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Finite Element Formulation for the Consolidation Process in poro-elasto-plastic Media

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Abstract

The mechanical behavior of soils can be strongly influenced by water present in the pores between the solid grains. This interaction can have significant influence on the design process in geotechnical engineering. An example would be the settlement of a foundation. Also in other areas of engineering the behavior of porous media is important, for instance when gas saturated foams or biological tissues are studied. Simulations of porous media are a difficult task for modeling as well as for numerical treatment. In this work, the numerical realization of Biot's consolidation model with the finite element method is presented. Additionally to Biot's model, the solid skeleton is allowed to deform elastic-plastic. Due to plastic deformations the problem gets nonlinear. For the implemented plasticity models of Drucker-Prager and von Mises a consistent linearization is performed analytically. The result is a super-linear convergent newton-like solver. The discretization of the coupled problem leads to a block-system of linear equations. This system is solved by an iterative solver of simple type suitable for saddle point problems relying on two linear solvers for the decoupled solid problem and fluid problem.

Zusammenfassung

In einem wassergesättigten Boden kann die Interaktion zwischen den Körnern und dem Wasser großen Einfluss auf das mechanische Verhalten haben und ist daher bei der Auslegung von Bauwerken zu berücksichtigen. Im Allgemeinen ist die Simulation von porösen Medien sowohl in Fragen bezüglich der Modellierung als auch deren numerischer Umsetzung herausfordernd. In dieser Arbeit wird das Konsolidierungsmodell von Biot mithilfe der Finiten Elemente Methode gelöst. Zusätzlich zu dem Model von Biot wird ein elastisch-plastisches Verhalten des Festkörperskeletts angenommen. Im Fall von tatsächlich auftretenden plastischen Verformungen wird das Problem nichtlinear, zu dessen Lösung ein quasi-Newtonverfahren angewandt wird. Eine konsistente Linearisierung des plastischen Anteils ergibt eine Methode mit super-linerarer Konvergenz. Das gekoppelte Fluid-Festkörperproblem führt auf ein Blocksystem an linearen Gleichungen welches iterativ gelöst wird. Dabei wird beachtet, dass im undrainierten Grenzfall das System ein Sattelpunktproblem darstellt.

CONTENTS

1. INTRODUCTION

What is consolidation? Imagine, standing at the sea and looking at the horizon. Your feet will sink in the wet sand. You are experiencing consolidation. Walking further and looking back you see that the footprints in the sand stay present. You have deformed the sand plastically. More engineering relevant consolidation processes in connection with plastic material behavior can be found in geotechnics. The settlement of foundations or the construction of dams are examples where the interaction of the soil and the infiltrated water can have significant influence on the behavior of the construction. Building a dam to fast can cause failure because of this interaction. Not only in geotechnics, also in other engineering fields like civil engineering or biomechanics simulation of porous materials is becoming more and more important.

A state of the art approach in modeling is to consider the material as continuum. This means that the body of interest is described as a set of continuously connected points in space. Physical quantities like displacements or pressure are treated as fields on a macroscopic level. For bodies undergoing reversible deformations the well established theory of elasticity exists. On the other hand to account for irreversible deformations different plasticity models has been developed. Beside the classical models of von Mises and Tresca complex models exist to describe the behavior of soils (see e.g. the textbook of Yu [38]). For continuum mechanics a huge amount of books and textbooks are available. The books Altenbach [1], Marsden and Hughes [20], Truesdell and Noll [30] and Truesdell and Toupin [31] are some examples.

In the last century, the classical theory of continua has been extended to account for porous structures in materials. Today, three main theories can be found for modeling porous materials. The earliest approach is Biot's Theory (BT) which is based on the work of von Terzaghi [32]. In a series of papers it has been developed and improved by Biot [5],[6], [7], [8]. It will be used in this work.

The Theory of Porous Media (TPM) is based on the theory of continua for mixtures, extended by the concept of volume fractions by Bowen [11], [10]. A historical review of the TPM is given by de Boer [14]. A large scale computation using the TPM with a complex plasticity model is given by Wieners et al. [33]. A comparison of the BT and the TPM can be found in Schanz and Diebels [26].

As a third theory, the Simple Mixture Model (SMM) of Wilmanski [35] is also able to describe porous media. Although, it is based on the fundamental laws of thermodynamics, it neglects some effects which are captured within the BT.

In case of two or three dimensional models analytic solutions are only available in special cases, e.g. see the review articles of Schanz [25] and Selvadurai [27]). Therefore, methods have been developed to obtain a numerical solution. Beside others, the Finite Difference Method (FDM), the Boundary Element Method (BEM) and the Finite Element Method (FEM) are well established methods. Within this work the FEM will be used. The books of Braess [12], Jung and Langer [18], Wriggers [37], Zienkiewicz [39] are only some examples of available textbooks on the topic.

The FEM is based on a variational formulation of the governing partial differential equations. Therefore, integration over the computational domain is necessary. Normally, these integrals are treated with a quadrature rule, e.g. Gauss integration. As a standard approach for the incorporation of plasticity, as a first step the stress response is calculated at the quadrature points. Second, on the basis of the calculated stress it is checked whether the response is elastic or plastic. In case of a plastic response a return mapping algorithm has to be applied. A large class of return map algorithms are analyzed in [22]. For the global nonlinear problem a newton like method has to be applied. The use of a so called consistent tangent operator introduced by Simo and Taylor [29] yields super-linear convergence. This is proven by Gruber et al. [23] and Sauter and Wieners [24] for different plasticity models.

Within this work a different approach is presented. The plastic strains are also approximated with shape functions, as the displacement field is. Instead of performing the return map algorithm at the quadrature points it is performed at the nodes of the plastic strain approximation. Furthermore, a consistent linearization yielding a super-linear convergent newton like method is presented.

1.1. Remarks on the notation

In this work symbolic notation as well as index notation will be used.

Notation. *(Index notation) The components of a first order tensor u are written as u_i whereas the components of a second order tensor T are written as T_{ij}. Higher order tensors are written analogously. In general, any index $i, I, j, J, k, K, \ldots$ take on the values* $1,2,3$.

Furthermore, Einstein summation convention will be used frequently.

Notation. *(Summation convention) If an index occurs twice in an product expression the summation over this index is taken from one to three. The summation sign $\sum$ is omitted,*

$$r = r_i e_i = r_1 e_1 + r_2 e_2 + r_3 e_3$$

The Kronecker delta δ_{ij} denotes the components of the second order identity tensor with the property

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \quad .$$

The derivative of a tensor T with respect to a space coordinate x_i is denoted by $T_{,i}$. It is assumed that x_i refers to an orthonormal Cartesian basis. The derivative with respect to time is denoted as $\dot{T}$.

1.2. Outline of the work

In chapter 2, the fundamentals of continuum mechanics and Biot's Theory are described. First considerations of geometric aspects of the behavior of bodies ending with the introduction of strain measures are given. This is followed by a short section on kinetics. Next, basic constitutive models for elasticity and plasticity are introduced. Finally, balance equations for mass, momentum and moment of momentum are stated. In chapter 3, the governing equations in connection with boundary data lead to boundary value problems. In chapter 4, the used algorithms and methods for solving the problem of porous elasto-plastic materials are described. Chapter 5 finishes this thesis with the validation of the implemented program and a numerical example.

2. FUNDAMENTALS OF CONTINUUM MECHANICS AND BIOT'S THEORY OF POROUS MEDIA

This chapter introduces the continuum mechanical background and Biot's theory of porous media. Firstly, the geometric description of a physical object is introduced. This is followed by a section on kinetics. Next, constitutive equations are given for elastic and plastic material behavior and fluid flow (Law of Darcy). Finally, the balance equations for mass, momentum and moment of momentum are given.

2.1. Geometry

This section introduces the geometric description of the physical objects which will be treated in this work. First, the concept of representing bodies as connected points is introduced. In order to describe local properties of the change in geometry, the so-called deformation gradient is introduced. Finally, with the motivation to connect geometry and force, which are responsible for a change in geometry, a quantity called strain is extracted from the deformation gradient. Basic considerations from classical mechanics about space of observation, frames and tensors are given in the appendix A.

2.1.1. Body

The physical objects of consideration in this work are continuous media with physical properties embedded in the three-dimensional Euclidean space. Whether the whole object or parts of it are called body, which is introduced as follows.

Definition 1. *(Body) A body is an open set* $\mathbb{S} \subset \mathbb{R}^3$ *and its boundary is as smooth as required.*

The final goal in this work is to study the behavior (the movement and the change of shape) of bodies which are influenced by forces. For now the reasons for the forces remains unmentioned. Therefore, a particular state of the object is chosen to be reference for subsequent behave of the object. This state is called the reference configuration and will be denoted by $\mathbb{B}$. In the following a point of the reference (material) configuration is denoted by $X \in \mathbb{B}$ and called a material point. The geometry of any state of the object can then be described by a mapping called deformation.

Definition 2. *(Deformation) A deformation is an invertible $\mathbb{B} \to \mathbb{S}$ map, which is as smooth as required. Therefore,*

$$\begin{aligned}\phi : \mathbb{B} &\to \mathbb{S} \\ x &= \phi(X), \\ \phi^{-1} : \mathbb{S} &\to \mathbb{B} \\ X &= \phi^{-1}(x),\end{aligned}$$

where ϕ^{-1} denotes the inverse.

The configuration at a fixed time $t > 0$ is called current (spatial) configuration and a point $x \in \mathbb{S}$ is called a spatial point. In the following, transformations which relate physical quantities of the reference configuration to the current configuration are investigated. If $\{X_I\}$ is a coordinate system on the reference configuration and $\{x_i\}$ is a coordinate system on the current configuration, then a quantity $Q_A...Za...z$ is called a two point tensor of order n. For the numerical treatment of problems in continuum mechanics it is convenient to introduce the displacement as the difference of the current configuration and the reference configuration,

$$u(X) = \phi(X) - X. \tag{2.1}$$

2.1.2. Deformation gradient

In the following, a tensor describing the deformation at a material point and its infinitesimal neighborhood is introduced. This tensor is called deformation gradient and plays a fundamental role in continuum mechanics. Before a definition is made, an example illustrating a deformation is given.

Example. *(Example Deformation) Choosing a rectangular Cartesian coordinate system for all following considerations. Let the deformation $\phi : \mathbb{R}^3 \to \mathbb{R}^3$*

$$\begin{aligned}x_1(X) &= X_1 + \frac{(e^{2X_2} - 1 + 0.7(X_1)^3)}{20} \\ x_2(X) &= X_2 - \frac{(e^{X_1} - 1 + 0.7(X_2)^3)}{10} \\ x_3(X) &= X_3\end{aligned}$$

be given. Consider a straight line $l(t) = A + (B - A)t$ from $A = [0,0,0]^\top$ to $B = [1,1,0]^\top$. Applying the deformation on l yields the curve $c(t) = \phi(l(t))$. A Taylor series of ϕ at A is made. The situation is visualized in figure 2.1. Obviously the approximating property of the Taylor's Series can be seen. Furthermore, the deformed curve and all its approximations share the same tangent at A. Therefore, to describe the deformation locally in a infinitesimal neighborhood of A the first derivative is sufficient.

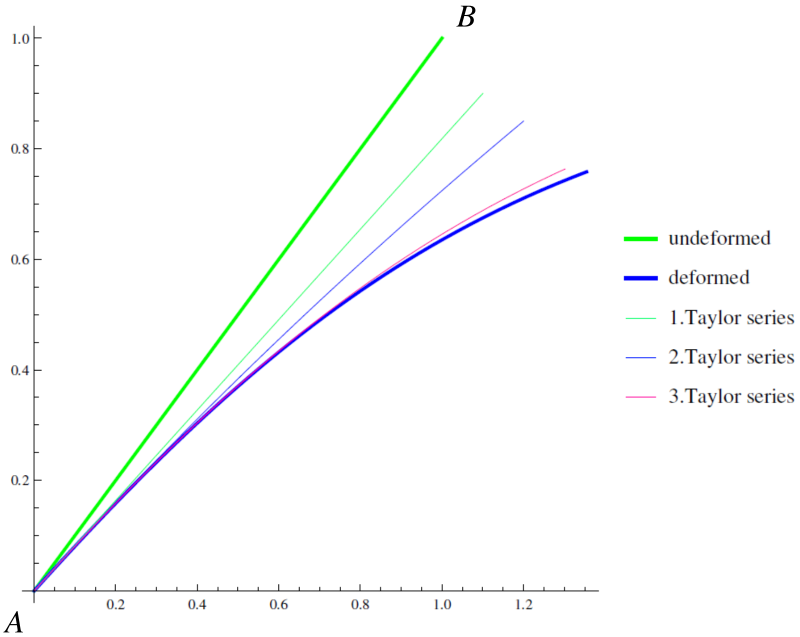

Figure 2.1.: Plot of the deformation and its approximations

With the motivation of the previous example one defines the deformation gradient as follows.

Definition 3. *(Deformation Gradient) Let ϕ be a deformation of the reference body $\mathbb{B}$. Then the two point tensor of order two,*

$$F_{iJ}(X) = \frac{\partial \phi_i(X)}{\partial X_J} \quad X \in \mathbb{B} \tag{2.2}$$

is the deformation gradient of ϕ [20, p.48].

The deformation gradient F maps therefore infinitesimal line elements $\mathrm{d}X$ emigrating from $X \in \mathbb{B}$ to its deformed configuration denoted by $\mathrm{d}x$,

$$\mathrm{d}x_i = F_{iJ}\mathrm{d}X_J. \tag{2.3}$$

Due to the deformation a body can change its volume. This is investigated in the following. Consider a particle X of the reference configuration and the three infinitesimal vectors $\mathrm{d}A$, $\mathrm{d}B$ and $\mathrm{d}C$. These vectors span the infinitesimal volume

$$\mathrm{d}V = (\mathrm{d}A \times \mathrm{d}B) \cdot \mathrm{d}C \tag{2.4}$$

Mapping the vectors into a deformed configuration they read as $F(X)\mathrm{d}a$, $F(X)\mathrm{d}b$ and $F(X)\mathrm{d}c$. The deformed volume is now

$$\mathrm{d}v = (F\mathrm{d}A \times F\mathrm{d}B) \cdot F\mathrm{d}C = \det(F)\mathrm{d}V \tag{2.5}$$

Therefore, the local change in volume can be measured by

$$\frac{\mathrm{d}v}{\mathrm{d}V} = \det(F(X)) =: J(X). \tag{2.6}$$

At the reference configuration the deformation gradient is the identity, $F = I$ and therefore J = 1. Furthermore, the existence of an inverse of the deformation requires $J \neq 0$. Together with the requirement that a sequence of deformations in time has to be continuous [31], the condition $J > 0$ follows. A similar consideration as for an infinitesimal volume element holds for an area $\mathrm{d}A = \mathrm{d}C \times \mathrm{d}D$ in the reference configuration. The corresponding area $\mathrm{d}a$ in the deformed configuration is obtained by

$$\mathrm{d}a = \det(F)F^{-\top}\mathrm{d}A. \tag{2.7}$$

For establishing constitutive relations it is convenient to have a strain measure which is invariant for rigid body motions. With the deformation gradient a quantity is given, which is invariant to rigid body translations but not to rigid body rotations. Applying the polar decomposition theorem [20, p. 51] the deformation gradient $F(X)$ can be decomposed as

$$F_{iJ}(X) = R_{iK}(X)U_{KJ}(X) = V_{ik}(X)R_{kJ}(X). \tag{2.8}$$

In (2.8), R is a orthogonal tensor and called rotational tensor, U is the right stretch tensor and V is the left stretch tensor. The tensors U and V are now appropriate quantities to define strain measures, because the information of rigid body rotations is solely contained in R. Using the fact that R is an orthogonal tensor one introduces the right Cauchy-Green tensor as

$$C(X) = F^T(X)F(X) = U^T(X)U(X), \tag{2.9}$$

where no information of R is contained.

2.1.3. Strain measures

A major task in modeling is to find a suitable relation between the deformation of a continuum and its stress state which will be introduced formally in section 2.2. Such relations are called constitutive relations and will be treated in section 2.3. As indicated earlier, to state an objective constitutive relation it must not depend on rigid body motions. As every strain measure is a purely mathematical quantity every strain measure is equivalent to each other. One of the most stated strain measures is the so called Green-Lagrange-Strain tensor which is introduced in the following definition.

Definition 4. *(Green-Lagrange-Strain) The Green-Lagrange strain tensor E is defined as*

$$E_{ij} = \frac{1}{2}(C_{ij} - \delta_{ij}) = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{i,k}u_{j,k}), \tag{2.10}$$

where C is the right Cauchy-Green tensor.

By neglecting nonlinear terms in the definition of the Green-Lagrange-Strain tensor the so called linearized strain tensor is obtained.

Definition 5. *(Linearized Strain) The linearized strain tensor is defined as*

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{2.11}$$

2.2. Kinetics

In section 2.1, the geometric description of a physical object is introduced. Thereby, no further specifications of the reasons responsible for the behavior has been made. In this chapter, the mechanical ones are introduced. Furthermore, the notion of stress and different stress measures are introduced.

2.2.1. Load

The reason for the deformation of matter can have different reasons. From the set of all possible reasons (for instance thermal, electromagnetic, chemical and others) only the mechanical ones are treated here. These can be grouped in

- body loads and
- surface loads.

Here, body loads are related to the mass density of the body. The force density due to body loads $f(x)$ is given by

$$f(x) = \rho k, \tag{2.12}$$

with the mass density ρ and a vector field k. The specialization for gravitation as the only one treated here reads

$$f(x) = -\rho g e_3 \tag{2.13}$$

with the gravitational acceleration g and the vector e_3 pointing away from the center of gravity. The resulting force F^b on a body due to body load can be computed by integration over the volume of the body,

$$F^b = \int_{\mathbb{S}} f(x)\mathrm{d}x. \tag{2.14}$$

The surface loads represent the interaction of two neighbored bodies on their common surface. To characterize this interaction the so called Cauchy traction vector as the respective force surface density is introduced.

Definition 6. *(Cauchy Traction Vector) Let $\mathbb{S}$ be a body with its boundary $\partial\mathbb{S}$. Consider an oriented surface with the unit normal vector $n(x)$ pointing outward of $\mathbb{S}$ and $x \in \partial\mathbb{S}$. Then the vector t(x) representing the mechanical interaction between $\mathbb{S}$ and its surrounding is called the Cauchy traction vector at x.*

The resulting force F^s on a body $\mathbb{S}$ due to surface loads can be computed by integration over the surface of the body,

$$F^s = \int_{\partial\mathbb{S}} t(x)\mathrm{d}a. \tag{2.15}$$

Additionally to the notion of a resulting force, the resulting torque of a force acting at point a with respect to the origin o is defined as

$$\Theta_o = a \times F. \tag{2.16}$$

2.2.2. Stress state and the Cauchy Stress Tensor

As a point on the boundary of one body can also be on the boundary of other bodies but with different surface normals the Cauchy traction vector is not unique. This motivates the definition of a stress state as follows.

Definition 7. *(Stress State) The set of all traction vectors at a point x define the stress state at that point.*

Fortunately, the dependency of the Cauchy traction vector on the surface normal is linear, which tells the well known theorem of Cauchy.

Theorem 1. *(Cauchy) The Cauchy traction vector $t(x,n)$ at a fixed point x but arbitrary surface normal vector n is given by*

$$t(x,n) = \sigma(x)n, \tag{2.17}$$

with the Cauchy stress tensor σ,which is a second order tensor on the current configuration.

In view of theorem 1 the stress state is uniquely defined if the Cauchy stress tensor is known.

2.2.3. Other stress measures

The Cauchy traction vector represents the actual force acting on an area in the current configuration. As in most applications the current configuration is not known beforehand, the integration to get the resulting force can not be performed. Therefore, the domain of integration is transformed to the reference configuration. Take a deformation ϕ and a reference body $\mathbb{B}$. Then the current configuration $\mathbb{S}$ is given with $\phi(\mathbb{B})$. The resulting force due to surface loads is

$$\mathbf{F}^s = \int_{\partial\mathbb{S}} t(x)\mathrm{d}a = \int_{\partial\mathbb{S}} \sigma(x)n(x)\mathrm{d}a = \int_{\partial\mathbb{B}} \sigma(\phi(X))n(\phi(X))J(\phi(X))\mathrm{d}A. \tag{2.18}$$

This motivates the following definition.

Definition 8. *(Kirchhoff Stress) Let ϕ be a deformation with the deformation gradient F and $J = det(F)$. Furthermore, let σ the Cauchy stress tensor, then*

$$\tau = J\sigma \tag{2.19}$$

defines the Kirchhoff stress tensor.

Not only the domain can be transformed also the normal vector involved in equation (2.18) can be transformed to the reference configuration. This motivates the definition of the First Piola-Kirchhoff stress tensor P as

$$P = J\sigma F^{-T} \tag{2.20}$$

As a last stress measure the Second Piola-Kirchhoff stress tensor S is introduced as

$$S = JF^{-1}\sigma F^{-T}. \tag{2.21}$$

2.3. Constitutive relations

As mentioned before, a constitutive relation closes the gap between stress state and strain state. First, a general form of elastic constitutive relations (hyperelastic) which allow to model large strain behavior of many materials is given. Next, the equations for the special case of the linear elastic isotropic constitutive relation are given. Then, a section about plasticity introduces the classical rate independent plasticity theory, which will be specialized to a non-associative Drucker-Prager model which contains the von Mises model. Finally, considerations about a two phase media are given. Thereby, one phase is a solid skeleton showing an elasto-plastic response, and the other phase is a fluid flowing through the pores of the solid. This section is based on [20] and [16] for elasticity, [17] for plasticity and on [5] for porous media.

In the following, the space of symmetric second order tensors is denoted by $\mathbb{T}^s$.

2.3.1. Elasticity

Here, the class of hyperelastic constitutive models containing the linear elastic model of Hooke is given. For a hyperelastic constitutive relation there exists a stored energy functional W,

$$W : \mathbb{T}^s \to \mathbb{R} \tag{2.22}$$

such that the stress response σ at $x \in \mathbb{S}$ due to a given strain state $\varepsilon \in \mathbb{T}^s$ can be computed by

$$\sigma(x,\varepsilon) = \frac{\partial W}{\partial \varepsilon}(x,\varepsilon) \quad x \in \mathbb{S}, \varepsilon \in \mathbb{T}^s. \tag{2.23}$$

The second partial derivative of the functional W with respect to the strain tensor gives the so called elasticity tensor C which is a tensor of order four,

$$C(x,\varepsilon) = \frac{\partial^2 W}{\partial \varepsilon \partial \varepsilon}(x,\varepsilon). \tag{2.24}$$

As the determination of material parameters can be a difficult task one try to use the easiest constitutive relation which gives sufficient results. The easiest elastic constitutive relation for three dimensional modeling is obtained when only allowing a linear dependency of the stresses on the strains and assuming that the material response is equal in every direction (isotropic). Then the elasticity tensor has constant values depending on only two parameters λ and μ,

$$C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{2.25}$$

The stress can be computed by

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} = \lambda \epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij}. \tag{2.26}$$

The inversion of (2.26) yields the compliance relation

$$\epsilon_{ij} = C^{-1}_{ijkl}\sigma_{kl} = -\frac{\lambda}{2\mu(3\lambda+2\mu)}\sigma_{kk}\delta_{ij} + \frac{1}{2\mu}\sigma_{ij}. \tag{2.27}$$

With the definition of the Young's modulus

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \tag{2.28}$$

and Poisson's ratio

$$\nu = \frac{\lambda}{2(\lambda+\mu)} \tag{2.29}$$

the compliance relation (2.27) can be rephrased as

$$\epsilon_{ij} = -\frac{\nu}{E}\sigma_{kk}\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij}. \tag{2.30}$$

Other useful constants are the bulk modulus

$$K = \lambda + \frac{2}{3}\mu = \frac{E}{3(1-2\nu)} \tag{2.31}$$

and the shear modulus $G = \mu$.

2.3.2. Plasticity

For the behavior of metals the following can be observed. Before reaching a certain limit the response can be modeled as linear elastic with high accuracy. Above this limit the response becomes nonlinear and inelastic, which means that after releasing the applied load some deformations stay present. To determine whether the material is locally below or above this limit, a yield criterion of the form

$$f(\sigma, q) = \Phi(\sigma, q) - \sigma_y(q) \leq 0 \tag{2.32}$$

is introduced. In equation (2.32) f is the yield function, Φ is a function which maps the stress state to a real number, σ_y is the yield strength and q is a set of internal variables to model hardening response. The yield function has to be negative or zero. If it has an negative value the material has an elastic behavior, if it is zero the material yields. As the yielding of materials can influence the yield strength, the set of internal variables are introduced to model the effects of the stress history. Recall that for a given deformation the linearized strain can be computed by

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{2.33}$$

In the theory of plasticity for small stains it is assumed that the total strain tensor in (2.33) can be additively decomposed into an elastic and a plastic part,

$$\varepsilon = \varepsilon^e + \varepsilon^p. \tag{2.34}$$

The constitutive relation for a hyperelastic material is adjusted in such a way that it only depends on the elastic strains,

$$\sigma_{ij} = \frac{\partial W(x, \varepsilon^e)}{\partial \varepsilon^e_{ij}}. \tag{2.35}$$

For the plastic strain an evolution equation of the form

$$\dot{\varepsilon}^p = \gamma \mathbf{r}(\sigma, q) \tag{2.36}$$

is assumed. This equation is called the flow rule. In (2.36), γ is the consistency parameter and $r : \mathbb{T}^s \times \mathbb{R}^m \to \mathbb{T}^s$ is a prescribed function defining the direction of the plastic flow.

For the special case $r = \frac{\partial f}{\partial \sigma}$ the flow rule is called associative. Furthermore, the hardening law

$$\dot{q} = -\gamma \mathbf{h}(\sigma, q) \tag{2.37}$$

is a evolution law for the internal variables, where $h : \mathbb{T}^s \times \mathbb{R}^m \to \mathbb{R}^m$ is a prescribed function defining the type of hardening. In the associative case $h = D\frac{\partial f}{\partial q}$ with D containing the hardening moduli. In the following, conditions for the yield function and the consistency parameter are given. In the case $f < 0$ the material behaves elastic, thus $\gamma = 0$. For $f = 0$ plastic flow occurs according to (2.36). Therefore, $\gamma \neq 0$. As states with $f > 0$ are inadmissible $\dot{f} = 0$ has to be valid in the case of plastic flow. Summarizing, the Kuhn-Tucker complementary condition

$$\gamma f(\sigma, q) = 0 \tag{2.38}$$

and the consistency condition

$$\gamma \dot{f}(\sigma, q) = 0 \tag{2.39}$$

have to be fulfilled for all states.

A special yield creterion for isotropic material behavior is given by the Drucker-Prager yield criterion

$$f(\sigma) = \sqrt{\frac{1}{2}}\|s\| + p\eta - \xi c, \tag{2.40}$$

where p and s are the hydrostatic stress state and the deviatoric stress state

$$\begin{aligned} p &= \frac{1}{3}\sigma_{ii}, \\ s_{ij} &= \sigma_{ij} - p\delta_{ij}. \end{aligned} \tag{2.41}$$

Notation. *(Tensor norm) The norm of the second order tensor s is defined as $\|s\| = \sqrt{s_{ij}s_{ij}}$.*

In equation (2.40), η, ξ and c are material parameters. In this work, they are computed in the way that an inner approximation of the Mohr-Coulomb yield criterion is achieved [15]. Therefore, the friction angle and the cohesion are the input parameters for the implemented code.

For the flow rule often the yield function is adopted to

$$\bar{f} = \sqrt{\frac{1}{2}}\|s\| + p\bar{\eta}, \tag{2.42}$$

where $\bar{\eta}$ is computed with the dilatancy angle instead of the friction angle. Then the non-associative flow rule reads

$$\dot{\varepsilon}^p_{kl} = \gamma \frac{\partial \bar{f}}{\partial \sigma_{kl}} = \gamma \left(\sqrt{\frac{1}{2}} \frac{s_{kl}}{\|s\|} + \frac{1}{3}\bar{\eta}\delta_{kl} \right). \tag{2.43}$$

Neglecting the term containing the hydrostatic pressure p in equation (2.40) yields the von Mises yield criterion. Instead of using c as a material parameter the uniaxial yield strength denoted by $\sigma_{y,0}$ is used,

$$f = \sqrt{\frac{3}{2}}\,\|s\| - \sigma_{y,0} - q, \tag{2.44}$$

and the scalar isotropic hardening variable q is introduced. This yield criterion is based on the assumptions that the occurrence of plastic flow is governed by the deviatoric stress energy. The associative evolution of q is given by

$$\dot{q} = -\gamma D \frac{\partial f}{\partial q} = \gamma D, \tag{2.45}$$

with the scalar isotropic hardening modulus D.

2.3.3. Theory of saturated porous media

Up to now, the physical body has been modeled as a single continuum. In this section, a superposition of two continua is introduced. One is a solid skeleton and the other is a fluid. The following assumptions are made in the subsequent developments:

- the solid skeleton behaves isotropic
- the constitutive modeling of the solid skeleton is elastic-plastic
- the stress- strain relationship is linear in the elastic range
- only small strains occur, therefore the linearized strain tensor (2.11) is a appropriate strain measure
- the fluid and the solid grains are incompressible
- the fluid flow through the pores of the solid skeleton is according to the Law of Darcy

To characterize the saturated porous media Coussy [13] distinguishes between the total porosity and the connected porosity. The total porosity is defined as the ration of non-solid volume to the total volume. In this work this quantity will not be used. The connected porosity e^f is the ratio of connected void space V^f to the total volume V

$$e^f = \frac{V^f}{V}. \tag{2.46}$$

It is assumed that the fluid can move freely through the connected void space, according to Darcy's law. In the considered case of full saturation with one fluid, the volume fraction of the solid is given as

$$e^s = 1 - e^f. \tag{2.47}$$

The mass density ρ of the bulk material is then given in terms of the mass densities of the single components

$$\rho = e^f \rho_f + e^s \rho_s, \tag{2.48}$$

where ρ_f is the mass density of the fluid and ρ_s is the mass density of the solid grains.

In the concept of partial stresses it is assumed that distinct stress tensors for the solid and fluid can be defined [6]. Then the total stress tensor σ^{tot} is defined as

$$\sigma^{\text{tot}} = \sigma^s + \sigma^f, \tag{2.49}$$

where σ^s is the stress tensor for the solid and σ^f is the stress tensor for the fluid. The constitutive equations for the solid

$$\sigma^s_{ij} = 2\mu\varepsilon^s_{ij} + \left(\lambda + \frac{Q^2}{R}\right)\varepsilon^s_{kk}\delta_{ij} + Q\varepsilon^f_{kk}\delta_{ij}, \tag{2.50}$$

and the fluid

$$\sigma^f_{ij} = Q\varepsilon^s_{kk}\delta_{ij} + R\varepsilon^f_{kk}\delta_{ij}. \tag{2.51}$$

are introduced. In [5], Biot shows that four independent physical constants are needed. In the equations (2.50) and (2.51) these are the two Lame parameters λ and μ describing the solid and Q and R describe the coupling between the two phases. The parameters Q and R can be expressed in terms of the compression modulus of the solid grains K^s, the compression modulus of the fluid K^f, the compression modulus of the solid skeleton and the porosity e^f as

$$R = \frac{(e^f)^2 K^f (K^s)^2}{K^f(K^s - K) + e^f K^s (K^s - K^f)} \tag{2.52}$$

$$Q = \frac{e^f(\alpha - e^f) K^f (K^s)^2}{K^f(K^s - K) + e^f K^s (K^s - K^f)} \tag{2.53}$$

with Biot's effective stress coefficient $\alpha = 1 - {}^{K}\!/_{K_s}$.

The stress state in the fluid is assumed to be hydrostatic. The relation between fluid stress and pore pressure p is defined by,

$$\sigma^f_{ij} = -e^f p\delta_{ij}. \tag{2.54}$$

The minus sign is introduced because in elasticity a tensile stress is denoted positive, but a positive pore pressure is usually associated with an expansion of the pores. Due to a tensile stress the pores contract. That is why the pressure is defined as the negative hydrostatic fluid stress. Using Biot's effective stress coefficient α the total stress is given as

$$\sigma^{\text{tot}}_{ij} = 2\mu\varepsilon^s_{ij} + \lambda\varepsilon^s_{kk}\delta_{ij} - \alpha p\delta_{ij}. \tag{2.55}$$

In the limiting case of incompressible solid grains $\alpha \to 1$. In view of equation (2.55), the effective stress on the solid governing the elastic-plastic behavior is introduced as

$$\sigma_{ij}^{\text{eff}} = \sigma_{ij}^{\text{tot}} + \alpha p \delta_{ij}. \tag{2.56}$$

Furthermore, the variation of fluid volume per unit reference volume ζ is introduced

$$\zeta = \alpha \varepsilon_{kk}^{s} + \frac{(e^f)^2}{R} p. \tag{2.57}$$

In case of incompressible constitutes it simplifies to

$$\zeta = \varepsilon_{kk}^{s}. \tag{2.58}$$

A relation between the velocity of the fluid and the pore water pressure is given by the law of Darcy and has the form [19]

$$q = e^f w_i = e^f k(\rho^f g_i - p_{,i}). \tag{2.59}$$

In equation (2.59), $w_i = \dot{u}_i^f - \dot{u}_i^s$ are the average relative velocity components, k is the isotropic permeability and g is the gravitational acceleration vector.

2.4. Balance laws

In section 2.2, the notion of force has been introduced. There different reasons for forces are mentioned. Now in this chapter a balance law connects these forces in an equation. Equations of the same structure can be defined for the mass, linear momentum, angular momentum, energy and an inequality of the same structure for entropy production. Furthermore, a balance law for the mass and the force in the fluid are given. This chapter is based on [1], [19], [20] and [36].

2.4.1. Preliminary theorems

As a first theorem the divergence theorem is given, which allows to transform an volume integral into a surface integral.

Theorem 2. *(Divergence Theorem) Let f be a vector field on $\mathbb{S}$, then*

$$\int_{\mathbb{S}} div\, f \,\mathrm{d}x = \int_{\partial\mathbb{S}} f\, n \,\mathrm{d}a \tag{2.60}$$

holds. $\partial\mathbb{S}$ denotes the boundary of $\mathbb{S}$ and n the outward normal vector of $\partial\mathbb{S}$.

The second theorem which will be needed in the further developments is the transport theorem. It considers the rate of change in time of an value obtained by integration, where not only the integrand but also the domain of integration depends on time.

Theorem 3. *(Reynold's Transport Theorem) Let f be a given differentiable vector field on $\mathbb{S}$, then*

$$\frac{d}{dt}\int_{\mathbb{S}} f \mathrm{d}x = \int_{\mathbb{S}} (\dot{f} + f div\, v)\mathrm{d}x = \int_{\mathbb{S}} \left(\frac{\partial f}{\partial t} + div(f\, v)\right) \mathrm{d}x = \int_{\mathbb{S}} \frac{\partial f}{\partial t}\mathrm{d}x + \int_{\partial\mathbb{S}} n\, f\, v \mathrm{d}a \quad (2.61)$$

holds. v denotes the velocity.

Equation (2.61) can be interpreted in that way that the rate of change of the quantity depends additive on two parts. The volume integral captures the change of the integrand and the surface integral the change of the integration domain.

2.4.2. Master balance law

Let $\gamma(x,t)$ be the density distribution of a mechanical quantity. The integration over the body yields an extensive quantity Y,

$$Y(\mathbb{S},t) = \int_{\mathbb{S}} \gamma(x,t)\mathrm{d}x = \int_{\mathbb{B}} \gamma(\phi(X),t)J(X,t)\mathrm{d}X = \int_{\mathbb{B}} \gamma_0(\phi(X),t)\mathrm{d}X \quad (2.62)$$

with $\gamma_0 = \gamma J$. This quantity can be measured in the laboratory. The rate of the mechanical quantity with respect to time has to be in equilibrium with the external influence on the respective body. Therefore,

$$\frac{d}{dt}Y(\mathbb{S},t) = \int_{\partial\mathbb{S}} \theta(x,n,t)\mathrm{d}a + \int_{\mathbb{S}} \chi(x,t)\mathrm{d}x \quad (2.63)$$

has to hold, where θ is the surface density and χ is the volume density of the external action. $\frac{d}{dt}$ denotes the material time derivative. For the surface densities θ a general version of the theorem of Cauchy (Theorem 1) holds. Therefore, the dependency of θ on the normal vector n can be expressed as

$$\theta(x,n,t) = \tilde{\theta}(x,t)n, \quad (2.64)$$

where $\tilde{\theta}$ is a tensor of one order higher than θ. Applying the transport theorem (2.61) yields the master balance law in integral form,

$$\int_{\mathbb{S}} \left(\frac{\partial \gamma}{\partial t} + \mathrm{div}(\gamma\mathbf{v})\right) \mathrm{d}x = \int_{\partial\mathbb{S}} \theta(x,n,t)\mathrm{d}a + \int_{\mathbb{S}} \chi(x,t)\mathrm{d}x. \quad (2.65)$$

Assuming all field variables in equation (2.65) are continuous. Then γ, $\tilde{\theta}$ and χ satisfy equation (2.65) if and only if they satisfy the local Eulerian master balance law

$$\frac{\partial \gamma}{\partial t} + \mathrm{div}(\gamma v) = \mathrm{div}\tilde{\theta} + \chi. \tag{2.66}$$

This local form is obtained by transforming the surface integral to an volume integral with the help of the divergence theorem. Since the master balance law holds for an arbitrary body $\mathbb{S}$ the local form follows.

Also a local Lagrangian version of the master balance law is available. Therefore, the external actions are transformed to the reference configuration,

$$\int_{\partial \mathbb{S}} \theta(x,n,t)\mathrm{d}a + \int_{\mathbb{S}} \chi(x,t)\mathrm{d}x = \int_{\partial \mathbb{B}} \theta_0(X,n,t)\mathrm{d}A + \int_{\mathbb{B}} \chi_0(X,t)\mathrm{d}X. \tag{2.67}$$

with

$$\theta_0 = \theta J F^{-T} \qquad \text{and} \qquad \chi_0 = \chi J.$$

The wanted result is obtained by applying the divergence theorem and using the arbitray integration domain

$$\frac{\partial \gamma_0(X,t)}{\partial t} = \mathrm{div}\tilde{\theta}_0(X,t) + \chi_o(X,t). \tag{2.68}$$

In the following, the master balance law are specialized for mass, momentum and moment of momentum.

2.4.3. Conservation of mass

The total mass m of a body $\mathbb{S}$ is given by integration over its mass density ρ,

$$m = \int_{\mathbb{S}} \rho(x,t)\mathrm{d}x = \int_{\mathbb{B}} \rho_0(X,t)\mathrm{d}X \tag{2.69}$$

Specifying the master balance law (2.68) and (2.66) for the mass and assuming no external action yields conservation of the mass in Lagrangian description

$$\frac{\partial \rho_0(X,t)}{\partial t} = 0 \tag{2.70}$$

and Eulerian description

$$\frac{\partial \rho(x,t)}{\partial t} + \mathrm{div}(\rho(x,t)\mathbf{v}(x,t)) = 0. \tag{2.71}$$

2.4.4. Balance of momentum

The momentum p of a body $\mathbb{S}$ is given as

$$p = \int_{\mathbb{S}} \rho(x,t)v(x,t)\mathrm{d}x = \int_{\mathbb{B}} \rho_0(X,t)v(\phi(X),t)\mathrm{d}X. \tag{2.72}$$

and relates velocity and mass density distribution. Postulating that force has an action on the momentum the master balance law (2.65) gets the form

$$\frac{\mathrm{D}p}{\mathrm{D}t} = \int_{\partial\mathbb{S}} t(x,n,t)\mathrm{d}a + \int_{\mathbb{S}} f(x,t)\mathrm{d}x \tag{2.73}$$

where t are the Cauchy traction vector (see definition 6 in chapter 2.2) and f are the body load density introduced in equation (2.12). The localization of equation (2.73) gives the well known relations [1]

$$\mathrm{div}\sigma + f = \rho\frac{\mathrm{D}v}{\mathrm{D}t}. \tag{2.74}$$

for the Eulerian description and

$$\mathrm{div}P + f_0 = \rho_0\frac{\partial v}{\partial t} \tag{2.75}$$

for the Lagragian description, where the first Piola-Kirchhoff stress tensor P introduced in equation (2.20) is used.

2.4.5. Balance of moment of momentum

The momentum l_o of a body $\mathbb{S}$ with respect to the origin o is given as

$$l_o = \int_{\mathbb{S}} x \times \rho v(x,t)\mathrm{d}x, \tag{2.76}$$

where $x \times \rho v(x,t)$ is the cross product of the spatial position x and the momentum density ρv. Similar to the balance law of momentum one postulates that torque (2.16) has an action on the moment of momentum. This gives the balance law,

$$\frac{\mathrm{D}l_o}{\mathrm{D}t} = \int_{\partial\mathbb{S}} x \times t(x,n,t)\mathrm{d}a + \int_{\mathbb{S}} x \times f(x,t)\mathrm{d}x \tag{2.77}$$

or equivalent in index notation

$$\frac{\mathrm{D}l_k^o}{\mathrm{D}t} = \int_{\partial\mathbb{S}} x_i t_j \epsilon_{ijk}\mathrm{d}a + \int_{\mathbb{S}} x_i f_j \epsilon_{ijk}\mathrm{d}x. \tag{2.78}$$

Using the mass conservation (2.70) and $v \times v = 0$ the left-hand side can be evaluated to

$$\frac{\mathrm{D}l_o}{\mathrm{D}t} = \int_{\mathbb{S}} x \times \rho \frac{\mathrm{D}}{\mathrm{D}t} v(x,t) \mathrm{d}x = \int_{\mathbb{S}} x_i \rho \frac{\mathrm{D}}{\mathrm{D}t} v_j \epsilon_{ijk} \mathrm{d}x, \tag{2.79}$$

where ϵ_{ijk} is the Levi-Civita symbol with the property

$$\epsilon_{ij} = \begin{cases} 1 & \text{if (i, j, k) is an even permutation of (1,2,3)} \\ -1 & \text{if (i, j, k) is an odd permutation of (1,2,3)} \\ 0 & else \end{cases} .$$

Using the Cauchy stress tensor and the divergence theorem, the surface integral on the right-hand side of (2.78) is transformed to a volume integral

$$\int_{\partial\mathbb{S}} x_i t_j \epsilon_{ijk} \mathrm{d}a = \int_{\mathbb{S}} (x_i \sigma_{jl,l} + x_{i,l} \sigma_{il} \epsilon_{ijk}) \mathrm{d}x \tag{2.80}$$

Now (2.78) is rephrased as

$$\int_{\mathbb{S}} \left[\rho \frac{\mathrm{D}}{\mathrm{D}t} v_j - \sigma_{jl,l} - f_j \right] x_i \epsilon_{ijk} \mathrm{d}x = \int_{\mathbb{S}} x_{i,l} \sigma_{jl} \epsilon_{ijk} \mathrm{d}x. \tag{2.81}$$

If the balance of momentum (2.74) holds, the left-hand side vanishes. It remains

$$\int_{\mathbb{S}} x_{i,l} \sigma_{jl} \epsilon_{ijk} \mathrm{d}x = \int_{\mathbb{S}} \delta_{il} \sigma_{jl} \epsilon_{ijk} \mathrm{d}x = \int_{\mathbb{S}} \sigma_{ji} \epsilon_{ijk} \mathrm{d}x. \tag{2.82}$$

This implies that the Cauchy stress tensor is symmetric.

2.4.6. Balance equations for a biphase medium

The same balance laws introduced have to hold for a biphase medium. For the mass balance law no exchange between the phases is added and therefor for the solid part equation (2.70) stays valid. For the fluid the rate with respect to time of variation of fluid volume per unit reference volume ζ introduced in equation (2.57) has to fulfill the continuity equation

$$\dot{\zeta} + w_{i,i} = 0. \tag{2.83}$$

Inserting the definitions for ζ in case of incompressible phases and the Law of Darcy (2.59) yields

$$\dot{\varepsilon}^s_{kk} + k(\rho^f g_j - p_{,j})_{,i} = 0. \tag{2.84}$$

As only consolidation processes are treated in this work the inertia terms are neglected from the outset for the porous medium. In the momentum balance law the total stress tensor plays the crucial role,

$$\sigma^{\text{tot}}_{ij,j} + f_i = 0 \tag{2.85}$$

where

$$f_i = e^f f_i^f + e^s f_i^s \tag{2.86}$$

accounts for body loads. Using the definition of the total stress tensor (2.55) gives

$$\left[\sigma^{\text{eff}}_{ij} - p\delta_{ij}\right]_{,j} + f_i = 0 \tag{2.87}$$

in case of incompressible phases.

3. BOUNDARY VALUE PROBLEMS

This chapter summarizes the concepts developed so far which leads in connection with boundary data to boundary value problems. Excluding problem 1 in section 3.1, which outlines a possible extension of this work, only the geometric linear case is treated. Therefore, the difference between reference and current configuration is negligible. Hence, in the following the domain which is occupied by the physical object is denoted by Ω and fixed during calculation. The boundary of Ω is Γ. Furthermore, no distinction between different stress or strain measures is made. The stress tensor will be σ and the strain tensor ε in the rest of this work.

For the subsequent derivations the definition of the following function spaces are needed. The space of p-times continuous differentiable functions is denoted

$$\mathbb{C}^p(\Omega,\mathbb{R}^d) = \{f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^d \mid f \text{ p-times continuously differentiable}\}. \tag{3.1}$$

Furthermore, the space of integrable functions is introduced as

$$\mathbb{L}^p(\Omega,\mathbb{R}^d) = \{f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^d \mid \int_\Omega [f(x)]^p \mathrm{d}x < \infty\}. \tag{3.2}$$

A generalization of the concept of differentiability leads to so called weak derivatives. The integrable function $w \in \mathbb{L}^2(\Omega,\mathbb{R})$ is called weak derivative of order $m = |\alpha| = \alpha_1 + \cdots + \alpha_d$ of the function u if

$$\int_\Omega u(x) \frac{\partial^\alpha \phi}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} \,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega w(x)\phi(x) \,\mathrm{d}x \tag{3.3}$$

holds for all $\phi \in \mathbb{C}^\infty(\Omega,\mathbb{R})$ with compact support [18]. The so called Sobolev space is then given as

$$\mathbb{H}^p(\Omega,\mathbb{R}^d) = \{f \in \mathbb{L}^2(\Omega,\mathbb{R}^d) \mid f \text{ has weak derivatives } w \in \mathbb{L}^2(\Omega,\mathbb{R}^d) \text{ up to order } p\}. \tag{3.4}$$

3.1. Differential form of boundary value problems

In this section, the differential or strong form of boundary value problems for nonlinear elasticity, linearized elasticity, elasto-plasticity and poro-elasto-plastic consolidation are given.

Problem 1. *(Nonlinear Elasticity) Find the displacement field* $u \in \mathbb{C}^2(\Omega, \mathbb{R}^3) \cap \mathbb{C}^1(\Omega \cup \Gamma_t, \mathbb{R}^3) \cap \mathbb{C}^0(\overline{\Omega}, \mathbb{R}^3)$ *such that the balance of momentum*

$$\sigma_{ij,j}(u(x)) + f_i(x) = 0 \quad hold \quad \forall x \in \Omega,$$

the Dirichlet boundary conditions

$$u(x) = u_\Gamma(x) \quad hold \quad \forall x \in \Gamma_u,$$

and the Neumann boundary conditions

$$\sigma(x)n(x) = t_\Gamma(x) \quad hold \quad \forall x \in \Gamma_t.$$

In the balance of momentum $f \in \mathbb{C}^0(\Omega, \mathbb{R}^3)$ *is a prescribed function for body loads and the Cauchy stress tensor* σ *is given through a hyperelastic constitutive relation*

$$\sigma_{ij} = \frac{\partial W(E)}{\partial E_{ij}} \quad with \quad E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{i,k}u_{j,k}).$$

$u_\Gamma \in \mathbb{C}^0(\Gamma_u, \mathbb{R}^3)$ *is a function of prescribed displacements on a part* Γ_u *of the boundary and* $t_\Gamma \in \mathbb{C}^1(\Gamma_t, \mathbb{R}^3)$ *is a function of prescribed traction vectors on the part* Γ_t *of the boundary. For* Γ_u *and* Γ_t *hold*

$$\Gamma_u \cup \Gamma_t = \Gamma \quad , \quad \Gamma_u \cap \Gamma_t = \emptyset.$$

Using the linearized displacement-strain relation (2.11) and the constitutive model for a linear elastic isotropic material (2.26) the boundary value problem 1 can be simplified.

Problem 2. *(Linearized Elasticity) Let* $f \in \mathbb{C}^0(\Omega, \mathbb{R}^3)$. *Find the displacement field* $u \in \mathbb{C}^2(\Omega, \mathbb{R}^3) \cap \mathbb{C}^1(\Omega \cup \Gamma_t, \mathbb{R}^3) \cap \mathbb{C}^0(\overline{\Omega}, \mathbb{R}^3)$ *such that*

$$\mu u_{i,jj}(x) + (\lambda + \mu)u_{j,ij}(x) + f_i(x) = 0 \quad hold \quad \forall x \in \Omega,$$

and the boundary conditions analogously to problem 1 are fulfilled.

Additionally to the elastic material response an plastic response is added. Since only rate independent plasticity is considered, time is considered in a pseudo sense. The boundary value problem assuming small strains reads as follows.

Problem 3. *(Elasto-Plasticity) Let* $f \in \mathbb{C}^0(\Omega, \mathbb{R}^3)$. *Find the displacement field* $u \in \mathbb{C}^2(\Omega, \mathbb{R}^3) \cap \mathbb{C}^1(\Omega \cup \Gamma_t, \mathbb{R}^3) \cap \mathbb{C}^0(\overline{\Omega}, \mathbb{R}^3)$ *such that*

$$\sigma_{ij,j}(x) + f_i(x) = 0 \quad hold \quad \forall x \in \Omega,$$

and the boundary conditions analogously to problem 1 are fulfilled. The stress tensor is given as

$$\sigma_{ij} = \lambda \varepsilon^e_{kk} \delta_{ij} + 2\mu \varepsilon^e_{ij}$$

with the elastic strain tensor

$$\varepsilon_{ij}^{e} = \frac{1}{2}(u_{i,j} + u_{j,i}) - \varepsilon_{ij}^{p}.$$

The evolution of the plastic strain tensor ε^p *is given as*

$$\dot{\varepsilon}^p = \begin{cases} 0, & \text{if } f(\sigma,q) \leq 0 \\ \gamma r(\sigma,q), & \text{if } f(\sigma,q) = 0, \end{cases}$$

where f *is an appropriate yield function,* q *is a set of appropriate internal variables whose evolution is determined by*

$$\dot{q} = \begin{cases} 0, & \text{if } f(\sigma,q) \leq 0 \\ \gamma h(\sigma,q), & \text{if } f(\sigma,q) = 0. \end{cases}$$

The functions r *and* h *determine the respective direction of the evolution. The consistency parameter* γ *and the yield function* f *have to fulfill*

$$\gamma f = 0 \text{ and if } f = 0 \ \rightarrow \ \gamma \dot{f} = 0.$$

Finally, the so call *u-p* form of the poro-elasto-plastic consolidation problem is given. In addition to problem 3 a fluid phase is present and the interaction with the solid is considered. As consolidation is a process where real time matters, the time interval $T = (0, t^{final})$ of interest is introduced. Other than the problems stated so far, the following problem is a initial boundary value problem.

Problem 4. *(Poro-Elasto-Plastic Consolidation) Let* $f \in \mathbb{C}^0(\Omega \times T, \mathbb{R}^3)$*. Find the pressure field* $p \in \mathbb{C}^2(\Omega \times T, \mathbb{R}) \cap \mathbb{C}^1(\Omega \cup \Gamma_q \times T, \mathbb{R}) \cap \mathbb{C}^0(\overline{\Omega} \times T, \mathbb{R})$ *and the displacement field* $u \in \mathbb{C}^2(\Omega \times T, \mathbb{R}^3) \cap \mathbb{C}^1(\Omega \cup \Gamma_t \times T, \mathbb{R}^3) \cap \mathbb{C}^0(\overline{\Omega} \times T, \mathbb{R}^3)$ *such that the fluid mass balance*

$$-\left[k(p_{,i} - f_i) + \frac{\partial u_i}{\partial t}\right]_{,i} = 0 \quad \text{hold} \quad \forall (x,t) \in \Omega \times T$$

the momentum balance

$$\left[\sigma_{ij}^{eff} - p\delta_{ij}\right]_{,j} + f_i = 0 \quad \text{hold} \quad \forall (x,t) \in \Omega \times T,$$

the boundary conditions

$$\begin{aligned} u(x,t) &= u_\Gamma(x,t) \quad \text{hold} \quad \forall (x,t) \in \Gamma_u \times T, \\ \sigma^{tot}(x,t)n(x) &= t_\Gamma(x,t) \quad \text{hold} \quad \forall (x,t) \in \Gamma_t \times T, \\ p(x,t) &= p_\Gamma(x,t) \quad \text{hold} \quad \forall (x,t) \in \Gamma_p \times T, \\ q(x,t)n(x) &= q_\Gamma(x,t) \quad \text{hold} \quad \forall (x,t) \in \Gamma_q \times T \end{aligned}$$

and the initial conditions

$$\begin{aligned} u(x,0) &= u_0 \quad \text{hold} \quad \forall x \in \Omega, \\ p(x,0) &= p_0 \quad \text{hold} \quad \forall x \in \Omega. \end{aligned}$$

For the effective stress tensor σ^{eff} *the same relations as in problem 3 hold.*

3.2. Weak form of boundary value problems

For a numerical solution it is preferred to recast the given boundary value problems into a weak (integral) form. This is done for problem 4. This section is according to the textbooks [18] and [37].

As a first step the spaces V_0^u and V_0^p of test functions are introduced as

$$\begin{aligned} V_0^u &= \{\xi(x) \in \mathbb{H}^1(\Omega,\mathbb{R}^3) \,|\, \xi(x) = 0 \quad \forall x \in \Gamma_u\}, \\ V_0^p &= \{\eta(x) \in \mathbb{H}^1(\Omega,\mathbb{R}) \,|\, \eta(x) = 0 \quad \forall x \in \Gamma_p\}. \end{aligned} \tag{3.5}$$

The weak equilibrium equations are obtained as follows. Starting from the local equilibrium equation and multiply it with a vector-valued test function $\xi \in V_0^u$ and integrate over the computational domain Ω,

$$\int_\Omega \left[\left(\sigma^{\text{eff}}_{ij,j} - p_{,j}\delta_{ij}\right)\xi_i + f_i\xi_i\right] \mathrm{d}x = 0. \tag{3.6}$$

Applying integration by parts and divergence theorem on the effective stress and the pore pressure yields

$$\int_\Omega \left[-\sigma^{\text{eff}}_{ij}\xi_{i,j} + p\xi_{j,j} + f_i\xi_i\right] \mathrm{d}x + \int_{\partial\Omega} t_i^{\text{tot}}\xi_i \,\mathrm{d}x = 0. \tag{3.7}$$

Remark. *As the stress tensor is symmetric ($\sigma_{ij} = \sigma_{ji}$) any antisymmetric part of $\xi_{i,j}$ has no contribution to $\sigma_{ij}\xi_{i,j}$. Therefore,*

$$\sigma_{ij}\xi_{i,j} = \sigma_{ij}\frac{1}{2}(\xi_{i,j} + \xi_{j,i}) + \sigma_{ij}\frac{1}{2}(\xi_{i,j} - \xi_{j,i}) = \sigma_{ij}\frac{1}{2}(\xi_{i,j} + \xi_{j,i})$$

holds. In the engineering literature (e.g. [39]) this relation is used to introduce the notion of virtual strain $\delta\varepsilon = 1/2(\xi_{i,j} + \xi_{j,i})$.

The space of the test functions is chosen in a way that it contains only functions of value zero on the Dirichlet boundary Γ_u. Therefore, the part of the boundary where displacements are prescribed and the traction are unknown are eliminated in the last integral of (3.7). The final weak equilibrium equations reads

$$\int_\Omega \sigma^{\text{eff}}_{ij}\xi_{i,j}\,\mathrm{d}x - \int_\Omega p\xi_{j,j}\,\mathrm{d}x = F(\xi) \tag{3.8}$$

with the linear form

$$F(\xi) = \int_\Omega f_i\xi_i\,\mathrm{d}x + \int_{\Gamma_t} t_i^{\text{tot}}\xi_i\,\mathrm{d}x$$

containing only known data.

The weak form of the balance of mass is obtained analogously. Multiplication with a scalar-valued testfunction $\eta^p \in V_0^p$ and integration over the domain gives

$$\int_\Omega [k(p_{,i} - f_i) + \dot{u}_i]_{,i} \, \eta \, \mathrm{d}x = 0. \tag{3.9}$$

Applying integration by parts and divergence theorem yields

$$-\int_\Omega k\big(p_{,i} - f_i\big)\,\eta^p_{,i}\,\mathrm{d}x + \int_{\partial\Omega} k\big(p_{,i} - f_i\big) n_i \eta \,\mathrm{d}x + \int_\Omega \dot{u}_{i,i} \eta \,\mathrm{d}x = 0. \tag{3.10}$$

Again, the restriction of the test functions η to functions with value zero on Γ_p and using the Law of Darcy $q = k\big(p_{,i} - f_i\big) n_i$ gives the final equation

$$\int_\Omega k\, p_{,i}\, \eta_{,i}\,\mathrm{d}x - \int_\Omega \dot{u}_{i,i} \eta \,\mathrm{d}x = \int_\Omega k\, f_i^f \,\eta_{,i}\,\mathrm{d}x + \int_{\Gamma_q} q\,\eta\,\mathrm{d}x. \tag{3.11}$$

The results of this section are summarized in the statement of the weak consolidation problem which will be solved in chapter 4.

Problem 5. *(Weak Poro-Elasto-Plastic Consolidation) Let $f \in \mathbb{L}^2(\Omega \times T, \mathbb{R}^3)$ be given. Find the pressure field $p \in \mathbb{H}^1(\Omega \times T, \mathbb{R})$ and the displacement field $u \in \mathbb{H}^1(\Omega \times T, \mathbb{R}^3)$ such that the weak fluid mass balance*

$$\int_\Omega k\, p_{,i}\, \eta_{,i}\,\mathrm{d}x - \int_\Omega \dot{u}_{i,i} \eta \,\mathrm{d}x = \int_\Omega k\, f_i^f \,\eta_{,i}\,\mathrm{d}x + \int_{\Gamma_q} q\,\eta\,\mathrm{d}x \quad \textit{hold} \quad \forall \eta \in V_0^p, \forall t \in T,$$

the weak momentum balance

$$\int_\Omega \sigma_{ij}^{\mathit{eff}} \xi_{i,j}\,\mathrm{d}x - \int_\Omega p \xi_{j,j}\,\mathrm{d}x = \int_\Omega f_i \xi_i \,\mathrm{d}x + \int_{\Gamma_t} t_i \xi_i \,\mathrm{d}x \quad \textit{hold} \quad \forall \xi \in V_0^u, \forall t \in T$$

the boundary conditions

$$\begin{aligned} u(x,t) &= u_\Gamma(x,t) \quad \textit{hold} \quad \forall (x,t) \in \Gamma_u \times T, \\ \sigma^{tot}(x,t) n(x) &= t_\Gamma(x,t) \quad \textit{hold} \quad \forall (x,t) \in \Gamma_t \times T, \\ p(x,t) &= p_\Gamma(x,t) \quad \textit{hold} \quad \forall (x,t) \in \Gamma_p \times T, \\ q(x,t) n(x) &= q_\Gamma(x,t) \quad \textit{hold} \quad \forall (x,t) \in \Gamma_q \times T \end{aligned}$$

and the initial conditions

$$\begin{aligned} u(x,0) &= u_0 \quad \textit{hold} \quad \forall x \in \Omega, \\ p(x,0) &= p_0 \quad \textit{hold} \quad \forall x \in \Omega. \end{aligned}$$

are fulfilled. For the effective stress tensor σ^{eff} the same relations as in problem 3 hold.

4. SOLUTION METHODS AND ALGORITHMS

In this chapter, the necessary solution methods for solving the boundary value problems of chapter 3 are developed. First, a general solution scheme for the plastic evolution problem is given. It is specialized for the von Mises yield criterion with linear hardening and the Drucker-Prager model. Next, as a discretization scheme for solving the boundary value problem the Finite Element Method is introduced. In the nonlinear context of plasticity, special treatment of a consistent linearization has to be taken into account to achieve a fast convergent method. Finally, the overall solution procedure is given.

4.1. Solution of the plastic evolution problem

The evolution of the plastic strain is governed by the equations (2.36) and (2.37) which contain time derivatives. As no viscous effects are considered for the plasticity time is considered as a pseudo time and time derivatives are treated with a difference scheme. Suppose at time t_n all state variables and an increment in the displacement field are known. Therefore, the displacements and the total strain at the time $t_{n+1} = t_n + \Delta t_n$ can be computed

$$
\begin{aligned}
u_{n+1} &= u_n + \Delta u_n, \\
\varepsilon_{n+1} &= \varepsilon_n + \frac{1}{2}((\Delta u_n)_{i,j} + (\Delta u_n)_{j,i}).
\end{aligned}
$$

4.1.1. General closest point projection

To solve the problem of the evolution of plastic strain an operator split method is applied. First, with an elastic predictor step a trial state is computed and second in a plastic corrector step the plasticity variables are updated. The trial elastic state is obtained by updating the total strain field, but leaving the plastic strain field and internal variables at the state at time t_n. The trial stress is then given by

$$
\sigma^{\text{trial}} = C(\varepsilon_{n+1} - \varepsilon_n^{\text{pl}}).
$$

Evaluating the yield function gives $f^{\mathrm{trial}} := f(\sigma^{\mathrm{trial}})$. In the case $f^{\mathrm{trial}} \leq 0$ no plastic flow occurs and the behavior is purely elastic. Therefore,

$$\begin{aligned}\sigma_{n+1} &= \sigma^{\mathrm{trial}},\\ \varepsilon_{n+1}^{\mathrm{pl}} &= \varepsilon_n^{\mathrm{pl}},\\ q_{n+1} &= q_n.\end{aligned}$$

Now considering the case $f^{\mathrm{trial}} > 0$. The yield function is now violated, which means that the stress state is not admissible and has to be mapped back to the yield surface. In general this is done iteratively by a newton scheme. Here the closest point projection [28] is described. The plastic strain and the internal variables have to be updated according to

$$\begin{aligned}\varepsilon_{n+1}^{\mathrm{pl}} &= \varepsilon_n^{\mathrm{pl}} + \Delta\gamma \partial_\sigma f_{n+1},\\ q_{n+1} &= q_n - \Delta\gamma D \partial_q f_{n+1}\end{aligned} \tag{4.1}$$

where $\Delta\gamma$ can be computed enforcing the stress state to be on the yield surface

$$f_{n+1} := f(\sigma_{n+1}) = 0.$$

In equations (4.1), $\partial_\sigma f$ denotes the partial derivative of the yield function with respect to the stress tensor and $\partial_q f$ with respect to the set of internal variables.

The newton scheme with iteration parameter k works as follows. Define the k-th residuum of the plastic flow, the hardening law and the yield criterion

$$\begin{aligned}R_{\varepsilon,n+1}^{(k)}(\varepsilon_{n+1}^{\mathrm{pl},(k)}, q_{n+1}^{(k)}, \Delta\gamma^{(k)}) &:= -\varepsilon_{n+1}^{\mathrm{pl},(k)} + \varepsilon_n^{\mathrm{pl}} + \Delta\gamma^{(k)} \partial_\sigma f(\sigma_{n+1}^{(k)}, q_{n+1}^{(k)})\\ R_{q,n+1}^{(k)}(\varepsilon_{n+1}^{\mathrm{pl},(k)}, q_{n+1}^{(k)}, \Delta\gamma^{(k)}) &:= -q_{n+1}^{(k)} + q_n - \Delta\gamma^{(k)} D \partial_q f(\sigma_{n+1}^{(k)}, q_{n+1}^{(k)})\\ R_{f,n+1}^{(k)}(\varepsilon_{n+1}^{\mathrm{pl},(k)}, q_{n+1}^{(k)}) &:= f_{n+1}^{(k)}(\sigma_{n+1}^{(k)}, q_{n+1}^{(k)})\end{aligned} \tag{4.2}$$

where

$$\sigma_{n+1}^{(k)}(\varepsilon_{n+1}^{\mathrm{pl},(k)}) = C(\varepsilon_{n+1} - \varepsilon_{n+1}^{\mathrm{pl},(k)}).$$

As the initial state ($k = 0$) serves the trial state, $\sigma_{n+1}^{(0)} = \sigma^{\mathrm{trial}}$, $\varepsilon_{n+1}^{\mathrm{pl},(0)} = \varepsilon_n^{\mathrm{pl}}$ and $\Delta\gamma^{(0)} = 0$. The plastic corrector step is solved if all residua are simultaneously zero. The following procedure is repeated until this goal is reached. The next step in solving the plastic evolution problem is to linearize the residuum equations above. Therefore, the directional derivative of the strain residuum with respect to the plastic strains are given as a showcase first,

$$\begin{aligned}\mathrm{D}R_\varepsilon[\Delta\varepsilon^{\mathrm{pl},(k)}] :&= \frac{\mathrm{d}}{\mathrm{d}\alpha} R(\varepsilon_{n+1}^{\mathrm{pl},(k)} + \alpha\Delta\varepsilon_{n+1}^{\mathrm{pl},(k)})\\ &= \frac{\mathrm{d}}{\mathrm{d}\alpha}[\varepsilon_{n+1}^{\mathrm{pl},(k)} + \alpha\Delta\varepsilon_{n+1}^{\mathrm{pl},(k)} + \varepsilon_n^{\mathrm{pl}} + \Delta\gamma^{(k)} \partial_\sigma f(\sigma_{n+1}^{(k)})]|_{\alpha=0}\\ &= \Delta\varepsilon_{n+1}^{\mathrm{pl},(k)} + \Delta\gamma^{(k)} \partial_{\sigma\sigma} f(\sigma_{n+1}^{(k)}) \Delta\sigma^{(k)}\end{aligned}$$

In the following, the index denoting the final time step $n+1$ is omitted. The linearization of the strain residuum gives,

$$
\begin{aligned}
&L_{R_\varepsilon}(\Delta\varepsilon^{\mathrm{pl},(k)},\Delta q^{(k)},\Delta\Delta\gamma^{(k)})\\
&= R_\varepsilon(\varepsilon^{\mathrm{pl},(k)},q^{(k)},\Delta\gamma^{(k)}) + \mathrm{D}R_\varepsilon[\Delta\varepsilon^{\mathrm{pl},(k)}] + \mathrm{D}R_\varepsilon[\Delta q^{(k)}] + \mathrm{D}R_\varepsilon[\Delta\Delta\gamma^{(k)}]\\
&= -\varepsilon^{\mathrm{pl},(k)} + \varepsilon_n^{\mathrm{pl}} + \Delta\gamma^{(k)}\partial_\sigma f^{(k)}\\
&\qquad -\Delta\varepsilon^{\mathrm{pl},(k)} + \Delta\gamma^{(k)}\partial_{\sigma\sigma} f^{(k)}\Delta\sigma^{(k)} + \Delta\gamma^{(k)}\partial_{q\sigma} f^{(k)}\Delta q^{(k)} + \Delta\Delta\gamma^{(k)}\partial_\sigma f^{(k)}.
\end{aligned}
$$

The linearization of the hardening residuum gives

$$
\begin{aligned}
&L_{R_q}(\Delta\varepsilon^{\mathrm{pl},(k)},\Delta q^{(k)},\Delta\Delta\gamma^{(k)}) =\\
&= -q_{n+1}^{(k)} + q_n - \Delta\gamma^{(k)} D\partial_q f(\sigma_{n+1}^{(k)},q_{n+1}^{(k)})\\
&\qquad -\Delta\gamma^{(k)} D\partial_{\sigma q} f^{(k)}\Delta\sigma^{(k)} - \Delta q^{(k)} - \Delta\gamma^{(k)} D\partial_{qq} f^{(k)}\Delta q^{(k)} - \Delta\Delta\gamma^{(k)} D\partial_q f^{(k)}.
\end{aligned}
$$

The linearization of the yield function residuum gives

$$
L_{R_f}(\Delta\varepsilon^{\mathrm{pl},(k)},\Delta q^{(k)},\Delta\Delta\gamma^{(k)}) = f^{(k)} + \partial_\sigma f^{(k)}\Delta\sigma^{(k)} + \partial_q f^{(k)}\Delta q^{(k)}.
$$

Taking into account that ε_{n+1} is constant during the plastic corrector step the increment in the stress is

$$
\Delta\sigma^{(k)} = -C\Delta\varepsilon^{\mathrm{pl},(k)}
$$

and with a rearrangement the increment in plastic strain

$$
\Delta\varepsilon^{\mathrm{pl},(k)} = -C^{-1}\Delta\sigma^{(k)}.
$$

Forcing the linearized equations to zero gives the following linear equations for the stress increment, the increment in the internal variables and the increment in the consistency parameter

$$
\begin{pmatrix} L_{R_\varepsilon}\\ L_{R_q}\\ L_{R_f}\end{pmatrix} = \mathbf{0},
$$

with the matrix representation

$$
\begin{pmatrix}
C^{-1}+\Delta\gamma^{(k)}\partial_{\sigma\sigma} f^{(k)} & \Delta\gamma^{(k)}\partial_{q\sigma} f^{(k)} & \partial_\sigma f^{(k)}\\
\Delta\gamma^{(k)}\partial_{\sigma q} f^{(k)} & -D^{-1}-\Delta\gamma^{(k)}\partial_{qq} f^{(k)} & \partial_q f^{(k)}\\
\partial_\sigma f^{(k)} & \partial_q f^{(k)} & 0
\end{pmatrix}
\begin{pmatrix}\Delta\sigma^{(k)}\\ \Delta q^{(k)}\\ \Delta\Delta\gamma^{(k)}\end{pmatrix}
= -\begin{pmatrix} R_\varepsilon\\ D^{-1}R_q\\ R_f\end{pmatrix}.
$$

Solving for the incremental consistency parameter gives

$$
\Delta\Delta\gamma^{(k)} = \frac{R_f - b^\top A^{-1}(R_\varepsilon R_q)^\top}{b^\top A^{-1} b} \tag{4.3}
$$

with the abbreviations

$$
\begin{aligned}
A &:= \begin{pmatrix} C^{-1} + \Delta\gamma^{(k)} \partial_{\sigma\sigma} f^{(k)} & \Delta\gamma^{(k)} \partial_{q\sigma} f^{(k)} \\ \Delta\gamma^{(k)} \partial_{\sigma q} f^{(k)} & -D^{-1} - \Delta\gamma^{(k)} \partial_{qq} f^{(k)} \end{pmatrix} \\
b &:= \begin{pmatrix} \partial_\sigma f^{(k)} \\ \partial_q f^{(k)} \end{pmatrix}.
\end{aligned} \tag{4.4}
$$

The increments in the stress and the internal variables are then obtained by

$$
\begin{pmatrix} \Delta\sigma^{(k)} \\ \Delta q^{(k)} \end{pmatrix} = A^{-1} \left[\begin{pmatrix} R_\varepsilon \\ R_q \end{pmatrix} + \Delta\Delta\gamma^{(k)} b \right].
$$

Finally, the k-th state is updated to the $(k+1)$-th state according to

$$
\begin{aligned}
\varepsilon^{\mathrm{pl},(k+1)} &= \varepsilon^{\mathrm{pl},(k)} + C^{-1} \Delta\sigma^{(k)} \\
q^{(k+1)} &= q^{(k)} + \Delta q^{(k)} \\
\Delta\gamma^{(k+1)} &= \Delta\gamma^{(k)} + \Delta\Delta\gamma^{(k)}
\end{aligned}
$$

and the procedure is repeated if the requirements of the residuum equations (4.2) are not meet yet.

4.1.2. Specialization to the von Mises yield criterion with linear isotropic hardening

For the special case of the von Mises yield criterion and linear isotropic hardening the algorithm described in section 4.1.1 simplifies to the radial return method proposed in [34]. The increment in the consistency parameter given in equation (4.3) can be computed analytically. Since in the first iteration the strain and hardening residuum are zero the nominator reduces to the evaluation of the yield function at the trial state. For the denominator the following calculations allow also a simple form. As a preliminary calculation the derivative of the von Mises yield function (2.44) with respect to the stress tensor is given by

$$
(\partial_\sigma f)_{kl} = \frac{1}{2} \frac{\frac{3}{2}\left(\frac{\partial s_{ij}}{\partial \sigma_{kl}} s_{ij} + s_{ij} \frac{\partial s_{ij}}{\partial \sigma_{kl}} \right)}{\sqrt{\frac{3}{2} s_{ij} s_{ij}}} = \sqrt{\frac{3}{2}} \frac{s_{kl}}{\sqrt{s_{ij} s_{ij}}}, \tag{4.5}
$$

where again s is the deviatoric part of the stress tensor,

$$
s_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{kk} \delta_{ij} \tag{4.6}
$$

with the property

$$
s_{kk} = \sigma_{kk} - \frac{1}{3} \sigma_{kk} \delta_{kk} = \sigma_{kk} (1 - \frac{1}{3} 3) = 0. \tag{4.7}
$$

The derivative of the linear hardening law gives $\partial_q f = 1$.

Since also the increment in the consistency parameter $\Delta\gamma$ is zero in the first iteration the inverse of (4.4) is simply

$$A^{-1} := \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}, \tag{4.8}$$

where D is in the present case of linear hardening the scalar hardening modulus D. The denominator $b^T A^{-1} b$ is additive composed of $\partial_q f^T D \partial_q f = D$ and

$$\partial_\sigma f^T C \partial_\sigma f = (\partial_\sigma f)_{ij} C_{ijkl} (\partial_\sigma f)_{kl} = \tag{4.9}$$

$$= \sqrt{\frac{3}{2}} \frac{s_{ij}}{\sqrt{s_{mn}s_{mn}}} \left(\lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right) \sqrt{\frac{3}{2}} \frac{s_{kl}}{\sqrt{s_{mn}s_{mn}}} = \tag{4.10}$$

$$= \frac{3}{2} \frac{1}{s_{mn}s_{mn}} (\lambda s_{ii}s_{kk} + 2\mu s_{ij}s_{ij}) = 3\mu = 3G. \tag{4.11}$$

Altogether, the final formula for the increment of the consistency parameter has the closed form

$$\Delta\gamma = \frac{f^{\text{trial}}}{3G + D}. \tag{4.12}$$

Finally, the plastic strains and the internal variable are updated according to

$$\varepsilon_{n+1}^{\text{pl}} = \varepsilon_n^{\text{pl}} + \Delta\gamma \sqrt{\frac{3}{2}} \frac{s_{ij}}{\sqrt{s_{mn}s_{mn}}},$$

$$q_{n+1} = q_n + \Delta\gamma D.$$

Note that in case of this simple model the equations above allow a closed form solution.

4.1.3. Specialization to the non-associative Drucker-Prager model

Again, the problem can be solved analytically. The derivative of the yield function (2.40) with respect to the stress tensor yields

$$(\partial_\sigma f)_{kl} = \sqrt{\frac{1}{2}} \frac{s_{kl}}{||s||} + \frac{1}{3}\eta \delta_{kl}. \tag{4.13}$$

Furthermore, the derivative of (2.42) with respect to the stress tensor yields

$$(\partial_\sigma \bar{f})_{kl} = \sqrt{\frac{1}{2}} \frac{s_{kl}}{||s||} + \frac{1}{3}\bar{\eta} \delta_{kl}. \tag{4.14}$$

The denominator of the formula for the increment of consistency parameter is computed to

$$
\begin{aligned}
\partial_\sigma \omega^T C \partial_\sigma f &= (\partial_\sigma \omega)_{ij} C_{ijkl} (\partial_\sigma f)_{kl} \\
&= \left(\sqrt{\frac{1}{2}} \frac{s_{ij}}{\sqrt{s_{mn}s_{mn}}} + \frac{1}{3}\bar{\eta}\delta_{ij} \right) \left(\lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right) \left(\sqrt{\frac{1}{2}} \frac{s_{kl}}{\sqrt{s_{mn}s_{mn}}} + \frac{1}{3}\eta\delta_{kl} \right) \\
&= \mu + K\bar{\eta}\eta = G + K\bar{\eta}\eta .
\end{aligned}
\tag{4.15}
$$

Therefore, the increment of the consistency parameter has the form

$$
\Delta\gamma = \frac{f(\sigma^{\text{trial}})}{G + K\bar{\eta}\eta} = \frac{\sqrt{\frac{1}{2}}\left\|s^{\text{trial}}\right\| + p^{\text{trial}}\eta - \xi c}{G + K\bar{\eta}\eta} . \tag{4.16}
$$

Finally, the update of the plastic strains reads

$$
\varepsilon_{n+1}^{\text{pl}} = \varepsilon_n^{\text{pl}} + \Delta\gamma \left(\sqrt{\frac{1}{2}} \frac{s_{kl}}{\|s\|} + \frac{1}{3}\bar{\eta}\delta_{kl} \right) . \tag{4.17}
$$

4.2. Basics of the Finite Element Method

For solving boundary problems as presented in chapter 3, different methods are available. The most common are the Finite Difference Method (FDM), the Boundary Element Method (BEM) and the Finite Element Method (FEM). The one used in this work is the FEM, for which a large body of literature exists, for instance see [12], [18], [37], [39].

The basic idea of the FEM is to approximate the infinite dimensional solution space $\mathbb{H}^1(\Omega, \mathbb{R}^d)$ of the weak problem by an appropriate finite dimensional subspace denoted as T^h. For the construction of T^h the domain Ω is decomposed into non-overlapping subdomains, so called finite elements τ,

$$
\Omega \approx \Omega^h = \bigcup_{e=1}^{M^\tau} \tau_e ,
$$

where M^τ is the total number of finite elements. As the shape of the finite elements τ are restricted to be simple, the domain Ω may only be approximated. In this work, tetrahedrons are used only. The superscript h has to be understood as a discretization parameter, describing the geometric size of the finite elements. In the special case of tetrahedrons every finite element consists of four faces, six edges and four nodes. Here, the decomposition has to fulfill the following criterion. For all pairs τ_i and τ_j of finite elements with

$i, j \in \{1,2,\ldots,M^\tau\}$ and $i \neq j$ holds

$$\tau_i \cap \tau_j = \begin{cases} \emptyset, \\ \text{common node} \\ \text{common edge} \\ \text{common face} \end{cases} .$$

For a first simple construction of T^h the number of geometric nodes M^n in Ω^h is needed. Using a linear ansatz (the shape functions are linear), it is also the number of shape functions N^i, $i \in \{1,2,\ldots,M^n\}$. The shape function corresponding to node i has the property

$$N^i(x_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

where x_j, $j \in \{1,2,\ldots,M^n\}$ is the spatial position of node j. Now T^h is given as

$$T^h = \text{span}\left\{N^i\right\}_{i=1}^{M^n}.$$

Adding new nodes for the shape functions on the faces, edges or inside the tetrahedrons yields a higher order ansatz. The shape functions introduced so far are continuous functions defined on the whole domain Ω, but have only on a small subset a value different from zero.

In problem 5 of chapter 3, the unknown quantities are the displacement field $u(x)$ and the pore pressure field $p(x)$. For the displacement field the approximation u^h is introduced

$$u(x) \approx u^h(x) = \sum_{a=1}^{n_u} N^{u,a}(x)\hat{u}^a, \tag{4.18}$$

where $\hat{u}^a$ denotes the displacement at node a and n_u the number of shape functions. As the displacement field is vector valued three values have to be stored for every node. The displacement in the coordinate direction i is given as

$$u_i^h(x) = \sum_{a=1}^{n_u} N^{u,a}(x)\hat{u}_i^a.$$

The discrete total strain field is then given through the relation

$$\varepsilon_{kl}^h(x) = \frac{1}{2}\sum_{a=1}^{n_u}\left(N_{,l}^{u,a}(x)\hat{u}_k^a + N_{,k}^{u,a}(x)\hat{u}_l^a\right). \tag{4.19}$$

For the scalar pore pressure the approximation p^h is introduced

$$p(x) \approx p^h(x) = \sum_{a=1}^{n_p} N^{p,a}(x)\hat{p}^a. \tag{4.20}$$

The Ritz-Galerkin method uses the same space for the test-functions as for the ansatz functions, $V_0 = T^h$. Therefore, any test-function associated with the displacements can be represented by

$$\xi(x) = \sum_{a=1}^{n_u} N^{u,a}(x)\hat{\xi}^a$$

and those associated with the pore pressure by

$$\eta(x) = \sum_{a=1}^{n_p} N^{p,a}(x)\hat{\eta}^a.$$

Inserting the approximations in the respective weak balance laws yields a system of linear equation. How the sparse system matrix is computed is shown in the next two sections.

4.3. A Newton-like FEM for the incremental elasto-plastic problem

In this section, only the incremental elasto-plastic-problem is considered. Here, incremental means that not only the response to one given load is of interest, but a sequence of load and deformation combinations fulfilling the equilibrium equations.

Additionally to the standard approximation of the displacement field, the plastic strains are approximated. This is done with

$$\varepsilon^{\mathrm{pl}}(x) \approx \varepsilon^{\mathrm{pl},h}(x) = \sum_{a=1}^{n_{pl}} N^{pl,a}(x)\hat{\varepsilon}^{pl,a}, \quad \varepsilon_{kl}^{\mathrm{pl},h}(x) = \sum_{a=1}^{n_{pl}} N^{pl,a}(x)\hat{\varepsilon}_{kl}^{pl,a},$$

where $N^{pl,a}$ are n_{pl} ansatz functions defined on Ω. In this work, linear ansatz functions are used. Recall the weak form of the momentum equation (3.8) with two modifications

$$\int_{\Omega} \sigma_{ij}(u, \varepsilon^{\mathrm{pl}}, q)\xi_{i,j}\,\mathrm{d}x = \lambda F(\xi). \tag{4.21}$$

The pore pressure term is neglected and the loadfactor λ is introduced. The goal will be to find a sequence of states $\kappa = (u, \varepsilon^{\mathrm{pl}}, q, \lambda)$.

For a general state κ equation (4.21) is not fulfilled and yields a residuum

$$\begin{aligned} R^{\xi}(\kappa) &= -\int_{\mathbb{S}} \xi_{i,j}\sigma_{ij}(u, \varepsilon^{\mathrm{pl}})\mathrm{d}x + \lambda F(\xi) \\ &= -\int_{\mathbb{S}} \xi_{i,j}C_{ijkl}(\varepsilon_{kl}(u) - \varepsilon_{kl}^{\mathrm{pl}})\mathrm{d}x + \lambda F(\xi) \\ &= -\int_{\mathbb{S}} \xi_{i,j}C_{ijkl}\varepsilon_{kl}(u)\mathrm{d}x + \int_{\mathbb{S}} \xi_{i,j}C_{ijkl}\varepsilon_{kl}^{\mathrm{pl}}\mathrm{d}x + \lambda F(\xi). \end{aligned}$$

Complementary to the continuous state κ the discrete state $\kappa^h = (u^h, \varepsilon^{\mathrm{pl},h}, q^h, \lambda)$ gives the discrete residuum

$$R^{h,\xi}(\kappa^h) = \int_{\mathbb{S}} \sum_{a=1}^{n_u} N^{u,a}_{,j}(x)\hat{\xi}^a_i C_{ijkl} \frac{1}{2} \sum_{b=1}^{n_u} \left(N^{u,b}_{,l}(x)\hat{u}^b_k + N^{u,b}_{,k}(x)\hat{u}^b_l \right) \mathrm{d}x$$
$$- \int_{\mathbb{S}} \sum_{a=1}^{n_u} N^{u,a}_{,j}(x)\hat{\xi}^a_i C_{ijkl} \sum_{b=1}^{n_{pl}} N^{pl,b}(x)\hat{\varepsilon}^{pl,b}_{kl} \,\mathrm{d}x + \lambda F(\xi).$$

Since the residuum is linear in the test function and the discrete space of test functions has $3n$ linear independent base functions a residuum vector

$$\begin{aligned} \hat{R}^a_i(\kappa^h) &= \int_{\mathbb{S}} N^{u,a}_{,j}(x) C_{ijkl} \frac{1}{2} \sum_{b=1}^{n_u} \left(N^{u,b}_{,l}(x)\hat{u}^b_k + N^{u,b}_{,k}(x)\hat{u}^b_l \right) \mathrm{d}x \\ &\quad - \int_{\mathbb{S}} N^{u,a}_{,j}(x) C_{ijkl} \sum_{b=1}^{n_{pl}} N^{pl,b}(x)\hat{\varepsilon}^{pl,b}_{kl} \,\mathrm{d}x + \lambda \hat{F}^a_i \qquad (4.22) \\ &= \sum_{b=1}^{n_u} K^{ab}_{if}\hat{u}^b_f + \sum_{b=1}^{n_{pl}} P^{ab}_{ikl}\hat{\varepsilon}^{pl,b}_{kl} + \lambda \hat{F}^a_i \end{aligned}$$

can be defined. In equation (4.22), a rule for computation of the a-th block and the i-th subindex is given. The index a runs from one to the number of degrees of freedom n_u, whereas i runs from one to three. Also the abbreviations

$$K^{ab}_{if} = \int_{\mathbb{S}} N^{u,a}_{,j}(x) C_{ijkl} \frac{1}{2} \left(N^{u,b}_{,l}(x)\delta_{kf} + N^{u,b}_{,k}(x)\delta_{lf} \right) \mathrm{d}x \qquad (4.23)$$

$$P^{ab}_{ikl} = \int_{\mathbb{S}} N^{u,a}_{,j}(x) C_{ijkl} N^{pl,b}(x) \mathrm{d}x \qquad (4.24)$$

are used. K is the standard stiffness matrix. The superscripts a and b refer to the block and the subscripts i and f describe the position within this block. For P the same notation holds, but note that the second subscript kl indicates a second order tensor. This means that P has in every block three second order tensors.

In the following, let κ^{eq} be a discrete state in equilibrium, hence

$$R^{h,\xi}(\kappa^{\mathrm{eq}}) = 0 \quad \forall \xi.$$

Furthermore, let $\kappa^{\mathrm{neq}} = \kappa^{\mathrm{eq}} + \Delta\kappa$ be a state not satisfying equilibrium. Then in case of von Mises the a-th entry of the coefficient vector $\hat{\varepsilon}^{pl,neq}$ can be computed by

$$\hat{\varepsilon}^{pl,neq,a}_{ij} = \hat{\varepsilon}^{pl,eq,a}_{ij} + \max\{0, \Delta\gamma(x^a)\} \sqrt{\frac{3}{2}} \frac{s^{\mathrm{trial}}_{ij}(x^a)}{\|s^{\mathrm{trial}}(x^a)\|} \qquad (4.25)$$

In equation (4.25), x^a is the space coordinate of the node corresponding to the a-th entry in the coefficient vector. s^{trial} denotes the deviatoric trial stress given by

$$s_{ij}^{\text{trial}}(x^a) = G\left[\sum_{c=1}^{n_u}\left(N_{,j}^{u,c}(x^a)\delta_{if} + N_{,i}^{u,c}(x^a)\delta_{jf} - \frac{2}{3}N_{,f}^{u,c}(x^a)\delta_{ij}\right)\bar{u}_f^c - 2\left(\hat{\varepsilon}_{ij}^{pl,eq,a} - \frac{1}{3}\hat{\varepsilon}_{mm}^{pl,eq,a}\right)\right].$$

The increment in the consistency parameter $\Delta\gamma(x^a)$ is given according to (4.12) as

$$\Delta\gamma(x^a) = \frac{f(s^{\text{trial}}(x^a))}{3G+D} = \frac{\sqrt{\frac{3}{2}}\,\|s^{\text{trial}}(x^a)\| - (\sigma_y + q(x^a)D)}{3G+D}$$

In case of elastic loading equation (4.25) gets the form

$$\hat{\varepsilon}_{ij}^{pl,neq,a} = \hat{\varepsilon}_{ij}^{pl,eq,a} \tag{4.26}$$

whereas for plastic loading

$$\hat{\varepsilon}_{ij}^{pl,neq,a} = \hat{\varepsilon}_{ij}^{pl,eq,a} + \frac{\sqrt{\frac{3}{2}}s_{ij}^{\text{trial}}(x^a)}{3G+D}\left[\sqrt{\frac{3}{2}} - \frac{\sigma_y + qD}{\|s^{\text{trial}}(x^a)\|}\right] \tag{4.27}$$

is obtained. To obtain a new state satisfying equilibrium, a linearization of the unbalanced residuum is performed. The linearization is done only in the discrete displacements and the load factor, as the discrete plastic strains are a function of the discrete displacements. This reads

$$L_{R^h} = R^{h,\xi}(\kappa^{\text{neq}}) + \mathrm{D}R^{h,\xi}(\kappa^{\text{neq}})[\Delta u^h] + \mathrm{D}R^{h,\xi}(\kappa^{\text{neq}})[\Delta\lambda]. \tag{4.28}$$

Since the residuum is already linear in the load factor the last derivative is simply given as

$$\mathrm{D}R^{h,\xi}(\kappa^{\text{neq}})[\Delta\lambda] = \Delta\lambda F(\xi^h). \tag{4.29}$$

In the same way the residuum contains a linear part in the displacements, the first integral in (4.22). Because of this fact the standard stiffness matrix K introduced in (4.23) enters the derivative with respect to the displacements. The crucial part to get a fast convergent Newton like-solver is the remaining derivative of the plastic strains with respect to the displacements. This will give an additional matrix B (see equation (4.42)) with the same size and sparsity pattern as K. Together, an expression of the form

$$\mathrm{D}R^{h,\xi}(\kappa^{\text{neq}})[\Delta u^h] = (K - B)\Delta\hat{u} \tag{4.30}$$

is obtained. In the following, only the plastic loading case is treated further, since elastic loading has no influence on the B matrix. The derivative of the b-th component of the plastic strain coefficient vector is given as

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\hat{\varepsilon}_{kl}^{pl,neq,b} = \frac{\sqrt{\frac{3}{2}}}{3G+D}\left[\sqrt{\frac{3}{2}}\frac{\mathrm{d}s_{kl}^{\text{trial}}}{\mathrm{d}\alpha} - (\sigma_y + qD)\left(\frac{\frac{\mathrm{d}s_{kl}^{\text{trial}}}{\mathrm{d}\alpha}}{\|s^{\text{trial}}\|} - \frac{s_{kl}^{\text{trial}}\frac{\mathrm{d}s_{ij}^{\text{trial}}}{\mathrm{d}\alpha}s_{ij}^{\text{trial}}}{\|s^{\text{trial}}\|^3}\right)\right], \tag{4.31}$$

where the derivative of the deviatoric trial stress reads as

$$\frac{\mathrm{d}s_{kl}^{\mathrm{trial}}(x^b)}{\mathrm{d}\alpha} = G\left[\sum_{c=1}^{n_u}\left(N_{,l}^{u,c}(x^b)\delta_{kf} + N_{,k}^{u,c}(x^b)\delta_{lf} - \frac{2}{3}\delta_{kl}N_{,f}^{u,c}(x^b)\right)\Delta u_f^c\right]. \tag{4.32}$$

A further calculation shows,

$$\frac{\mathrm{d}s_{ij}^{\mathrm{trial}}}{\mathrm{d}\alpha}s_{ij}^{\mathrm{trial}} = 2G\left[\sum_{c=1}^{n_u} N_{,l}^{u,c}(x^b)s_{lf}^{\mathrm{trial}}\Delta u_f^c\right]. \tag{4.33}$$

Using the definition of the elasticity tensor given in equation (2.25), an entry in the B matrix is given as

$$\begin{aligned} B^{a\,c}_{i\,f} = \int_{\mathbb{S}} 2G^2 \sum_{b=1}^{n_{pl}} N^{pl,b}(x)\Big[Z^1\Big(N_{,j}^{u,a}(x)N_{,j}^{u,c}(x^b)\delta_{if} + N_{,f}^{u,a}(x)N_{,i}^{u,c}(x^b)\delta_{if} \\ -\frac{2}{3}N_{,i}^{u,a}(x)N_{,f}^{u,c}(x^b)\Big) + 2Z^2 N_{,j}^{u,a}(x)s_{ij}(x^b)N_{,d}^{u,c}(x^b)s_{df}(x^b)\Big]\mathrm{d}x, \end{aligned} \tag{4.34}$$

with the abbreviations

$$\begin{aligned} Z^0 &= \frac{\sqrt{\frac{3}{2}}}{3G+D}, \\ Z^1 &= Z^0\left[\sqrt{\frac{3}{2}} - \frac{\sigma_y + qD}{\|s^{\mathrm{trial}}\|}\right], \\ Z^2 &= Z^0\frac{\sigma_y + qD}{\|s^{\mathrm{trial}}\|^3}. \end{aligned}$$

Using all results achieved so far and forcing the linearization of the residuum given in (4.28) to zero, yields

$$\begin{bmatrix} (K-B) & \hat{F} \end{bmatrix}\begin{bmatrix} \Delta\hat{u} \\ \Delta\lambda \end{bmatrix} = \hat{R} \tag{4.35}$$

which are $3n$ equations for $3n$ displacements increments $\Delta\hat{u}$ plus a scalar increment for the load factor $\Delta\lambda$. To obtain a solvable system of linear equations a constraint equation is added. Therewith, either displacement increment Δu_i^b for selected b, i or the increment in the load factor $\Delta\lambda$ is set to zero. Finally, we obtain

$$\begin{bmatrix} (K-B) & \hat{F} \\ c^T & d \end{bmatrix}\begin{bmatrix} \Delta\hat{u} \\ \Delta\lambda \end{bmatrix} = \begin{bmatrix} \hat{R} \\ 0 \end{bmatrix} \tag{4.36}$$

with

$$\begin{aligned} &\text{case A:} \quad d = 1 \quad \text{and} \quad c = 0, \qquad \text{or} \\ &\text{case B:} \quad d = 0 \quad \text{and} \quad c = 0 \quad \text{but} \quad c_i^b = 1. \end{aligned}$$

In case A, only the displacements are adjusted to bring a state with fixed load factor to equilibrium. In case B one single displacement component is held fixed and the others together with the load factor are adjusted to get the state to equilibrium.

In case of the Drucker-Prager plasticity model the consistent linearization is given as follows. Recall the yield criterion

$$f = \sqrt{\frac{1}{2}}\|s\| + p\eta - \xi c, \tag{4.37}$$

and the non-associative flow rule

$$\frac{\partial \omega}{\partial \sigma_{kl}} = \sqrt{\frac{1}{2}}\frac{s_{kl}}{\|s\|} + \frac{1}{3}\bar{\eta}\delta_{kl}. \tag{4.38}$$

The increment in the consistency parameter is then given as

$$\Delta\gamma(x^a) = \frac{f(\sigma^{\text{trial}}(x^a))}{G + K\bar{\eta}\eta} = \frac{\sqrt{\frac{1}{2}}\left\|s^{\text{trial}}(x^a)\right\| + p(x^a)\eta - \xi c}{G + K\bar{\eta}\eta}. \tag{4.39}$$

The new plastic strain can be computed with

$$\begin{aligned}\hat{\hat{\varepsilon}}^{p,a} = \hat{\hat{\varepsilon}}^{p,a} + \frac{1}{G + K\bar{\eta}\eta}\Bigg[\frac{1}{2}s^{\text{trial}}(x^a) + \sqrt{\frac{1}{2}}\frac{s_{kl}^{\text{trial}}(x^a)p\eta - s_{kl}^{\text{trial}}(x^a)\xi c}{\|s^{\text{trial}}\|} \\ + \sqrt{\frac{1}{2}}\left\|s^{\text{trial}}\right\|\frac{1}{3}\bar{\eta}\delta_{kl} + \frac{1}{3}\bar{\eta}\delta_{kl}(p\eta - \xi c)\Bigg]\end{aligned} \tag{4.40}$$

The derivative yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\alpha}\hat{\hat{\varepsilon}}_{kl}^{p,a} &= \frac{1}{G + K\bar{\eta}\eta}\Bigg[\frac{1}{2}\frac{\mathrm{d}s_{kl}^{\text{trial}}}{\mathrm{d}\alpha} + \sqrt{\frac{1}{2}}\eta\left(\frac{\frac{\mathrm{d}s_{kl}^{\text{trial}}}{\mathrm{d}\alpha}p^{\text{trial}} + s_{kl}^{\text{trial}}\frac{\mathrm{d}p^{\text{trial}}}{\mathrm{d}\alpha}}{\|s^{\text{trial}}\|} - \frac{s_{kl}^{\text{trial}}p^{\text{trial}}\frac{\mathrm{d}s_{ij}^{\text{trial}}}{\mathrm{d}\alpha}s_{ij}^{\text{trial}}}{\|s^{\text{trial}}\|^3}\right) \\ &\quad - \sqrt{\frac{1}{2}}\xi c\left(\frac{\frac{\mathrm{d}s_{kl}^{\text{trial}}}{\mathrm{d}\alpha}}{\|s^{\text{trial}}\|} - \frac{s_{kl}^{\text{trial}}\frac{\mathrm{d}s_{ij}^{\text{trial}}}{\mathrm{d}\alpha}s_{ij}^{\text{trial}}}{\|s^{\text{trial}}\|^3}\right) + \frac{1}{3}\bar{\eta}\delta_{kl}\left(\sqrt{\frac{1}{2}}\frac{\frac{\mathrm{d}s_{ij}^{\text{trial}}}{\mathrm{d}\alpha}s_{ij}^{\text{trial}}}{\|s^{\text{trial}}\|} + \frac{\mathrm{d}p^{\text{trial}}}{\mathrm{d}\alpha}\eta\right)\Bigg] = \\ &= \frac{1}{G + K\bar{\eta}\eta}\Bigg[\frac{\mathrm{d}s_{kl}^{\text{trial}}}{\mathrm{d}\alpha}\left(\frac{1}{2} + \sqrt{\frac{1}{2}}\frac{\eta p - \xi c}{\|s^{\text{trial}}\|}\right) + \frac{\mathrm{d}s_{ij}^{\text{trial}}}{\mathrm{d}\alpha}s_{ij}^{\text{trial}}\sqrt{\frac{1}{2}}\left(s_{kl}^{\text{trial}}\frac{\xi c - \eta p}{\|s^{\text{trial}}\|^3} + \frac{\bar{\eta}\delta_{kl}}{3\|s^{\text{trial}}\|}\right) \\ &\quad + \frac{\mathrm{d}p}{\mathrm{d}\alpha}\eta\left(\sqrt{\frac{1}{2}}\frac{s_{kl}^{\text{trial}}}{\|s^{\text{trial}}\|} + \frac{\delta_{kl}}{3}\bar{\eta}\right)\Bigg].\end{aligned} \tag{4.41}$$

Using the definition of the elasticity tensor given in equation (2.25), a entry in the B matrix is given as

$$B^{a\,c}_{i\,f} = \int_{\mathbb{S}} \sum_{b=1}^{n_{pl}} N^{pl,b}(x) \Big[Y^1 \Big(N^{u,a}_{,j}(x) N^{u,c}_{,j}(x^b) \delta_{if} + N^{u,a}_{,f}(x) N^{u,c}_{,i}(x^b) \delta_{if} - \frac{2}{3} N^{u,a}_{,i}(x) N^{u,c}_{,f}(x^b) \Big)$$
$$+ Y^2 N^{u,a}_{,j}(x) s^{\text{trial}}_{ij}(x^b) N^{u,c}_{,d}(x^b) s^{\text{trial}}_{df}(x^b) + Y^3 N^{u,a}_{,i}(x) N^{u,c}_{,d}(x^b) s^{\text{trial}}_{df}(x^b)$$
$$+ Y^4 N^{u,a}_{,j}(x) s^{\text{trial}}_{ij}(x^b) N^{u,c}_{,f}(x^b) + Y^5 N^{u,a}_{,i}(x) N^{u,c}_{,f}(x^b) \Big] \mathrm{d}x, \tag{4.42}$$

with the abbreviations

$$\begin{aligned}
Y^0 &= \frac{1}{\sqrt{2}(G + K\bar{\eta}\eta)}, \\
Y^1 &= 2Y^0 G^2 \left[\sqrt{\frac{1}{2}} + \frac{\bar{\eta} p - \xi c}{\|s^{\text{trial}}\|} \right], \\
Y^2 &= 4Y^0 G^2 \frac{\xi c - \bar{\eta} p}{\|s^{\text{trial}}\|^3}, \\
Y^3 &= 2Y^0 G\, K \frac{\bar{\eta}}{\|s^{\text{trial}}\|}, \\
Y^4 &= 2Y^0 G\, K \frac{\eta}{\|s^{\text{trial}}\|}, \\
Y^5 &= \sqrt{2} Y^0 K^2 \bar{\eta}\eta.
\end{aligned}$$

4.4. Solution procedure for the elasto-plastic consolidation problem

Using the discretizations (4.18) and (4.20) for problem 5 in chapter 3 a semi-discrete problem is achieved. As a last step before the solution procedure can be summarized a time discretization is necessary. Instead of considering the continuous time interval T, discrete times with a fixed distance are used. The weak fluid balance of problem 5 in section 3 contains a time derivative of the divergence of the displacement field, which is approximated by

$$\dot{u}_{i,i} \approx \frac{1}{\Delta t} \big(u^{k+1}_{i,i} - u^{k}_{i,i} \big), \tag{4.43}$$

where the superscript k denotes the time step.

The final procedure looks as follows. Let $(u^k, p^k, \varepsilon^{\text{pl},k}, q^k)$ be a given equilibrium state. Then, the repeated application of the following steps lead to the solution at timestep $k+1$. The index i denotes the iteration step.

1. Compute the residuum vectors R^i_u and R^i_p.

2. Update the matrix B according to the actual state.
3. If $\left\|R_u^i\right\| \leq \epsilon_u$ and $\left\|R_p^i\right\| \leq \epsilon_p$ accept the current state as solution and proceed with the next time step.
4. Solve the system of linear equations

$$\begin{bmatrix} (K_{uu}-B) & K_{up} \\ K_{pu} & K_{pp} \end{bmatrix} \begin{bmatrix} \Delta\hat{u} \\ \Delta\hat{p} \end{bmatrix} = \begin{bmatrix} \hat{R}_u^i \\ \hat{R}_p^i \end{bmatrix}. \tag{4.44}$$

5. Add the increments $\Delta\hat{u}$ and $\Delta\hat{p}$,

$$\begin{aligned} \hat{u}^{k+1,i+1} &= \hat{u}^{k+1,i} + \Delta\hat{u}, \\ \hat{p}^{k+1,i+1} &= \hat{p}^{k+1,i} + \Delta\hat{p}. \end{aligned}$$

6. Perform the return mapping algorithm given in section 4.1 to obtain $\varepsilon^{\mathrm{pl},k+1,i+1}$ and $q^{k+1,i+1}$.
7. Set $i = i+1$ and proceed with step one.

The values ϵ_u and ϵ_p are user defined error bounds. They are set to $\epsilon_u = \epsilon_p = 10^{-10}$ for all calculations given in this work. The residuum vectors R_u^i and R_p^i are given as

$$\begin{aligned} R_u^i &= K_{uu}\hat{u}^{k+1,i} + P\hat{\varepsilon}^{p,k+1,i} + K_{pu}\hat{p}^{k+1,i} + \hat{F}^{u,k+1}, \\ R_p^i &= K_{pp}\hat{p}^{k+1,i} + K_{up}\hat{u}^{k+1,i} + \hat{F}^{p,k+1}, \end{aligned} \tag{4.45}$$

where K_{uu} is the stiffness matrix (4.23), P is given with (4.24). The other matrices are

$$\begin{aligned} (K_{up})^{ab}_{i} &= \int_{\mathbb{S}} N^{p,b}(x) N^{u,a}_{,i}(x)\mathrm{d}x, \\ (K_{pp})^{ab} &= \int_{\mathbb{S}} k N^{p,a}_{,i}(x) N^{p,b}_{,i}(x)\mathrm{d}x \quad \text{and} \\ (K_{pu})^{ab}_{i} &= \frac{1}{\Delta t}(K_{up})^{\top}. \end{aligned}$$

The vectors $\hat{F}^{u,k+1}$ and $\hat{F}^{u,k+1}$ are computed with prescribed data,

$$\begin{aligned} (\hat{F}^{u,k+1})^a_i &= \int_{\Omega} f_i^{k+1}(x) N^{u,a}(x)\,\mathrm{d}x + \int_{\Gamma_t} t_i^{k+1}(x) N^{u,a}\,\mathrm{d}x, \\ (\hat{F}^{p,k+1})^a &= \int_{\Omega} k\, f_i^{k+1}(x)\, N^{p,a}_{,i}(x)\,\mathrm{d}x + \int_{\Gamma_q} q^{k+1}(x)\, N^{p,a}(x)\,\mathrm{d}x. \end{aligned}$$

The solution of the block system (4.44) is obtained iteratively [33]. A new iterate is obtained as follows. The iteration parameter is j.

1. Compute the defects

$$
\begin{aligned}
d_u &= R_u^i - (K_{uu} - B)\Delta\hat{u}^j - K_{up}\Delta\hat{p}^j, \\
d_p &= R_p^i - K_{pu}\Delta\hat{u}^j - K_{pp}\Delta\hat{p}^j.
\end{aligned}
$$

2. Compute the primal correction $\Delta\Delta\hat{u}^j$ from

$$(K_{uu} - B)\Delta\Delta\hat{u}^j = d_u.$$

   Here, a bi-cg-stab solver is used.

3. Compute the new defect

$$d_p = d_p - K_{pu}\Delta\Delta\hat{u}^j.$$

4. Compute the dual correction $\Delta\Delta\hat{p}^j$ from

$$K_{pp}\Delta\Delta\hat{p}^j = d_p.$$

   Here, a bi-cg-stab solver is used.

5. The new iterate is given by

$$
\begin{bmatrix} \Delta\hat{u}^{j+1} \\ \Delta\hat{p}^{j+1} \end{bmatrix} = \begin{bmatrix} \Delta\hat{u}^j \\ \Delta\hat{p}^j \end{bmatrix} + \alpha_j \begin{bmatrix} \Delta\Delta\hat{u}^j \\ \Delta\Delta\hat{p}^j \end{bmatrix}
$$

   where α_j is determined by minimizing the new Euclidian defect,

$$\alpha_j = \frac{(d_u)^\top a + (d_p)^\top b}{a^\top a + b^\top b}$$

   with

$$
\begin{aligned}
a &= (K_{uu} - B)\Delta\Delta\hat{u}^j + K_{up}\Delta\Delta\hat{p}^j, \\
b &= K_{pu}\Delta\Delta\hat{u}^j + K_{pp}\Delta\Delta\hat{p}^j.
\end{aligned}
$$

5. EXAMPLES

The solution methods for the presented problems haven been implemented in a computer code. This was done within the framework of the Dune-Project [2, 3, 4, 9]. To verify the implementation of the plasticity a circular tube undergoing plastic strains is considered. For the validation of the poro-elastic interaction problem a column under vertical loading is considered. In this test, the material parameters are chosen such that no plastic yielding occurs. Finally, the full plastic consolidation process under a footing is studied. For all calculations a quadratic ansatz for the displacements is used. The pore pressure and the plastic strains are approximated linear.

5.1. Circular tube

As an example for the validation of the implemented code an internally pressurized circular tube as depicted in figure 5.1 is considered. An analytical solution to the plain strain problem with the perfectly plastic von Mises model is given in [17]. The parameters used

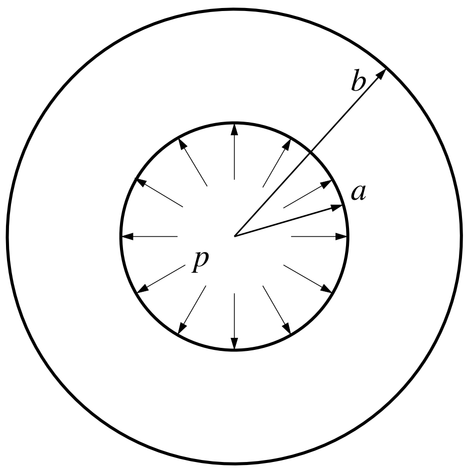

Figure 5.1.: Definition of the circular tube

for the analysis are given in table 5.1. Due to the radial symmetrie of the problem only one quater has been discretized. The mesh for the numerical analysis consists of 1560 elements and 588 nodes. In case of the associative von Mises model, as it is considered here, the plastic strains are volume preserving. In combination of the simple elements used here, this

property	symbol	value	unit
inner radius	a	0.1	m
outer radius	b	0.2	m
Young modulus	E	200000	N/m^2
Poisson's ratio	ν	0.3	–
yield strength	σ_y	400	N/m^2
hardening modulus	D	200	N/m^2

Table 5.1.: Parameters for the internally pressurized tube

	iteration					
step	0	1	2	3	4	5
1	0.014131	5.9662e-15				
2	1.1932e-14					
3	0.0072457	0.00015706	4.8816e-06	2.2617e-10	3.3466e-15	
4	0.0080283	0.00031447	1.3031e-05	1.0715e-07	1.8088e-13	
5	0.01226	0.0010036	4.7221e-05	5.0109e-06	4.4616e-10	5.8174e-15
6	0.0090735	0.00051594	1.081e-05	8.2934e-08	9.357e-14	
7	0.0070128	0.00059659	1.1666e-06	1.5184e-11		
8	0.0054439	0.00051928	3.3568e-06	7.221e-11		
9	0.0044958	0.00023467	1.7651e-06	9.0492e-11		
10	0.0036713	0.00046022	1.4525e-05	1.351e-06	5.8372e-11	
11	0.003625	4.2625e-05	3.4497e-06	5.2422e-10	3.2603e-11	
12	0.0022817	1.9971e-05	2.6467e-08	1.3233e-11		
13	0.00015271	2.0042e-08	4.9444e-14			
14	0.00013202	1.6967e-08	4.4476e-13			
15	0.0001151	1.38e-08	3.7983e-13			

Table 5.2.: Residua of the Newton-like method

leads to wrong solutions if the plastic strains are large in comparison to the elastic strains. Therefor, a small hardening modulus is chosen for the numerical stability. In general with ongoing yielding, the number of iterations of the iterative solver increases. Nevertheless, table 5.2 shows the super-linear convergence of the Newton-like method. The obtained pressure-displacement curve is plotted in figure 5.2. It has a good a agreement with the closed-form solution.

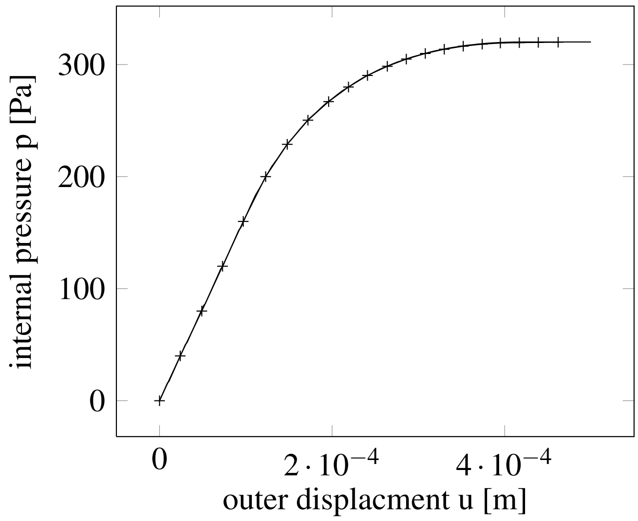

Figure 5.2.: Comparison of the analytic solution and numeric solution

5.2. Poro-elastic column

To verify the solution methods for the poro-elastic interaction within the elastic range, a column is considered. An analytical solution to the one dimensional problem with bound-

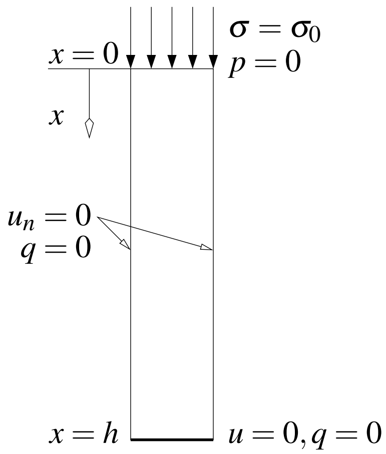

Figure 5.3.: Definition of the column problem

ary conditions as depicted in figure 5.3 is given in [21]. The bottom of the column is fixed and impermeable, the top is loaded with the traction σ_0 but the fluid can escape. For the three dimensional case the walls are also impermeable and fixed horizontally, i.e. in normal direction. Homogeneous initial conditions are assumed. The analytic distribution of

property	symbol	value	unit
height	h	10	m
lame parameters	λ	5.56	kN/m^2
	μ	8.33	kN/m^2
permeability	k	10^{-9}	m^4/Ns
applied load	σ_0	100	kN/m^2

Table 5.3.: Parameters for the column problem

the effective stress

$$\sigma^{\text{eff}} = \sigma_0 \left(-1 + \sum_{i=0}^{\infty} \frac{2}{M} \sin\left(M\frac{x}{h}\right) \mathrm{e}^{-M^2\hat{t}} \right), \tag{5.1}$$

the displacements

$$u = \frac{h\sigma_0}{\lambda + 2\mu} \left(1 + \sum_{i=0}^{\infty} \frac{2}{M} \sin\left(M\frac{x}{h}\right) \mathrm{e}^{-M^2\hat{t}} \right), \tag{5.2}$$

and the pore pressure

$$p = \sum_{i=0}^{\infty} \frac{2}{M} \sin\left(M\frac{x}{h}\right) \mathrm{e}^{-M^2\hat{t}}, \tag{5.3}$$

are compared to the results obtained from the tree dimensional analysis. A mesh consisting of 300 elements and 130 nodes has been used. In (5.1), (5.2) and (5.3) the abbreviations

$$\begin{aligned} \hat{t} &= (\lambda + \mu)kt, \\ M &= \frac{\pi(2i+1)}{2} \end{aligned} \tag{5.4}$$

have been used. In table 5.3 chosen parameters are summarized. The maximum displacement occurs on the top of the column. From figure 5.4 and 5.5 a good agreement of the numerical results and the analytic one dimensional solution can be seen.

5.3. Footing

As a last example a footing problem is considered. The soft ground of 6 m height is modeled with the linear isotropic elasticity model in combination with the non-associative Drucker-Prager plasticity model. This represents only a rough approximation of real soil behavior. On a 6 m $\times$ 6 m square a load of 100 $\mathrm{kN/m^2}$ is applied within two different time ranges. Any gravity loading is neglected and homogeneous initial conditions are assumed. Due to the symmetry of the problem, only one quarter is simulated. The mesh consists of 17666 elements and 3718 nodes. The boundary conditions are set as follows. The

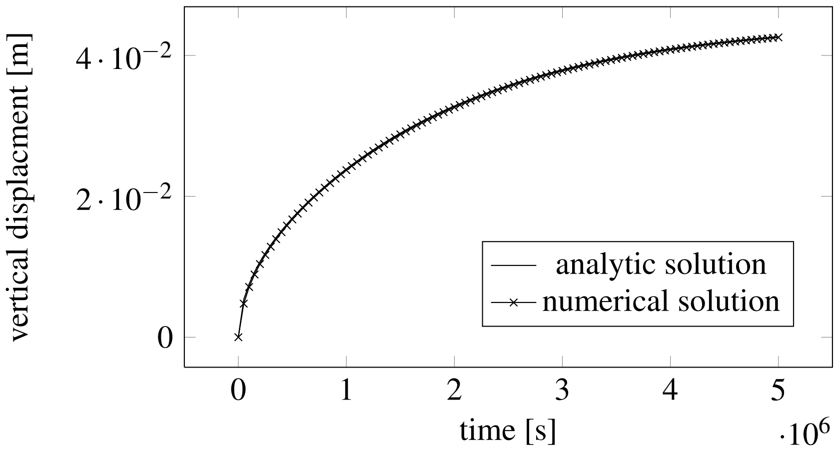

Figure 5.4.: Displacement at the top of the column plotted over time

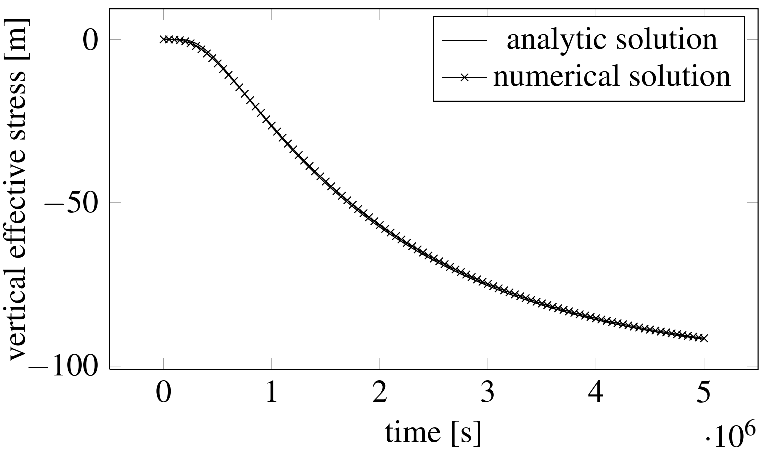

Figure 5.5.: Effective stress at the bottom of the column plotted over time

property	symbol	value	unit
lame parameters	λ	2.02	$^{kN}/_{m^2}$
	μ	1.35	$^{kN}/_{m^2}$
friction angle	ϕ	30	$\circ$
dilatancy angle	χ	15	$\circ$
cohesion	c	30	$^{kN}/_{m^2}$
permeability	k	10^{-8}	$^{m^4}/_{kNs}$

Table 5.4.: Parameters for the footing problem

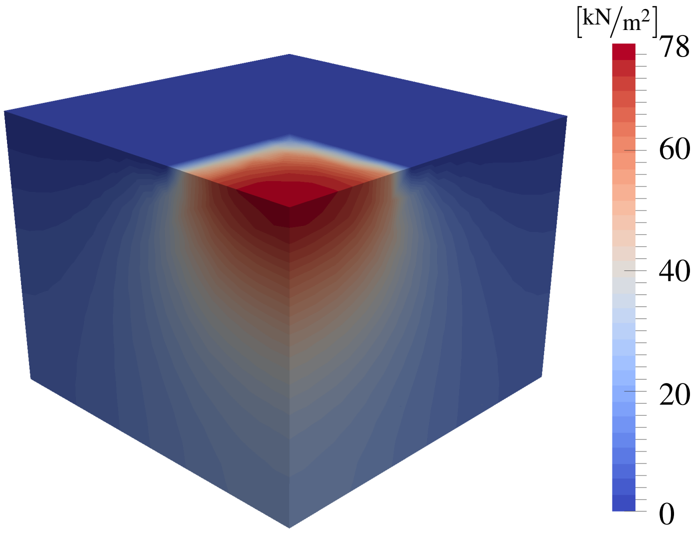

Figure 5.6.: Distribution of the excessed pore pressure after applying the load

bottom at 6 m depth is vertically fixed and impermeable. Both, the symmetry planes and the cutoff planes at 8 m from the center are fixed for the normal displacements and are also impermeable. The top is impermeable on the 3 m $\times$ 3 m square which is loaded. The rest of the top is permeable.

The used material parameters are summarized in table 5.4. They are chosen in order to show some effects qualitatively. Therefore, the parameters are not suitable for a simulation of real soil behavior. The size of time-step Δt is one day. The calculation of 100 time-steps has been performed once applying the load within one day and once within thirty days.

The distribution of the excessed pore pressure after applying the load within one day is visualized in figure 5.6. The vertical displacements are shown in figure 5.7 on the deformed geometry (ten times raised) and plotted over a horizontal line at the upper surface in figure 5.8. Due to the nearly incompressible undrained response an uplift beside the loaded area

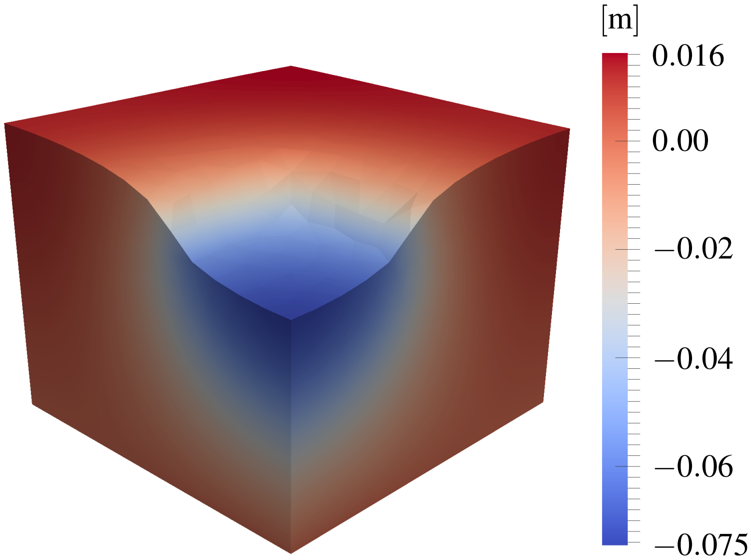

Figure 5.7.: Vertical displacements plotted on the deformed configuration after applying the load

happens.

The excessed pore pressure and the vertical displacement at the center of the loading area are plotted over time in figures 5.9 and 5.10. When applying the load within one day the maximum excessed pore pressure is higher than applying the load within thirty days. This is because already some consolidation happens during loading. If the pressure is carried from the water, less stabilizing pressure acts on the solid skeleton. Therefore, the yield strength is lower. This explains why some plastic strains occur when loading happens within one day. In case of loading within thirty days no plastic reactions occur. Therefore, the final vertical displacements are slightly lower. A qualitative distribution of the plastic strains is given in figure 5.11.

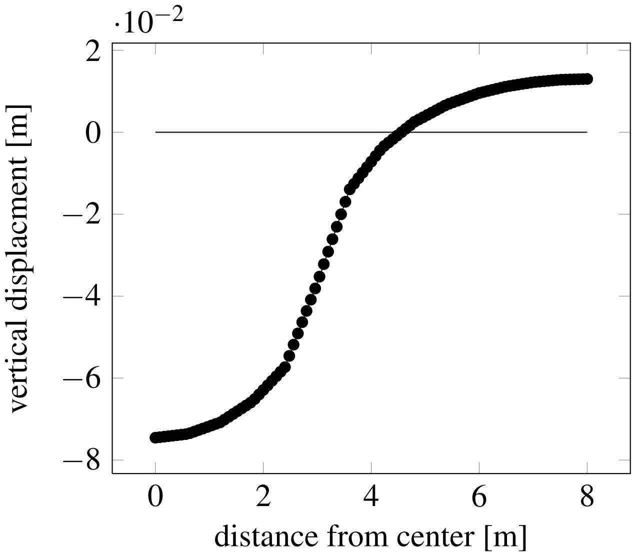

Figure 5.8.: Vertical displacements along a line at the top after applying the load

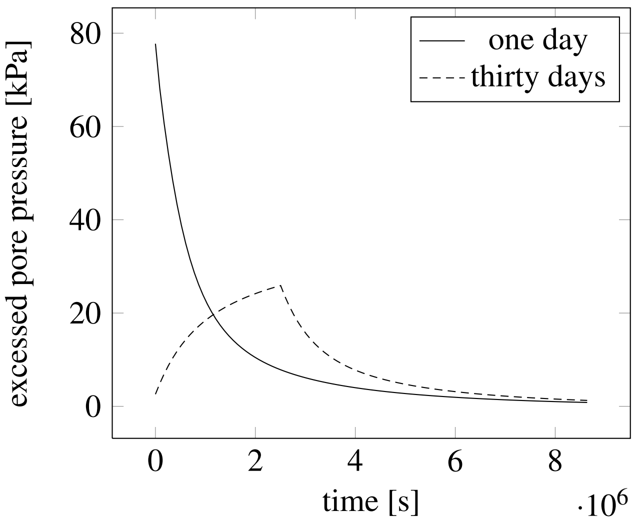

Figure 5.9.: Excessed pore pressure at the center plotted over time

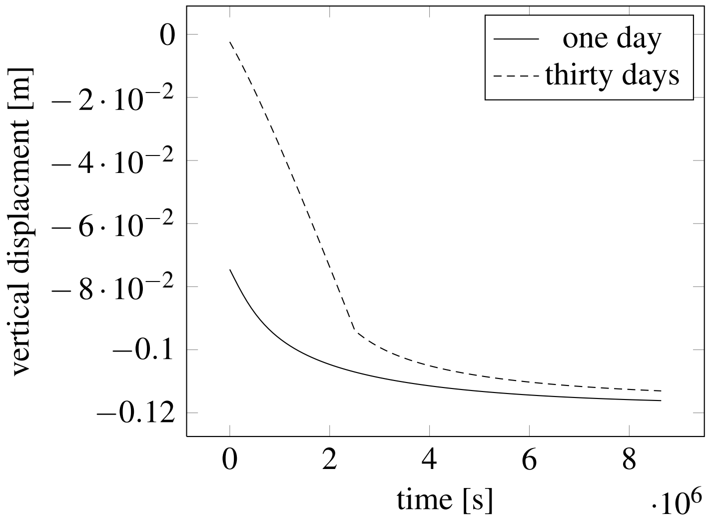

Figure 5.10.: Vertical displacement at the center plotted over time

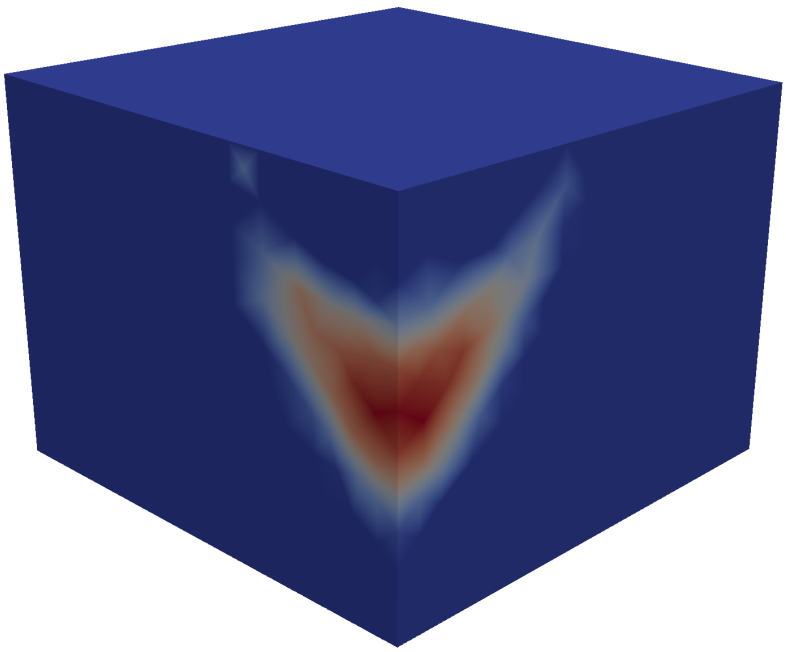

Figure 5.11.: Qualitative distribution of the plastic strains for loading within one day

6. CONCLUSION

In the present work we deal with a theoretical approach to model consolidation phenomena and its numerical realization with a finite element method. Thereby, we focused on Biot's model for a two-phase porous media relaying on an effective stress approach. Arising inertia terms and therefor dynamical effects are neglected from the outset. Both, the solid grains and the fluid are assumed to be incompressible. The constitutive equations for the effective stresses are a linear isotropic elasticity model in combination with a non-associative Drucker-Prager model or an associative von Mises model with linear isotropic hardening. The restriction on a linearized geometry description allows an additive split of the strain tensor in an elastic and a plastic strain tensor. Additionally to the discretization of the primal field variables, here the displacement field and the pore pressure field, we introduced a discretization of the plastic strain field. As the nonlinear problem is solved with a Newton-like method, we derived a consistent linearization of the underlying equations and provide closed-form formulas for the tangential stiffness matrix. This leads to a super-linear convergent method. This was verified in a numerical example. As an open issue remains the treatment of the plastic strains on partly plastified elements. As a standard approach adaptive refinement of the mesh is proposed in the literature. Due to the discretization of the plastic strain field another approach seems fruitful. The plastic response could be evaluated in each node whether the yield criterion is violated or not. Then the region where a plastic response occures could be determined approximately.

Other possible extensions of the presented work are obvious. When taking inertia terms into account also wave propagation phenomena could be studied. The porous medium could be infiltrated by more than one fluid and compressible constitutes could be analyzed. The elasticity model could be extended to account for non-linear and anisotropic behave. Also more complex plasticity models could be used. The consideration of the geometric exact description in combination with a multiplicative decomposition of the deformation gradient in an elastic and a plastic part would be possible.

A. SPACE, FRAMES AND TENSORS

The space of consideration for any physical object in this work is the three-dimensional Euclidean space $\mathbb{E}^3$. No distinction between $\mathbb{E}^3$ and $\mathbb{R}^3$ is made. With the choice of a fixed specific reference point in the space, distances can be measured. Any point in the space can then be described by three real numbers, its coordinates. These coordinates only make sense in connection with a coordinate system, mapping the numbers to the right point in space.

Definition 9. *(Coordinate system) A coordinate system $\{z_i\}$ on $\mathbb{R}^3$ is a C^∞ map of an open set $O_z \in \mathbb{R}^3 \to \mathbb{R}^3$, whose range is an open set $O_x \in \mathbb{R}^3$ and the inverse of the $O_z \to O_x$ mapping is also a C^∞ map.*

With definition of a coordinate system also coordinate lines are defined.

Definition 10. *(Coordinate lines) The coordinate lines of $\{z_i\}$ are the curves $c_1(z_1)$, $c_2(z_2)$, $c_3(z_3)$ whose components in the common frame are*

$$
\begin{aligned}
&\textit{for the first coordinate line} \\
&\qquad x_i(c_1(z_1)) = \{z_i\}(z_1, z_2, z_3) \quad \textit{where } z_2 \textit{ and } z_3 \textit{ are fixed}, \\
&\textit{for the second coordinate line} \\
&\qquad x_i(c_2(z_2)) = \{z_i\}(z_1, z_2, z_3) \quad \textit{where } z_1 \textit{ and } z_3 \textit{ are fixed}, \\
&\textit{and for the third coordinate line} \\
&\qquad x_i(c_3(z_3)) = \{z_i\}(z_1, z_2, z_3) \quad \textit{where } z_1 \textit{ and } z_2 \textit{ are fixed}.
\end{aligned}
$$

As the reference point and the coordinate system can be chosen arbitrarily, a transformation law between the coordinates referring to different coordinate systems is necessary. The assertion of an Euclidean space implies that there is an rectangular Cartesian coordinate system, to which all points refer [31, p. 241]. This common frame is fixed at the beginning but the choice is not unique as mentioned before. All other frames are derived from the common frame. A point r in space is expressed in the common frame through its real coordinates r_i

$$r = r_i e_i = r_1 e_1 + r_2 e_2 + r_3 e_3 \tag{A.1}$$

where e_i is the i-th unit coordinate vector of the common frame.Taking an arbitrarily chosen coordinate system $\{z_a\}$ the coordinates of the same point r as in (A.1) are given by

$$r_a^z = \{z_a\}^{-1}(r_i^e), \tag{A.2}$$

where r_i^e are the coordinates with respect to the common frame.

To describe physical properties explicitly, the reference to base vectors is necessary.

Definition 11. *(Coordinate base vectors) Let e_i be the base vectors of the common frame, which are independent from the point in space. The tangents to the coordinate lines of a coordinate system $\{z_a\}$*

$$a_b^z = \frac{\partial x_i(c_b(z_b))}{\partial z_b} e_i \qquad b = 1\dots 3, \tag{A.3}$$

are then the coordinate base vectors of $\{z_a\}$, which depend in general on the position in space.

As indicated before, in general one can establish as many coordinate systems (with different base vectors) as wanted and transfer the representation of the physical quantity to the new coordinate system. As there are physical properties depending on a different number of sets of base vectors, the transformation law for each property has to be adjusted to the number of sets of base vectors needed. For instance the mass density of a matter is modeled with a scalar field which has no dependency on a base vector, the force on a body is modeled by a vector with three components which have reference to one set of base vectors. For the generalization of the force concept, the notion of stress is introduced in chapter 2.2. As the quantity describing the stress relates two vectors with each other, it will depend on two sets of base vectors and will therefore have nine components. In general, such quantities are called tensors.

Consider the quantity Q, which has the components $Q_{ij\dots kl}$ with n indices $\{i, j\dots k, l\}$, which all refer to the base vectors of the same coordinate system $\{z_i\}$. Therefore, 3^n components are needed to describe the quantity. If the coordinate system $\{y_i\}$ is changed to $\{z_\alpha\}$ and, therefore, the base vectors (see A, Definition 11) are transformed according to

$$\begin{aligned} a_\alpha^z &= \frac{\partial x_k}{\partial z_\alpha} e_k \rightarrow \quad e_k = \frac{\partial z_\alpha}{\partial x_k} a_\alpha^z \\ a_i^y &= \frac{\partial x_k}{\partial y_i} e_k \rightarrow \quad a_i^y = \frac{\partial x_k}{\partial y_i}\frac{\partial z_\alpha}{\partial x_k} a_\alpha^z = \frac{\partial z_\alpha}{\partial y_i} a_\alpha^z, \end{aligned}$$

then Q transforms according to

$$Q = Q_{ij\dots kl} a_i^y a_j^y \dots a_k^y a_l^y = Q_{\alpha\beta\dots\gamma\delta} \frac{\partial z_\alpha}{\partial y_i}\frac{\partial z_\beta}{\partial y_j} \dots \frac{\partial z_\gamma}{\partial y_k}\frac{\partial z_\delta}{\partial y_l} a_\alpha^z a_\beta^z \dots a_\gamma^z a_\delta^z \tag{A.4}$$

and its components transform like

$$\begin{aligned} Q_{ij\dots kl}^y &= Q_{\alpha\beta\dots\gamma\delta}^z \frac{\partial z_\alpha}{\partial y_i}\frac{\partial z_\beta}{\partial y_j} \dots \frac{\partial z_\gamma}{\partial y_k}\frac{\partial z_\delta}{\partial y_l} \\ Q_{\alpha\beta\dots\gamma\delta}^z &= Q_{ij\dots kl}^y \frac{\partial y_i}{\partial z_\alpha}\frac{\partial y_j}{\partial z_\beta} \dots \frac{\partial y_k}{\partial z_\gamma}\frac{\partial y_l}{\partial z_\delta}. \end{aligned}$$

Physical quantities have to be invariant against a change of the coordinate system and, therefore, transform its representation as given above. Because of the dependency on n sets of base vectors Q is called a tensor of order n.

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