use of this approach shows that for all $k \geq 0$ the Hamilton circle C_k contains subpaths bP_kb and bQ_kb with endvertices p_1, p_2 and q_1, q_2 respectively and P_k and Q_k being adjacent with the same a-colored n-cycle F_k.

As a result, the limit graph C_∞ has at least one end with degree ≥ 4.

To obtain a Hamilton circle for $(n, m) \in \{(4,4), (5,3), (6,3)\}$, it is necessary to modify the construction, such that the situation mentioned above can be avoided.

Proposition 5.2.5. The graph $G = \mathrm{Cay}\left\langle a, b \mid b^2, a^8, (a^2b)^4 \right\rangle$ is Hamiltonian.

Proof. G is the twist-amalgamation of $H = \mathrm{Cay}\left\langle a, b \mid b^2, a^4, (ab)^4 \right\rangle$.



(a) The inner part of a Hamilton circle D_1 in G. (b) A replacement for the inner part of D_1.

Figure 5.5.: Two different Hamilton circles in $G = \mathrm{Cay}\left\langle a, b \mid b^2, a^4, (ab)^4 \right\rangle$.

Figure 5.5(a) shows the part of a Hamilton circle D_1 in H that is created after 25 steps of the instructions in the proof of lemma 4.1.4 (see [DJW95, theorem 3.1]).

Replace the part of D_1 which is shown in figure 5.5(a) by the graph shown in figure 5.5(b) and do not modify the other steps of the construction to obtain a Hamilton circle D of H.

Consider the green-filled 4-cycle C in figure 5.5(b). The Hamilton circle D can be expressed as $D = babQbabP$ where the two a-edges belong to C. There is no a-colored 4-cycle F in H such that both P and Q contain an edge of F.

Thus, applying the construction of theorem 5.2.3 and corollary 5.2.4 results in a Hamilton circle of G. □

Proposition 5.2.6. The graph $G = \mathrm{Cay}\left\langle a, b \mid b^2, a^{10}, (a^2b)^3 \right\rangle$ is Hamiltonian.

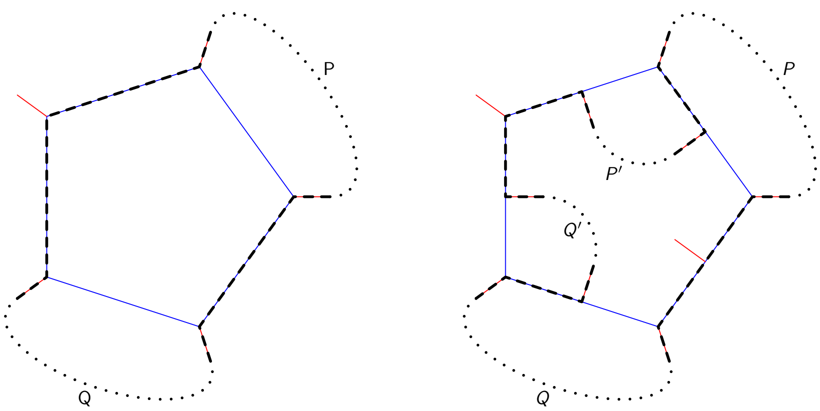
PROOF. G is the twist-amalgamation of $H = \mathrm{Cay}\left\langle a, b \mid b^2, a^5, (ab)^3 \right\rangle$. Figure 4.2(f) shows a Hamilton circle D of H. We will apply the construction of theorem 5.2.3 with some modifications.

Start with a Hamilton circle C_0 in $H = H^{\underline{0}}$. In every step, the Hamilton circle C_k satisfies the conditions of lemma 5.2.1

Consider an a-colored 5-cycle C of $H^{\underline{k}}$. Then C_k uses either exactly 4 edges or exactly 3 edges of C.

- If C_k uses 4 edges of C, it can be expressed as $C_k = ba^4bP$, where P is a subarc of C_k and a^4 is a subpath of both C and C_k. Let D be a Hamilton cycle in H, such that $D = ba^4bP'$, where P' is a subarc of D. Replace $C_k = ba^4bP$ by $ba^7bP'ba^{-1}bP$ to obtain a Hamilton circle in the graph $H^{\underline{k}}$ with H embedded inside C.
- If C_k uses 3 edges of C, it can be expressed as $C_k = ba^2bQbabP$, where P and Q are subarcs of C_k and a^2 and a are subpaths of C. (see figure 5.6(a)).

  Let D be a Hamilton cycle in H, such that $D = ba^2bQ'babP'$, where P' and Q' are subarcs of D and a^2 and a are subpaths of the same a-cycle. Replace $C_k = ba^2bQbabP$ by $ba^{-1}bP'ba^2bQ'ba^{-1}bQba^2bP$ to obtain a Hamilton circle in the graph $H^{\underline{k}}$ with H embedded inside C (see figure 5.6(b)).



(a) The Hamilton circle C_k.

(b) The Hamilton circle after embedding H in the blue cycle C.

Figure 5.6.: The transition in the twist-graph construction if $(m, n) = (3, 5)$.

Repeat this transformation for all 5-cycles of C_k to get a Hamilton circle C_{k+1} of $H^{\underline{k+1}}$.

These modifications avoid the critical situation that leads to ends with degree ≥ 4 in C_∞. Consider a common a-colored cycle C' of the subpaths P' and Q' of C_k for some $k \geq 0$. Let p_1, p_2 be the endpoints of $bP'b$ and q_1, q_2 be the endpoints of $bQ'b$.

By construction either $P' \cap C$ or $Q' \cap C$ remains unchanged in C_{k+1}. W.l.o.g let $R = P' \cap C$ be the (unchanged) subpath of C_{k+1}. Since no other a-colored cycle than C' is adjacent to R, the path R does not change in any further step. Hence, for all $K > k$ there are no subpaths bP_Kb, bQ_Kb of C_K with endpoints p_1, p_2 and q_1, q_2 respectively, such that P_K and Q_K are adjacent with a common a-colored n-cycle.

As in corollary 5.2.4, P' and Q' are separated in the following steps. As a result, all ends of C_∞ have degree 2, so C_∞ is a Hamilton cycle of G. $\square$

Proposition 5.2.7. The graph $G = \mathrm{Cay}\left\langle a, b \mid b^2, a^{12}, (a^2b)^3 \right\rangle$ is Hamiltonian.

Proof. G is the twist-amalgamation of $H = \mathrm{Cay}\left\langle a, b \mid b^2, a^6, (ab)^3 \right\rangle$. We will apply the construction of theorem 5.2.3 with some modifications. Start with a Hamilton circle C_0 in $H = H^{\underline{0}}$ as constructed in lemma 4.1.5. In every step, the Hamilton circle C_k satisfies the conditions of lemma 5.2.1. Furthermore, H contains two or three different types of a-colored 6-cycles.

Consider one side F of the Hamilton circle C_0 in H. F is the union of hexagons $g_0, g_1, g_2, \ldots$ and g_i and g_j intersect in exactly one edge if $i = j \pm 1$ and are disjoint otherwise. If the border of g_0 is an a-colored 6-cycle C, then C_k uses 5 edges of C and C is of type 1. From all other 6-cycles, C_k uses exactly 4 edges. Type 2 denotes those a-colored 6-cycles C, through which C_k runs as shown in figure 5.8(a), type 3 is shown in figure 5.8(c). The construction of C_0 as described in lemma 4.1.5 yields 6-cycles of all three types, as can be easily seen in figure 5.7.

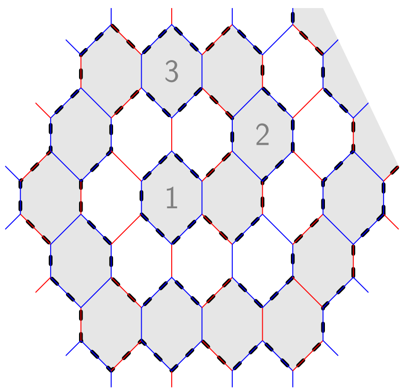


Figure 5.7.: A part of the Hamilton circle C_0 in $H = \mathrm{Cay}\left\langle a, b \mid b^2, a^6, (ab)^3 \right\rangle$.

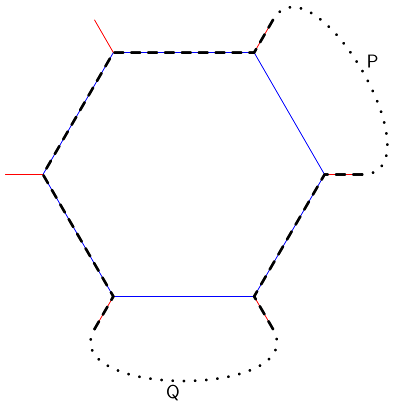
Let C be an a-colored 6-cycle of $H^{\underline{k}}$.

- If C_k uses 5 edges of C, it can be expressed as $C_k = ba^5bP$ (type 1). Let D be a Hamilton cycle in H, such that $D = ba^5bP'$, where P' is a subarc of D. Replace $C_k = ba^5bP$ by $ba^9bP'ba^{-1}bP$ to obtain a Hamilton circle in the graph $H^{\underline{k}}$ with H embedded inside C.

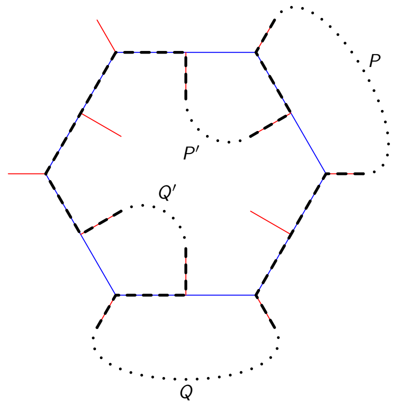
- If C_k uses 4 edges of C, it can be expressed as $C_k = ba^3bQbabP$ (figure 5.8(a), type 2) or $C_k = ba^2bQba^2bP$ (figure 5.8(c), type 3).

  ⋆ If $C_k = ba^3bQbabP$, let $D = ba^2bQ'ba^2bP'$ be a Hamilton cycle in H, where both a^3 and a are subpaths of the same a-colored 6-cycle. Replace C_k by $ba^{-1}bP'ba^4bQ'ba^{-1}bQba^2bP$. The transition is depicted in the figures 5.8(a) and 5.8(b).

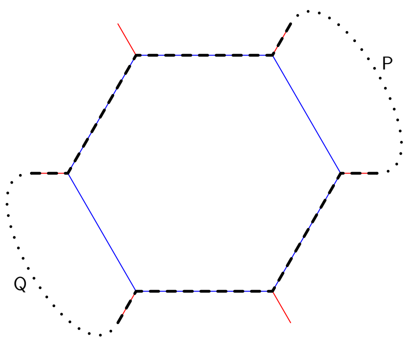
  ⋆ If $C_k = ba^2bQba^2bP$, let $D = ba^3bQ'babP'$ be a Hamilton cycle in H, where both copies of a^2 are subpaths of the same a-colored 6-cycle. Replace C_k by $ba^{-1}bQ'ba^2bP'ba^{-1}bQba^4bP$. The transition is depicted in the figures 5.8(c) and 5.8(d).



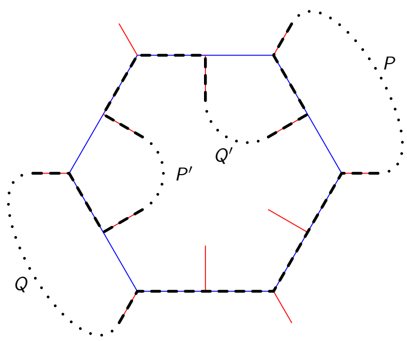
(a) The Hamilton circle $C_k = ba^3bQbabP$.



(b) The Hamilton circle after embedding H inside C.



(c) The Hamilton circle $C_k = ba^2bQba^2bP$.



(d) The Hamilton circle after embedding H inside C.

Figure 5.8.: The transitions in the twist-graph construction if $(m, n) = (3, 6)$.

Repeat this transformation for all 6-cycles of C_k to get a Hamilton circle C_{k+1} of $H^{\underline{k+1}}$.

Consider a common a-colored cycle C' of the subpaths P' and Q' of C_k for some $k \geq 0$. As in the previous proposition either $P' \cap C$ or $Q' \cap C$ remains unchanged in C_{k+1}. W.l.o.g let $R = P' \cap C$ be the (unchanged) subpath of C_{k+1}. Since no other a-colored cycle than

C' is adjacent to R, the path R does not change in any further step. This assures that P' and Q' are separated in the following steps and all ends have degree 2. □

Combining corollary 5.2.4 and the propositions 5.2.5, 5.2.6 and 5.2.7 yields the main result of this section.

Theorem 5.2.8. Every graph $G = \mathrm{Cay}\left\langle a, b \mid b^2, a^{2n}, (a^2b)^m \right\rangle$, $n \geq 3$, $m \geq 2$ is Hamiltonian.

Corollary 5.2.9. The graphs

$$\begin{aligned} G_1 &= \mathrm{Cay}\left\langle a, b \mid b^2, a^{2n}, (a^2ba^{-2}b)^m \right\rangle, \ n \geq 3, \ m \geq 1, \\ G_2 &= \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bcd)^m, (bc)^n \right\rangle, \ n \geq 3, \ m \geq 2, \\ G_3 &= \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bcdcbd)^m, (bc)^n \right\rangle, \ n \geq 3, \ m \geq 1 \end{aligned}$$

are Hamiltonian.

Proof. G_1 is the twist-amalgamation $H_1^{\overline{\infty}}$ of

$$H_1 = \mathrm{Cay}\left\langle a, b \mid b^2, a^n, (aba^{-1}b)^m \right\rangle, \ n \geq 3, \ m \geq 1.$$

H_1 is isomorphic to

$$H_0 = \mathrm{Cay}\left\langle a, b \mid b^2, a^n, (ab)^{2m} \right\rangle,$$

see proposition 4.2.2. The twist-amalgamation $G_0 = H_0^{\overline{\infty}}$ has a Hamilton circle. Hence, $G_1 = H_1^{\overline{\infty}} \cong H_0^{\overline{\infty}} = G_0$ and G_1 is Hamiltonian.

G_2 and G_3 are isomorphic to G_0 and G_1 if m, n are chosen suitably, see [Geo11a, theorem 9.4]. □

5.3. Twist-squeeze-amalgamations

Proving the Hamiltonicity of twist-squeeze-amalgamations (see definition 2.6.4) is slightly more complicated than in the case of twist-amalgamations.

The twist-squeeze-amalgamation of a graph

$$G = \mathrm{Cay}\left\langle a, b \mid b^2, a^n, (ab)^m \right\rangle, \ n \geq 3, \ m \geq 2$$

(see proposition 4.2.1) is

$$G^{\overline{\infty}} = \mathrm{Cay}\left\langle a, b \mid b^2, (a^2b)^n, (ab)^{2m} \right\rangle, \ n \geq 3, \ m \geq 2.$$

Consider the case $(n, m) = (4, 2)$ first. As in section 5.2, start with a Hamilton circle C_0 in $G = G^{\overline{0}}$. In every step, a copy of G is embedded inside every 4-cycle of $G^{\overline{k}}$ with alternating colors a, b to obtain $G^{\overline{k+1}}$. Using the construction below, the Hamilton circle

C_k of $G^{\overline{k}}$ can be transformed into a Hamilton circle C_{k+1} of $G^{\overline{k+1}}$. By induction over k the Hamiltonicity of $G^{\overline{k}}$ follows.

For all $k \geq 0$ and every 4-cycle C of $G^{\overline{k}}$ with alternating colors a, b one of the following conditions is satisfied (if necessary, reverse the orientation of C_k which means that a is replaced by a^{-1} and vice versa).

1. $C_k = aba^{-1}baP$, where P is a subarc of C_k and $ba^{-1}b$ is a subpath of C,
2. $C_k = aaba^{-1}a^{-1}P$, where P is a subarc of C_k and aba^{-1} is a subpath of C,
3. $C_k = aaaPa^{-1}a^{-1}a^{-1}Q$, where P, Q are subarcs of C_k and a, a^{-1} are edges of C as displayed in figure 5.10(c),
4. $C_k = abaPabaQ$, where P, Q are subarcs of C_k and the two b-edges lie on C,

It can be easily tested, that C_0, as defined in proposition 4.2.1, fulfills these requirements (see figure 5.9).

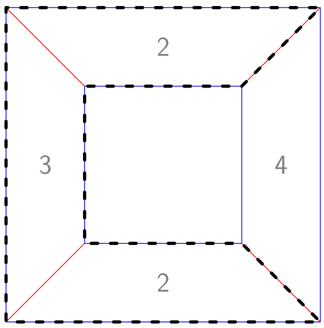


Figure 5.9.: The Hamilton circle C_0 in $G^{\overline{0}} = \mathrm{Cay}\left\langle a, b \mid b^2, a^4, (ab)^2 \right\rangle$.

For all cycles C of

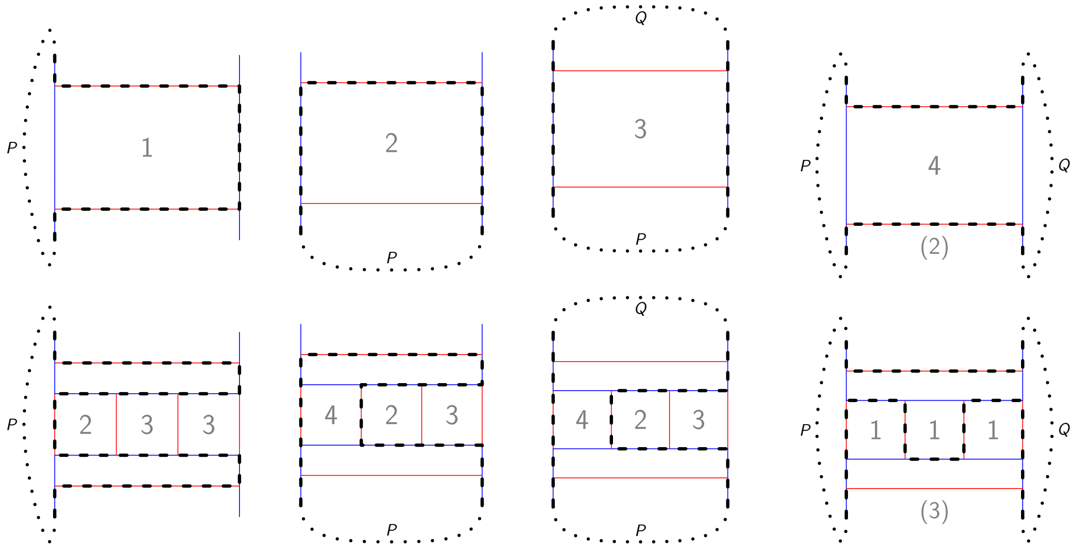
- type 1, replace $aba^{-1}baP$ by $aba^{-4}ba^{-4}baP$ (see figure 5.10(a)),
- type 2, replace $aaba^{-1}a^{-1}P$ by $a(ab)^2a^{-3}ba^{-3}a^{-1}P$ (see figure 5.10(b)),
- type 3, replace $aaaPa^{-1}a^{-1}a^{-1}Q$ by $aa^3ba^3aPa^{-1}a^{-1}ba^{-1}a^{-1}Q$ (see figure 5.10(c)),
- type 4, replace $abaPabaQ$ by $aaba^{-1}bab^{-1}baaPabaQ$ (see figure 5.10(d)).

Apply this transformation to all 4-cycles C of $G^{\overline{k}}$. The resulting graph C_{k+1} is a Hamilton circle of $G^{\overline{k+1}}$. All of the 4-cycles of $G^{\overline{k+1}}$ are one of the 4 types, as can be seen in figure 5.10.

By induction $G^{\overline{k}}$ is Hamiltonian for all $k \geq 0$.

Similarly to the arguments in the proofs of section 5.2, it follows that

- C_∞ visits all vertices of $G^{\overline{\infty}}$,
- C_∞ is connected (it meets every finite vertex cut),



(a) Type 1. (b) Type 2. (c) Type 3. (d) Type 4.

Figure 5.10.: The transitions in the twist-squeeze-graph construction if $(m, n) = (2, 4)$.

- every vertex of C_∞ has degree 2,
- there is a basis B of open neighbourhoods for the topology of $|G|$, such that every element of B is separated by exactly 4 edges from the rest of the graph, two of which are contained in C_∞. Therefore, every end of C_∞ has degree 2.

Finally, C_∞ is a Hamilton circle of $G^{\overline{\infty}}$.

Remark. Depending on the order in which the 4-cycles C are considered and the replacements are applied, different Hamilton circles C_k may be obtained.

Proposition 5.3.1. Every graph

$$G = \text{Cay}\left\langle a, b \mid b^2, (a^2b)^n, (ab)^4 \right\rangle, \ n \geq 3, \ n \equiv 0 \mod 2$$

is Hamiltonian.

Proof. G is the twist-squeeze-amalgamation of

$$H = \text{Cay}\left\langle a, b \mid b^2, a^n, (ab)^2 \right\rangle.$$

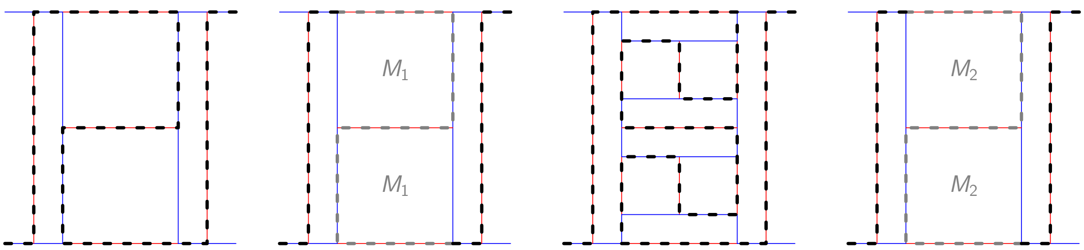
Start with a Hamilton circle C_0 of $H = H^{\overline{0}}$. As in the previous case $n = 4$ there are the same four different types of alternately-colored 4-cycles C in $H^{\overline{k}}$ and suitable transitions. Thus we can repeat the operations of figure 5.10 to obtain a Hamilton circle C_{k+1} of $H^{\overline{k+1}}$. Again, C_∞ is a Hamilton circle in $G = H^{\overline{\infty}}$. □

Now consider $G = \text{Cay}\left\langle a, b \mid b^2, (a^2b)^3, (ab)^4 \right\rangle$, that is the twist-squeeze-amalgamation of $H = \text{Cay}\left\langle a, b \mid b^2, a^3, (ab)^2 \right\rangle$, which is shown in figure 5.12(a). Applying similar transitions as in proposition 5.3.1 causes some problems.

The Hamilton cycle C_0 of $H = H^{\overline{0}}$ is unique up to isomorphism. One of the 4-cycles C of H contains two b-edges which also lie on C_0. Hence, $C_0 = abaPabaQ$. If the outside (containing P and Q) of C has to remain unchanged, the transition for the cycle C is also unique: $abaPabaQ$ is replaced by a^6Pa^6Q to obtain a Hamilton cycle in the graph which embeds H in C. The transitions for the new 4-cycles inside C are not unique. In any case they result in a 4-cycle D of $H^{\overline{2}}$, such that two b-edges of D lie on the Hamilton cycle C_2.

This construction leads to a similar situation as in the remark after corollary 5.2.4. Two loops of C_k converge to a common end as k goes towards infinity. This end has degree ≥ 4 and hence C_∞ is no Hamilton circle of G.

To avoid those circumstances, the construction of a Hamilton circle in G is done the other way around. Start with a 4-cycle of $H^{\overline{k}}$, in which a copy of H is embedded (see figure 5.11(a)). The course of the Hamilton circle $(aba^{-1}b)^2$is displayed as a dashed black line. This configuration can be used as a module M_1 to build new configurations one level above (in $H^{\overline{k-1}}$).



(a) The module M_1. (b) Two copies of M_1 are combined to yield M_2. (c) The module M_2. (d) Two copies of M_2 are combined to yield M_3.

Figure 5.11.: The transitions in the twist-squeeze-graph construction if $(m, n) = (2, 4)$.

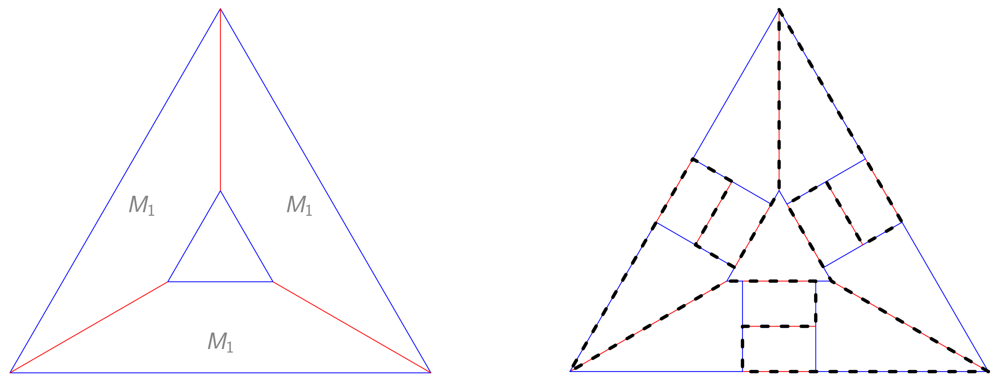
The module M_1 in figure 5.11(a) is used to construct a Hamilton cycle $C_1 = (aba^{-1}b)^6$ in $H^{\overline{1}}$ (see figure 5.12) or to get a new module $M_2 = (aba^{-1}b)^5$ (see figures 5.11(b) and 5.11(c)). Since, viewed from the outside, M_1 and M_2 look identical, M_2 can be used to build a Hamilton cycle $C_2 = (aba^{-1}b)^{15}$ in $H^{\overline{2}}$ or to construct a module $M_3 = (aba^{-1}b)^{11}$ (see figure 5.11(d)).

After repeating the described steps k times, a new Hamilton cycle

$$C_k = (aba^{-1}b)^{-3+9\cdot(2^k-1)}$$

of $H^{\overline{k}}$ is obtained.

Note that the set of alternately-colored 4-cycles yields a basis B of open neighbourhoods for the ends of G, such that every element of B is separated by exactly 4 edges from the rest of the graph, of which two of these edges are contained in C_∞. Therefore, every end has degree 2 in C_∞. Because in addition all vertices have degree 2 and C_∞ is a connected subspace that contains all vertices and ends of G, C_∞ is a Hamilton circle in G.



(a) Three copies of M_1 are combined to a Hamilton cycle C_1.

(b) The Hamilton cycle C_1.

Figure 5.12.: The construction of a Hamilton circle C_1 of $H^{\overline{1}}$.

Proposition 5.3.2. Every graph

$$G = \mathrm{Cay}\left\langle a, b \mid b^2, (a^2b)^n, (ab)^4 \right\rangle, \ n \geq 3, \ n \equiv 1 \mod 2$$

is Hamiltonian.

Proof. G is the twist-squeeze-amalgamation of $H = \mathrm{Cay}\left\langle a, b \mid b^2, a^n, (ab)^2 \right\rangle$.

The construction of a Hamilton circle in G is analogous to the above case $n = 3$. Start with a 4-cycle of $H^{\overline{k}}$, in which a copy of H is embedded. The course of the Hamilton circle is $(aba^{-1}b)^{\frac{n+1}{2}}$. This module M_1 can be used to build a Hamilton cycle

$$C_1 = (aba^{-1}b)^{\frac{n^2+n}{2}}$$

in $H^{\overline{1}}$ or to build a new module $M_2 = (aba^{-1}b)^{A(n,2)}$ and further modules

$$M_k = (aba^{-1}b)^{A(n,k)}$$

where $k \geq 1$. To calculate the exponent $A(n, k)$, solve the recurrence

$$A(n, 1) = \frac{n+1}{2},$$
$$A(n, k) = 1 + (n-1)A(n, k-1),$$

which gives the explicit representation

$$A(n, k) = \frac{n(n-1)^k - 2}{2(n-2)}.$$

As before, the described construction leads to Hamilton cycles

$$C_k = (aba^{-1}b)^{n\frac{n(n-1)^k-2}{2(n-2)}}$$

of $H^{\overline{k}}$ and finally to a Hamilton circle C_∞ in G. □

The ideas of the propositions 5.3.1 and 5.3.2 also work for twist-squeeze-amalgamations of infinite graphs. Choose $(n,m) = (4,4)$. The infinitely-ended graph

$$G = \text{Cay}\left\langle a, b \mid b^2, (a^2b)^4, (ab)^8 \right\rangle$$

is the twist-squeeze-amalgamation of the one-ended graph

$$H = \text{Cay}\left\langle a, b \mid b^2, a^4, (ab)^4 \right\rangle.$$

Proposition 5.3.3. The graph

$$G = \text{Cay}\left\langle a, b \mid b^2, (a^2b)^4, (ab)^8 \right\rangle,$$

is Hamiltonian.

Because the figures that are used in the following proof cover more than 10 pages, some of them are contained in appendix A.

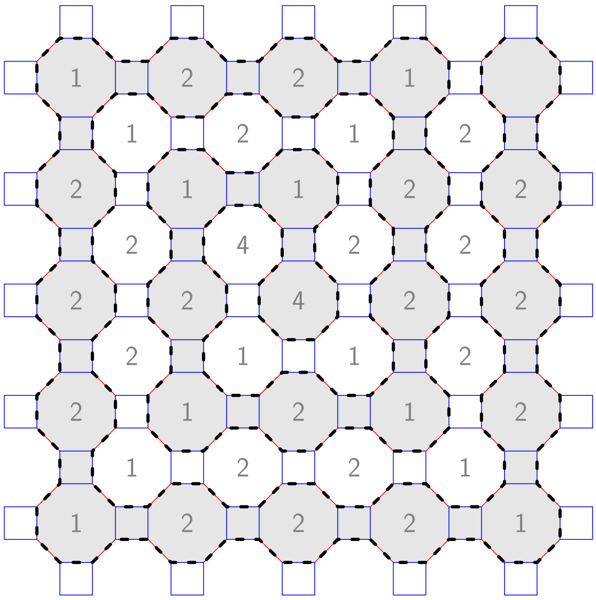
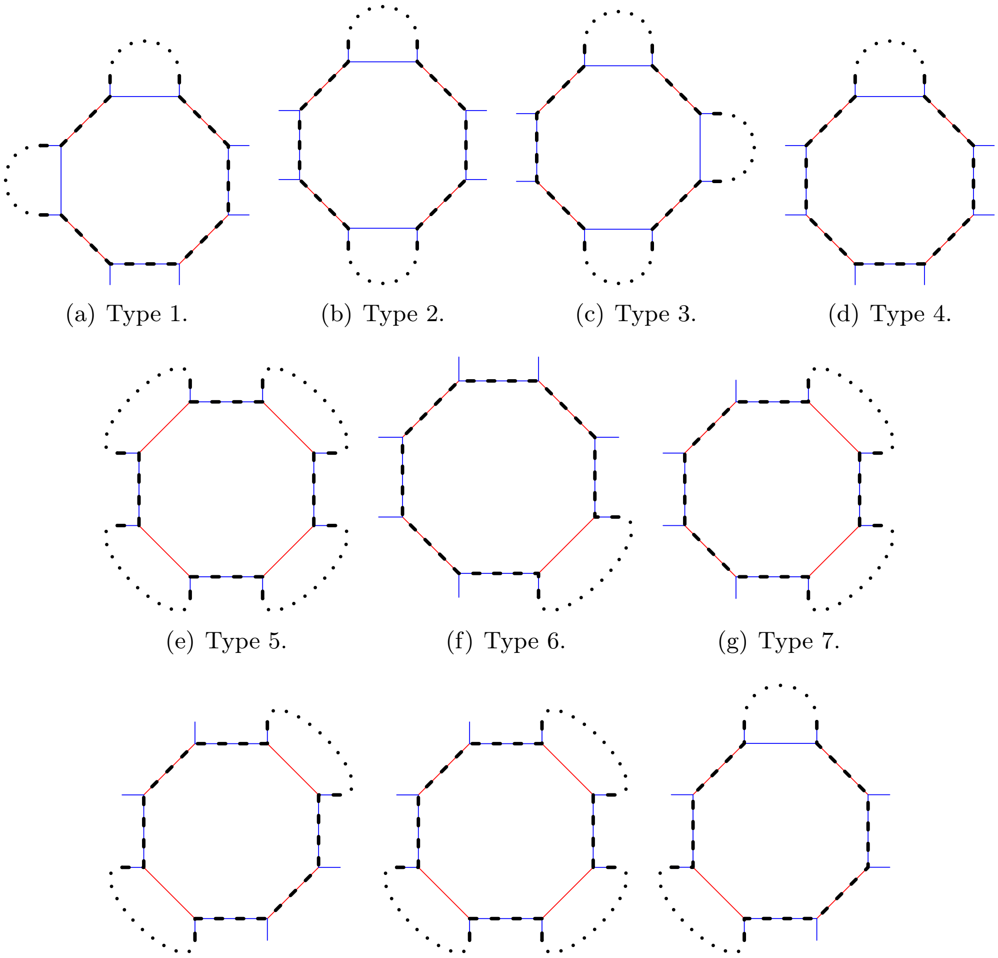


Figure 5.13.: A part of the Hamilton circle C_0 in $H = H^{\overline{0}} = \text{Cay}\left\langle a, b \mid b^2, a^4, (ab)^4 \right\rangle$.

Proof. Start with a Hamilton circle C_0 in $H^{\overline{0}}$ as outlined in figure 5.13 (see lemma 4.1.4). As in proposition 5.3.1, embed a copy of H in every alternately-colored 8-cycle of $H^{\overline{k}}$ to build $H^{\overline{k+1}}$. Applying the transitions described below, the circle C_k in $H^{\overline{k}}$ will be transformed into a Hamilton circle C_{k+1} of $H^{\overline{k+1}}$. By induction, $H^{\overline{k}}$ is Hamiltonian for every $k \geq 0$.

For all $k \geq 0$, every alternately-colored 8-cycle C of $H^{\overline{k}}$ is of one of the ten types displayed in figure 5.14. It can be easily examined that the Hamilton circle C_0 contains only the types $1, 2, 3$ and 4 (see figure 5.13).



(a) Type 1. (b) Type 2. (c) Type 3. (d) Type 4.

(e) Type 5. (f) Type 6. (g) Type 7.

(h) Type 8. (i) Type 9. (j) Type 10.

Figure 5.14.: The ten different types of alternately-colored cycles C.

Consider an 8-cycle C of type 1 (figure 5.14(a) and figure A.1(a) in appendix A). To find a suitable transition for the present Hamilton circle D when embedding a copy of H into C, exchange the inside and outside of C. The graph H is now lying in the outside of C while D is in the inside of C (see figure A.1(b)). As a next step, find suitable (infinite) paths through H as outlined in figure A.1(c) to receive a new circle. Such paths through H are shown in figure A.1(e). Since H is an infinite graph, not the full paths are shown in this figure. The paths have to be extended to all vertices of H in a similar way as explained in the proof of proposition 5.2.5, which is not a difficult task (extend the paths in the manner of a spiral towards the outside). Now exchange inside and outside of C again to yield a new Hamilton circle of the graph which embeds H into C (see figure A.1(d)).

Analogously, the transitions for the types 2–10 are shown in appendix A. Applying those transitions for all 8-cycles of $H^{\overline{k}}$, a Hamilton circle C_{k+1} in $H^{\overline{k+1}}$ is constructed.

Consider the limit graph C_∞.

As in some other proofs of this chapter,

- C_∞ visits all vertices of $G^{\overline{\infty}}$,
- C_∞ is connected (it meets every finite vertex cut),
- every vertex of C_∞ has degree 2,

To argue why every end has degree 2 is more difficult than in the proofs before since the transitions are more complex. Transitions like the types $2, 3, 4, 6, 7, 8, 9, 10$ do not cause problems, since they draw only one loop to the inside of a octagon. Type 1 draws two loops to the inside, but those loops are separated in the following steps (as in proposition 5.2.6). Type 5 draws four loops to the inside. They are also separated in the following steps in the style of proposition 5.2.6. As a result, any (standard subspace) neighbourhood of an arbitrary end is connected by exactly two edges to the remaining graph and therefore every end of C_∞ has degree 2. Thus, C_∞ is a Hamilton circle of G. $\square$

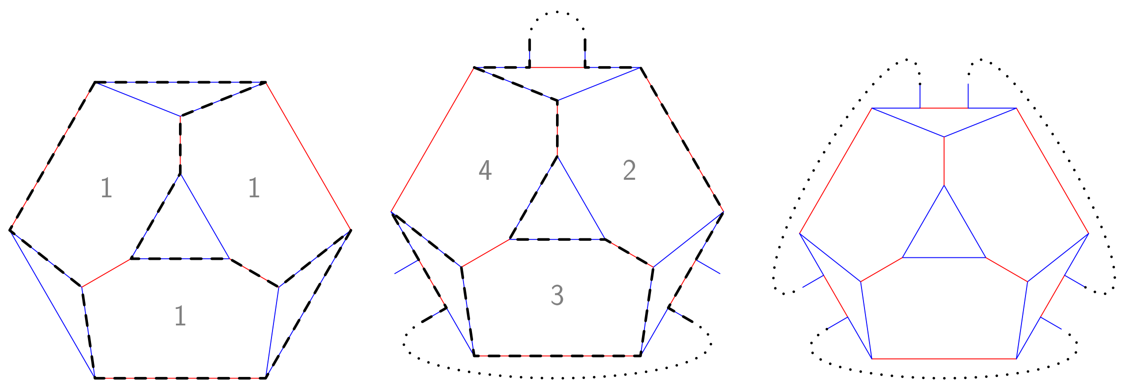
The results of section 5.3 are combined in the following theorem.

Theorem 5.3.4. Every graph

$$G = \mathrm{Cay}\left\langle a, b \mid b^2, (a^2b)^n, (ab)^{2m} \right\rangle, \ n \geq 3, \ m \geq 2$$

is Hamiltonian for $m = 2$ or $(m, n) = (4, 4)$.

The construction of a Hamilton circle in $\mathrm{Cay}\left\langle a, b \mid b^2, (a^2b)^4, (ab)^8 \right\rangle$ has been rather complex. Surprisingly, the case $(n, m) = (3, 3)$ that leads to $G = \mathrm{Cay}\left\langle a, b \mid b^2, (a^2b)^3, (ab)^6 \right\rangle$, which is the twist-squeeze-amalgamation of a finite graph, seems to be even more complicated.



(a) The unique Hamilton cycle in $H = \mathrm{Cay}\left\langle a, b \mid b^2, a^3, (ab)^3 \right\rangle$.

(b) The transition for type 1.

(c) There is no transition for type 2.

Figure 5.15.: Complications in the case $(n, m) = (3, 3)$.

The Hamilton cycle in the underlying finite graph $H = \mathrm{Cay}\left\langle a, b \mid b^2, a^3, (ab)^3 \right\rangle$ is unique up to isomorphism. All alternately-colored cycles in H are of the same type (outlined in figure 5.15(a)). If a transition as in proposition 5.3.1 or in proposition 5.3.3 is to be applied, the transition is also unique (see figure 5.15(b)). In $\overline{H^2}$, there are three different types of alternately-colored cycles. Consider the type that is marked 2 in figure 5.15(b). For a cycle C of this type, it is impossible to find a transition which covers all vertices of the copy of H embedded inside C without changing the path outside C (see figure 5.15(c)).

Hence, the approach of proposition 5.3.1 fails in this case. The method of proposition 5.3.2 also does not provide an immediate solution. It is possible to build 8 different modules in the way of proposition 5.3.2 where all vertices of H are covered. The use of these modules allows the construction of different non-isomorphic Hamilton cycles in $\overline{H^2}$. However, it is apparently difficult to put the modules together in order to get new ones which can be applied in the same way as the already considered 8 modules. Therefore, the case $(n, m) = (3, 3)$ remains unsolved.

CHAPTER 6

Concluding remarks

Of the 37 types of planar cubic planar Cayley graphs in [Geo11a, table 1], we proved 2 to be nonhamiltonian, 22 to be Hamiltonian and 1 to be Hamiltonian if and only if $n = 2$.

In particular, a planar cubic Cayley graph G turns out to be Hamiltonian, if

- G is finite or one-ended or
- G is two-ended and has two generators or
- $\kappa(G) = 2$ and $G \neq \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (b(cb)^n d)^m \right\rangle$ or
- G is the twist-amalgamation of a finite or one-ended graph.

Appendix B contains an overview of all graphs treated in this master's thesis sorted by connection number, number of ends, generators and relations of the underlying group and spin-behavior.

6.1. Limitations

For the twist-squeeze-amalgamations, we can not give a final answer. We showed that

$$\mathrm{Cay}\left\langle a, b \mid b^2, (a^2b)^n, (ab)^{2m} \right\rangle,\ n \geq 3,\ m \geq 2$$

is Hamiltonian if $m = 2$ or $(m, n) = (4, 4)$ are chosen. For other values of n and m, the question of the Hamiltonicity remains open.

The infinitely-ended graphs

$$G = \mathrm{Cay}\left\langle a, b \mid b^2, (a^2ba^{-2}b)^m, (baba^{-1})^n \right\rangle,\ n \geq 2,\ m \geq 2$$

are the twist-squeeze-amalgamations of

$$H = \mathrm{Cay}\left\langle b, c, d \mid b^2, c^2, d^2, (bc)^n, (cd)^m, (db)^n \right\rangle,\ n \geq 2,\ m \geq 2$$

along the alternately b- and c-colored cycles. Those graphs are also not covered in this thesis.

Furthermore, we did not study the multi-ended graphs G with $\kappa(G) = 3$ and 3 generators, which can not be expressed as twist-amalgamation or twist-squeeze-amalgamation of a finite or one-ended graph.

6.2. Open problems

This thesis supports the conjecture that all planar cubic 3-connected Cayley graphs are Hamiltonian, which is a subcase of conjecture 2.7.6, see [Geo11a, conjecture 1.3]. The question if all finitely generated 3-connected planar Cayley graphs have a Hamilton circle remains open.

It might be worthwhile to extend the approaches of chapter 5 to other classes of multi-ended graphs, especially those which can be expressed as twist-squeeze-amalgamation of a finite or one-ended graph. However, this analysis could turn into excessive distinction of cases, as can be seen in section 5.3.

It could be another interesting problem to characterize the planar cubic Cayley digraphs which admit a (directed) Hamilton circle.

APPENDIX A

Transitions in proposition 5.3.3

Proposition 5.3.3 states that every graph

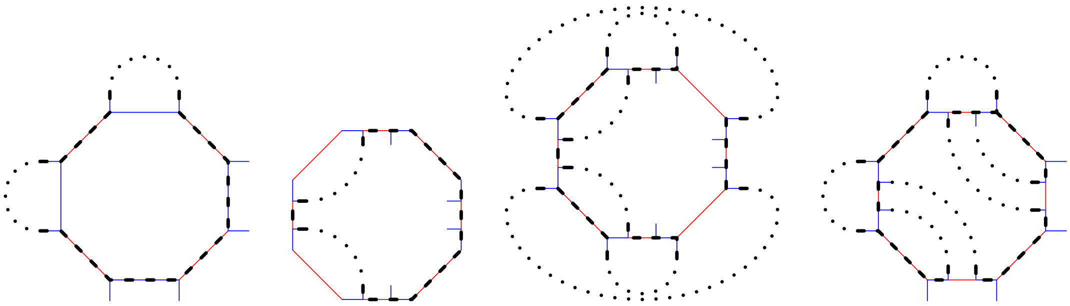
$$G = \mathrm{Cay}\left\langle a, b \mid b^2, (a^2b)^4, (ab)^8 \right\rangle,$$

is Hamiltonian.

Because the figures that are used in the proof cover more than ten pages, some of them are printed here.

Figure 5.14 shows ten different types of 8-cycles in the construction of a Hamilton circle.

On the following pages, the transitions for the ten types of cycles are provided.

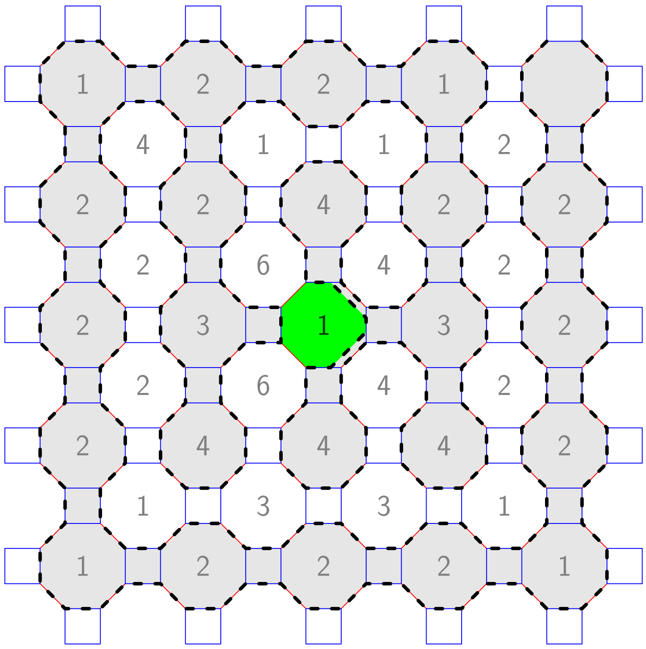


(a) Type 1.

(b) After exchanging inside and outside.

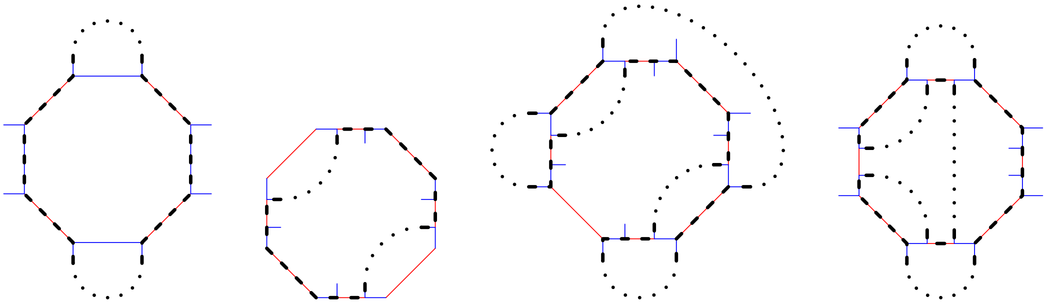
(c) The transition.

(d) After changing inside and outside again.



(e) The path in the neighbourhood of C.

Figure A.1.: The transition for type 1.

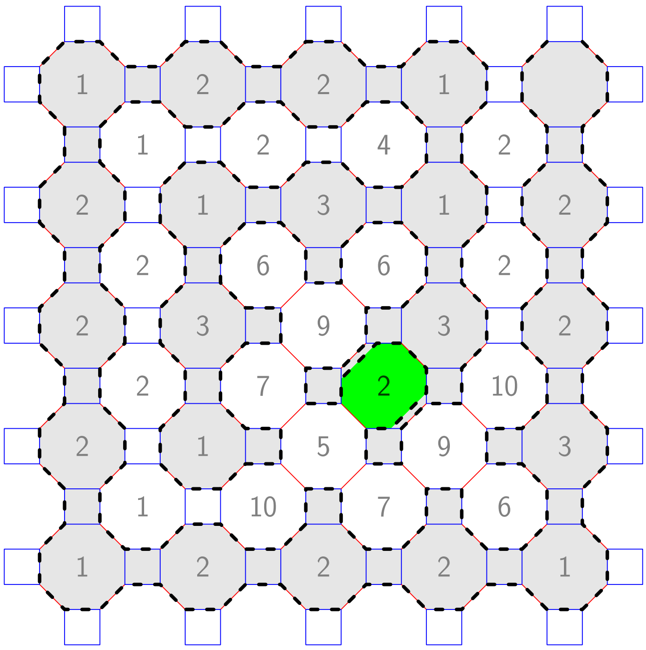


(a) Type 2.

(b) After exchanging inside and outside.

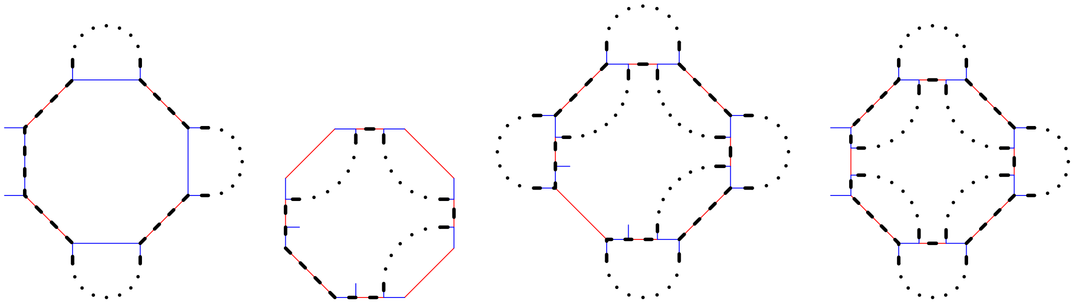
(c) The transition.

(d) After changing inside and outside again.



(e) The path in the neighbourhood of C.

Figure A.2.: The transition for type 2.

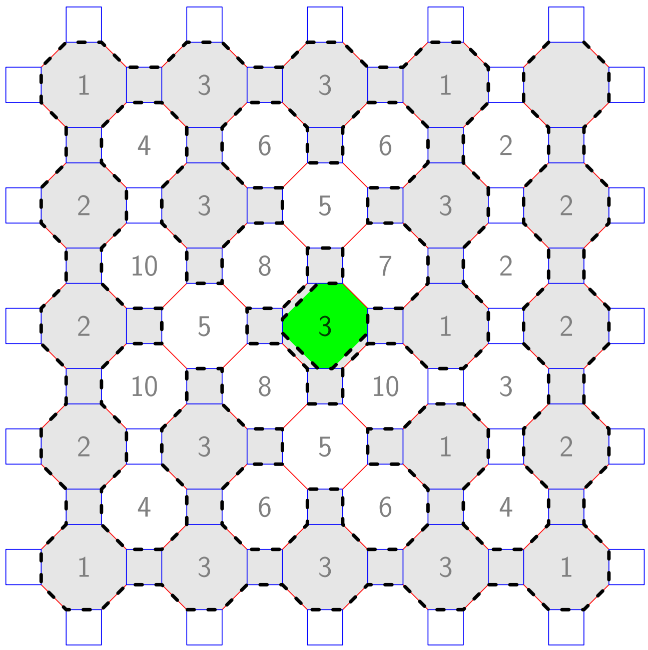


(a) Type 3.

(b) After exchanging inside and outside.

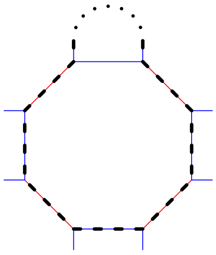
(c) The transition.

(d) After changing inside and outside again.

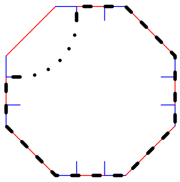


(e) The path in the neighbourhood of C.

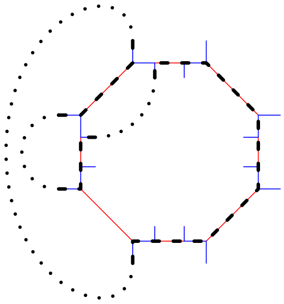
Figure A.3.: The transition for type 3.



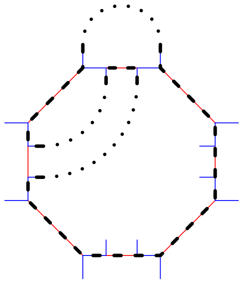
(a) Type 4.



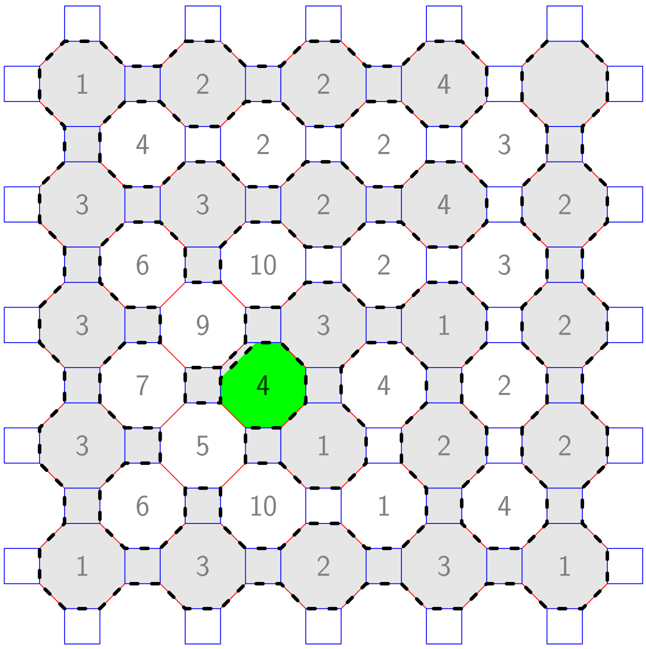
(b) After exchanging inside and outside.



(c) The transition.

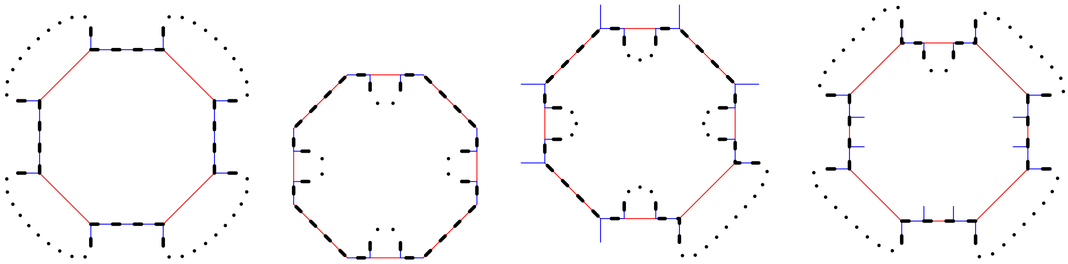


(d) After changing inside and outside again.



(e) The path in the neighbourhood of C.

Figure A.4.: The transition for type 4.

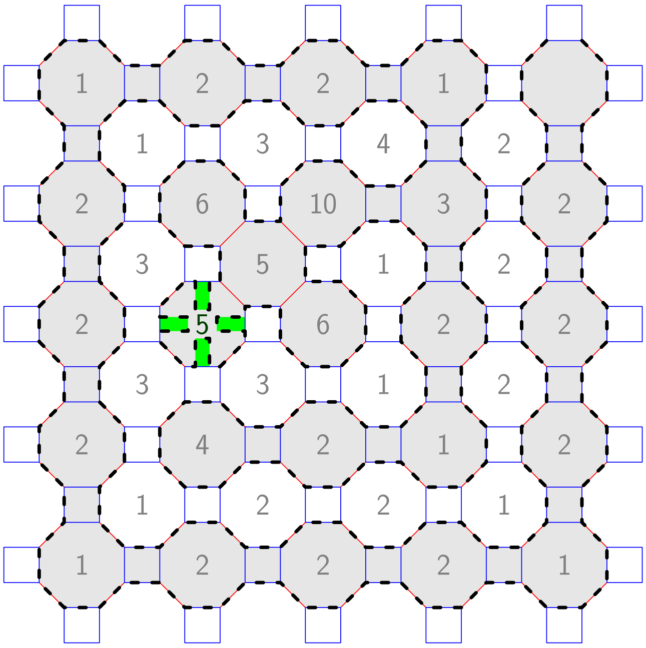


(a) Type 5.

(b) After exchanging inside and outside.

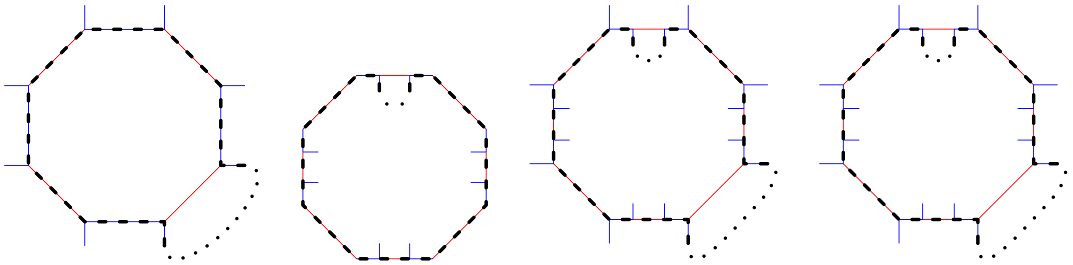
(c) The transition.

(d) After changing inside and outside again.

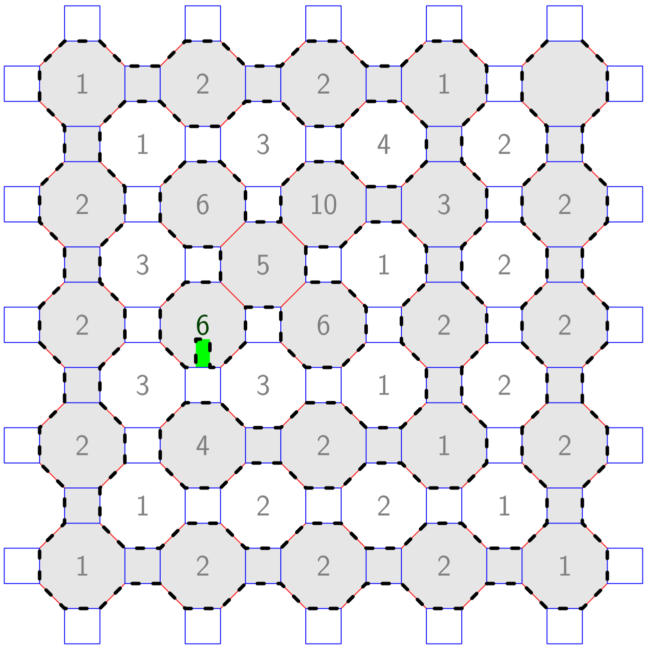


(e) The path in the neighbourhood of C.

Figure A.5.: The transition for type 5.

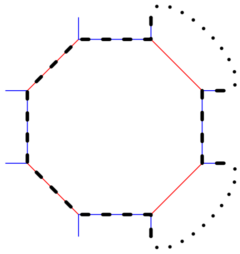


(a) Type 6. (b) After exchanging inside and outside. (c) The transition. (d) After changing inside and outside again.

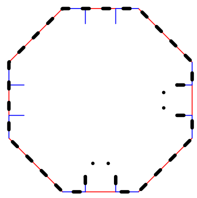


(e) The path in the neighbourhood of C.

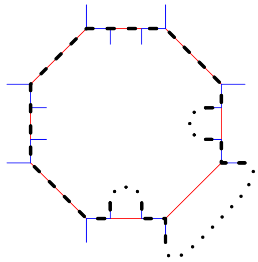
Figure A.6.: The transition for type 6.



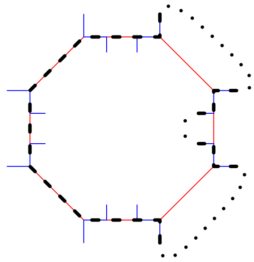
(a) Type 7.



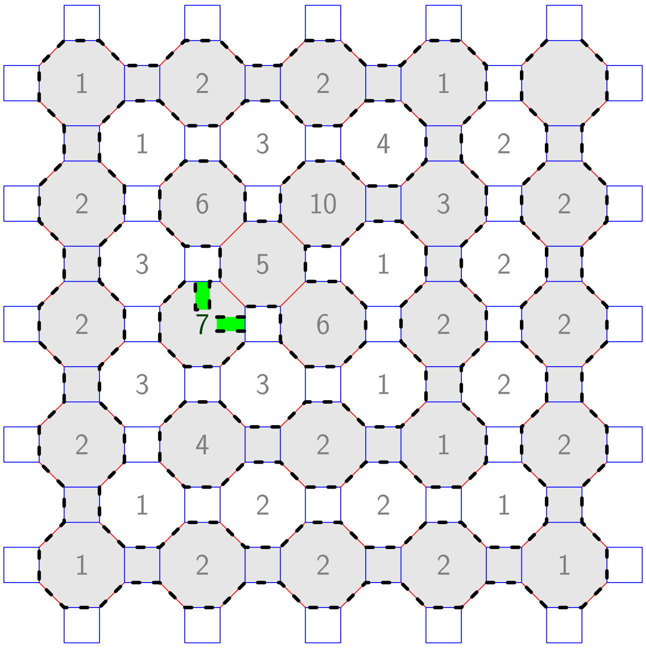
(b) After exchanging inside and outside.



(c) The transition.

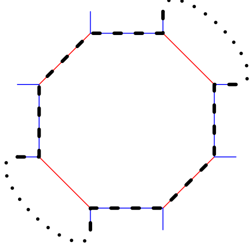


(d) After changing inside and outside again.

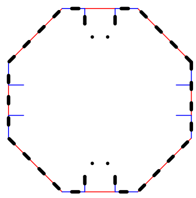


(e) The path in the neighbourhood of C.

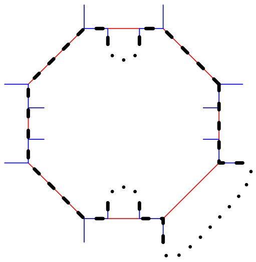
Figure A.7.: The transition for type 7.



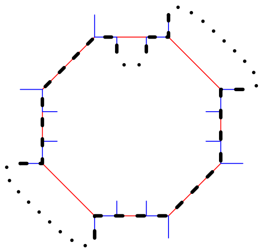
(a) Type 8.



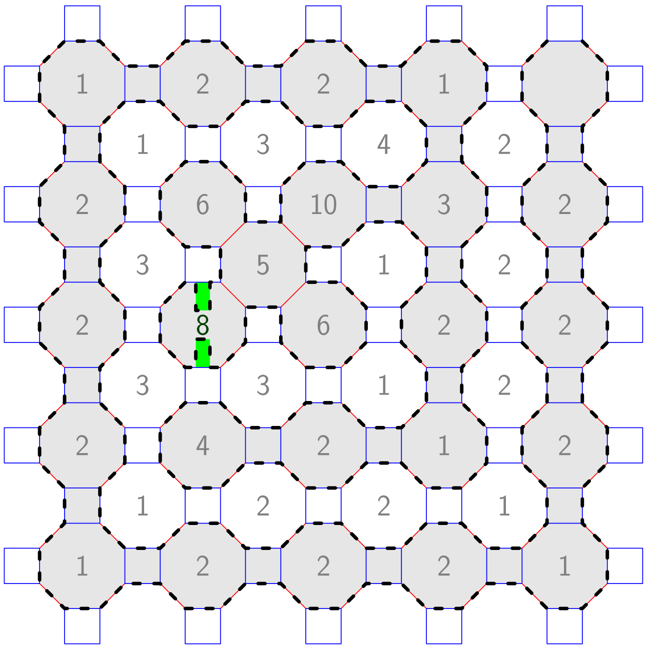
(b) After exchanging inside and outside.



(c) The transition.

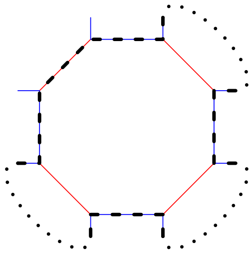


(d) After changing inside and outside again.

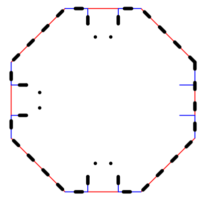


(e) The path in the neighbourhood of C.

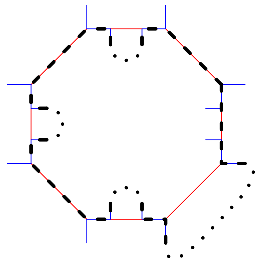
Figure A.8.: The transition for type 8.



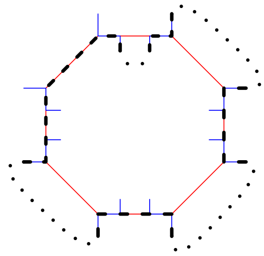
(a) Type 9.



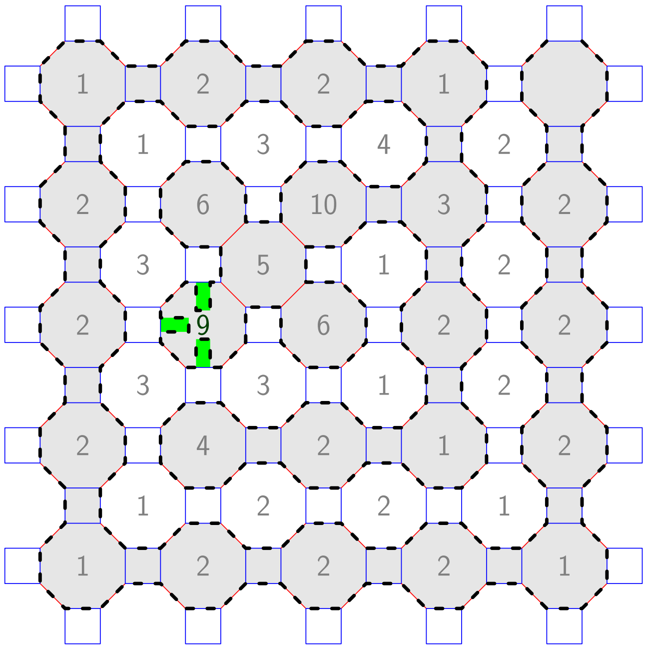
(b) After exchanging inside and outside.



(c) The transition.

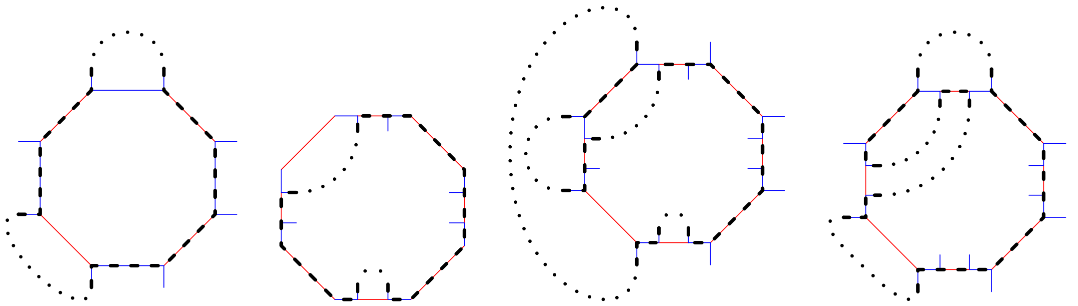


(d) After changing inside and outside again.



(e) The path in the neighbourhood of C.

Figure A.9.: The transition for type 9.

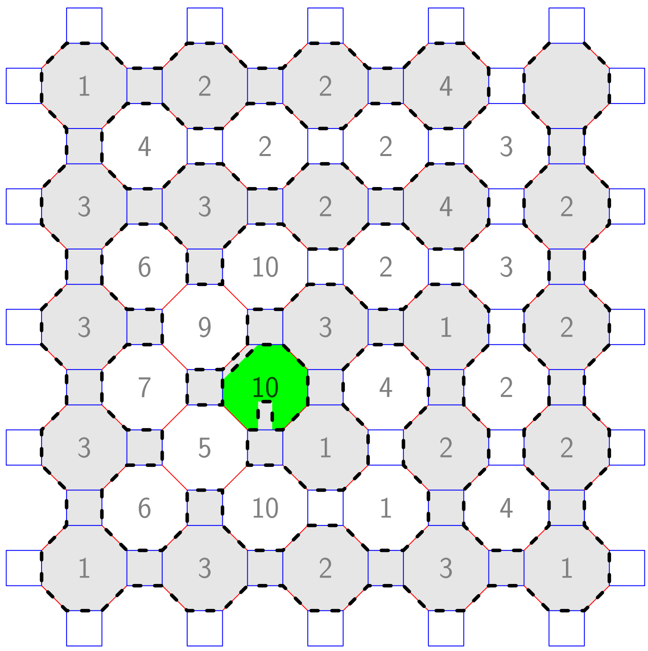


(a) Type 10.

(b) After exchanging inside and outside.

(c) The transition.

(d) After changing inside and outside again.



(e) The path in the neighbourhood of C.

Figure A.10.: The transition for type 10.

APPENDIX B

Classification of the graphs

The following table lists the graphs $G = \mathrm{Cay}(L)$ studied in this thesis. $L = \langle X \mid W \rangle$ is a presentation of the underlying group (see section 2.1) and $S \subseteq X$ is the set of spin-preserving colors (see definition 2.8.3).

$\kappa(G)$	$\lvert\Omega(G)\rvert$	X	W	S	Ham.	discussed in
1	∞	a, b	b^2, a^n	–[1]	no	chapter 3
1	∞	b, c, d	$b^2, c^2, d^2, (bc)^n$	–[1]	no	chapter 3
2	2 or ∞	a, b	$b^2, (ab)^n$	a, b	yes	proposition 3.1.2
2	2 or ∞	a, b	$b^2, (aba^{-1}b^{-1})$	a	yes	proposition 3.1.3
2	2	a, b	$b^2, a^4, (a^2b)^n$	b	yes	proposition 3.1.1
2	2 or ∞	b, c, d	$b^2, c^2, d^2, (bc)^2, (bcd)^m$	b, c, d	yes	proposition 3.2.1
2	2 or ∞	b, c, d	$b^2, c^2, d^2, (bc)^{2n}, (cbcd)^m$	c	yes	proposition 3.2.2
2	2 or ∞	b, c, d	$b^2, c^2, d^2, (bc)^n, (bd)^m$	$\emptyset$	yes	proposition 3.2.3
2	∞	b, c, d	$b^2, c^2, d^2, (b(cb)^n d)^m$	b, c, d	iff $n = 2$	proposition 3.2.4
2	2 or ∞	b, c, d	$b^2, c^2, d^2, (bcbd)^m$	b	yes	proposition 3.2.5
2	0	b, c, d	$b^2, c^2, d^2, (bc)^n, cd$	–[1]	yes	proposition 3.2.6
3	0 or 1	a, b	$b^2, a^n, (ab)^m$	a, b	yes	proposition 4.2.1
3	0 or 1	a, b	$b^2, a^n, (aba^{-1})^m$	a	yes	proposition 4.2.2
3	0 or 1	a, b	$b^2, (a^2b)^m$	b	yes	proposition 4.2.3
3	0 or 1	a, b	$b^2, (a^2ba^{-2}b)^m$	$\emptyset$	yes	proposition 4.2.4
3	0 or 1	b, c, d	$b^2, c^2, d^2, (bcd)^n$	b, c, d	yes	proposition 4.3.1
3	0 or 1	b, c, d	$b^2, c^2, d^2, (cbcdbd)^n$	c, d	yes	proposition 4.3.2
3	0 or 1	b, c, d	$b^2, c^2, d^2, (bc)^n, (bdcd)^m$	d	yes	proposition 4.3.3

[1] This graph does not have a unique consistent embedding.

$\kappa(G)$	$\lvert\Omega(G)\rvert$	X	W	S	Ham.	discussed in
3	0 or 1	b, c, d	$b^2, c^2, d^2, (bc)^n, (cd)^m, (db)^p$	$\emptyset$	yes	proposition 4.3.4
3	2	a, b	$b^2, a^2ba^{-2}b, (baba^{-1})^n$	$\emptyset$	yes	proposition 5.1.3
3	2	a, b	$b^2, (a^2b)^2, (ab)^{2m}$	b	yes	proposition 5.1.2
3	∞	a, b	$b^2, (a^2b)^m a^{2n}$	b	yes	theorem 5.2.8
3	∞	a, b	$b^2, (a^2ba^{-2}b)^m a^{2n}$	$\emptyset$	yes	corollary 5.2.9
3	∞	b, c, d	$b^2, c^2, d^2, (bcd)^m (bc)^n$	b, c, d	yes	theorem 5.2.9
3	∞	b, c, d	$b^2, c^2, d^2, (bcdcbd)^m, (bc)^n$	b, c	yes	theorem 5.2.9
3	∞	a, b	$b^2, (a^2b)^n, (ab)^{2m}$	b	?	section 5.3

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