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The price of derivatives: Liquidity risk and price impacts

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Abstract

In financial theory market liquidity refers to how easy it is to buy or sell an asset within little time and without losing money due to price impacts. In contrast to classical arbitrage theory, which assumes that the market is perfectly liquid, this model deals with the risk coming from the illiquidity of an asset in terms of the following aspects: The price per share of an asset depends on the order size and every trade has a lasting impact on the price process. The price impact and the resulting supply curve are correlated to the level of liquidity of the asset. As a consequence the value of a trading strategy differs from the value in classical theory. We start by determining the additional costs of a trading strategy which are due to the illiquidity of the asset. We show that the model is free of arbitrage and that the market is appoximately complete. We then investigate the impact of the illiquidity on the replication value of contingent claims. To obtain the minimal replication strategy one has to solve a backward stochastic differential equation. Only if the process which describes the liquidity of the stock is a martingale, the minimal replication value is equal to the classical fair price. In general the minimal replication value is a non linear functional of the contingent claim, e.g. it depends in a non linear way on the order volume of the contingent claim. We compute this minimal replication value in simple models and compare the replication values of european options to their prices in classical theory.

Contents

1. Introduction

In classical arbitrage pricing theory it is assumed that the market is infinitely liquid. In that case the price of an asset is independent of the number of shares that are purchased. Furthermore, market participants act like price takers which means that their trades do not have an impact on the price of the asset. While famous models created by Black-Scholes or Merton deal with market and credit risk, they neglect the risk coming from the illiquidity of an asset.

In financial theory market liquidity refers to how easy it is to buy or sell stocks within little time and without losing money due to price impacts. If a number of stock shares is bought and can immediately be sold at the same price, the stock is called perfectly liquid. Whereas, if the shares can not be sold at all for any price the stock is illiquid. In reality the liquidity of an asset can be found somewhere in between these extremes. Liquidity becomes a risk factor if its impact on the price of an asset changes randomly over time. We think of liquidity risk as additional risk due to the timing and size of a trade. As a consequence of liquidity risk, the value of a trading strategy differs from the value in classical theory. We refer to this difference as liquidity costs. The additional costs can be substantially higher if large quantities are traded. Consequently this issue is of special interest for moderate and large traders. In this thesis we are interested in the liquidity costs of a trading strategy and the impact of illiquidity on the replication costs of financial derivatives.

The literature on liquidity risk can be divided into two categories. In the first category of models the price of an asset depends on the quoted price and the size of the transaction. In that case, there is no long term influence of a trade on the quoted price process. Examples of these models are Cetin and Rogers [21] and Rogers and Singh [9]. In the second category of liquidity risk models, trades have a lasting impact on the quoted price of an asset. The impact of these trades depends on the number of purchased shares. If this impact is more than linear in the number of shares, large trades have a proportional stronger influence on the price. These models are therefore known as larger trader models. Examples are Bank and Baum [12] , Frey [18] and Jarrow [19]. In this thesis we use a model which combines both of these approaches. The price of an asset depends on the transaction size and each trade triggers a price impact.

Cetin, Jarrow and Protter [22] created a time continuous model, where the price of an asset is determined by the quoted price and an increasing supply curve. They assume that trades do not cause any fluctuations of the quoted price. The liquidity costs in this model is determined by the quadratic variation of the trading strategy. The costs can be completely avoided if the strategy is continuous and of finite variation and the price of a financial derivative is therefore equal to the price in classical theory. Different studies done by Weber and Rosenow [17] and Farmer, Gillemot, Lillo [5] have shown that there is a correlation between price fluctuations and the level of liquidity of a stock. Blais and Protter [10] could show that for highly liquid stocks the supply curve is approximately linear with a randomly changing slope. This slope depends on the level of liquidity and it can therefore be used to model liquidity. This observation is used in Roch [3]. In his model the price of a stock is given by a linear supply curve and therefore depends on the size of the transaction. In addition each trade has an impact, which is directly related to the amount of liquidity of the asset, measured by the slope of the supply curve. The exact mechanism of the price impact of a trade is motivated by an investigation of the limit order book of a stock. This way A. Roch combines both notions of liquidity risk in one model.

This thesis is based on the work of A. Roch [3]. Similar to his paper we are interested in the impact of liquidity on the replication costs of a contingent claim. Roch uses a specific stochastic volatility model for the stock price. We derive the theoretical results in a more general setting. The minimal replication costs of a contingent claim can be obtained by solving a backward stochastic differential equation. Using simple models we are able to calculate these costs explicitly and compare the values with prices in classical theory. In the original model of A. Roch explicit solutions are not available. Moreover, even numerical solutions seem to be very hard to obtain in the full model.

This thesis is organized as follows. In Chapter 2 we describe the limit order book of a stock and investigate the impact of liquidity on the stock price. We show that market orders trigger price impacts which depend on the size of the transaction and the density of unexecuted limit orders. Assuming that the supply curve of a stock is linear, we derive the density of the order book and determine the impact on the quoted price. We use these observations in Chapter 3 to calculate the actual observed stock price, which includes all price impacts and therefore depends on the full history of trades. For a self financing trading strategy we deduce the liquidity costs. We define admissible trading strategies and show that by merely trading these strategies we can not generate arbitrage with vanishing risk. In addition we define the notion approximate replication and show that the minimal replication value of a claim is equal or bigger to its expected output.

In Chapter 4 we consider the underlying processes in a general model and investigate the minimal replication value of a claim. We complete the market by adding sufficiently many assets (e.g. variance swaps) and prove that every integrable claim can be approximately replicated. The approximate hedges are continuous trading strategies of finite variation, which converge to the solution of a backward stochastic differential equation. The minimal replication value is determined by the minimal solution of this equation.

We show some analytical properties of the solution, including an approximation of the price of a claim which is linear (in the order size).
In Chapter 5 we consider models where the liquidity process M is a martingale. This case turns out to be an extension of the classical theory as the price of an asset is given by its expectation.
In the last Chapter 6 we compute the minimal replication value in simple models. We derive analytical and numerical solutions of the corresponding backward stochastic differential equations.

2. Background: The limit order book

To understand the impact of liquidity on the price of an asset, we first explain how assets are traded on the stock exchange. The standard mechanism for price formation in most modern financial markets is called **double continuous auction**. We then consider the supply curve of a stock. Assuming it has a linear shape we deduce the density of the limit order book. For further information on the double auction mechanism see Farmer, Gillemot, Lillo [5].

2.1. Double auction mechanism

In economic theory the price of an asset is determined by its demand and supply. If a person A wants to sell a stock at a specific price P or higher and a trader B is willing to buy the stock for the same price P or less, their interests match. Given that nobody else wants to buy or sell the stock, the trade will be executed at price P. But what happens if A offers the stock at a price P_A which is higher than the maximum price P_B the person B is willing to pay. In that case the trader has two possiblities: Either he waits until B or someone else wants to buy the stock for the price P_A or higher, or A sells the stock immediately for the lower price P_B. This introduces two different ways how agents place orders in financial markets.

- Market orders: A market order is a request to buy or sell a given number of shares immediately at the best available price. (i.e.: A sells the stock at price P_B)
- Limit orders: If the trader is more patient he might submit a limit order (also called quote), where he waits until someone is willing to buy (resp. sell) a number of shares for a price equal or higher (resp. lower) to the corresponding limit price. If the trader wants to sell the stock, the limit order is called ask and if he wants to buy the stock, it is called bid. (i.e. A places an ask-order if he waits until someone wants to buy the stock at price P_A or higher.)

Of course not all of the limit orders can be executed immediately, therefore they are stored in a queue, which we refer to as limit order book. This means the limit order book contains all unexecuted bid and ask orders. At every time t we define the best ask with price p_A to be the lowest ask order in the limit order book and the best bid order with price p_B as the highest bid order. Of coures the prices satisfy $p_B < p_A$, otherwise they would be executed immediately. This means there is a gap between the highest bid and the lowest ask price. This gap is called the bid-ask spread $\Delta p = p_A - p_B$.

The number of shares of a order is called size or volume of the order. From now on a negative volume corresponds to a sell order and a positive volume to a buy order. Therefore, we can identify a order by its type (limit or market order), the price and the number of shares. In financial markets the prices are not continuous, but rather change in discrete quanta called ticks. Figure 2.1 illustrates the shape of a order book, the best prices and the bid-ask spread at time $t_1 : t_1 < t_2 < t_3$.

If somebody places a market order of x shares, the limit orders of opposite sign

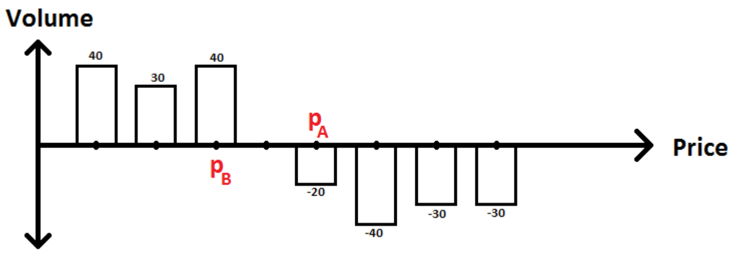

Figure 2.1.: The limit or book at time t_1

$-sign(x)$ will be executed starting with the best price. If there are several limit orders offered at the same price those with earlier arrival time will be served first. This way limit orders vanish until a total volume of x is executed. Of course every limit and market order can have a different volume. Therefore it is not possible to match limit and market orders one to one. It might happen that only a part of the volume of the limit order is executed by the first market order. The resulting rest of the volume remains in the order book and might be used to match the next order. If a market order arrives with a volume which is bigger than the total number of shares offered at the best price, all limit orders at that price will be executed in the limit order book. This means the best price with opposite sign as the market order will be removed. This impact on the price is called market impact or price impact. Figure 2.2 shows the limit order book at time t_2 when a sell market order is placed and Figure 2.3 represents the limit order book after the market order was executed and illustrates how changes in the best prices occur.

It is easy to see that the direction of these impacts remains the same: A buy market order increases the lowest ask price, while a sell market order decreases the highest bid. However every price impact makes the bid-ask spread bigger. The spread keeps growing until new limit orders are placed with prices in between the best ask price p_A and the best bid p_B.

The impact of a market order depends on the density of the limit order book $p_t(z)$ for a stock price z. As the tick size is very small (down to 1 cent) we will from now on assume that prices are continuous, therefore $z \in \mathbb{R}^+$. If the density per price is high, the stock is called liquid and a market order will only trigger a small movement. On the

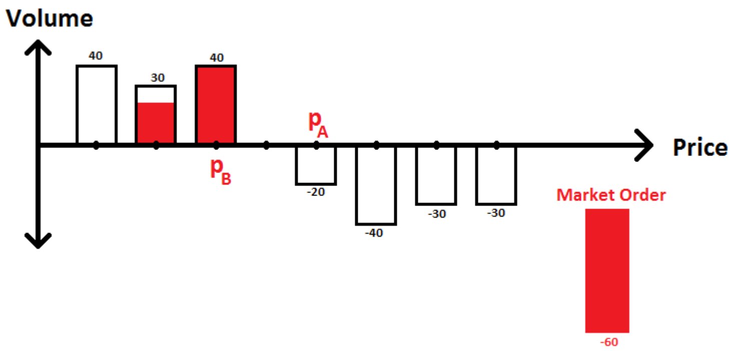

Figure 2.2.: The limit or book at time t_2

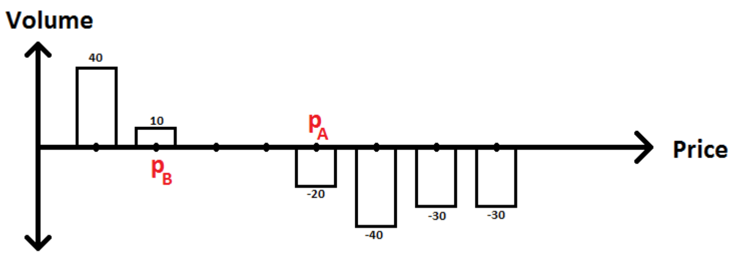

Figure 2.3.: The limit or book at time t_3

other side if the liquidity is low, it has a big impact on the best price.

In this thesis we are interested in the impact of liquidity on the price of financial derivatives. We consider a trader who wants to hedge a contingent claim. The trader has to place orders at specific times and is not patient enough to wait until someone accepts his or her offered price. Hence, the trader will only place market orders to hedge the contingent claim. It will be shown, that the liquidity of the underlying, which is given by the density of the order book, has an influence on the price of the claim.

2.2. The supply curve

Let us consider a stock which is actively traded through a limit order book. We take the point of view from a hedger who observes the order book and makes market orders. As stated above, the order book stores all unexecuted limit orders and this way determines the supply of a stock. We have seen that the price of a stock might be influenced by the size x of an order. A positive order $x > 0$ represents a buy, a negative order $x < 0$ a sell of x shares and $x = 0$ corresponds to the marginal trade. We are interested in the price per share for a transaction of a specific size. Let us define

$$S(t,x,\omega) \quad \text{for} \quad t \in [0,T], x \in \mathbb{R}, \omega \in \Omega$$

as the average price per share if we buy x shares at time t. Given the stage ω in the probability space Ω and fixing the time t we get a function of the order size $x \mapsto S(t,x,\omega)$ for $x \in \mathbb{R}$. In the following we supress the dependence on ω and write $S(t,x)$ for the random variable $\omega \mapsto S(t,x,\omega)$. We assume that a supply curve $S(t,x,\omega)$ satisfies the following conditions.

Definition 2.1. *A Supply-Curve is a function $S(t,x,\omega) : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}^+$ which satisfies*

1. *$S(t,x)$ is F_t-measurable and non-negative*
2. *Given stage ω and time t the function $x \mapsto S(t,x,\omega)$ is non-decreasing.*
3. *Given stage ω the function $(t,x) \mapsto S(t,x,\omega)$ is in C^2 in x, $\frac{\partial S(t,x,\omega)}{\partial x}$ is continuous in t and $\frac{\partial^2 S(t,x,\omega)}{\partial x^2}$ is continuous in t.*
4. *$t \mapsto S(t,0)$ is a semimartingale*
5. *$t \mapsto S(t,x)$ has continuous sample paths (including time 0) for all x*

Of course the number of shares of a stock that are offered at time t is limited. Therefore a buy-order of a stock leads to an decrease in supply, while a sell-order leads to an increase. From an economic perspective, it is clear that a decrease in supply drives the price up and an increase in supply decreases the price. Therefore it is reasonable to assume that the price of $x > 1$ stocks is higher than x times the price of one. This is considered in condition 2 of the above definition. As a special case this definition includes constant supply-curves. In that case the size of an order does not have an impact on the price. In a liquidity risk model we want the trader's transactions to have an affect on the executed price. Therefore a horizontal supply-curve, as in the classical theory, is not appropriate in our model. So what should the supply curve look like?

When we are purchasing x shares of a stock, the average price per share could be obtained by observing the limit order book and taking the average price of all executed limit orders. Unfortunately limit order book data of a stock is not accessible to the public. Therefore we rely on a study by Blais and Protter [10] from 2010. They analyzed the supply curve models of liquidity issues in stock and option market trading. They considered the most liquid stocks on the New York Stock Exchange and could show that a supply curve really exists and is non trivial ($S(t,x) \neq S(t,0)$). Furthermore, they determined that supply curves for highly liquid stocks are linear in the number of shares with a dynamically changing random slope. Figure 2.4 is taken from Blais and Protter [10]. It shows the price of the British Petroleum (BP) stock on the New York Stock Exchange depending on the trade size and the corresponding linear regression line.

Note that the average price per share $S_t(x) = S(t,x)$ of x shares of a stock is a

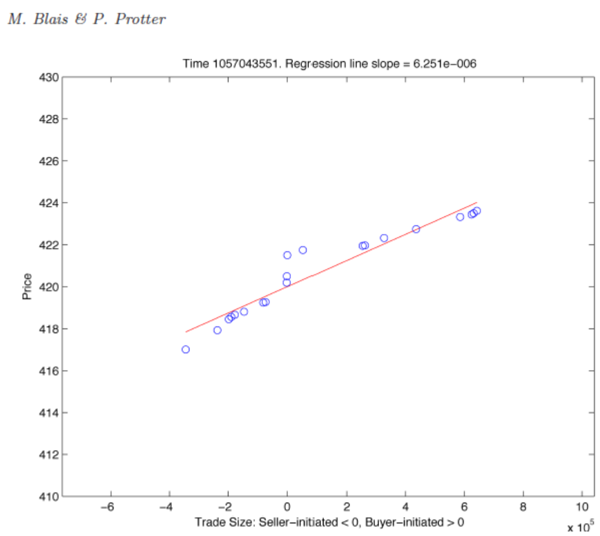

Figure 2.4.: Linear supply curve: British Petroleum

stochastic process. From now on we assume that we are dealing with a stock which has a linear supply curve $S(t, x, \omega)$ in the number of shares x. Hence, the average stock price $S_t(x)$ is given by

$$S_t(x) = S_t + M_t x \quad \text{for } x \in \mathbb{R}, \tag{2.1}$$

where $(S_t)_{t \geqslant 0}$ and the random slope $(M_t)_{t \geqslant 0}$ are continuous and positive processes. Since $x \mapsto S_t(x)$ is continuous at $x = 0$, there is no bid-ask spread in the limit order book. A stock is considered to be liquid if it can be converted into cash quickly, without having a big impact on the received price. This will be satisfied if the slope of the supply curve is small. By requiring that the process $(M_t)_{t \geqslant 0}$ takes small values and that the price for a marginal trade is positive, we also ensure that the price process $S_t(x)$ is positive (at least with high probability). In the next chapter we show that the linear structure of the supply curve corresponds to a constant density of the limit order book.

We call the linear supply curve (2.1) *the unaffected supply curve*, since a trade of x shares, executed at price $xS_t(x)$, does not influence the price of future trades. In the next chapter we determine the mechanism how the trades of a trader influence future prices.

2.3. The density of the order book

We deduce the density of the limit order book, which corresponds to the linear structure of the unaffected supply curve and use it to identify the price impact of a trade.

The limit order book can be defined by its density function $\rho_t(z)$ which denotes the density of the number of shares that are offered at price z at time t. This means that the number of total shares offered between the prices z_1 and z_2 ($z_1 \leqslant z_2$) is

$$\int_{z_1}^{z_2} p_t(z)dz.$$

If the trader orders a market order of size x, the order starts at the quoted price S_t and therefore $\rho_t(S_t)dz$ shares are obtained at that price. Then the order moves up in the limit order book until the total amount of purchased shares is x. This means that the trader pays a price of z_x for the last $\rho_t(z_x)dz$ shares, where z_x solves the equation

$$\int_{S_t}^{z_x} \rho_t(z)dz = x. \tag{2.2}$$

This way we obtain the total price for a market order of size x. If the trader wants to buy x shares, the total amount he has to pay is

$$\int_{S_t}^{z_x} z\rho_t(z)dz,$$

which is equivalent to $xS_t(x)$. Because of our assumption on the structure of the supply curve, we deduce

$$\int_{S_t}^{z_x} z\rho_t(z)dz = xS_t(x) = xS_t + x^2M_t$$

and therefore

$$z_x\rho_t(z_x)\frac{dz_x}{dx} = S_t + 2xM_t. \tag{2.3}$$

Because of equation (2.2) we know that $\rho_t(z_x)\frac{dz_x}{dx} = 1$. Then,

$$z_x = S_t + 2xM_t$$

and $\frac{dz_x}{dx} = 2M_t$. Equation (2.3) yields $\rho_t(S_t + 2xM_t)2M_t = 1$ for all x. Hence, the density ρ_t is constant

$$\rho_t(z) = \frac{1}{2M_t}. \tag{2.4}$$

Since we think of ρ_t as a measure of liquidity, we may also consider M_t to be a measure of illiquidity. In the following chapters we show that the larger M_t, the higher is the liquidity cost compared to the quoted price S_t.

As mentioned before, whenever a trade occurs in our model, it happens due to a market order and every market order is executed against the existing limit orders, decreasing the liquidity and triggering price impacts. If market orders (i.e. buy) are ordered, then limit orders in the limit order book are executed starting with the cheapest and then continuing with more expensive limit orders until the required number of shares is purchased. Because of the density of the order book, this means that after the market order of size x at time t all limit orders within the price interval $G = [S_t, S_t + 2M_tx]$ are used up. If no additional limit orders were placed, there would be a bid-ask spread of $\Delta p = 2M_tx$ in the limit order book immediately after the order, such that

$$p_{t+}(z) = \begin{cases} 0, & \text{for } z \in [S_t, S_t + 2M_tx] \\ p_t(z), & \text{otherwise.} \end{cases}$$

In that case the lowest ask price would move up to $S_t + 2M_tx$, while the highest bid price would remain the same. What happens to this gap after the order is executed? If the market order is a bid, the number of offered shares is decreasing, whereas the number of people who are interested in buying the stock remains the same. In that case the demand per share is increasing and people are willing to pay a higher price than S_t for the stock. Furthermore, as long as there is a gap between the best ask and bid price and as long as there are no transaction costs, there will be traders who make use of this situation by placing limit orders in the gap. This way it is reasonable to think that the limit order book immediately fills up after a trade. But should the gap be filled by bid or ask orders?

If the limit order book was entirely filled up by bid orders after an market order of size x, the quoted price of the stock would go up to $S_t + 2M_t x$. This would be the biggest price impact that could occur. Otherwise, if the entire gap was filled with ask orders, there would not be any price impact at all. Weber and Rosenow [17] showed that the truth is somewhere in between: The price impact exists, but is less than the full impact. This statement is based on the following observation. There is a negative correlation between price changes and the volume of incoming limit orders, which suggest that market participants respond to a market order by adding new limit orders in the opposite way. If a market-order (buy) is placed the lowest ask price of the asset instantly moves up. Informed traders, who determine the fundamental price of the stock, take advantage of the price impact by selling limit orders at the temporarily disadvantaged price. Therefore we assume that the upper part of the gap in the limit order book disappears immediately after the trade. This phenomenon is called short-term resiliency effect. To model this effect we define a parameter

$$\lambda \in [0, 1]$$

which indicates the percentage of the gap in the limit order book that is filled up with bid orders. Hence, depending on λ the price of the asset after a market order of size x moves up to

$$S_{t+}(x) = S_t + 2\lambda M_t x.$$

We assume that the density level of the order book remains unaffected by the trade.

3. The impact of liquidity

Let us consider a simple economy which consists of a financial asset (typically a stock) and a risk free asset. We use a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{0 \leqslant t \leqslant T}, \mathbb{P})$, which satisfies the usual conditions. For simplicity we assume that the interest rate is constant and we always work with discounted price processes.

We start by defining an endogenously given adapted semimartingale

$$S = (S_t)_{t \geqslant 0}$$

and call it the **unaffected quoted price process** of the stock. We assume that S is the price process that occurs due to actions of all market participants except the single trader who wants to replicate a contingent claim. This process is independent of the trader's strategy. As discussed in the last chapter we assume that the **unaffected average price per share** $S_t(x)$ at time t is a linear function of the order size $x \in \mathbb{R}$ and can be written as

$$S_t(x) = S_t + M_t x$$

where $M = (M_t)_{t \geqslant 0}$ is an adapted semimartingale. $S_t(0) = S_t$ is the (unaffected) quoted price. We want M to be continuous and non-negative and think of it as a measure of illiquidity. M=0 refers to a stock which is infinitely liquid and the stock becomes less liquid if M increases. For simplicity we call M the liquidity process. In addition to S we define a second price process

$$S^X = (S_t^X)_{t \geqslant 0}$$

which is called the **observed quoted price process**. This price process S^X gives the actual observed quoted price including the price impacts of the trader's hedge X. Consequently, S_t^X is dependent on all historic actions of the trader prior to time t. The X in the notation emphasizes the dependence on the traders's strategy. In addition it depends on the process M and the given resiliency paramter $\lambda \in [0, 1]$. Since we assume that the level of liquidity is not affected by X, the **observed average price per share** $S_t^X(x)$ at time t is again given by the linear supply curve

$$S_t^X(x) = S_t^X + M_t x.$$

$S_t^X(0) = S_t^X$ is the observed quoted price. The process $(S_t^X)_{t \geqslant 0}$ and its properties will be derived in Section 3.1.1.

Note: The basic unaffected price process S could as well be interpreted as the fundamental value of a company and the process S^X as the observed price that occures due to all trades on the market. In that case X indicates the total order flow of the stock and the model can be used to explain asset price bubbles. See Jarrow, Protter, Roch [20] in this context. But as the goal of this thesis is to replicate and price contingent claims, we are interested in the impact of the hedge and therefore in the impact that a single trader has on the price process. Hence, the unaffected quoted price process S results from the limit and market orders of all other market participants. We assume that everyone is allowed to place market orders and by doing this every participant has an impact on the price S, which is proportional to the liquidity process M. Then it is reasonable to conclude that the volatility of S is in part correlated to the process M. This is supported by Farmer, Gillemot, Lillo [5]. There it is also shown that fluctuations caused by individual market orders are driven by liquidity fluctuations, variations in the market's ability to absorb new orders. This means that both illiquidity and the size of the transaction have to be taken into account in order to explain price changes. Also have a look at Weber, Rosenow [17] in this context.

Throughout this thesis we use the following definition of a stochastic integral. For a semimartingale X and a predictable X-integrable process Y (e.g. every caglad process) we write $Y \circ X = \int Y dX$ with

$$(Y \circ X)_t = \int_0^t Y dX = \int_{(0,t]} Y dX.$$

Note that this definition is slightly different from the stochastic integral used by Cetin, Jarrow, Protter [22] and Roch [3], who include the 0 and use $\int_{[0,t]} Y dX$. We make use of the following theorems and definitions.

Definition 3.1. *A random partition π is a finite sequence of finite stopping times*

$$0 = \tau_0 \leqslant \tau_1 ... \leqslant \tau_k < \infty.$$

A sequence of random partitions $(\pi_n)_{n \geqslant 1}$ with $\pi_n : \quad 0 = \tau_0^n \leqslant ... \leqslant \tau_{k_n}^n$ is said to tend to identity if $\sup_{k=1,...,k_n} \tau_k^n \to \infty$ and

$$|| \pi_n ||= \sup_{0 \leqslant k \leqslant k_n} | \tau_{k+1}^n - \tau_k^n | \to 0 \tag{3.1}$$

almost surely for $n \to \infty$. A sequence of random partitions $(\pi_n)_{n \geqslant 1}$ of $[0,t]$ with $\pi_n : \quad 0 = \tau_0^n \leqslant ... \leqslant \tau_{k_n}^n \leqslant t$ is said to tend to identity if $\tau_{k_n}^n$ converges to t almost surely and (3.1) is satisfied.

Let $\mathbb{D}$ denote the space of all adapted processes with cadlag paths and $\mathbb{L}$ the space of all adapted processes with caglad paths.

Theorem 3.2. *X is a semimartingale and Y is a process in $\mathbb{D}$ or in $\mathbb{L}$. Let $(\pi_n)_{n\geqslant 1}$ be a sequence of random partitions tending to the identity. Then,*

$$\int Y^{\pi_n} dX = \sum_i Y_{\tau_i^n}\left(X^{\tau_{i+1}^n} - X^{\tau_i^n}\right) \quad \overset{ucp}{\to} \quad \int Y_- dX.$$

Here ucp means uniformly on compact sets in probability and $Y_- = (Y_{t-})_{t\geqslant 0}$.

Proof. See Theorem 21 of Chapter II in Protter [16] ♦

Theorem 3.3. *Y is an adapted cadlag process, X and Z are semimartingales. Let $(\pi_n)_{n\geqslant 1}$ be a sequence of random partitions tending to identity. Then*

$$\sum Y_{\tau_i^n}\left(X^{\tau_{i+1}^n} - X^{\tau_i^n}\right)\left(Z^{\tau_{i+1}^n} - Z^{\tau_i^n}\right) \quad \overset{ucp}{\to} \quad \int Y_- d[X,Z].$$

Proof. See Theorem 30 of Chapter II in Protter [16] ♦

For a cadlag process Y consider the process $\overline{Y}$ with $\overline{Y}_s = Y_s \mathbb{1}_{[0,t)}(s)$. Then for $i = 1, ..., k_n$ and $s \leqslant t$

$$\overline{Y}_{\tau_i^n}\left(X_s^{\tau_{i+1}^n} - X_s^{\tau_i^n}\right) = Y_{\tau_i^n}\left(X_s^{(\tau_{i+1}^n \wedge t)} - X_s^{(\tau_i^n \wedge t)}\right),$$

such as

$$\overline{Y}_{\tau_i^n}\left(X_s^{\tau_{i+1}^n} - X_s^{\tau_i^n}\right)\left(Z_s^{\tau_{i+1}^n} - Z_s^{\tau_i^n}\right) = Y_{\tau_i^n}\left(X_s^{(\tau_{i+1}^n \wedge t)} - X_s^{(\tau_i^n \wedge t)}\right)\left(Z_s^{(\tau_{i+1}^n \wedge t)} - Z_s^{(\tau_i^n \wedge t)}\right).$$

The sequence $(\overline{\pi}_n)_{n\geqslant 1}$ with $\pi_n : 0 = \overline{\tau}_k^n \leqslant ... \leqslant \overline{\tau}_{k_n}^n \leqslant t$ and $\overline{\tau}_k^n = (\tau_k^n \wedge t)$ is a sequence of random partitions of $[0,t]$ tending to identity. Hence, the statements of Theorem 3.2 and Theorem 3.3 remain true for a sequence of random partitions of $[0,t]$ tending to identity. In that case the convergence is uniformly on compact subsets of $[0,t]$ in probability. We write *ucp* on $[0,t]$.

Corollary 3.4. *Let Y be an adapted caglad (or cadlag) process and let X, Z be two semimartingales defined on the intervall $[0,t]$. $(\pi_n)_{n\geqslant 1}$ denotes a sequence of random partitions of $[0,t]$ tending to identity. Then*

$$\sum_i Y_{\tau_i^n}\left(X^{\tau_{i+1}^n} - X^{\tau_i^n}\right) \quad \xrightarrow{ucp} \quad \int Y_- dX \tag{3.2}$$

and

$$\sum Y_{\tau_i^n}\left(X^{\tau_{i+1}^n} - X^{\tau_i^n}\right)\left(Z^{\tau_{i+1}^n} - Z^{\tau_i^n}\right) \quad \xrightarrow{ucp} \quad \int Y_- d[X,Z]. \tag{3.3}$$

3.1. Trading strategies in a model with liquidity risk

3.1.1. The observed stock price

We consider a semimartingale $X = (X_t)_{0 \leqslant t \leqslant T}$ which represents the trading strategy in the risky asset. X_t denotes the number of shares hold by the trader after the trade at time t. In contrast to the classical arbitrage theory, where a trading strategy is a predictable process, we assume that X is cadlag. It can therefore be used as an integrator for stochastic integrals. Moreover, we assume that the trader already holds X_{0-} shares of stocks before time 0, where X_{0-} is a $\mathcal{F}_0$-measureable random variable.

Let $\pi_n : 0 = \tau_0^n \leqslant ... \leqslant \tau_{k_n}^n$ be a random partition of $[0,T]$ tending to identity. For a process P we define operators

$$\Delta_k^n : \qquad \Delta_k^n P = P_{\tau_k^n} - P_{\tau_{k-1}^n}$$

for $k = 1, ..., k_n$ and $\Delta_0^n P = \Delta P_0 = P_0 - P_{0-}$. X can be approximated by a simple cadlag process X^{π_n}, which is equal to $X_{\tau_k^n}$ on $[\tau_k^n, \tau_{k+1}^n)$,

$$X_t^{\pi_n} = X_{0-} + \sum_{k=0}^{k_n} \Delta_k^n X \mathbb{1}_{[\tau_k^n, \infty)}(t).$$

We trade the stock according to the strategy X^{π_n} and are interested in the observed quoted price at time $t \leqslant T$. At every stopping time $\tau_i^n \leqslant t$, we buy $\Delta_i^n X$ shares of the stock, causing a price impact of $2\lambda M_{\tau_i^n} \Delta_i^n X$ at time $\tau_i^n +$. We sum up all price impacts until time $t \in (\tau_{k-1}^n, \tau_k^n]$ and get the observed quoted price associated to the strategy X^{π_n}

$$S_t^{\pi_n} = S_t + 2\lambda \sum_{i=0}^{k-1} M_{\tau_i^n} \Delta_i^n X. \tag{3.4}$$

Let $t = \tau_k^n$, where $k \leqslant k_n$. We add the price impact of the trade at time t to obtain

$$\begin{aligned} S_{t+}^{X^{\pi_n}} &= S_t + 2\lambda \sum_{i=0}^{k} M_{\tau_i^n} \Delta_i^n X \\ &= S_t + 2\lambda \sum_{i=0}^{k-1} M_{\tau_i^n} \Delta_i^n X + \left(2\lambda \sum_{i=1}^{k} M_{\tau_{i-1}^n} \Delta_i^n X - 2\lambda \sum_{i=1}^{k} M_{\tau_{i-1}^n} \Delta_i^n X \right) \\ &= S_t + 2\lambda M_0 \Delta_0^n X + 2\lambda \sum_{i=1}^{k} M_{\tau_{i-1}}^n \Delta_i^n X + 2\lambda \sum_{i=1}^{k} (\Delta_i^n M)(\Delta_i^n X). \end{aligned} \tag{3.5}$$

By (3.2) and (3.3) this converges for $n \to \infty$ in *ucp* to the semimartingale and hence right continuous process

$$\begin{aligned} S_{t+}^X &= S_t + 2\lambda \left(M_0 \Delta_0^n X + \int_0^t M_{u-} dX_u + \int_0^t d\left[M, X\right]_u \right) \\ &= S_t + 2\lambda \left(X_t M_t - \int_0^t X_{u-} dM_u - X_{0-} M_0 \right). \end{aligned} \tag{3.6}$$

In the last step we use integration by parts. This defines the cadlag process $(S_{t+}^X)_{0 \leqslant t \leqslant T}$. Furthermore, we obtain the observed quoted price by subtracting the price impact at time t.

Definition 3.5. *For $0 \leqslant t \leqslant T$ the* ***observed quoted price*** *is*

$$S_t^X = S_{t+}^X - 2\lambda M_t \Delta X_t, \tag{3.7}$$

where S_{t+}^X is given by (3.6) and $\Delta X_t = X_t - X_{t-}$.

Note that both $(S_{t+}^X)_{t \leqslant T}$ and $(S_t^X)_{t \leqslant T}$ depend on the strategy X, the process M and the resiliency parameter λ. As discussed in Chapter 2, we assume that the level of liquidity is not affected by any trades.

Definition 3.6. *For $0 < t \leqslant T$ and $x \in \mathbb{R}$ the* ***observed average price per share*** *is*

$$S_t^X(x) = S_t^X + M_t x, \tag{3.8}$$

with S_t^X by (3.7).

Remember that λ indicates the part of the order book which is renewed with bid orders. Let us define the price impact of a trading strategy $X = (X_t)_{t \geqslant 0}$ at time t as the difference between the observed quoted price and the unaffected quoted price,

$$S_t^X - S_t = 2\lambda \left(X_{t-} M_t - \int_0^t X_{u-} dM_u - X_{0-} M_0 \right).$$

The full price impact occurs in case $\lambda = 1$. For $\lambda \in [0, 1)$ the impact is λ times the full impact.

The effect of trading the stock with the strategy X can be summarized as follows. At time $t \in [0, T]$ we observe the price process S_t^X and buy ΔX_t shares of the stock for an average price per share $S_t^X(\Delta X_t) = S_t^X + M_t \Delta X_t$ (hence a total of $\Delta X_t(S_t^X + M_t \Delta X_t)$). At time $t+$ the observed quoted stock price jumps to $S_{t+}^X = S_t^X + 2\lambda M_t \Delta X_t$.

3.1.2. Self financing trading strategies

A trading strategy (X,Y) consists of a semimartingale $X=(X_t)_{0\leqslant t\leqslant T}$ and an adapted cadlag process $Y=(Y_t)_{0\leqslant t\leqslant T}$. At every moment t we have a portfolio (X_t,Y_t), which means that after the trade at time t the hedger holds X_t shares in the stock and Y_t in the (discounted) risk free asset.

In financial mathematics a trading strategy is called self financing if the value of the portfolio is obtained via merely trading the stock and the risk free asset without adding or removing any money. If the trader owns a portfolio (X_{t_1},Y_{t_1}) at time t_1, he does not trade until time $t_2>t_1$ and then changes the portfolio to (X_{t_2},Y_{t_2}) without additional costs, the value of the portfolio must satisfy

$$Y_{t_1}+X_{t_1}S^X_{t_2}(X_{t_2}-X_{t_1})=Y_{t_2}+X_{t_2}S^X_{t_2}(X_{t_2}-X_{t_1})$$

or equivalently

$$Y_{t_2}-Y_{t_1}+(X_{t_2}-X_{t_1})S^X_{t_2}(X_{t_2}-X_{t_1})=0.$$

Similarly at time 0 a self-financing strategy satisfies $\Delta Y_0+\Delta X_0 S^X_0(\Delta X_0)=0$. To define self-financing strategies we first define the (discounted) total costs $R^{(X,Y)}_t$ of a trading strategy (X,Y) at time t. To do this we consider a sequence $(\pi_n)_{n\geqslant 1}$ of random partitions $0=\tau^n_0\leqslant\ldots\leqslant\tau^n_{k_n}=t$ of $[0,t]$ and the associated discrete trading strategies (X^{π_n},Y^{π_n}), which are equal to $(X_{\tau^n_k},Y_{\tau^n_k})$ on $[\tau^n_k,\tau^n_{k+1})$ and $X^{\pi_n}_{0-}=X_{0-}$. The total costs of (X^{π_n},Y^{π_n}) at time t are

$$\begin{aligned}R_t(X^{\pi_n},Y^{\pi_n})&=Y_0-Y_{0-}+\Delta^n_0X^{\pi_n}S^{X^{\pi_n}}_0(\Delta^n_0X^{\pi_n})\\&\quad+\sum_{k=1}^{k_n}Y_{\tau^n_k}-Y_{\tau^n_{k-1}}+\left(X_{\tau^n_k}-X_{\tau^n_{k-1}}\right)\left(S^{X^{\pi_n}}_{\tau^n_k}+\left(X_{\tau^n_k}-X_{\tau^n_{k-1}}\right)M_{\tau^n_k}\right)\\&=Y_t-Y_{0-}+\Delta^n_0XS^X_0(\Delta^n_0X)+\sum_{k=1}^{k_n}\Delta^n_kX\left(S^{X^{\pi_n}}_{\tau^n_k}+\Delta^n_kXM_{\tau^n_k}\right).\end{aligned}\tag{3.9}$$

We define the total costs of a trading stragegy (X,Y) as the limit of the total costs of the associated discrete strategy (X^{π_n},Y^{π_n}).

Definition 3.7. *The **total costs** $R_t(X,Y)$ of the trading strategy (X,Y) at time t are*

$$\begin{aligned}R_t(X,Y)=&Y_t+X_t\left(S^X_{t+}-\lambda M_tX_t\right)-Y_0-X_0\left(S^X_{0+}-\lambda M_0X_0\right)+\Delta Y_0+\Delta X_0S^X_0(\Delta X_0)\\&-\int_0^tX_{u-}dS_u+\lambda\int_0^tX^2_{u-}dM_u+(1-\lambda)\int_0^tM_{u-}d[X,X]_u\,.\end{aligned}$$

Proposition 3.8. *Let (X,Y) be a trading strategy and $(\pi_n)_{n\geqslant 1}$ a sequence of random partitions of $[0,t]$ tending to identity, then*

$$R_t(X^{\pi_n},Y^{\pi_n}) \quad \xrightarrow{ucp} \quad R_t(X,Y).$$

Proof. Let π_n: $0=\tau_0^n \leqslant \tau_1^n \leqslant \quad ... \quad \leqslant \tau_{k_n}^n = t$ be a random partition . Then

$$\begin{aligned}
& \sum_{k=1}^{k_n} \Delta_k^n X (S_{\tau_k^n}^{X^{\pi_n}} + M_{\tau_k^n}\Delta_k^n X) && (3.10)\\
= \quad & \sum_{k=1}^{k_n} \left(X_{\tau_k^n} S_{\tau_k^n}^{X^{\pi_n}} - X_{\tau_{k-1}^n}\left(S_{\tau_k^n}^{X^{\pi_n}} - S_{\tau_{k-1}^n}^{X^{\pi_n}} + S_{\tau_{k-1}^n}^{X^{\pi_n}}\right)\right) + \sum_{k=1}^{k_n} M_{\tau_k^n}(\Delta_k^n X)^2 \\
= \quad & \sum_{k=1}^{k_n} \left(X_{\tau_k^n} S_{\tau_k^n}^{X^{\pi_n}} - X_{\tau_{k-1}^n} S_{\tau_{k-1}^n}^{X^{\pi_n}}\right) - \sum_{k=1}^{k_n} X_{\tau_{k-1}^n}\Delta_k^n S^{X^{\pi_n}} + \sum_{k=1}^{k_n} M_{\tau_k^n}(\Delta_k^n X)^2 \\
\overset{(3.5)}{=} \quad & X_t S_t^X - X_0 S_0^X - \sum_{k=1}^{k_n}\left(X_{\tau_{k-1}^n}\Delta_k^n S + 2\lambda M_{\tau_{k-1}^n} X_{\tau_{k-1}^n}\Delta_{k-1}^n X\right) \\
& + \sum_{k=1}^{k_n} M_{\tau_k^n}(\Delta_k^n X)^2 \\
= \quad & X_t S_t^X - X_0 S_0^X - 2\lambda M_0 X_0 \Delta X_0 + 2\lambda M_t X_t \Delta X_t - \sum_{k=1}^{k_n} X_{\tau_{k-1}^n}\Delta_k^n S \\
& - 2\sum_{k=1}^{k_n} \lambda M_{\tau_k^n} X_{\tau_k^n}\Delta_k^n X + \sum_{k=1}^{k_n}\lambda M_{\tau_k^n}(\Delta_k^n X)^2 + \sum_{k=1}^{k_n}(1-\lambda) M_{\tau_k^n}(\Delta_k^n X)^2 \\
= \quad & X_t S_{t+}^X - X_0 S_{0+}^X - \sum_{k=1}^{k_n} X_{\tau_{k-1}^n}\Delta_k^n S - \sum_{k=1}^{k_n}\lambda M_{\tau_k^n}\Delta_k^n X^2 + \sum_{k=1}^{k_n}(1-\lambda) M_{\tau_k^n}(\Delta_k^n X)^2 \\
= \quad & X_t S_{t+}^X - X_0 S_{0+}^X - \lambda M_t X_t^2 + \lambda M_0 X_0^2 \\
& - \sum_{k=1}^{k_n} X_{\tau_{k-1}^n}\Delta_k^n S + \sum_{k=1}^{k_n}\lambda X_{\tau_{k-1}^n}^2 \Delta_k^n M + \sum_{k=1}^{k_n}(1-\lambda) M_{\tau_k^n}(\Delta_k^n X)^2.
\end{aligned}$$

By (3.2) we get

$$\begin{aligned}
\sum_{k=1}^{k_n} X_{\tau_{k-1}^n}\Delta_k^n S \quad &\xrightarrow{ucp} \quad \int_0^t X_{u-}dS_u, \\
\sum_{k=1}^{k_n} \lambda X_{\tau_{k-1}^n}^2\Delta_k^n M \quad &\xrightarrow{ucp} \quad \lambda\int_0^t X_{u-}^2 dM_u.
\end{aligned}$$

Furthermore,

$$\sum_{k=1}^{k_n} M_{\tau_k^n}(\Delta_k^n X)^2 = \sum_{k=1}^{k_n} M_{\tau_{k-1}^n}(\Delta_k^n X)^2 + \sum_{k=1}^{k_n}(\Delta_k^n M)(\Delta_k^n X)^2.$$

By (3.3) the first term converges in *ucp* to $\int_0^t M_{u-}d[X,X]_u$. Whereas

$$\left|\sum_{i=1}^{k_n}(\Delta_i^n M)(\Delta_i^n X)^2\right| \leqslant \sup_i |\Delta_i^n M| \sum_{i=1}^{k_n}(\Delta_i^n X)^2.$$

Because M is continuous this converges to zero for $n \to \infty$. Hence, for $n \to \infty$ the sum (3.10) converges to

$$\begin{aligned}&X_t\left(S_{t+}^X - \lambda M_t X_t\right) - X_0\left(S_{0+}^X - \lambda M_0 X_0\right) - \int_0^t X_{u-}dS_u \\ &+ \int_0^t X_{u-}^2 dM_u + \int_0^t M_{u-}d[X,X]_u.\end{aligned}$$

Furthermore,

$$\begin{aligned}&\Delta X_0 S_0^X(\Delta X_0) - Y_{0-} - X_0\left(S_{0+}^X - \lambda M_0 X_0\right) \\ &= -Y_0 - X_0\left(S_{0+}^X - \lambda M_0 X_0\right) + \Delta Y_0 + \Delta X_0 S_0^X(\Delta X_0)\end{aligned}$$

Substituting these results in (3.9) yields the statement.

♦

Definition 3.9. *A trading strategy (X,Y) is **self-financing** (s.f.t.s.) on $[0,T]$ if*

$$R_t(X,Y) = 0 \quad \text{for all } t \in [0,T].$$

A s.f.t.s. satisfies $R_0(X,Y) = 0$ or equivalently $\Delta Y_0 + \Delta X_0 S_0^X(\Delta X_0) = 0$. This term is equal to the costs of transforming the portfolio (X_{0-}, Y_{0-}) to (X_0, Y_0). Hence, the initial value in the cash account is $Y_{0-} = Y_0 + \Delta X_0 S_0^X(\Delta X_0)$.

Definition 3.10. *For a trading strategy (X, Y) let us define the process $V^M(X,Y)$ by*

$$V_t^M = V_t^M(X,Y) = Y_t + X_t\left(S_{t+}^X - \lambda M_t X_t\right).$$

As will be discussed in Section 3.1.4 the value V_t^M is equal to the asymptotically realizeable wealth of the portfolio (Y_t, X_t). This means if we try to liquidate the shares of the stock until time $t+$ the value we obtain is smaller or equal to the realizeable wealth. We make use of this notation to obtain the following representation of self-financing strategies.

Corollary 3.11. *A trading strategy (X,Y) is self-financing on $[0,T]$ if and only if*

$$Y_0 = Y_{0-} - \Delta X_0 S_0^X(\Delta X_0)$$

and for $t \in (0,T]$

$$V_t^M = V_0^M + \int_0^t X_{u-}dS_u - \lambda\int_0^t X_{u-}^2 dM_u - (1-\lambda)\int_0^t M_{u-}d\left[X,X\right]_u.$$

Proof. This follows from Definition 3.9. ♦

When it comes to the replicating of contingent claims, we assume that the trader does not hold any stock shares before time 0 and hence only consider strategies with $X_{0-} = 0$. This guarantees that no price impacts where triggered before time 0 and therefore $S_0^X = S_0$. Then, $Y_{0-} = Y_0 + X_0 S_0(X_0)$ is the initial value of the trading strategy (X, Y) and

$$V_0^M = Y_0 + X_0(S_0^X + \lambda M_0 X_0) = Y_{0-} - (1-\lambda) M_0 X_0^2.$$

By Corollary 3.11 if $X_{0-} = 0$ a s.f.t.s. satisfies

$$\begin{aligned} Y_t + X_t \left(S_{t+}^X - \lambda M_t X_t\right) = Y_{0-} &+ \int_0^t X_{u-} dS_u - \lambda \int_0^t X_{u-}^2 dM_u \\ &- (1-\lambda) \left(\int_0^t M_u d\left[X, X\right]_u + M_0 X_0^2 \right). \end{aligned} \tag{3.11}$$

Furthermore, if all stock shares have been liquidated at time T, e.g. $X_T = 0$, the realizeable wealth is equal to the value of the risk free asset $V_T^M = Y_T$. By (3.11) a s.f.t.s. with $X_{0-} = X_T = 0$ satisfies

$$Y_T = Y_{0-} + \int_0^T X_{u-} dS_u - \lambda \int_0^T X_{u-}^2 dM_u - (1-\lambda) \left(\int_0^T M_u d\left[X, X\right]_u + M_0 X_0^2 \right). \tag{3.12}$$

Remark. A s.f.t.s. with $X_{0-} = 0$ is determined by the strategy in the risky asset $(X_t)_{0 \leqslant t \leqslant T}$ and the initial value in the cash account Y_{0-}. The process $(Y_t)_{0 \leqslant t \leqslant T}$ is given by (3.11).

3.1.3. Liquidity costs of s.f.t.s.

In addition to V^M we define the following processes.

Definition 3.12. *For a trading strategy (X, Y) we denote by $V^C(X, Y)$ the value of the portfolio (X_t, Y_t) at time t in an infinitely liquid market (classical theory)*

$$V_t^C = V_t^C(X, Y) = Y_t + X_t S_t, \qquad 0 \leqslant t \leqslant T.$$

Definition 3.13. *For a trading strategy (X, Y) let $V^L(X, Y)$ be the value of the portfolio (X_t, Y_t) at time t if it is immediately liquidated*

$$V_t^L = V_t^L(X, Y) = Y_t + X_t S_t^X, \qquad 0 \leqslant t \leqslant T.$$

Remark. In an infinitely liquid market a s.f.t.s. $(X, \widehat{Y})$ satisfies

$$V_t^C(X, \widehat{Y}) - V_s^C(X, \widehat{Y}) = \int_s^t X_{u-} dS_u. \tag{3.13}$$

We refer to liquidity cost as additional cost a trader has to pay because the traded assets are not infinitely liquid.

Definition 3.14. *The **liquidity costs** L_t of a s.f.t.s. (X, Y) are given as the difference between the value $V_t^C(X_t, \widehat{Y}_t)$, where $(X, \widehat{Y})$ is a self financing trading strategy in an infinitely liquid market (M=0), and the value $V_t^C(X, Y)$. By (3.13) for $t \in [0, T]$*

$$L_t = L_t(X, Y) = Y_{0-} + \int_0^t X_{u-} dS_u - V_t^C(X, Y).$$

From the definition of L_t it follows that

$$\left(V_t^C + L_t\right) - \left(V_s^C + L_s\right) = \int_s^t X_{u-} dS_u, \qquad 0 \leqslant s \leqslant t \leqslant T.$$

Note that this is consistent with the definition of a s.f.t.s in classical arbitrage theory if there are no liquidity costs.

Proposition 3.15. *The liquidity costs of a s.f.t.s (X, Y) with $X_{0-} = 0$ are*

$$L_t = \lambda \left(\int_0^t (X_t - X_{u-})^2 \, dM_u + M_0 X_t^2 \right) + (1 - \lambda) \left(\int_0^t M_{u-} d\left[X, X\right]_u + M_0 X_0^2 \right).$$

Proof. By (3.11) with $X_{0-} = 0$,

$$\begin{aligned} Y_t + X_t \left(S_{t+}^X - \lambda M_t X_t\right) = Y_{0-} + \int_0^t X_{u-} dS_u - \lambda \int_0^t X_{u-}^2 dM_u \\ - (1 - \lambda) \left(\int_0^t M_{u-} d\left[X, X\right]_u + M_0 X_0^2 \right). \end{aligned}$$

Using (3.6) we obtain

$$\begin{aligned} Y_t + X_t S_t + \lambda \left((M_t - M_0) X_t^2 - 2 X_t \int_0^t X_{u-} dM_u + \int_0^t (X_{u-})^2 dM_u \right) + \lambda M_0 X_t^2 \\ + (1 - \lambda) \left(\int_0^t M_{u-} d\left[X, X\right]_u + M_0 X_0^2 \right) = Y_{0-} + \int_0^t X_{u-} dS_u \end{aligned}$$

or equivalently

$$\begin{aligned} \lambda \left(\int_0^t (X_t - X_{u-})^2 \, dM_u + M_0 X_t^2 \right) + (1 - \lambda) \left(\int_0^t M_{u-} d\left[X, X\right]_u + M_0 X_0^2 \right) \\ = Y_{0-} + \int_0^t X_{u-} dS_u - V_t^C(X, Y) = L_t. \end{aligned}$$

♦

While the second term of these costs, which is due to the quadratic variation of X ,

$$(1-\lambda)\left(\int_0^t M_{u-}d\left[X,X\right]_u + M_0 X_0^2\right)$$

is positiv, we can not make any statement about the sign of the term

$$\lambda\left(\int_0^t (X_t - X_{u-})^2\, dM_u + M_0 X_t^2\right).$$

Note that the investor always has a certain benefit if the process M decreases, which means that the stock becomes more liquid. It might happen that the liquidity costs are negative in total. In that case the investor benefits from the stock being not perfectly liquid.

The liquidity costs are not linear. To see this, consider a s.f.t.s. (X,Y). For a constant $a>0$ we build a trading strategy $(aX_t)_{t\leqslant T}$, such that for every time $0\leqslant t\leqslant T$ we invest aX_t shares in the risky asset. This leads to the following liquidity costs

$$\begin{aligned}
L_t(aX) &= \lambda\left(\int_0^t (aX_t - aX_{s-})^2\, dM_s + a^2 M_0 X_t^2\right)\\
&\quad + (1-\lambda)\left(\int_0^t M_{s-}d\left[aX,aX\right]_s + a^2 M_0 X_0^2\right)\\
&= a^2 L_t(X).
\end{aligned}$$

In this sense the additional costs due to liquidity are quadratic in the number of stock shares we purchase. An informed trader, could exploit this situation. To illustrate this think of an investor, who buys x shares at time s. The price of the stock goes up higher after his purchase ($S_{s+} = S_s^X + 2\lambda M_s x$). Now the trader waits until time t when the conditions are good to sell the shares. Assuming that the trader does not place any orders within the time (s,t) this leads to a benefit of

$$x(S_t - S_s) + x^2(2\lambda M_s - M_t - M_s).$$

If the trader is patient enough to wait until $M_t \leqslant M_s(2\lambda - 1)$, the benefit of an investment is of quadratic growth in the number of purchased shares. This suggests that large traders who are willing to buy big amounts of shares can manipulate the market and manage to obtain a proportional higher benefit. To make sure this does not happen, we want M to be a submartingale. Then, $\mathbb{E}[M_t \mid \mathcal{F}_s] \geqslant M_s \geqslant M_s(2\lambda-1)$. In Section 3.2 we show that this condition is sufficient to rule out arbitrage opportunities.

Example: *Liquidity costs in a model with constant liquidity*

We assume that the liquidity is constant,

$$M_t = M, \qquad 0\leqslant t\leqslant T.$$

By (3.7) with $X_{0-} = 0$ the observed quoted price is $S_t^X = S_t + 2\lambda M X_{t-}$ and by (3.8) the average price for ΔX_t shares is

$$S_t^X(\Delta X_t) = S_t + MX_{t-}(2\lambda - 1) + MX_t. \tag{3.14}$$

A trading strategy (X, Y) with $X_{0-} = 0$ is self-financing if and only if

$$(V_t^C + L_t) - (V_0^C + L_0) = \int_0^t X_{u-} dS_u, \qquad 0 \leqslant t \leqslant T.$$

The corresponding liquidity costs are

$$L_t = \lambda M X_t^2 + (1 - \lambda)\left(M[X, X]_t + X_0^2\right). \tag{3.15}$$

We observe that in this model the liquidity costs at every time t are positiv,

$$L_t \geqslant 0.$$

Note that any self-financing trading strategy with $X_0 = 0$ which is continuous and of finite variation and liquidates all stock shares at maturity T, e.g. $X_T = 0$, has final liquidity costs $L_T = 0$.

3.1.4. The cost of turning around a position

As stated by A. Kyle [1] a liquidity model should deal with the following dimensions:

- Resiliency: The rate at which prices bounce back from a shock.
- Depth: The size of order flow required to change prices a given amount.
- Tightness: The cost of turning around a position over a short period of time.

This liquidity-risk model of A. Roch deals with all of these dimensions. The (short-term) resiliency is given by $(1 - \lambda)$. The depth is defined as the size of the order flow, which is required to change the price by one unit. This means the price impact that occurs immediately after a trade has to be $2\lambda M_t X_t = 1$ and the depth therefore is $\frac{1}{2\lambda M_t}$.

The tightness is related to the best trading strategy of the model. Later on we show that continuous trading strategies with finite variation play a key role when it comes to the replication of contingent claims. As a motivation, have a look at a trader who wants to purchase x shares of a stock within the time $[t, t + \Delta t]$. If the trader decides to buy x shares immediately at time t the costs are

$$S_t^X(x) = S_t^X x + M_t x^2.$$

The resulting cost due to liquidity risk are less if the trader places n smaller orders at time $t + i\frac{\Delta t}{n}$ $(i = 1, ..., n)$ with size of $\frac{1}{n}X$ shares per order instead. The processes M and S are continuous and for Δt small $M_t \approx M_{t+\Delta t}$ and the price

$$S^X_{t+i\frac{\Delta t}{n}}(\frac{x}{n}) \approx S_t + (i-1)2\lambda M_t\frac{x}{n} + M_t\frac{x}{n}.$$

Then the total cost for this series of trades is

$$\begin{aligned}
&\sum_{i=1}^{n}\frac{x}{n}\left(S_t + (i-1)2\lambda M_t\frac{x}{n} + M_t\frac{x}{n}\right)\\
&= xS_t + \frac{n(n-1)}{2n^2}2\lambda M_t x^2 + M_t\frac{x^2}{n}\\
&\longrightarrow xS_t + \lambda M_t x^2 \quad \text{ for } n \to \infty.
\end{aligned}$$

As a consequence the trader is better off if he splits the trade into smaller parts. This remains true if we want to sell the shares. The trader keeps a portfolio (X_t, Y_t) at time t and he or she wants to liquidate it immediately after time t the value is $Y_t + X_t(S^X_{t+} - M_t X_t)$. But if the trader splits the liquidation process into several trades of small size the maximum value that can be obtained at time $t+$ is $V^M_t = Y_t + X_t(S^X_{t+} - \lambda M_t X_t)$. Because $S^X_{t+} - 2\lambda M_t x$ is the quoted stock price if x shares are liquidated after the last trade at time t, the limit of the proceeds V^M_t may be represented as

$$Y_t + X_t(S^X_{t+} - \lambda M_t X_t) = Y_t + \int_0^{X_t} S^X_{t+} - 2\lambda M_t x\,dx.$$

This value is called **asymptotically realizeable wealth** and is higher than the immediate liquidation value. See Bank and Baum [2] in this context.

No matter whether the position is long or short, the trader is better off if he splits the trades into smaller parts. This way we may assume that the best way of trading is to use continuous trading strategies of finite variation. We take up this idea in Chapter 4.

3.2. Arbitrage-Theory

In a real economy the price of a product is determined by its demand and supply. As a consequence arbitrage opportunities vanish within little time. An arbitrage opportunity is a trading strategy which leads to a benefit without taking any risk. Of course we also want to eliminate these opportunities in the liquidity risk model. We therefore assign our market with trading restrictions, such that only specific trading strategies are allowed which rule out arbitrage.

A benefit without taking any risk, means that the trader uses a s.f.t.s such that the portfolio at time t converted into cash is bigger than the initial value of the cash account. The benefit is measured in cash. We therefore only consider trading strategies where all stock shares have been liquidated before or at time T or equivalently strategies with

$$X_T = 0.$$

Hence,

$$V_T^L(X,Y) = V_T^C(X,Y) = V_T^M(X,Y) = Y_T.$$

Similar to the definition of a *Free lunch with vanishing risk* in Delbaen and Schachermayer [6], we also want to eliminate sequences of trading strategies which cause a benefit with asymptotically vanishing risk, which we refer to as asymptotic arbitrage.

Definition 3.16. (Arbitrage)

- *A self-financing trading strategy (X, Y) with $Y_{0-} = X_{0-} = 0$ and $X_T = 0$ is called* ***arbitrage*** *if*

$$\mathbb{P}(Y_T \geqslant 0) = 1 \quad \text{and} \quad \mathbb{P}(Y_T > 0) > 0.$$

- *A sequence of self-financing trading strategies $((X^n, Y^n))_{n \geqslant 0}$ with $Y^n_{0-} = X^n_{0-} = 0$ and $X^n_T = 0$ is called* ***asymptotic arbitrage*** *if*

$$Y_T^n \xrightarrow{L^1} H_T$$

with convergence in L^1, where

$$\mathbb{P}(H_T \geqslant 0) = 1 \quad \text{and} \quad \mathbb{P}(H_T > 0) > 0.$$

If there are no (asymptotic) arbitrage opportunities in a set of trading strategies, we call this set ***free of (asymptotic) arbitrage****.*

We show that the following assumption are sufficient to rule out arbitrage opportunities.

Assumption 3.17. From now on we assume that there exists an equivalent measure $\mathbb{Q} \sim \mathbb{P}$ such that

(M1) the unaffected price process $(S_t)_{t \geqslant 0}$ is a $\mathbb{Q}$-local martingale,

(M2) the illiquidity process M is a $\mathbb{Q}$-local submartingale.

By Doob-Meyer decomposition for a a $\mathbb{Q}$-local submartingale M there exists a local martingale $\overline{M}$ and a predictable increasing process $\overline{A}$ with $A_0 = 0$, such that

$$M = \overline{M} + \overline{A}.$$

Using the following definition of admissible trading strategies, it turns out that the set of admissible trading strategies is *free of arbitrage* and even *free of asymptotic arbitrage.*

Definition 3.18. *Let $\mathbb{Q}$ be an equivalent martingale measure which satisfies the assumption 3.17 and let $M = \overline{M} + \overline{A}$ be the Doob-Meyer decomposition of the $\mathbb{Q}$-local submartingale M. A trading strategy (X, Y) is called **admissible** (with respect to $\mathbb{Q}$) if the process*

$$U_t = U_t(X) = \int_0^t X_{u-} dS_u - \lambda \int_0^t X_{u-}^2 d\overline{M}_u, \quad 0 \leqslant t \leqslant T,$$

is a $\mathbb{Q}$-Supermartingal. A sequence $((X^n, Y^n))_{n \geqslant 1}$ is called admissible if (X^n, Y^n) is an admissible trading strategy for every n.

The following lemma shows that this definition of admissible trading strategies is consistent with the definition in classical theory, where the stock is infinitely liquid ($M = 0$). In classical arbitrage theory an admissible trading strategy X satisfies

$$\int_0^t X_{u-} dS_u \geqslant -K$$

for some $K \geqslant 0$. The payoff in classical theory is bounded from below, which suggests that not every amount of money can be borrowed.

Lemma 3.19. *A trading strategy (X, Y) is admissible, if the process U is bounded from below.*

Proof. From the general theory of stochastic integrals we know that U is a $\mathbb{Q}$-local martingale. Every local martingale which is bounded from below is a supermartingale. ♦

Theorem 3.20. *Given assumption 3.17, there exists*

- *no admissible arbitrage opportunity,*
- *no admissible asymptotic arbitrage opportunity.*

Proof. Let us consider an admissible s.f.t.s (X,Y). Because $Y_{0-} = 0$ and $X_T = 0$ by (3.12) its payoff at time T can be written as

$$Y_T = \int_0^T X_{u-} dS_u - \lambda \int_0^T X_{u-}^2 dM_u - (1-\lambda)\left(\int_0^T M_{u-} d[X,X]_u + M_0 X_0^2\right).$$

This yields

$$\begin{aligned} U_T &= \int_0^T X_{u-} dS_u - \lambda \int_0^T X_{u-}^2 d\overline{M}_u \\ &= Y_T + \lambda \int_0^T X_{u-}^2 d\overline{A}_u + (1-\lambda)\left(\int_0^T M_{u-} d[X,X]_u + M_0 X_0^2\right). \end{aligned}$$

Because (X,Y) is admissible, U is a $\mathbb{Q}$-Supermartingale and consequently $\mathbb{E}^{\mathbb{Q}}[U_T] \leqslant U_0 = 0$. Therefore,

$$\mathbb{E}^{\mathbb{Q}}[Y_T] = \mathbb{E}^{\mathbb{Q}}\Bigg[U_T - \underbrace{\lambda \int_0^T X_{u-}^2 d\overline{A}_u}_{\geqslant 0} - \underbrace{(1-\lambda)\left(\int_0^T M_{u-} d[X,X]_u + M_0 X_0^2\right)}_{\geqslant 0}\Bigg] \leqslant 0$$

Since $\mathbb{Q}(Y_T \geqslant 0) = \mathbb{P}(Y_T \geqslant 0) = 1$, we know that

$$\mathbb{P}(Y_T = 0) = \mathbb{Q}(Y_T = 0) = 1.$$

Therefore (X,Y) is not an arbitrage. For the second part let $((X^n, Y^n))_{n \geqslant 1}$ be a sequence of self-financing admissible trading strategies with $Y_{0-}^n = 0$ and $X_T^n = 0$ for every n. Similar to the first part of the proof we get

$$\mathbb{E}^{\mathbb{Q}}[Y_T^n] \leqslant 0.$$

If $Y_T^n \to H_T$ in L^1, it follows that

$$E^{\mathbb{Q}}[Y_T^n] \to E^{\mathbb{Q}}[H_T] \leqslant 0$$

where $H_T \geqslant 0$ $\mathbb{Q}$-a.s.. We conclude

$$\mathbb{P}(H_T = 0) = \mathbb{Q}(H_T = 0) = 1,$$

which contradicts asymptotic arbitrage.

♦

3.3. The approximate replication of a contingent claim

We are interested in the impact of liquidity of the underlying asset on the price of contingent claims. We therefore try to replicate a contingent claim by trading the basic assets of our economy.

Definition 3.21. *A* ***contingent claim*** *is a F_T-measureable random variable H which is integrable with respect to $\mathbb{Q}$,*

$$\mathbb{E}^{\mathbb{Q}}\left[|H|\right] < \infty.$$

We want the value of the trading strategy at time T to be equal to the payoff of the claim H.

Definition 3.22. *The* ***exact replication*** *problem of a contingent claim H consists of finding an admissible s.f.t.s. (X, Y) with $X_{0-} = 0$ which satisfies*

$$V_T^L(X, Y) = H \quad \mathbb{P}\text{-a.s.}.$$

Depending on the delivery of a claim (cash or physical), we have to impose requirements on X_T and Y_T, which lead to different replication problems. In the following definition of an approximating sequence of trading strategies we require $X_T^n = 0$, which corresponds to a cash delivery problem. In Section 3.3.1 we justify this assumption.

Definition 3.23. *A claim H is* ***approximately replicable****, if there exists an admissible sequence of s.f.t.s. $((X^n, Y^n))_{n\geqslant 1}$, with $X_{0-}^n = X_T^n = 0$ for every n, which satisfies*

$$V_T^L(X^n, Y^n) = Y_T^n \xrightarrow{L^1} H.$$

The sequence $((X^n, Y^n))_{n\geqslant 1}$ is called ***(admissible) approximating sequence****. Denote by $\Phi^a(H)$ the set of all (admissible) approximating sequences for H.*

Definition 3.24. *A market is called* ***approximately complete****, if every $\mathbb{Q}$-integrable contingent claim H is approximately replicable.*

We can not show that every claim H can be replicated, but it turns out that every $\mathbb{Q}$-integrable claim can be approximately replicated using a sequence of s.f.t.s. (see Chapter 4).

Remark. One would like to consider claims which are functions of the observed quoted stock price at maturity T, $H = h(S_T^X)$. Note that in our model the observed quoted stock price itself depends on the trading strategy $X = (X_t)_{0\leqslant t\leqslant T}$. As a consequence the true replication of H is an implicit problem. We do not consider such implicit problems. As an approximation we deal with claims of the form

$$H = h(S_T^{\widehat{X}})$$

where $\widehat{X} = (\widehat{X}_t)_{0\leqslant t\leqslant T}$ is the hedge of the claim H in a model without liquidity risk. $\widehat{X}$ and therefore the claim H are independent of the replication strategy X. With the exception of Chapter 4.4 the claim H is an arbitrary $\mathbb{Q}$-integrable random variable independent of X.

3.3.1. Physical- vs Cash-Delivery

Because of the non-trivial supply curve, the portfolio (X_T, Y_T) can not be liquidated without causing additional costs. In fact the asymptotically realizeable wealth, which results if we piecewise sell our shares after time T as fast as possible, is given by $V_T^M(X,Y) = Y_T + X_T(S_{T+}^X - \lambda M_T X_T)$. In general this value differs form the value of the portfolio $V_T^L(X,Y) = Y_T + X_T S_T^X$. As a consequence the replication of a financial derivative, which claims stock shares at maturity, is not identical to the replication of a derivative, where cash is delivered. In fact a general contigent claim consists of two positions

$$(H^C, H^P)$$

where H^C is the position of the claim which is delivered in cash and H^P is the number of stock shares that are delivered at maturity. When replicating this claim, we want to find a s.f.t.s with $X_T = H^P$ and

$$V_T^L(X,Y) = H^C + H^P S_T^{\hat{X}}.$$

For instance, consider a European-Call option. The owner of the option is delivered a stock share if the stock price $S_T^{\hat{X}}$ is bigger than the strike price K at maturity and therefore has to pay the strike price K in cash (e.g. $H^P = \mathbb{1}_{\left\{S_T^{\hat{X}} > K\right\}}$ and $H^C = -K\mathbb{1}_{\left\{S_T^{\hat{X}} > K\right\}}$).

Depending on the delivery, we have to face different replication problems:

- **Cash-Delivery:** If the claim has cash-delivery the long position in the option receives cash at maturity and $H^P = 0$. As a consequence all shares in the underlying stock must be liquidated before or at maturity. Thus, we want to find a s.f.t.s. (X,Y) such that $X_T = 0$ and $Y_T = H^C$.
- **Physical- or Mixed-Delivery:** We consider a claim which has both physical and cash delivery. At maturity the owner of the claim is delivered a certain amount H^P of stock shares and H^C in cash. We have already mentioned that the cheapest way of trading is to split trades into tiny packages. Hence, let us assume that $\Delta X_T = 0$. The owner of the stock position changes, but no stock shares are purchased or sold at time T. The replication problem consists of finding a s.f.t.s. (X,Y) with $X_T = H^P$ and $V_T(X,Y) = H^C + H^P S_T^{\hat{X}}$.

We show that the mixed delivery problem has an asymptotic solution if an associated cash delivery problem has an asymptotic solution.

Therefore, consider a claim (H^C, H^P) which has both, a physical and a cash position. We assume that H^P is bounded and define $H = H^C + H^P S_T^{\hat{X}}$. We build a new claim $(\widetilde{H}^C, \widetilde{H}^P)$ with $\widetilde{H}^P = 0$ and

$$\widetilde{H}^C = H - \lambda M_T (H^P)^2.$$

In Theorem 4.9 we show that if $\widetilde{H}^C$ is $\mathbb{Q}$-integrable, there exists an approximating sequence $((X^n, Y^n))_{n\geqslant 1}$ of bounded continuous trading strategies with $X_0^n = X_T^n = 0$. Thus,

$$V_T^L(X^n, Y^n) = Y_{0-}^n + \int_0^T X_{u-}^n dS_u - \lambda \int_0^T (X_{u-}^n)^2 dM_u \xrightarrow{L^1} \widetilde{H}^C. \tag{3.16}$$

We assume that $((X^n, Y^n))_{n\geqslant 1}$ is known. For every n sufficiently large we construct a sequence of trading strategies $((\widetilde{X}^{n,m}, \widetilde{Y}^{n,m}))_{m\geqslant 1}$ with $\widetilde{Y}_{0-}^{n,m} = Y_{0-}^n$ and

$$\widetilde{X}_t^{n,m} = \begin{cases} X_t^n & \text{for } 0 \leqslant t < T - \frac{1}{m}, \\ \mathbb{E}^{\mathbb{Q}}[H^P \mid \mathcal{F}_t] + (X_{T-\frac{1}{m}}^n - \mathbb{E}^{\mathbb{Q}}[H^P \mid \mathcal{F}_t])(T-t)m & \text{for } T - \frac{1}{m} \leqslant t \leqslant T. \end{cases}$$

Note that $\widetilde{X}_T^{n,m} = H^P$ for every n. We only consider continuous processes S and M in $\mathcal{H}^2(0,T)$ (See Definition A.2). For every (n,m) we apply Theorem A.4 with k=1 to get a sequence of continuous trading strategies of finite variation $((\widetilde{X}^{n,m,l}, \widetilde{Y}^{n,m,l}))_{l\geqslant 1}$ with $\widetilde{Y}_{0-}^{n,m,l} = Y_{0-}^n$ and $\widetilde{X}_T^{n,m,l} \to H^P$ for $l \to \infty$. By Theorem A.4 and Theorem A.5 with k=1

$$\int_0^t \widetilde{X}_{u-}^{n,m,l} dS_u \xrightarrow{L^2} \int_0^t X_{u-}^n dS_u, \tag{3.17}$$

for $l, m \to \infty$. The same theorems with k=2 yield

$$\int_0^t (\widetilde{X}_{u-}^{n,m,l})^2 dM_u \xrightarrow{L^2} \int_0^t (X_{u-}^n)^2 dM_u, \tag{3.18}$$

for $l, m \to \infty$. By (3.11) with $\Delta\widetilde{X}_T^{n,m,l} = 0$, $\widetilde{X}_0^{n,m,l} = 0$ and $[\widetilde{X}^{n,m,l}, \widetilde{X}^{n,m,l}] = 0$,

$$V_T^L(\widetilde{X}^{n,m,l}, \widetilde{Y}^{n,m,l}) = \widetilde{Y}_{0-}^{n,m,l} + \int_0^T \widetilde{X}_{u-}^{n,m,l} dS_u - \lambda \int_0^T (\widetilde{X}_{u-}^{n,m,l})^2 dM_u + \lambda M_T (\widetilde{X}_T^{n,m,l})^2.$$

For $l, m \to \infty$ by (3.17) and (3.18)

$$\begin{aligned} V_T^L(\widetilde{X}^{n,m,l}, \widetilde{Y}^{n,m,l}) &\xrightarrow{L^2} Y_{0-}^n + \int_0^T X_{u-}^n dS_u - \lambda \int_0^T (X_{u-}^n)^2 dM_u + \lambda M_T (H^P)^2 \\ &= V_T^L(X^n, Y^n) + \lambda M_T (H^P)^2. \end{aligned}$$

Together with (3.16) we get

$$V_T^L(\widetilde{X}^{n,m,l}, \widetilde{Y}^{n,m,l}) \xrightarrow{L^1} \widetilde{H}^C + \lambda M_T (H^P)^2 = H$$

for $l, m, n \to \infty$ with $\widetilde{X}_T^{n,m,l} \to H^P$. Hence, there exists an approximate replication strategy for the contingent claim (H^C, H^P).

We therefore assume that the contingent claim H requires cash at maturity. Thus, $\widetilde{H}^P = 0$, $\widetilde{H}^C = H$. Consequently, we restrict our trades to strategies with $X_T = 0$.

3.3.2. The minimal replication value

Definition 3.25. *The* ***minimal replication value*** *(of admissible sequences) of an approximately replicable contingent claim H is*

$$\pi_0^a(H) = \inf\left\{\liminf Y_{0-}^n,\ \text{where } ((X^n, Y^n))_{n\geqslant 1} \in \Phi^a(H)\right\}.$$

Here $\Phi^a(H)$ denotes the set of all admissible sequences of trading strategies which asymtotically replicate H.

In classical arbitrage theory the fair price of a product is given by its expectation. This is in general not true for the minimal replication value in the model with liquidity risk. For an admissible trading strategy the expected additional costs due to the liquidity of an asset are positive. As a consequence, the minimal replication value is equal or bigger to the expected output.

Lemma 3.26. *Let H be a contigent claim. If H is approximately replicable, then the minimal replication value satisfies*

$$\pi_0^a(H) \geqslant \mathbb{E}^{\mathbb{Q}}\left[H\right].$$

Proof. Let $((X^n, Y^n))_{n\geqslant 1}$ be an approximating sequence such that $\lim_{n\to\infty} V_T^L(X^n, Y^n) = H$. By (3.12) and $U(X^n)$ in Definition 3.18,

$$\begin{aligned} V_T^L(X^n, Y^n) &= Y_{0-}^n + \int_0^T X_{u-}^n dS_u - \lambda \int_0^T (X_{u-}^n)^2 dM_u \\ &\quad - (1-\lambda)\left(\int_0^T M_u d\left[X^n, X^n\right]_u + M_0 (X_0^n)^2\right) \\ &\leqslant Y_{0-}^n + U_T(X^n). \end{aligned}$$

Because (X^n, Y^n) is admissible, the process $U(X^n)$ is a supermartingale. Hence for every n

$$Y_{0-}^n \geqslant \mathbb{E}^{\mathbb{Q}}[H].$$

This is true for all approximating sequences and therefore for the infimum.

♦

In classical arbitrage theory the fair price of a contingent claim is unique and independent of the replication strategy. The price is determined such that the economy extended with the trade of the contigent claim is free of arbitrage and it can be shown that every other price causes arbitrage opportunities. In the model with liquidity risk where $((X^n, Y^n))_{n\geqslant 1}$ is an approximating sequence for a claim H, the sequence

$((-X^n, -Y^n))_{n\geqslant 1}$ does in general not replicate $(-H)$. Hence, $-\pi_t(-H) \neq \pi_t(H)$ and because of Lemma 3.26 we get that

$$-\pi_0^a(-H) \leqslant \mathbb{E}^{\mathbb{Q}}[H] \leqslant \pi_0^a(H).$$

In Section 3.1.3 we have seen that the impact of a trading strategy X on the liquidity costs is not linear. As a consequence the replication costs if we purchase x shares of a financial derivative are in general not equal to x times the replication costs of one share. For a claim $H = h(S_T^{\hat{X}})$ we are also interested in the average replication costs depending on the number of purchased shares x. We therefore consider the claim

$$H^x = h(S_T^{x\hat{X}})$$

where $\hat{X}$ is the (continuous) hedge in a model without liquidity risk, $x \in \mathbb{R}$ and $S_T^{x\hat{X}}$ by (3.7). Since the stock is not infinitely liquid, it is reasonable to deduce that the contingent claim, depending on the stock, is not infinitely liquid too. Similar to the price of the stock, the average price of the contingent claim in the extended economy is given by a supply curve. In Section 4.4 we show that if h is continuous and bounded, the average price per share converges to $\mathbb{E}^{\mathbb{Q}}[h(S_T)]$ for $x \to 0$.

4. The replication in a general model

Let $B = (B_1, ..., B_d)$ be a d-dimensional Brownian motion with respect to $\mathbb{Q}$ and $\mathcal{F}$ the canonical filtration of B which satsifies the usual conditions. We specify the dynamics of the unaffected quoted stock price S and the liquidity process M in terms of B and extend our economy with $(d-1)$ additional assets. We then show that the market is approximately complete. The approximating sequence converges to the solution of a backward stochastic differential equation and the minimal replication value is determined by the minimal solution of this equation.

4.1. The extended economy

The dynamcis of the price process and the liquidity process M are defined by given continuous processes

$$\psi_j^i = (\psi_{j,t}^i)_{0 \leqslant t \leqslant T} \qquad (0 \leqslant i \leqslant d,\, 1 \leqslant j \leqslant d),$$

which are adapted to $\mathcal{F}$, integrable with respect to B and $\psi_1^k = 0$ for $k = 2, ..., d$. For $0 \leqslant u \leqslant T$

$$dS_u = \sum_{j=1}^{d} \psi_{j,u}^1 dB_{j,u} \quad \text{with } S_0 > 0$$

and

$$dM_u = \psi_{1,u}^0 du + \sum_{j=2}^{d} \psi_{j,u}^0 dB_{j,u} \quad \text{with } M_0 > 0.$$

The contingent claim depends on d independent processes $B_1, ..., B_d$. In order to hedge the risk coming from these processes we have to extend our economy with $(d-1)$ additional assets. We denote their (unaffected) quoted price process by G^k $(2 \leqslant k \leqslant d)$ and assume

$$dG_u^k = \sum_{j=2}^{d} \psi_{j,u}^k dB_{j,u} \quad \text{with } G_0^k > 0.$$

Roch uses variance swaps to complete the market, which fit into the above pattern (see Proposition 3.1 in [3]). We assume that the above stochastic differential equations have a unique strong solution. Furthermore the process M is non-negative and

$$\psi_1^0 \geqslant 0.$$

Then M is a positive $\mathbb{Q}$-submartingale. We assume that the average price per share $G_t^k(x)$, when x shares of the product k at time t are purchased, depends on the slope M_t^k of its supply curve, e.g.

$$G_t^k(x) = G_t^k + M_t^k x.$$

A trade of x shares leads to a bid-ask spread, which is refilled with both bid and ask orders. Denoting by λ^k the resiliency parameter of the k-th asset, this causes a price impact of $2\lambda^k M_t^k x$ immediately after the trade. We only use financial derivatives G^k which are frequently traded. For simplicity we assume that the liquidity process is constant. Thus for $k = 2, ..., d$

$$M_t^k = M^k, \qquad 0 \leqslant t \leqslant T.$$

We are trading the product k due to a semimartingale $\chi^k = (\chi_t^k)_{t\leqslant T}$. The observed quoted price process $G_t^{\chi^k}$ depends on the trader's history χ^k. By (3.6) and (3.14)

$$G_t^{\chi^k}(\Delta\chi_t^k) = G_{t+}^{\chi^k} - 2\lambda^k \Delta\chi_t^k \qquad \text{with} \qquad G_{t+}^{\chi^k} = G_t^k + 2\lambda^k M^k \chi_t^k.$$

A trading strategy in the extended market is a $(d+1)$-dimensional process

$$(X_t, \chi_t^2, ..., \chi_t^d, Y_t)_{t\leqslant T}.$$

At every time t the hedger holds Y_t in the risk free asset, X_t shares in the stock and χ^k in the additional assets (k=2,...,d). For simplicity we refer to it as (X, χ, Y), where $\chi = (\chi^2, ..., \chi^d)$. We again assume that the trader does not hold any shares before time 0, e.g. $X_{0-} = \chi_{0-}^k = 0$ and since we want to replicate a claim which delivers cash at maturity we require $X_T = \chi_T = 0$. Consequently, the value of a trading strategy at maturity is

$$V_T^L(X, \chi, Y) = Y_T.$$

We have to adapt the definition of a self-financing trading strategy and admissibility to the extended model. Similar to Definition 3.7 we call a trading strategy (X, χ, Y) self-financing if its total costs are 0. Analogously to Proposition 3.8 it can be shown that this is equivalent to the following definition.

Definition 4.1. *A trading strategy (X, χ, Y) with $X_{0-} = \chi_{0-}^k = X_T = \chi_T^k = 0$ ($k = 2, ..., d$) is called **self-financing** (s.f.t.s.) if and only if*

$$\begin{aligned} Y_t + X_t(S_{t+}^X - \lambda M_t X_t) + \sum_{k=2}^d \chi_t^k (G_{t+}^{\chi^k} - \lambda^k M^k \chi_t^k) \\ = Y_{0-} + \int_0^t X_{u-} dS_u + \sum_{k=2}^d \int_0^t \chi_{u-}^k dG_u^k - \lambda \int_0^t X_{u-}^2 dM_u - QV(X, \chi)_t \end{aligned} \tag{4.1}$$

where

$$QV(X,\chi)_t = (1-\lambda)\left(\int_0^t M_u d\left[X,X\right]_u + M_0 X_0^2\right) + \sum_{k=2}^{d}(1-\lambda^k)M_u^k\left(\left[\chi^k,\chi^k\right]_t + (\chi_0^k)^2\right). \tag{4.2}$$

By (4.1) the initial value in the cash account is

$$Y_{0-} = Y_0 + X_0(S_0 + M_0 X_0) + \sum_{k=2}^{d}\chi_0^k(G_0^k + M^k\chi_0^k).$$

The process

$$V_t^M = V_t^M(X,\chi,Y) = Y_t + X_t(S_{t+}^X - \lambda M_t X_t) + \sum_{k=2}^{d}\chi_t^k(G_{t+}^{\chi^k} - \lambda^k M^k \chi_t^k)$$

denotes the maximal realizeable wealth of the portfolio (X_t, χ_t, Y_t). See Section 3.1.4. $QV(X,\chi)$ are the costs coming from the quadratic variation. Note that QV is zero if the trading strategy is continuous and of finite variation with $X_0 = \chi_0^2 = ... = \chi_0^d = 0$. In terms of B this reads

$$\begin{aligned} V_t^M(X,\chi,Y) &= Y_{0-} + \int_0^t X_{u-}\psi_{1,u}^1 dB_{1,u} - \lambda\int_0^t X_{u-}^2\psi_{1,u}^0 du \\ &\quad + \sum_{j=2}^{d}\int_0^t\left(X_{u-}\psi_{j,u}^1 + (X_{u-})^2\psi_{j,u}^0 + \sum_{k=2}^{d}\chi_{u-}^k\psi_{j,u}^k\right)dB_{j,u} - QV(X,\chi)_t \\ &= Y_{0-} + \sum_{j=1}^{d}\int_0^t Z_{j,u}dB_{j,u} - \lambda\int_0^t Z_{1,u}^2\Theta_u du - QV(X,\chi)_t, \end{aligned} \tag{4.3}$$

where $\Theta_u = \frac{\psi_{1,u}^0}{(\psi_{1,u}^1)^2} \geqslant 0$ and

$$Z_{1,u} = X_{u-}\psi_{1,u}^1, \tag{4.4}$$

$$Z_{j,u} = \left(X_{u-}\psi_{j,u}^1 + (X_{u-})^2\psi_{j,u}^0 + \sum_{k=2}^{d}\chi_{u-}^k\psi_{j,u}^k\right), \quad 2 \leqslant j \leqslant d. \tag{4.5}$$

In matrix notation

$$\begin{pmatrix} Z_{2,u} \\ \vdots \\ Z_{d,u} \end{pmatrix} = \underbrace{\begin{pmatrix} \psi_{2,u}^2 & \cdots & \psi_{2,u}^d \\ \vdots & \ddots & \vdots \\ \psi_{d,u}^2 & \cdots & \psi_{d,u}^d \end{pmatrix}}_{\Psi_u^{2,d}} \begin{pmatrix} \chi_{u-}^2 \\ \vdots \\ \chi_{u-}^d \end{pmatrix} + \begin{pmatrix} \psi_{2,u}^1 X_{u-} + \psi_{2,u}^0(X_{u-})^2 \\ \vdots \\ \psi_{d,u}^1 X_{u-} + \psi_{d,u}^0(X_{u-})^2 \end{pmatrix}$$

This transformation has to be invertible. We therefore assume that the matrix

$$\Psi_u = (\psi^k_{j,u})_{1\leqslant k,j\leqslant d} = \begin{pmatrix} \psi^1_{1,u} & 0 \\ 0 & \Psi^{2,d}_u \end{pmatrix}$$

is invertible for $0 \leqslant u \leqslant T$. Then, for a given $Z = (Z_1, ..., Z_d)$ we obtain processes X and χ with

$$X_{u-} = \frac{Z_{1,u}}{\psi^1_{1,u}} \tag{4.6}$$

and

$$\begin{pmatrix} \chi^2_{u-} \\ \vdots \\ \chi^d_{u-} \end{pmatrix} = \begin{pmatrix} & & \\ & \Psi^{2,d}_u & \\ & & \end{pmatrix}^{-1} \begin{pmatrix} Z^2_u - \psi^1_{2,u}X_{u-} - \psi^0_{2,u}(X_{u-})^2 \\ \vdots \\ Z^d_u - \psi^1_{d,u}X_{u-} - \psi^0_{d,u}(X_{u-})^2 \end{pmatrix}. \tag{4.7}$$

Definition 4.2. *Let $M = \overline{M} + \overline{A}$ be the Doob-Meyer decomposition of the $\mathbb{Q}$-local submartingale M. For $t \leqslant T$ we define the process U by*

$$\begin{aligned} U_t = U_t(X,\chi) &= \int_0^t X_{u-}dS_u + \sum_{k=2}^d \int_0^t \chi^k_{u-}dG^k_u - \lambda \int_0^t X^2_{u-}d\overline{M}_u. \\ &= \sum_{j=1}^d \int_0^t Z_{j,u}dB_{j,u}. \end{aligned}$$

*with Z by (4.4) and (4.5). A trading strategy (X,χ,Y) is called L^2-**admissible** (with respect to $\mathbb{Q}$) if the process U is square integrable,*

$$\mathbb{E}^{\mathbb{Q}}\left[\sup_{0\leqslant t\leqslant T} U^2_t\right] < \infty.$$

A sequence $((X^n,\chi^n,Y^n))_{n\geqslant 1}$ is called L^2-admissible if (X^n,χ^n,Y^n) is an L^2-admissible trading strategy for every n. and Z^n by (4.4) and (4.5) converges to Z almost surely with

$$\mathbb{E}^{\mathbb{Q}}[\int_0^T Z^2_u du] < \infty.$$

Remark. To make sure the market is free of arbitrage, it is sufficient to require that the process U is a supermartingale, e.g. the corresponding strategy is admissible. The additional restrictions to L^2-admissible trading strategies and L^2-admissible sequences are necessary to make statements about the minimal replication value. See Theorem 4.12 and Theorem 4.13.

The process U of an L^2-admissible trading strategy is a square integrable $\mathbb{Q}$-Martingale. The results of Chapter 3.2, in particular Theorem 3.20, can easily be adapted to the extended model.

Theorem 4.3. *Since S and G^k are $\mathbb{Q}$-local martingales and M is a $\mathbb{Q}$-local submartingale, there exists*

- *no admissible arbitrage opportunity,*
- *no admissible asymptotical arbitrage opportunity.*

Proof. Analogously to Theorem 3.20. ♦

Example: *The stochastic volatility model of A. Roch*

A. Roch [3] uses a stochastic volatility model to specify the dynamics of the unaffected stock price S and the liquidity process M. His model deals with three different sources of risk:

- The unaffected stock price S,
- The volatility Σ of the stock,
- The level of liquidity M.

Roch defines the quoted price process S as the solution of

$$dS_t = \Sigma_t S_t dW_{1,t}, \qquad S_0 > 0.$$

The processes Σ and M are defined in terms of U and V, which satisfy

$$\begin{aligned} dU_t &= \gamma(U_t + \eta)dt + \Gamma(U_t)dW_{2,t}, \qquad U_0 > 0, \\ dV_t &= \alpha(V_t + a)dt + \Lambda(V_t)dW_{3,t}, \qquad V_0 > 0. \end{aligned}$$

Here $\gamma, \eta, \alpha, a \in \mathbb{R}$ and $W = (W_{i,t})_{0 \leqslant t \leqslant T, i=1,2,3}$ is a Brownian motion with correlated components. The functions Λ and Γ are chosen such that the solutions of the above stochastic differential equations are well defined. Then,

$$\Sigma_t = \sqrt{U_t + V_t} \quad \text{and} \quad M_t = \sqrt{\frac{U_t}{\kappa}},$$

for some $\kappa > 0$. To hedge the risks which are induced by the processes M, Σ and S, Roch expands the model with two additional assets. Empirical works have shown that the liquidity is in part correlated to the variance of the log-returns of a stock. Therefore he uses two variance swaps

$$G^i_{T_i} = \int_0^{T_i} \Sigma_u^2 du - K_i, \quad (i = 1, 2),$$

with maturity $T_1 \neq T_2$ and strike price K_i, as intruments to complete the market. He shows that the corresponding matrix Ψ_t is invertible at every time $t \in [0, T]$. In addition Roch proves that the constructed replication strategy is a viscosity solution of a partial differential equation. For details see Roch [2].

4.2. Smooth trading strategies and the replication problem

We want to approximately replicate a $\mathbb{Q}$-integrable claim H. This means we try to find a sequence of (L^2) admissible s.f.t.s. $((X^n, \chi^n, Y^n))_{n\geqslant 1} = ((X^n, \chi^{2,n}, ..., \chi^{d,n}, Y^n))_{n\geqslant 1}$, with $X^n_{0-} = X^n_T = \chi^{k,n}_{0-} = \chi^{k,n}_T = 0$ for $k = 2, ..., d$, which satisfies

$$V^L_T(X^n, \chi^n, Y^n) = Y^n_T \xrightarrow{L^1} H.$$

Costs can be avoided when we are using continuous trading strategies of finite variation (smooth trading strategies). Theorem A.6 highlights the special nature of these strategies when it comes to the replication of contingent claims.

In order to show the existence of asymptotic solutions for an integrable claim H, we deal with the following backward stochastic differential equation (BSDE).

Definition 4.4. *For a stopping time $\tau \leqslant T$ and a $\mathcal{F}_\tau$ measureable random variable H, which is integrable with respect to $\mathbb{Q}$, consider the BSDE*

$$dV_t = \sum_{j=1}^{d} Z_{j,t} dB_{j,t} - \lambda f(t, Z_{1,t})dt, \quad 0 \leqslant t \leqslant \tau, \tag{4.8}$$

with continuous generator $f(t,z) = \lambda z^2 \Theta_t$ and terminal condition $V_\tau = H$. This is equivalent to the stochastic integral equation

$$V_t = H + \lambda \int_t^T \mathbb{1}_{\{u\leqslant\tau\}} Z^2_{1,u} \Theta_u du - \sum_{j=1}^{d} \int_t^T \mathbb{1}_{\{u\leqslant\tau\}} Z_{j,u} dB_{j,u}, \quad 0 \leqslant t \leqslant T.$$

A $(d+1)$-dimensional progessively measureable process $(Z, V) = (Z_{1,t}, ..., Z_{d,t}, V_t)_{0\leqslant t\leqslant T}$, which is adapted to the filtration $\mathcal{F}$ and satisfies equation (4.8), is called ***solution*** *of the BSDE if $Z_t = 0$ and $V_t = V_\tau$ for $t \in [\tau, T]$ and*

$$\mathbb{E}^{\mathbb{Q}}[\int_0^\tau |Z^2_u| \, du] < \infty.$$

A solution (Z^*, V^*) is called ***minimal*** *if for any solution (Z, V) of (4.8) and $t \geqslant 0$*

$$V^*_{t\wedge\tau} \leqslant V_{t\wedge\tau}.$$

Theorem 4.5. *The BSDE (4.8) has at least one solution if*

(i) $\exists \quad \gamma > 0 : \quad \mathbb{1}_{\{t\leqslant\tau\}} \Theta_t \leqslant \gamma$, *for* $0 \leqslant t \leqslant T$ *and*
(ii) $\exists \quad \lambda > \gamma : \quad \mathbb{E}^{\mathbb{Q}}[\exp(\lambda |H|)] < \infty$.

Proof. See Theorem 2 in P. Briand and Y. Hu [13].

♦

Theorem 4.6. *Let H and $H^{'}$ with $H \leqslant H^{'}$ be $\mathcal{F}_\tau$ measureable random variables and*

(i) $\exists \quad \gamma > 0 : \quad \mathbb{1}_{\{t \leqslant \tau\}} \Theta_t \leqslant \gamma$, *for* $0 \leqslant t \leqslant T$,

(ii) $\mathbb{E}^{\mathbb{Q}}\left[\exp(\lambda \left|H\right| + \lambda \left|H^{'}\right|)\right] < \infty$ *for all* $\lambda > 0$.

If (Z, V) is a solution of (4.8) with terminal condition H and $(Z^{'}, V^{'})$ a solution with terminal condition $H^{'}$, then

$$V_0 \leqslant V_0^{'}.$$

Proof. See Theorem 5 in [14].

♦

Corollary 4.7. *If Θ is uniformly bounded, the solution of the BSDE (4.8) is unique amongst all solutions which have moments of all order.*

Proof. By Theorem 4.6 and Corollary 4 in [14]. ♦

If Θ is uniformly bounded and H is bounded, the conditions of Corollary 4.7 are satisfied and the equation has a unique solution. Also see Delbaen, Hu and Richou [7] on the uniqueness of quadratic BSDEs with convex generators and unbounded terminal conditions.

Remark. For a solution (Z, V) of the BSDE (4.8) we get processes X and χ by (4.6) and (4.7). If X and χ are continuous and of finite variation, then $V = V^M(X, \chi, Y)$ is the maximal realizeable wealth of the s.f.t.s. (X, χ, Y) with Y determined by (4.1). See (4.3).

In case there exists a solution of the BSDE (4.8) with terminal condition H we construct a sequence of smooth trading strategies with self-financing and L^2-admissible elements which approximately replicates H

Lemma 4.8. *Let $\tau \leqslant T$ be a stopping time and H a $\mathbb{Q}$-integrable, $\mathcal{F}_\tau$ measureable random variable. We assume there exists a solution (Z, V) of the BSDE (4.8), the matrix Ψ_t is invertible for $t \in [0, T]$ and for a constant $K > 0$*

(i) $0 < \mathbb{1}_{\{t \leqslant \tau\}} \Theta_t = \mathbb{1}_{\{t \leqslant \tau\}} \frac{\psi_{1,t}^0}{(\psi_{1,t}^1)^2} < K$ *for* $t \in [0, T]$ *and*

(ii) $\mathbb{1}_{\{t \leqslant \tau\}} \left|\psi_{j,t}^i\right| < K$ *for* $t \in [0, T]$ *and* $(i, j) \in \{0, .., d\} \times \{1, ..., d\}$.

Then there exists an L^2-admissible sequence $((X^n, \chi^n, Y^n))_{n\geqslant 1}$ of s.f.t.s. such that $Y_0^n = V_0$ and for $n \to \infty$

$$Y_T^n \xrightarrow{L^1} H.$$

Proof. Let $(Z_t, V_t)_{0\leqslant t\leqslant T}$ be a solution of the BSDE (4.8). With Z in (4.6) and (4.7) we transform this solution to $(X, \chi, V) = (X_t, \chi_t, V_t)_{0\leqslant t\leqslant T}$, with $X_t = \chi_t = 0$ and $V_t = V_\tau$ for $t \in [\tau, T]$. Then (X, χ, V) solves the equation

$$H = V_0 + \int_0^T X_{u-} dS_u + \sum_{k=2}^d \int_0^T \chi_{u-}^k dG_u^k - \lambda \int_0^T X_{u-}^2 dM_u. \tag{4.9}$$

Let us define the bounded processes

$$\overline{X}^m = X \mathbb{1}_{\{|X|\leqslant m \ \cap \ |\chi|\leqslant m\}} \qquad \text{and} \qquad \overline{\chi}^m = \chi \mathbb{1}_{\{|X|\leqslant m \ \cap \ |\chi|\leqslant m\}}. \tag{4.10}$$

Because of (ii) these processes satisfy

$$\mathbb{E}[\int_0^T (\overline{X}_{u-}^m)^2 d[S,S]_u] < \infty \quad \text{and} \quad \mathbb{E}[\int_0^T (\overline{\chi}_{u-}^m)^2 d[G,G]_u] < \infty.$$

By Theorem A.6 we get a sequence of continuous, bounded processes $\overline{X}^{m,n}, \overline{\chi}^{m,n}$, with $\overline{X}_0^{m,n} = \overline{X}_T^{m,n} = \overline{\chi}_0^{m,n} = \overline{\chi}_T^{m,n} = 0$, such that for $n \to \infty$

$$\int_0^T \overline{X}_{u-}^{m,n} dS_u \xrightarrow{L^2} \int_0^T \overline{X}_{u-}^m dS_u, \tag{4.11}$$

$$\int_0^T \overline{\chi}_{u-}^{m,n} dG_u \xrightarrow{L^2} \int_0^T \overline{\chi}_{u-}^m dG_u. \tag{4.12}$$

Because the processes are bounded, $\overline{X}^{m,n}$ (resp. $\overline{\chi}^{m,n}$) is square-integrable with respect to $[S,S]$ (resp. $[G,G]$). Let us construct a s.f.t.s. $(\overline{X}^{m,n}, \overline{\chi}^{m,n}, \overline{Y}^{m,n})$, where the process $\overline{Y}^{m,n}$, with $\overline{Y}_{0-}^{m,n} = \overline{Y}_0^{m,n} = V_0$, is determined by (4.1).

By equation (4.4) and (4.5) with $(\overline{X}^m, \overline{\chi}^m)$ instead of X and χ, we obtain a process $\overline{Z}^m = (\overline{Z}_1^m, ...\overline{Z}_d^m)$ and analogously we get a process $\overline{Z}^{m,n} = (\overline{Z}_1^{m,n}, ..., \overline{Z}_d^{m,n})$ due to $\overline{X}^{m,n}, \overline{\chi}^{m,n}$. $(\overline{X}^m, \overline{\chi}^m)$ converges to $(\overline{X}, \overline{\chi})$ almost surely and consequently $Z^m \xrightarrow{a.s.} Z$. Because of (4.10) we have that $|\overline{Z}^m| \leqslant |Z|$ and since (Z, V) is a solution of the BSDE (4.8), we know that $\mathbb{E}^{\mathbb{Q}}[\int_0^T Z_u^2 du] < \infty$. The dominated convergence theorem proves

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T (Z_u - \overline{Z}_u^m)^2 du\right] \to 0, \qquad (m \to \infty)$$

By Theorem A.6 we get $X^{m,n} \to X^m$ almost surely (resp. $\chi^{m,n} \to \chi^n$ a.s.) and consequently by (4.4) and (4.5) with bounded processes ψ, $Z^{m,n} \to Z^m$ almost surely. The dominated convergence theorem states

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T (\overline{Z}_u^m - \overline{Z}_u^{m,n})^2 du\right], \qquad (n \to \infty).$$

Together we get,

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T (Z_u - \overline{Z}_u^{m,n})^2 du\right] \leqslant \mathbb{E}\left[\int_0^T (Z_u - \overline{Z}_u^{m})^2 du\right] + \mathbb{E}\left[\int_0^T (\overline{Z}_u^{m} - \overline{Z}^{m,n})^2 du\right] \to 0.$$

By (4.9) and (4.1) with $[X^{m,n}, X^{m,n}] = [\chi^{m,n}, \chi^{m,n}] = 0$ and $X_0^{m,n} = \chi_0^{m,n} = 0$,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\mid \overline{Y}_T^{m,n} - H \mid\right] \\
&= \mathbb{E}\Big[\Big| \int_0^T (\overline{X}_{u-}^{m,n} - X_{u-}) dS_u + \int_0^T (\overline{\chi}_{u-}^{m,n} - \chi_{u-}) dG_u - \lambda \int_0^T ((\overline{X}_{u-}^{m,n})^2 - (X_{u-})^2) dM_u \Big|\Big] \\
&= \mathbb{E}\left[\left|\int_0^T (\overline{Z}_u^{m,n} - Z_u) dB_u - \lambda \int_0^T ((\overline{Z}_{1,u}^{m,n})^2 - (Z_{1,u})^2)\Theta_u du\right|\right] \\
&\leqslant \mathbb{E}\left[\int_0^T \left|\overline{Z}_u^{m,n} - Z_u\right|^2 du\right] + \mathbb{E}\left[\lambda K \int_0^T \left|(\overline{Z}_{1,u}^{m,n})^2 - (Z_{1,u})^2\right| du\right] \longrightarrow 0
\end{aligned}$$

for $n \to \infty$ and $m \to \infty$. Hence, we find a sequence of bounded, continuous trading strategies (X^n, χ^n, Y^n) with finite variation, such that $Y_T^n \to H$ in L^1.

It remains to show that the constructed sequence is L^2-admissible. X^n, χ^n, Y^n are bounded for every n and by (ii) the processes ψ are uniformly bounded. Consider

$$\begin{aligned}
U_t(X^n, \chi^n) &= \int_0^t X_{u-}^n dS_u + \int_0^T \chi_{u-}^n dG_u - \lambda \int_0^t (X_{u-}^n)^2 d\overline{M}_u \\
&= \int_0^t Z_u^n dB_u
\end{aligned}$$

with $Z^n = (Z_1^n, ..., Z_d^n)$ via equation (4.4) and (4.5) with (X^n, χ^n) instead of (X, χ). Then Z^n is bounded too. Consequently $U_t(X^n, \chi^n)$ is a square integrable martingale for every n and the strategy (X^n, χ^n, Y^n) is L^2-admissible. Furthermore, $Z^n \to Z$ almost surely by construction.

♦

The next theorem states that every integrable claim H can be approximately replicated if the matrix Ψ_t is invertible at every time t. The market therefore is approximately complete.

Theorem 4.9. *If Ψ_t is invertible $(0 \leqslant t \leqslant T)$, then every integrable contingent claim H is approximately replicable. There exists a sequence of continuous, L^2-admissible s.f.t.s. (X^n, χ^n, Y^n) with finite variation such that*

$$Y_T^n \xrightarrow{L^1} H, \qquad (n \to \infty).$$

Proof. Consider the claim

$$H^N = \begin{cases} N & \text{for } H \geqslant N, \\ H & \text{for } -N \leqslant H \leqslant N, \\ -N & \text{for } H \leqslant -N. \end{cases}$$

We define the stopping time

$$\tau_L = \inf\left\{ t \leqslant T \middle| \psi^1_{1,t} \leqslant \frac{1}{L} \text{ or } \left|\psi^i_{j,t}\right| \geqslant L \text{ for } (i,j) \in \{0,...,d\} \times \{1,...,d\} \right\}. \tag{4.13}$$

Furthermore, let

$$H^N_L = \mathbb{E}^{\mathbb{Q}}\left[H^N \mid \mathcal{F}_{\tau_L}\right].$$

The random variable $H^N_L \leqslant N$ is $\mathcal{F}_{\tau_L}$-measureable and bounded. By (4.13) $\mathbb{1}_{\{t \leqslant \tau_L\}} \left|\psi^i_{j,t}\right| \leqslant L$ and $\mathbb{1}_{\{t \leqslant \tau_L\}} \Theta_t \leqslant L^2$ for $0 \leqslant t \leqslant T$. Theorem 4.5 states that there exists a solution to the BSDE (4.8). By Theorem 4.8 we get an L^2-admissible sequence of smooth trading strategies $(X^{n,L,N}, \chi^{n,L,N}, Y^{n,L,N})$ with $Y^{n,L,N}_T \to H^N_L$ in L^1 for $n \to \infty$. Then,

$$\mathbb{E}^{\mathbb{Q}}\left[\left|Y^{n,L,N}_T - H\right|\right] \leqslant \mathbb{E}\left[\mid Y^{n,L,N}_T - H^N_L \mid\right] + \underbrace{\mathbb{E}\left[\mid H^N - H^N_L \mid\right]}_{(2)} + \underbrace{\mathbb{E}\left[\mid H - H^N \mid\right]}_{(3)}.$$

Using the martingale convergence for (2) and dominated convergence theorem for (3) this converges to 0 for $n, L, N \to \infty$. Thus, we find a sequence of smooth L^2-admissible s.f.t.s. which asymptotically replicates H.

♦

4.3. The minimal replication value

In contrast to the minimal replication value (of admissible sequences) defined in Section 3.3.2, we now restrict our trade to L^2-admissible s.f.t.s.. In addition we want the approximating sequences to be L^2-admissible (see Definition 4.2).

Definition 4.10. *An L^2-**admissible approximating sequence** is an L^2-admissible sequence of s.f.t.s. which asymptotically replicates the claim H. The set $\Phi(H)$ consists of all L^2-admissible approximating sequences.*

In the definition of the minimal replication value in the extended economy, we only consider L^2-admissible approximating sequences $((X^n, \chi^n, Y^n))_{n \geqslant 1}$ with

$$\Delta X^n_T = \Delta \chi^n_T = 0, \qquad (n \geqslant 1).$$

We now justify this restriction. Therefore, consider an arbitrary L^2-admissible sequence $((X^n, \chi^n, Y^n))_{n\geqslant 1}$ $(\chi^n = (\chi^{2,n}, ..., \chi^{d,n}))$. By (4.4) and (4.5) we get a processes $Z^n = (Z_1^n, ..., Z_d^n)$. We apply Theorem A.5 to obtain processes $\widetilde{Z}^{n,m}$, with $\Delta\widetilde{Z}_T^{n,m} = 0$. Then $\widetilde{Z}^{n,m}$ satisfies

$$\int_0^t \widetilde{Z}_u^{n,m} dB_u \xrightarrow{L^2} \int_0^t (Z_u^n) dB_u, \qquad (m \to \infty),$$

and if Θ is uniformly bounded

$$\int_0^t \Theta_u (\widetilde{Z}_u^{n,m})^2 du \xrightarrow{L^1} \int_0^t \Theta_u (Z_u^n)^2 du, \qquad (m \to \infty).$$

We assume that Ψ is invertible. By (4.6) and (4.7) we construct $\widetilde{X}^{n,m}$ and $\widetilde{\chi}^{n,m}$ with $\widetilde{X}_t^{n,m} = X_t^n$ and $\widetilde{\chi}_t^{n,m} = \chi_t^n$ for $t \in [0, T - \frac{1}{m}]$ and $\Delta\widetilde{X}_T^{n,m} = \Delta\widetilde{\chi}_T^{n,m} = 0$. Then

$$QV(\widetilde{X}^{n,m}, \widetilde{Y}^{n,m})_{T-\frac{1}{m}} = QV(X^n, \chi^n)_{T-\frac{1}{m}} \xrightarrow{a.s.} QV(X^n, \chi^n)_{T-}, \qquad (m \to \infty).$$

Since V_T^n is integrable and by the definition of L^2-admissible strategies, $\mathbb{E}^{\mathbb{Q}}[QV(X^n, \chi^n)_T] < \infty$. The monotone convergence theorem yields $QV(\widetilde{X}^{n,m}, \widetilde{Y}^{n,m})_{T-\frac{1}{m}} \to QV(X^n, \chi^n)_{T-}$ in L^1 for $m \to \infty$. Furthermore note that by the definition of QV in (4.2) and (3.3)

$$QV(X^n, \chi^n)_T - QV(X^n, \chi^n)_{T-} = (1-\lambda) M_T (\Delta X_T^n)^2 + \sum_{k=2}^d (1-\lambda^k) M^k (\Delta \chi_T^{k,n})^2.$$

With $\widetilde{Y}^{n,m}$ determined by $\widetilde{Y}_{0-}^{n,m} = Y_{0-}^n$ and (4.1), the process $(\widetilde{X}^{n,m}, \widetilde{\chi}^{n,m}, \widetilde{Y}^{n,m})$ is self-financing. Then by (4.3) for $m \to \infty$

$$\begin{aligned}
&V_{T-\frac{1}{m}}^M (\widetilde{X}^{n,m}, \widetilde{\chi}^{n,m}, \widetilde{Y}^{n,m}) \\
&= \widetilde{Y}_{0-}^{n,m} + \int_0^{T-\frac{1}{m}} \widetilde{Z}_u^{n,m} dB_u - \lambda \int_0^{T-\frac{1}{m}} \Theta_u (\widetilde{Z}_{1,u}^{n,m})^2 du - QV(\widetilde{X}^{n,m}, \widetilde{\chi}^{n,m})_{T-\frac{1}{m}} \\
&\xrightarrow{L^1} Y_{0-}^n + \int_0^T Z_u^n dB_u - \lambda \int_0^T \Theta_u (Z_{1,u}^n)^2 du \\
&\qquad - QV(X^n, \chi^n)_T + (1-\lambda) M_T (\Delta X_T^n)^2 + \sum_{k=2}^d (1-\lambda^k) M^{k,n} (\Delta \chi_T^{k,n})^2.
\end{aligned}$$

This limit is equal or bigger $V_T^M(X^n, \chi^n, Y^n) = Y_T^n$ for every n. Because $\Delta\widetilde{X}_T^{n,m} = \Delta\widetilde{\chi}_T^{n,m} = 0$ we get $QV(\widetilde{X}^{n,m}, \widetilde{\chi}^{n,m})_{T-} = QV(\widetilde{X}^{n,m}, \widetilde{\chi}^{n,m})_T$. Then $V_{T-\frac{1}{k}}^M (\widetilde{X}^{n,m}, \widetilde{\chi}^{n,m}, \widetilde{Y}^{n,m})$ converges to $V_T^M (\widetilde{X}^{n,m}, \widetilde{\chi}^{n,m}, \widetilde{Y}^{n,m})$ in L^1 for k to infinity.

Consequently, for every L^2-admissible sequence $((X^n, \chi^n, Y^n))_{n\geqslant 1}$ we find an L^2-admissible sequence $((\widetilde{X}^n, \widetilde{\chi}^n, \widetilde{Y}^n))_{n\geqslant 1}$ with $\widetilde{Y}_{0-}^n = Y_{0-}^n$ and

$$\lim_{n\to\infty} \widetilde{Y}_T^n \geqslant \lim_{n\to\infty} Y_T^n \qquad (a.s.).$$

Definition 4.11. *The* ***minimal replication value*** *(of L^2-admissible sequences) is*

$$\pi_0(H) = \inf\left\{\liminf Y_{0-}^n,\ \text{where } ((X^n, \chi^n, Y^n))_{n\geqslant 1} \in \Phi(H) \text{ with } \Delta X_T^n = \Delta\chi_T^n = 0\right\}.$$

Since G^k are $\mathbb{Q}$-local martingales, the process U of an L^2-admissible strategy is a $\mathbb{Q}$-martingale and Lemma 3.26 remains true. Hence,

$$\pi_0(H) \geqslant \mathbb{E}^{\mathbb{Q}}[H].$$

For a claim H let $\Phi^s(H) \subseteq \Phi(H)$ denote the set of all approximating sequences which are continuous and of finite variation (smooth). It turns out that to calculate the minimal replication value it is sufficient to consider smooth sequences in $\Phi^s(H)$.

Theorem 4.12. *Let Θ_u be uniformly bounded and the matrix Ψ_t is invertible for $t \in [0,T]$. Then the minimal replication value of a contingent claim H satisfies*

$$\pi_0(H) = \inf\left\{\liminf Y_0^n,\ \text{where } ((X^n, \chi^n, Y^n))_{n\geqslant 1} \in \Phi^s(H)\right\}.$$

Proof. We show that for every approximating sequence in $\Phi(H)$ we find a sequence in $\Phi^s(H)$ with equal or lower replication costs. Consider an L^2-admissible self-financing sequence $((X^n, \chi^n, Y^n))_{n\geqslant 1}$ in $\Phi(H)$ with $\Delta X_T^n = \Delta\chi_T^n = 0$. Then by (4.3) with $V_t^n = V_t^M(X^n, \chi^n, Y^n)$

$$\begin{aligned} V_t^n &= Y_{0-}^n + \int_0^t X_{u-}^n dS_u + \sum_{k=2}^d \int_0^t \chi_{u-}^{k,n} dG_u^k - \lambda \int_0^t (X_{u-}^n)^2 dM_u - QV(X^n, \chi^n)_t \\ &= Y_{0-}^n + \sum_{j=1}^d \int_0^t Z_{j,u}^n dB_{j,u} - \lambda \int_0^t (Z_{1,u}^n)^2 \Theta_u du - QV(X^n, \chi^n)_t. \end{aligned} \tag{4.14}$$

Note that $V_T^n = Y_T^n \to H$ in L^1. For every pair (N, L) we introduce a stopping time

$$\tau_{N,L} = \inf\left\{t \leqslant \tau_L \,\middle|\, |V_t^n| > N \text{ or } QV(X^n, \chi^n)_t > N\right\} \tag{4.15}$$

with τ_L as in (4.13). V_T^n is integrable, Θ bounded and $\mathbb{E}^{\mathbb{Q}}[\int_0^T Z_{j,u}^2 du] < \infty$ for $j = 1, ..., d$ by the definition of an L^2-admissible strategy. By (4.14) $\mathbb{E}^{\mathbb{Q}}[QV(X^n, \chi^n)_T] < \infty$ and consequently $\tau_{N,L} \to T$ almost surely for N and L to infinity. Because QV and V^n are cadlag, we get that $QV(X^n, \chi^n)_{\tau_{N,L}-} \leqslant N$ and $V_{\tau_{N,L}-}^n \leqslant N$. Then with $\overline{H} = V_{\tau_{N,L}-}^n$

$$\begin{aligned} \overline{H} &= Y_{0-}^n + \sum_{j=1}^d \int_0^{\tau_{N,L}-} Z_{j,u}^n dB_{j,u} - \lambda \int_0^{\tau_{N,L}-} (Z_{1,u}^n)^2 \Theta_u du - QV(X^n, \chi^n)_{\tau_{N,L}-} \\ &\leqslant Y_{0-}^n + \sum_{j=1}^d \int_0^{\tau_{N,L}-} Z_{j,u}^n dB_{j,u} - \lambda \int_0^{\tau_{N,L}-} (Z_{1,u}^n)^2 \Theta_u du = \widehat{H} \leqslant 2N. \end{aligned}$$

Consequently $(Z^n, \widehat{V})$ with $\widehat{V}_0 = Y^n_{0-}$ is a solution of the BSDE (4.8) with bounded terminal condition $\widehat{V}_T = \widehat{H}$ and $\tau = \tau_{N,L}-$. Furthermore, by Theorem 4.5 there exists a solution $(\overline{Z}, \overline{V})$ of the same equation, but with terminal condition $\overline{H} = \overline{V}_T$. Since $\overline{H} \leqslant \widehat{H}$ Theorem 4.5 yields

$$\overline{V}_0 \leqslant \widehat{V}_0 = Y^n_{0-}.$$

By Lemma 4.8 there exists an L^2-admissible approximating sequence of smooth trading strategies with initial value $\overline{V}_0$ which converges to $\overline{H} = V^n_{\tau_{N,L}-}$ in L^1 at maturity T. By (4.14)

$$\mathbb{E}^{\mathbb{Q}}\left[\left|V^n_T - V^n_{\tau_{N,L}-}\right|\right] \leqslant \mathbb{E}^{\mathbb{Q}}\left[\left|\int_{\tau_{N,L}-}^T Z^n_u dB_u\right|\right] + \mathbb{E}^{\mathbb{Q}}\left[\left|\lambda \int_{\tau_{N,L}-}^T (Z^n_{1,u})^2 \Theta_u du\right|\right] + \mathbb{E}^{\mathbb{Q}}\left[\left|QV(X^n,\chi^n)_T - QV(X^n,\chi^n)_{\tau_{N,L}-}\right|\right].$$

Since Θ is bounded and Z^n square integrable, the first two integrals converge to zero for $N, L \to \infty$ by dominated convergence. In general QV is not continuous. But since $\Delta X^n_T = \Delta \chi^n_T = 0$ we get that $QV(X^n,\chi^n)_{\tau_{N,L}-} = QV(X^n,\chi^n)_T$. Then, the last term converges to zero by monotone convergence. Hence, we find a sequence of smooth self-financing and admissible trading strategies which approximately replicates V^n_T and with initial value equal or smaller Y^n_{0-}. This is true for every n and consequently we find an approximating sequence of smooth trading strategies $((\widetilde{X}^n, \widetilde{\chi}^n, \widetilde{Y}^n))_{n\geqslant 1}$ in $\Phi^s(H)$ with

$$\lim_{n\to\infty} \widetilde{Y}^n_{0-} \leqslant \lim_{n\to\infty} Y^n_{0-}.$$

♦

Theorem 4.13. *Let Θ_u be uniformly bounded and Ψ_u invertible on $[0,T]$. If the BSDE (4.8) (with $\tau = T$ and terminal condition H) has a minimal solution (Z^*, V^*), then the minimal replication value of a claim H satisfies*

$$\pi_0(H) = V_0^*.$$

Proof. By Lemma 4.8 for every solution (Z,V) of the BSDE (4.8) with $\tau = T$ and terminal condition H we find an approximating sequence in $\Phi^s(H)$. Hence,

$$\pi_0(H) \leqslant V_0^*.$$

On the other hand we consider an L^2-admissible approximating sequence $((X^n,\chi^n,Y^n))_{n\geqslant 1}$ in $\Phi^s(H)$. Since Θ is uniformly bounded and by the definition of an L^2-admissible sequence of trading strategies

$$\sum_{j=1}^d \int Z^n_{j,u} dB_u \xrightarrow{ucp} \sum_{j=1}^d \int Z_{j,u} dB_u$$

and

$$\int (Z^n_{1,u})^2 \Theta_u d_u \xrightarrow{ucp} \int Z^2_{1,u} \Theta_u d_u.$$

Then by (4.3)

$$\begin{aligned} V^n_t &= Y^n_0 + \sum_{j=1}^{d} \int_0^t Z^n_{j,u} dB_u - \int_0^t (Z^n_{1,u})^2 \Theta_u d_u \\ &\xrightarrow{L^1} V_0 + \sum_{j=1}^{d} \int_0^t Z_{j,u} dB_u - \int_0^t Z^2_{1,u} \Theta_u d_u = V_t, \end{aligned}$$

with $V_0 = \lim_{n\to\infty} Y^n_0$. We get $V^n_T \to H$ in L^1 and $V^n_T \to V_T$ in probability. Consequently $H = V_T$ almost surely or equivalently

$$H = V_0 + \sum_{j=1}^{d} \int_0^T Z_{j,u} dB_u - \int_0^T Z^2_{1,u} \Theta_u d_u.$$

Hence, (Z, V) is a solution of the BSDE (4.8) with $\tau = T$ and terminal condition H. Together with Theorem 4.12 we get

$$\pi_0(H) \geqslant V^*_0.$$

♦

4.4. Analytical properties of the average price per share

In this section we investigate claims of the form

$$H = h(S^{\widehat{X}}_T)$$

where $(\widehat{X}, \widehat{\chi}, \widehat{Y}) = (\widehat{X}_t, \widehat{\chi}_t, \widehat{Y}_t)_{0\leqslant t\leqslant T}$ is the perfect continuous hedge of an arbitrary claim $\widehat{H}$ in the model without illiquidity (the case $M = 0$) and $\widehat{X}_T = 0$. This means at maturity T all stock shares are sold. Then by (3.7)

$$S^{\widehat{X}}_T = S_T - 2\lambda \int_0^T \widehat{X}_{u-} dM_u. \tag{4.16}$$

In a model without illiquidity $(x\widehat{X}_t, x\widehat{\chi}, x\widehat{Y}_t)_{0\leqslant t\leqslant T}$ is the hedge of the claim $x\widehat{H}$ and $\mathbb{E}^{\mathbb{Q}}[x\widehat{H}] = x\mathbb{E}^{\mathbb{Q}}[\widehat{H}]$ its price. In our model with illiquidity the minimal replication value $\pi_0(H)$ is not linear in H. We therefore consider the claim

$$H^x = h(S^{x\widehat{X}}_T)$$

and investigate the dependence of $\pi_0(xH^x)$ on x. Our first aim is to show that the average price per share converges to $\mathbb{E}^{\mathbb{Q}}[h(S_T)]$ when the number of purchased shares x goes to zero.

Theorem 4.14. *If h is continuous then*

$$\frac{\pi_0(xH^x)}{x} \to \mathbb{E}^{\mathbb{Q}}[h(S_T)], \qquad (x \to 0).$$

Proof. For the claim H^x consider

$$(H^x)^N = h^N(S_T^{x\hat{X}})$$

with $h^N(x) = \max(\min(h(x), N), -N)$ and

$$(H^x)^N_L = \mathbb{E}^{\mathbb{Q}}\left[(H^x)^N | \mathcal{F}_{\tau_L}\right]$$

with stopping time τ_L defined in (4.13). Then

$$\left|x(H^x)^N_L\right| \leqslant xN, \tag{4.17}$$

$$\mathbb{1}_{\{t \leqslant \tau_L\}}\Theta_t \leqslant L^2, \quad 0 \leqslant t \leqslant T. \tag{4.18}$$

By Theorem 4.5 for every x, L, N there exists a solution (V^x, Z^x), with $V_t^x = V_{\tau_L}^x$ and $Z_t^x = 0$ for $t \in [\tau_L, T]$, which solves the stochastic integral equation

$$x(H^x)^N_L = V_t^x + \int_t^T Z_u^x dB_u - \lambda \int_t^T (Z_{1,u}^x)^2 \Theta_u du. \tag{4.19}$$

Lemma 4.8 states that there exists an L^2-admissible, approximating sequence of continuous s.f.t.s. $(X^{x,n}, \chi^{x,n}, Y^{x,n})$ with $Y_0^{x,n} = V_0^x$.

We want to show the result for $x \to 0$ and therefore consider small values x with

$$\mid x \mid < \frac{1}{4\lambda L^2 N}. \tag{4.20}$$

By the maximum principle (Proposition 2.1 in Kobylanski [11])

$$\mid V_t^x \mid \leqslant |x| \, N \leqslant \frac{1}{4\lambda L^2}, \quad 0 \leqslant t \leqslant T. \tag{4.21}$$

The formula of Ito applied on $f(V_t^x) = (V_t^x)^2$, with V_t^x by (4.19), yields

$$\begin{aligned}
(x(H^x)^L)^2 &= (V_t^x)^2 - 2\int_t^T \left(\lambda\Theta_u (Z_{1,u}^x)^2 V_u^x - \frac{1}{2} \mid Z_u^x \mid^2\right) du + 2\int_t^T V_u^x Z_u^x dB_u \\
&\overset{(4.18)}{\geqslant} (V_t^x)^2 + \int_t^T (1 - \lambda 2L^2 \mid V_u^x \mid) \mid Z_u^x \mid^2 du + 2\int_t^T V_u^x Z_u^x dB_u \\
&\overset{(4.21)}{\geqslant} (V_t^x)^2 + \int_t^T \frac{1}{2} \mid Z_u^x \mid^2 du + 2\int_t^T V_u^x Z_u^x dB_u.
\end{aligned} \tag{4.22}$$

$V_t^x Z_t^x$ is square integrable. Then

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \Theta_u(Z_{1,u}^x)^2 du \mid \mathcal{F}_t\right] &\overset{(4.18)}{\leqslant} \mathbb{E}^{\mathbb{Q}}\left[\int_t^T L^2 \mid Z_u^x \mid^2 du \mid \mathcal{F}_t\right] \\ &\overset{(4.22)}{\leqslant} 2L^2 \mathbb{E}^{\mathbb{Q}}\left[(x(H^x)^L)^2 \mid \mathcal{F}_t\right] \\ &\overset{(4.17)}{\leqslant} 2x^2L^2N^2. \end{aligned} \tag{4.23}$$

Furthermore, by (4.19) $\frac{1}{x}V_t^x = \mathbb{E}^{\mathbb{Q}}\Big[\mathbb{E}^{\mathbb{Q}}[h(S_T^{x\widehat{X}}) \mid \mathcal{F}_{\tau_L}] + \frac{\lambda}{x}\int_t^T \Theta_u(Z_{1,u}^x)^2 du \mid \mathcal{F}_t\Big]$. This together with (4.23) yields

$$\left|\frac{1}{x}V_t^x - \mathbb{E}^{\mathbb{Q}}[h^N(S_T^{x\widehat{X}}) \mid \mathcal{F}_{\tau_L \wedge t}]\right| = \mathbb{E}^{\mathbb{Q}}\left[\left|\frac{\lambda}{x}\int_t^T \Theta_u(Z_{1,u}^x)^2 du\right| \Big| \mathcal{F}_t\right] \leqslant 2xL^2N^2.$$

The continuity of h^N implies $h^N(S_T^{x\widehat{X}}) \to h^N(S_T)$ for $x \to 0$. Since h^N is bounded the dominated convergence theorem yields $\mathbb{E}^{\mathbb{Q}}[h^N(S_T^{x\widehat{X}}) \mid \mathcal{F}_{\tau_L \wedge t}] \to \mathbb{E}^{\mathbb{Q}}[h^N(S_T) \mid \mathcal{F}_{\tau_L \wedge t}]$ for $x \to 0$. Hence,

$$\lim_{x\to 0} \frac{1}{x}V_t^x = \mathbb{E}^{\mathbb{Q}}\left[h^N(S_T) \mid \mathcal{F}_{\tau_L \wedge t}\right]. \tag{4.24}$$

By Lemma 3.26 and $\mathbb{E}^{\mathbb{Q}}[(H^x)_L^N] = \mathbb{E}^{\mathbb{Q}}[(H^x)^N]$ we get $\mathbb{E}^{\mathbb{Q}}[x(H^x)^N] \leqslant \pi_0(x(H^x)_L^N)$ and consequently for $x \neq 0$

$$\mathbb{E}^{\mathbb{Q}}[(H^x)^N] \leqslant \frac{\pi_0(x(H^x)_L^N)}{x}.$$

Then,

$$\mathbb{E}^{\mathbb{Q}}[(H^x)^N] \leqslant \frac{\pi_0(x(H^x)_L^N)}{x} \leqslant \frac{1}{x}V_0^x$$

and by (4.24) for every $N > 0$ and $L > 0$

$$\frac{\pi_0(x(H^x)_L^N)}{x} \longrightarrow \mathbb{E}^{\mathbb{Q}}[h^N(S_T)], \quad (x \to 0).$$

Since $H_L^N \overset{L^1}{\to} H$ for $N, L \to \infty$

$$\frac{\pi_0(xH^x)}{x} \longrightarrow \mathbb{E}^{\mathbb{Q}}[h(S_T)], \quad (x \to 0).$$

♦

For the remaining part of this section $(\widehat{X}_t, \widehat{\chi}_t, \widehat{Y}_t)_{0\leqslant t\leqslant T}$, with $\widehat{X}_t = \widehat{\chi}_t = 0$ and $\widehat{Y}_t = Y_{\tau_L}$ for $t \in [\tau_L, T]$, denotes the perfect hedge of the claim

$$H_L^N = \mathbb{E}^{\mathbb{Q}}\left[h^N(S_T) \middle| \mathcal{F}_{\tau_L}\right]$$

in a model without illiquidity. Then by (4.3) with $M = 0$

$$H_L^N = \widehat{V}_t + \sum_{j=1}^{d} \int_t^T \widehat{Z}_{j,u} dB_{j,u} \tag{4.25}$$

where $\widehat{V}_0 = \widehat{Y}_0$ and $\widehat{Z}_t = (\widehat{Z}_{1,t}, ..., \widehat{Z}_{d,t})$ is given by (4.4) and (4.5). Because H_L^N is bounded, we get that $\mathbb{E}^{\mathbb{Q}}[\int_0^T \widehat{Z}_u^2 du] < \infty$.

Let (V^x, Z^x) be the solution of the BSDE (4.19), with $\mathcal{F}_{\tau_L}$ measureable terminal condition

$$(H^x)_L^N = \mathbb{E}^{\mathbb{Q}}\left[h^N(S_T^{x\widehat{X}_t}) \middle| \mathcal{F}_{\tau_L}\right].$$

By Theorem 4.5 this solution is unique with $Z_t^x = 0$ for $t \in [\tau_L, T]$. With Z^x in (4.6) we define a (caglad) process $X^x = (X_t^x)_{0 \leqslant t \leqslant T}$ with $X_t^x = 0$ for $t \in [\tau_L, T]$.

Lemma 4.15. *If h is Lipschitz continuous then*

$$\frac{1}{x} X^x \xrightarrow{L^2} \widehat{X}, \qquad (x \to 0),$$

and

$$S_{T+}^{X^x} \xrightarrow{L^2} S_{T+}^{x\widehat{X}},, \qquad (x \to 0).$$

Remark. Since h is Lipschitz continuous this is equivalent to $h(S_{T+}^{X^x}) \xrightarrow{L^2} h(S_{T+}^{x\widehat{X}})$. In addition we show that for small x the quadratic error is

$$\mathbb{E}^{\mathbb{Q}}[(h(S_{T+}^{X^x}) - h(S_{T+}^{x\widehat{X}}))^2] = O(x^3).$$

Proof. By (4.19) and (4.25)

$$(H^x)_L^N - H_L^N = (\frac{1}{x} V_t^x - \widehat{V}_t) - \frac{\lambda}{x} \int_0^T \Theta_u (Z_{1,u}^x)^2 du + \int_0^T (\frac{1}{x} Z_u^x - \widehat{Z}_u) dB_u. \tag{4.26}$$

The formula of Ito yields

$$\begin{aligned}\left((H^x)_L^N - H_L^N\right)^2 &= (\frac{1}{x} V_0^x - \widehat{V}_0)^2 - 2\frac{\lambda}{x} \int_0^T \Theta_u (Z_{1,u}^x)^2 (\frac{1}{x} V_u^x - \widehat{V}_u) du \\ &+ \int_0^T (\frac{1}{x} Z_u^x - \widehat{Z}_u)^2 du + 2 \int_0^T (\frac{1}{x} Z_u^x - \widehat{Z}_u)(\frac{1}{x} V_u^x - \widehat{V}_u) dB_u.\end{aligned}$$

The integrand of the last integral is square integrable and its expectation is 0. Then,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \left|\frac{1}{x}Z_u^x - \widehat{Z}_u\right|^2 du\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\left|\mathbb{E}^{\mathbb{Q}}\left[h^N(S_T^{x\widehat{X}}) - h(S_T) \mid \mathcal{F}_{\tau_L}\right]\right|^2\right] - \mathbb{E}^{\mathbb{Q}}\left[(\frac{1}{x}V_0^x - \widehat{V}_0)^2\right] \\
&\quad + \mathbb{E}^{\mathbb{Q}}\left[2\lambda\frac{1}{x}\int_0^T \Theta_u (Z_{1,u}^x)^2 (\frac{1}{x}V_u^x - \widehat{V}_u) du\right] \\
&\leqslant \mathbb{E}^{\mathbb{Q}}[\left|h^N(S_T^{x\widehat{X}}) - h(S_T)\right|^2] + \frac{2\lambda L^2}{x}\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \mid Z_u^x \mid^2 \left|\frac{1}{x}V_u^x - \widehat{V}_u\right| du\right].
\end{aligned}$$

By Proposition 2.1 in Kobylanski [11] $\left|\widehat{V}_t\right| \leqslant N$ and $\left|\frac{1}{x}V_t^x\right| \leqslant N$. Therefore, $\left|\frac{1}{x}V_t^x - \widehat{V}_t\right| \leqslant 2N$. By (4.23)

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \mid Z_u^x \mid^2 \left|\frac{1}{x}V_u^x - \widehat{V}_u\right| du\right] \leqslant 4x^2 L^2 N^3.$$

By (3.7) and because h is Lipschitz continuous with Lipschitz constant K

$$\left|h(S_T^{x\widehat{X}}) - h(S_T)\right| \leqslant 2K\lambda x \left|\int_0^T \widehat{X}_{u-} dM_u\right| \overset{(4.4)}{\leqslant} 2KL\lambda x \left|\int_0^T \widehat{Z}_{1,u} dM_u\right|. \tag{4.27}$$

The processes ψ_u are bounded for $u \leqslant \tau_L$, $\widehat{Z}$ is square-integrable and $dM_u = \psi_1^0 du + \sum_{j=2}^d \psi_{j,u}^0 dB_{j,u}$. Hence, we find constants C_1 and C_2 such that

$$\mathbb{E}^{\mathbb{Q}}\left[\int_t^T \left|\frac{1}{x}Z_u^x - \widehat{Z}_u\right|^2 du\right] \leqslant C_1 x^2 + C_2 x = O(x). \tag{4.28}$$

As $\left|\frac{1}{x}X_u^x - \widehat{X}_u\right| = \left|\frac{\frac{1}{x}Z_{1,u}^x}{\psi_{1,u}^1} - \frac{\widehat{Z}_{1,u}}{\psi_{1,u}^1}\right| \leqslant L\left|Z_{1,u}^x - \widehat{Z}_{1,u}\right|$ we get that $\frac{1}{x}X^x$ converges to $\widehat{X}$ in L^2. By (3.6) with $X_T^x = \widehat{X}_T = 0$

$$\begin{aligned}
S_{T+}^{X^x} - S_{T+}^{x\widehat{X}} &= 2\lambda \int_0^T \left(x\widehat{X}_u - X_u^x\right) dM_u \\
&= 2\lambda \int_0^T \Theta_u \left(x\widehat{Z}_{1,u} - Z_{1,u}^x\right) du + 2\lambda \sum_{j=2}^d \int_0^T \frac{\psi_{j,u}^0}{\psi_{1,u}^1} \left(x\widehat{Z}_{1,u} - Z_{1,u}^x\right) dB_{j,u}.
\end{aligned}$$

$\widehat{Z}_{1,t} = Z_{1,t}^x = 0$ for $t \in [\tau_L, T]$ and by (4.13) $\frac{\psi_{j,t}^0}{\psi_{1,t}^1} \leqslant L^2$ and $\Theta_t \leqslant L^2$ for $t \leqslant \tau_L$. Then for some constant $C > 0$

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\left|S_{T+}^{X^x} - S_{T+}^{x\widehat{X}}\right|^2\right] \\
&\quad \leqslant 8\lambda^2 L^2 \mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^T \left(x\widehat{Z}_{1,u} - Z_{1,u}^x\right) du\right)^2\right] + 8\lambda^2 L^2 \mathbb{E}^{\mathbb{Q}}\left[\left(\sum_{j=2}^d \int_0^T \left(x\widehat{Z}_{1,u} - Z_{1,u}^x\right) dB_{j,u}\right)^2\right] \\
&\quad \leqslant Cx^2 \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \left(\widehat{Z}_{1,u} - \frac{1}{x} Z_{1,u}^x\right)^2 du\right] \overset{(4.28)}{=} O(x^3).
\end{aligned}$$

♦

The average price per share depends on the number of purchased shares x and is determined by the supply curve

$$S_t^X(x) = S_t^X + M_t x.$$

The liquidity premium per share M_t gives the additional costs for every purchased stock. We are interested in a similar presentation for the average price $\frac{1}{x}\pi_0((H^x)_L^N)$ for x units of the bounded contingent claim $(H^x)_L^N$. By Theorem 4.13, with $\mathbb{1}_{\{t \leqslant \tau_L\}}\Theta_t$ uniformly bounded, we get that

$$\frac{d}{dx}\frac{1}{x}\pi_0\left((H^x)_L^N\right) = \frac{d}{dx}\left(\frac{1}{x}V_0^x\right), \qquad (x \neq 0).$$

Furthermore by (4.24) $\frac{1}{x}V_0^x \to \widehat{V}_0 = \mathbb{E}^{\mathbb{Q}}[H_L^N]$. Consequently, we define

$$\frac{d}{dx}\left(\frac{1}{x}V_0^x\right)\Big|_{x=0} = \lim_{x \to 0} \frac{1}{x}\left(\frac{1}{x}V_0^x - \widehat{V}_0\right).$$

If this value exists, we use

$$\frac{1}{x}\pi_0\left((H^x)_L^N\right) \approx \mathbb{E}^{\mathbb{Q}}[H_L^N] + M^H x \tag{4.29}$$

with $M^H = \frac{d}{dx}\left(\frac{1}{x}V_0^x\right)\Big|_{x=0}$ as an approximation for the average price if the number of units x is small. In that case M^H can be interpreted as the additional cost per unit for the replication of the claim, which is caused by illiquidity.

The following proposition gives a condition when this derivative M^H exists and shows how it can be computed in terms of the solution of the replication problem without trade impacts.

Theorem 4.16. *If h^N is Lipschitz continuous and differentiable except at a finite number of points, then $\frac{1}{x}V_0^x$ is differentiable with respect to x and the derivative is*

$$\frac{d}{dx}\left(\frac{1}{x}V_0^x\right)\Big|_{x=0} = \lambda\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \Theta_u(\widehat{Z}_{1,u})^2 du\right] - 2\lambda\mathbb{E}^{\mathbb{Q}}\left[(h^N)'(S_T)\int_0^T \frac{\widehat{Z}_{1,u}}{\psi^1_{1,u}} dM_u\right].$$

Proof. We want to compute

$$\lim_{x\to 0}\frac{1}{x}(\frac{1}{x}V_0^x - \widehat{V}_0).$$

By the definition of τ_L we have that $\mathbb{1}_{\{u\leqslant\tau_L\}}\Theta_u \leqslant L^2$ and by construction $Z^x_{1,u} = \widehat{Z}_{1,u} = 0$ for $u \in [\tau_L, T]$. By (4.26) we get

$$\begin{aligned}
&\left|\frac{1}{x}(\frac{1}{x}V_0^x - \widehat{V}_0) - \lambda\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \Theta_u(\widehat{Z}_{1,u})^2 du\right] + 2\lambda\mathbb{E}^{\mathbb{Q}}\left[(h^N)'(S_T)\int_0^T \frac{\widehat{Z}_{1,u}}{\psi^1_{1,u}} dM_u\right]\right| \\
&= \left|\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{x}\left(h^N(S_T^{x\hat{X}}) - h^N(S_T)\right)\Big|\mathcal{F}_{\tau_L}\right] + \frac{\lambda}{x^2}\int_0^T \Theta_u(Z^x_{1,u})^2 du\right]\right. \\
&\quad \left. - \mathbb{E}^{\mathbb{Q}}\left[\lambda\int_t^T \Theta_u\widehat{Z}^2_{1,u} du\right] + \mathbb{E}^{\mathbb{Q}}\left[2\lambda(h^N)'(S_T)\int_0^T \frac{\widehat{Z}_{1,u}}{\psi^1_{u,1}} dM_u\right]\right| \\
&\leqslant \mathbb{E}^{\mathbb{Q}}\left[\left|\frac{1}{x}\left(h^N(S_T^{x\hat{X}}) - h^N(S_T)\right) + 2\lambda(h^N)'(S_T)\int_0^T \frac{\widehat{Z}_{1,u}}{\psi^1_{1,u}} dM_u\right|\right] \\
&\quad + \lambda L^2\left|\mathbb{E}^{\mathbb{Q}}\left[\lambda\int_0^T\left(\left(\frac{Z^x_{1,u}}{x}\right)^2 - \widehat{Z}^2_{1,u}\right) du\right]\right|.
\end{aligned} \tag{4.30}$$

Because of (4.28) the second term converges to 0 for $x \to 0$. Furthermore, since h^N is differentiable

$$\begin{aligned}
\lim_{x\to 0}\frac{1}{x}\left(h^N(S_T^{x\hat{X}}) - h^N(S_T)\right) &= \lim_{x\to 0}\frac{1}{x}\left(h^N\left(S_T - 2\lambda x\int_0^T \frac{\widehat{Z}_{1,u}}{\psi^1_{1,u}} dM_u\right) - h^N(S_T)\right) \\
&= \frac{d}{dx}h^N\left(S_T - 2\lambda x\int_0^T \frac{\widehat{Z}_{1,u}}{\psi^1_{1,u}} dM_u\right) \\
&= -2\lambda(h^N)'(S_T)\int_0^T \frac{\widehat{Z}_{1,u}}{\psi^1_{1,u}} dM_u.
\end{aligned}$$

By (4.27) with $\widehat{Z}$ square integrable and processes ψ bounded, we get

$$\mathbb{E}^{\mathbb{Q}}\left[\left|h^N(S_T^{x\hat{X}}) - h^N(S_T)\right|\right] \leqslant 2\lambda x K L\mathbb{E}^{\mathbb{Q}}\left[\left|\int_0^T \widehat{Z}_{1,u} dM_u\right|\right] < \infty.$$

The dominated convergence theorem yields that the first part in (4.30) converges to 0 for $x \to 0$.

♦

Remark. If (Z^x, V^x) is the solution of (4.19) with terminal condition $xH_L^N = x\mathbb{E}^{\mathbb{Q}}[h^N(S_T) \mid \mathcal{F}_{\tau_L}]$ instead of $x(H^x)_L^N = x\mathbb{E}^{\mathbb{Q}}[h^N(S_T^{x\widehat{X}}) \mid \mathcal{F}_{\tau_L}]$, we get

$$\mathbb{E}^{\mathbb{Q}}\left[\lambda \int_0^T \left(\left(\frac{Z^x_{1,u}}{x}\right)^2 - \widehat{Z}^2_{1,u}\right) du\right] \to 0$$

similar to (4.28) and similar to (4.24)

$$\frac{1}{x} V_0^x \to \widehat{V}_0.$$

Analogously to Theorem 4.16 the slope of the supply curve (in $x = 0$) can be calculated in terms of $\widehat{Z}$ and satisfies

$$\frac{1}{x}\left(\frac{1}{x} V_0^x - \widehat{V}_0\right) \to \lambda \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \Theta_u (\widehat{Z}_{1,u})^2 du\right]. \tag{4.31}$$

5. The replication if M is a martingale

In general the minimal replication costs of a claim is not its expected outcome. The expected liquidity costs in the presented model, where M is a submartingale, are non-negative ($\mathbb{E}^{\mathbb{Q}}[L_T] \geqslant 0$) and the minimal replication value therefore bigger or equal to the expectation. We now prove that if the liquidity M is a $\mathbb{Q}$-local martingale, the price of the claim is as in classical theory given by

$$p^H = \mathbb{E}^{\mathbb{Q}}[H].$$

We assume that M is a $\mathbb{Q}$-local martingale, i.e. $\psi^0_{1,u} = 0$ for $0 \leqslant u \leqslant T$ and

$$dM_u = \sum_{j=2}^{d} \psi_i^{0,j} dB_{j,u}. \tag{5.1}$$

Theorem 5.1. *If M is a $\mathbb{Q}$-local martingale and the matrix Ψ_t invertible for $0 \leqslant t \leqslant T$, then the minimal replication value $\pi_0(H)$ of a contingent claim H at time 0 is equal to the expected outcome*

$$\pi_0(H) = \mathbb{E}^{\mathbb{Q}}\left[H\right].$$

Proof. Due to Lemma 3.26 we already know

$$\pi_0(H) \geqslant \mathbb{E}[H].$$

Consider the approximating sequence of smooth trading strategies $(X^n, \chi^n, Y^n)_{n\geqslant 1}$ constructed in the proof of Theorem 4.9 with $X^n_T = X^n_0 = \chi^n_T = \chi^n_0 = 0$ and $[X^n, X^n]_t = [\chi^n, \chi^n]_t = 0$. Hence, $QV(X^n, \chi^n)_T = 0$ and since M is a $\mathbb{Q}$-local martingale $\overline{M} = M$. By (4.1) and the definition of the process U

$$\begin{aligned} Y^n_T &= Y^n_{0-} + \int_0^T X^n_{u-} dS_u + \sum_{k=2}^{d} \int_0^T \chi^{n,k}_{u-} dG^k_u - \lambda \int_0^T (X^n_{u-})^2 dM_u \\ &= Y^n_{0-} + U_T(X^n, \chi^n). \end{aligned}$$

Because (X^n, χ^n, Y^n) is L^2-admissible the process $U_T(X^n, \chi^n)$ is a martingale and $\mathbb{E}^{\mathbb{Q}}\left[Y^n_T\right] = Y^n_0$. Since Y^n_T converges to H in L^1

$$\mathbb{E}^{\mathbb{Q}}\left[H\right] = \lim_{n\to\infty} \mathbb{E}^{\mathbb{Q}}\left[Y^n_T\right] = \lim_{n\to\infty} Y^n_0.$$

This implies $\pi_0(H) = \mathbb{E}^{\mathbb{Q}}\left[H\right]$.

♦

Theorem 5.2. *Under the same conditions as in Theorem 5.1 every time-0 value $p^H \neq \mathbb{E}^{\mathbb{Q}}[H]$ of a $\mathbb{Q}$-integrable claim H leads to arbitrage opportunities in a model extended with the trade of H.*

Proof. For the claim H let $((X^n, \chi^n, Y^n))_{n \geqslant 1}$ be an approximating sequence. In general the elements of the sequence $((-X^n, -\chi^n, -Y^n))_{n \geqslant 1}$ are not self-financing and the sequence is therefore no asymptotic solution of the claim $(-H)$. But since $(-H)$ is integrable and $\pi_0(-H) = \mathbb{E}^{\mathbb{Q}}[-H] = -\mathbb{E}^{\mathbb{Q}}[H]$, there exists an approximating sequence $((\widetilde{X}^n, \widetilde{\chi}^n, \widetilde{Y}^n))_{n \geqslant 1}$ with $\widetilde{Y}^n_T \to (-H)$ in L^1 and $\widetilde{Y}^n_0 = -\mathbb{E}^{\mathbb{Q}}[H]$. If $p^H \neq \mathbb{E}[H]$, we can generate arbitrage:

- $p^H > -\mathbb{E}^{\mathbb{Q}}[H]$: At time 0 we sell the claim H at price p^H. Then $p^H - \mathbb{E}[H] = c > 0$. The sequence of trading strategies $((X^n, \chi^n, Y^n + c))_{n \geqslant 1}$ is an asymptotic arbitrage opportunity.

- $p^H < \mathbb{E}^{\mathbb{Q}}[H]$: We buy the claim H at price p^H and invest $\mathbb{E}^{\mathbb{Q}}[H] - p^H = c > 0$ in the risk free asset. The sequence $((\widetilde{X}^n, \widetilde{\chi}^n, \widetilde{Y}^n + c))_{n \geqslant 1}$ generates asymptotical arbitrage.

♦

Corollary 5.3. *The equivalent martingale measure $\mathbb{Q}$ is unique.*

Proof. Consider the claim $H = \mathbb{1}_A$ where $A \in F_T$. Then

$$\mathbb{Q}(A) = \mathbb{E}^{\mathbb{Q}}[\mathbb{1}_A] = \pi_0^H.$$

Since the minimal replication value is independent of the measure $\mathbb{Q}$, $\mathbb{Q}$ is unique.

♦

Remark. If M is a $\mathbb{Q}$-local martingale, $\mathbb{Q}$ is unique and the price of a contingent claim H is equal to the classical price $\mathbb{E}^{\mathbb{Q}}[H]$. Nevertheless, the replication strategy in a model with liquidity risk is different from the hedge in classical theory.

We illustrate this in the following example.

Example: ***- Constant liquidity in a Black-Scholes economy***:

Consider a specific model, where B is a one dimensional Brownian motion (d=1) and

$$\begin{aligned}\psi^1_{u,1} &= \sigma S_u\\ \psi^0_{u,1} &= 0\end{aligned}$$

with $\sigma > 0$, $S_0 > 0$ and $M_0 = M > 0$ for $0 \leqslant t \leqslant T$. Consequently, the liquidity process M is constant and the unaffected quoted price process is a geometric Brownian motion

$$dS_u = \sigma S_u dB_u,$$

or equivalently

$$S_t = \sigma S_0 \exp\left(\sigma B_t - \frac{\sigma^2}{2}t\right).$$

By (3.15) the liquidity costs of a s.f.t.s are

$$L_t = \lambda M X_t^2 + (1-\lambda) M \left([X,X]_t + X_0^2\right) \geqslant 0.$$

We immediately see that $L_T = 0$ if the process X is continuous with finite variation and satisfies $X_T = X_0 = 0$. In that case $S^X_t = S_t + 2\lambda M X_t$ and $S^X_T = S_T$.

Consider a European call option with strike K on the quoted stock price S_T, which delivers cash at maturity T, i.e.

$$H = (S_T - K)_+.$$

Because M is a martingale, by Theorem 5.1 the fair price of this claim is equal to its expectation

$$\pi_0(H) = \mathbb{E}\left[(S_T - K)_+\right],$$

which is well known and given by the Black-Scholes formula

$$\begin{aligned}\pi^C_t &= S_t \Phi(d_t) - K\Phi(d_t - \sigma\sqrt{T-t})\\ d_t &= \frac{\log(\frac{S_t}{K}) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\end{aligned}$$

where Φ is the standard cumulative normal distribution function. The classical Δ-Hedge of the call option is $(\widehat{X}, \widehat{Y}) = (\widehat{X}_t, \widehat{Y}_t)_{0\leqslant t\leqslant T}$ with

$$\widehat{X}_t = \Phi(d_t) \quad \text{and} \quad \widehat{Y}_t = -K\Phi(d_t - \sigma\sqrt{T-t}).$$

This trading strategy is continuous, but not of finite variation. In a model with liquidity risk this strategy does not replicate the payoff $(S_T - K)_+$. In fact it leads to the following liquidity costs

$$\begin{aligned} L_T &= \lambda M \widehat{X}_T^2 + (1-\lambda) M \left([\widehat{X}, \widehat{X}]_T + \widehat{X}_0^2 \right) \\ &= \lambda M \left(\Phi(d_T)\right)^2 + (1-\lambda) M \left(\Phi(d_0)\right)^2 + (1-\lambda) M \int_0^T d\left[\Phi(d_u), \Phi(d_u)\right]_u \\ &= \lambda M \left(\Phi(d_T)\right)^2 + (1-\lambda) M \left(\Phi(d_0)\right)^2 + (1-\lambda) M \int_0^T \left(\Phi'(d_u)\right)^2 d\left[d_u, d_u\right]_u \\ &= \lambda M \left(\Phi(d_T)\right)^2 + (1-\lambda) M \left(\Phi(d_0)\right)^2 + (1-\lambda) M \int_0^T \frac{\left(\Phi'(d_u)\right)^2}{T-u} du \end{aligned}$$

since

$$d\left[d_u, d_u\right]_u = \frac{1}{\sigma^2 (T-u)} d\left[\log(\frac{S_u}{K}), \log(\frac{S_u}{K})\right] = \frac{1}{T-u} du.$$

See Cetin, Jarrow, Protter, Warachka [23] in this context.

Figure 5.1 shows the value of the Δ-Hedge in the Black-Scholes model (BS) and the value of the same hedge in the model with liquidity risk (LR). In this plot we observe the hedge of a call option with $S_0 = 80$, $\sigma = 0.3$, strike price $K = 75$ and a maturity of $T = 0.5$ years. In addition we use the following model parameters in the model with liquidity risk: $\lambda = 0.5$. The parameter M usually takes very small values, but to demonstrate a significant difference between the models we choose a big value M=10. For a trading strategy $X = (X_t)_{0 \leqslant t \leqslant T}$ and initial value $Y_0^{BS} = Y_0^{LR} = \mathbb{E}^{\mathbb{Q}}[H]$, let (X, Y^{BS}) be a self-financing strategy in the Black Scholes model (3.13) and by (3.11) let (X, Y^{LR}) be a s.f.t.s in the model with liquidity risk. Furthermore, let us define $V_t^{BS}(X) = V_t(X, Y^{BS})$ as the value of (X, Y^{BS}) in the BS model $(M = 0)$ and $V_t^{LR}(X) = V_t^L(X, Y^{LR})$ as the value of (X, Y^{LR}) in the model with liquidity risk. In the observed trajectory the stock price takes the value $S_T = 90.69$ and the Δ-Hedge $\widehat{X}$ leads to the following values:

$$\begin{aligned} V_0^{BS}(\widehat{X}) - V_0^{LR}(\widehat{X}) &= 0 \\ V_{\frac{T}{4}}^{BS}(\widehat{X}) - V_{\frac{T}{4}}^{LR}(\widehat{X}) &= 5.09 \\ V_{\frac{T}{2}}^{BS}(\widehat{X}) - V_{\frac{T}{2}}^{LR}(\widehat{X}) &= 5.68 \\ V_{\frac{3T}{4}}^{BS}(\widehat{X}) - V_{\frac{3T}{4}}^{LR}(\widehat{X}) &= 5.51 \\ V_T^{BS}(\widehat{X}) - V_T^{LR}(\widehat{X}) &= 6.35. \end{aligned}$$

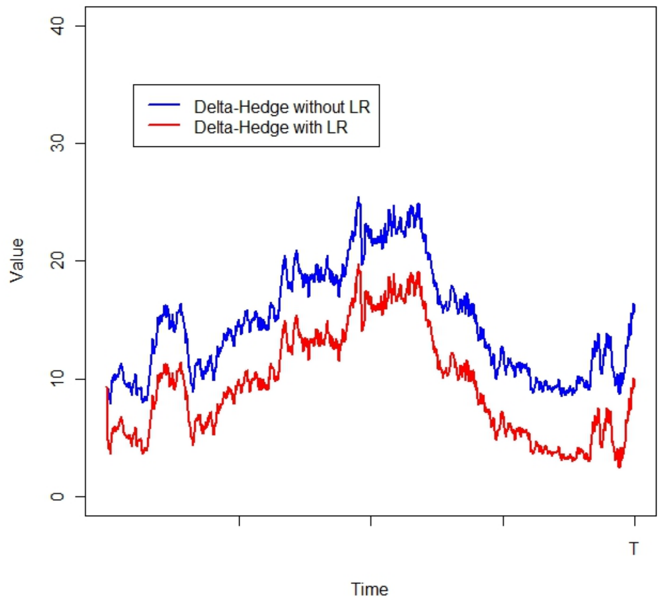

Figure 5.1.: The Delta-Hedge in the BS and LR model

Note that the value of the trading strategy in the model with illiquidity (LR) in Figure 5.1 has a jump immediately after time 0. This is due to the term $(1-\lambda)MX_0^2$ in the liquidity costs (3.15).

We now construct a smooth version of the Δ-Hedge and show that the trading strategy converges to the claim C at maturity. Therefore, let $(\widetilde{X}^n, \widetilde{Y}^n)$, with $\widetilde{Y}_0^n = \widehat{Y}_0 = \mathbb{E}^{\mathbb{Q}}[H]$, be a s.f.t.s in the model with liquidity risk and

$$\widetilde{X}_t^n = \begin{cases} n\int_{(t-\frac{1}{n})\vee 0}^t \widehat{X}_u du, & 0 < t \leqslant T - \frac{1}{n}, \\ n^2(T-t)\widehat{X}_{T-\frac{1}{n}}, & T - \frac{1}{n} < t \leqslant T. \end{cases}$$

The continuous process $\widetilde{X}^n$ is bounded and of finite variation with $\widetilde{X}_T^n = 0$. By (3.12) with M constant and Theorem A.6

$$\widetilde{Y}_T^n = \widetilde{Y}_0^n + \int_0^T \widetilde{X}_{u-}^n dS_u \xrightarrow{L^2} \widehat{Y}_0 + \int_0^T \widehat{X}_{u-} dS_u = H, \qquad (n \to \infty).$$

For a fixed n the error

$$H - \widetilde{Y}_T^n = \int_0^T (\widetilde{X}_u^n - \widehat{X}_u)\sigma S_u dB_u$$

is normally distributed with a variance converging to zero. Figure 5.2 compares the Δ-Hedge with the constructed smooth trading strategy (with $n \approx 8$). For the observed

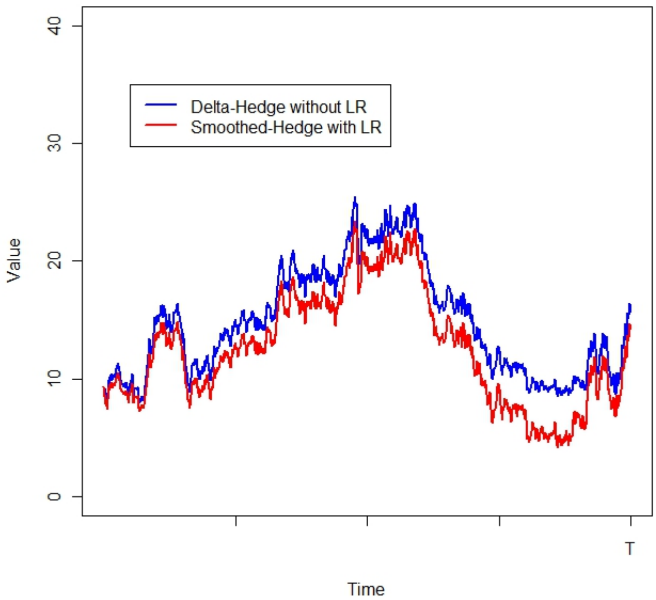

Figure 5.2.: The replication strategy in the BS and LR model

trajectory the following values are obtained,

$$\begin{aligned}
V_0^{BS}(\widehat{X}) - V_0^{LR}(\widetilde{X}^n) &= 0\\
V_{\frac{T}{4}}^{BS}(\widehat{X}) - V_{\frac{T}{4}}^{LR}(\widetilde{X}^n) &= 1.96\\
V_{\frac{T}{2}}^{BS}(\widehat{X}) - V_{\frac{T}{2}}^{LR}(\widetilde{X}^n) &= 2.31\\
V_{\frac{3T}{4}}^{BS}(\widehat{X}) - V_{\frac{3T}{4}}^{LR}(\widetilde{X}^n) &= 3.28\\
V_T^{BS}(\widehat{X}) - V_T^{LR}(\widetilde{X}^n) &= 1.50
\end{aligned}$$

We see that in the liquidity risk model the smooth hedge results in a better replication than the Δ-Hedge.

This model is an extension of the liquidity risk model of Cetin, Jarrow, and Protter [22]. In their model the stock price is determined by an increasing supply curve, but it is assumed that the trader acts as a price taker and consequently does not trigger any price impacts.

From now on we deal with a more sophisticated liquidity process. If M is a strict submartingale the replication value of an asset differs from the expected payoff of a claim. In order to calculate this value we have to reconstruct the claim using the basic assets of our economy.

6. Calculation of the minimal replication value

In Section 4.1 we create a sequence of smooth trading strategies, which approximately replicates the $\mathbb{Q}$-integrable claim H. Furthermore we have seen that in order to calculate the minimal replication costs we only consider smooth admissible sequences. To obtain the minimal replication value we solve the BSDE

$$dV_u = -\lambda Z_{1,u}^2 \Theta_u du + \sum_{j=1}^{d} Z_{j,u} dB_{j,u}, \quad 0 \leqslant u \leqslant T, \tag{6.1}$$

with terminal condition

$$V_T = H.$$

By Theorem 4.6 if

(i) Ψ is invertible,
(ii) Θ is uniformily bounded,
(iii) there exists a minimal solution (Z^*, V^*) of the BSDE (6.1),

then the minimal replication value satisfies

$$\pi_0(H) = V_0^*.$$

In this chapter we consider simple economies. For contingent claims which depend on the unaffected quoted stock price we calculate the minimal replication costs $\pi_0(H)$.

We assume that the liquidity process M is deterministic. To make sure the model is free of arbitrage we want M to be a submaringale. Hence, $(M_t)_{0 \leqslant t \leqslant T}$ is a deterministic non-decreasing function. We consider models, where the unaffected stock price is given by a Bachelier and a Black-Scholes economy.

6.1. A Bachelier model with deterministic liquidity

Let us consider the model

$$\begin{aligned} dS_t &= \sigma dB_t, & S_0 &> 0, \\ dM_t &= c dt, & M_0 &> 0, \end{aligned}$$

where $\sigma > 0$, $c > 0$ and B is a one dimensional Brownian motion. The unaffected stock price is given by a Bachelier process

$$S_t = S_0 + \sigma B_t$$

and the supply curve is a linear function in time with a positive slope. In the notation of the general model of Section 4.1 this is equivalent to

$$\psi_1^0 = c$$
$$\psi_1^1 = \sigma$$

and $\psi_j^i = 0$ otherwise. $\Theta_u = \frac{c}{\sigma^2}$ is constant and $\Psi_u = \sigma$ is invertible (d=1). To obtain the minimal replication costs $\pi_0(H)$, we want to find a minimal solution of the BSDE (6.1) or equivalently a minimal solution (Z, V) of the equation

$$H = V_t - \frac{k}{2}\int_t^T Z_u^2 du + \int_t^T Z_u dB_u, \qquad 0 \leqslant t \leqslant T, \tag{6.2}$$

with $k = \frac{2\lambda c}{\sigma^2}$. We derive a closed form solution for this equation and show that it is minimal.

Theorem 6.1. *Let $H \geqslant 0$ be a $\mathcal{F}_t$-measureable random variable with $\mathbb{E}^{\mathbb{Q}}[\exp(2kH)] < \infty$. Then there exists a minimal solution (Z, V) of (6.2) with*

$$V_t = \frac{1}{k}\log\left(\mathbb{E}^{\mathbb{Q}}\left[\exp(kH)\middle|\mathcal{F}_t\right]\right).$$

Proof. The process $N_t = \mathbb{E}^{\mathbb{Q}}\left[\exp(kH)\middle|\mathcal{F}_t\right]$ is a square-integrable martingale. By the martingale representation theorem there exists a predictable process X with

$$N_t = N_0 + \int_0^t X_u dB_u.$$

Let us define

$$V_t = \frac{1}{k}\log(N_t) \qquad \text{and} \qquad Z_t = \frac{X_t}{kN_t}.$$

Then $V_T = \frac{1}{k}\log(\exp(kH)) = H$ and the formula of Ito yields

$$dV_t = \frac{1}{kN_t}dN_t - \frac{1}{2kN_t^2}(dN_t)^2 = Z_t dB_t - \frac{k}{2}Z_t^2 dB_t.$$

This is equivalent to equation (6.2). By martingale presentation theorem $\mathbb{E}^{\mathbb{Q}}\left[\int_0^T (kN_u Z_u)^2 du\right] < \infty$. Since $N_t \geqslant 1$ we conclude $\mathbb{E}^{\mathbb{Q}}\left[\int_0^T Z_u^2 du\right] < \infty$. Hence, (Z, V) is a solution of the BSDE (6.2).

It remains to show that the solution is minimal. For an arbitrary solution $(\widetilde{Z}, \widetilde{V})$ consider $E_t = \exp(k\widetilde{V}_t)$. Then $E_T = \exp(kH)$ and by Ito

$$dE_t = kE_t d\widetilde{V}_t + \frac{1}{2}k^2 E_t (d\widetilde{V}_t)^2 = kE_t \widetilde{Z}_t dB_t.$$

The process E is a $\mathbb{Q}$-local martingale which is bounded from below. Hence, E is a supermartingale and

$$\exp(k\widetilde{V}_t) = E_t \geqslant \mathbb{E}^{\mathbb{Q}}\left[\exp(kH)\Big|\mathcal{F}_t\right] = N_t = \exp(kV_t).$$

Consequently,

$$V_t \leqslant \widetilde{V}_t, \qquad 0 \leqslant t \leqslant T.$$

♦

Lemma 6.2. *The minimal replication value of the European option $H = (S_T - K)_+$ (resp. $H = (K - S_T)_+$) is*

$$\pi_0(H) = \frac{1}{k}\log\left(\mathbb{E}^{\mathbb{Q}}\left[\exp(kH)\right]\right)$$

where $k = \frac{2\lambda c}{\sigma^2}$ and

$$\mathbb{E}^{\mathbb{Q}}\left[\exp(k(S_T - K)_+)\right] = \exp\left(k(S_0 - K) + \frac{1}{2}k^2\sigma^2 T\right)\Phi(-d^B + \sigma k\sqrt{T}) + \Phi(d^B)$$
$$\mathbb{E}^{\mathbb{Q}}\left[\exp(k(K - S_T)_+)\right] = \exp\left(k(K - S_T) + \frac{1}{2}k^2\sigma^2 T\right)\Phi(d^B + \sigma k\sqrt{T}) + \Phi(-d^B)$$

where Φ denotes the cumulative distribution function of a standard normal distribution and

$$d^B = \frac{K - S_0}{\sigma\sqrt{T}}.$$

Proof. The claim $H = (S_T - K)_+$ satisfies

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[\exp(2kH)] &\leqslant \exp\left(2k(S_0 - K)\right)\mathbb{E}^{\mathbb{Q}}\left[\exp(2k\sigma W_T)\right] \\ &= \exp\left(2k(S_0 - K)\right)\exp(2k\sigma^2 T) < \infty.\end{aligned}$$

Note that $S_T > K$ is equivalent to $W_T > \frac{K-S_0}{\sigma}$. Then,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[\exp(k(S_T-K)_+)\right] \\
&\quad= \exp(k(S_0-K))\mathbb{E}^{\mathbb{Q}}\left[\exp(k\sigma W_T \mathbb{1}_{\{S_T>K\}})\right] \\
&\quad= \exp(k(S_0-K))\frac{1}{\sqrt{2\pi T}}\int_{\frac{K-S_0}{\sigma}}^{\infty}\exp\left(-\frac{1}{2T}(x^2-2Tk\sigma x)\right)dx \\
&\qquad+\frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{\frac{K-S_0}{\sigma}}\exp\left(-\frac{x^2}{2\sqrt{(T)}}\right)dx \\
&\quad= \exp\left(k(S_0-K)+\frac{1}{2}\sigma^2k^2T\right)\frac{1}{\sqrt{2\pi T}}\int_{\frac{K-S_0}{\sigma}}^{\infty}\exp\left(-\frac{1}{2}(\frac{x-Tk\sigma}{\sqrt{T}})^2\right)dx \\
&\qquad+\frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{\frac{K-S_0}{\sigma}}\exp\left(-\frac{x^2}{2\sqrt{(T)}}\right)dx \\
&\quad= \exp\left(k(S_0-K)+\frac{1}{2}\sigma^2k^2T\right)\Phi\left(\frac{K-S_0-k\sigma^2T}{\sigma\sqrt{(T)}}\right)+\Phi\left(\frac{K-S_0}{\sigma\sqrt{(T)}}\right).
\end{aligned}$$

The calculation for $H = (S_T-K)_+$ is similar. ♦

To compare the replication costs with the values in classical arbitrage theory, let us consider the price of a European option in the Bachelier model without liquidity risk (LR). The price of a European call option in a Bachelier model without LR is

$$C_0^B = (S_0-K)\Phi\left(-d^B\right)+\sigma\sqrt{T}\phi(d^B)$$

where Φ is the CDF and ϕ is the density function of a standard normal distribution and $d^B = \frac{K-S_0}{\sigma\sqrt{T}}$. By the Call-Put parity the price of a put option is

$$P_0^B = K-S_0-C_0^B.$$

We compare this price with the minimal replication costs of smooth strategies in the model with deterministic liquidity. We consider a European call option $H = (S_T-K)_+$ with strike price $K = 100$ and maturity $T = 0.5$. In addition we use the following model parameters: $\lambda = 1$, $\sigma = 3$ and $c = 0.5$. Figure 6.1 shows the price of the option for different initial values of the underlying stock price.

As expected the replication costs in the model with liquidtiy risk is slightly bigger than the price in classical theory. This is due to the positive liquidity costs

$$L_T(X^n) \to \lambda\int_0^T X_{u-}^2 dM_u,$$

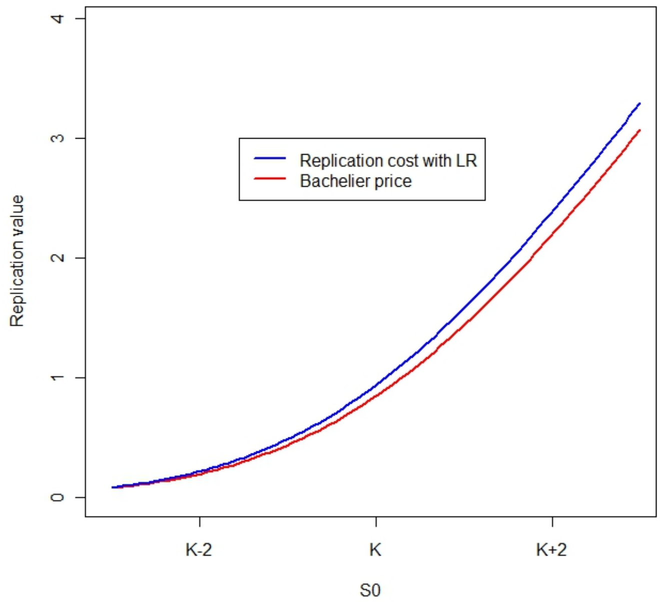

Figure 6.1.: Replication value: $S_0 \approx K$

where X^n is the smooth replication strategy in the model with LR and $X_t = \frac{Z_{1,t}}{\sigma^2}$ where (Z, V) is the minimal solution of the BSDE. See Proposition (3.15). The replication costs of a claim are an increasing function in the size of the transaction. This will be further discussed in Section 6.2.

We are interested in the impact of λ on the minimal replication value. Remember that the bid-ask spread is filled up immediately after a trade. The parameter $\lambda \in [0, 1]$ is the part of the spread which is refilled with ask orders and therefore corresponds to the impact on the stock price. Then, $\lambda = 0$ corresponds to no price impact and $\lambda = 1$ indicates the biggest impact. Figure 6.2 illustrates the impact of the parameter λ on the replication costs of the call option with $S_0 = 98$.

The slope of the supply curve c has the same impact on the replication costs as λ. This is reasonable since c determines the size of the bid-ask spread after a trade. See Figure 6.3.

As a second example consider the claim

$$H^x = (S_T^{x\hat{X}} - K)_+$$

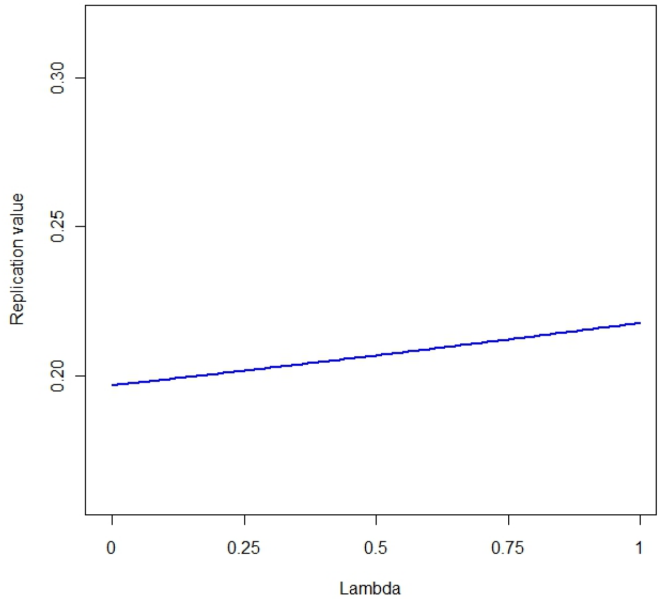

Figure 6.2.: Impact of parameter $\lambda \in [0, 1]$

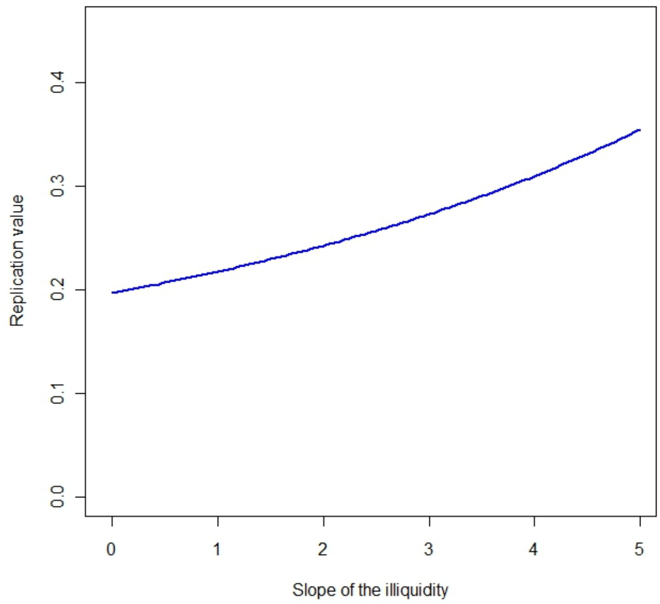

Figure 6.3.: Impact of the slope $c > 0$

where $S_T^{x\widehat{X}}$ is the observed quoted price and $\widehat{X}$ denotes the stock part of the hedge $(\widehat{X}, \widehat{Y})$ in the model without illiquidity, e.g. $\widehat{X}_t = \Phi(\frac{S_t - K}{\sigma\sqrt{T-t}})$.

We use a Monte-Carlo simulation to calculate the expectation $\mathbb{E}^{\mathbb{Q}}[\exp(kH^x)]$. Since $H^x \leqslant H$, where $H = (S_T - K)_+$, the replication costs of H^x are less or equal to the costs of H. In Figure 6.4 we consider the replication costs of the claim H^x and H with the same model parameters. For $\lambda = 1$ and a high slope of the illiquidity $c = 0.5$, the replication costs of H^x are lower than the price in classical theory.

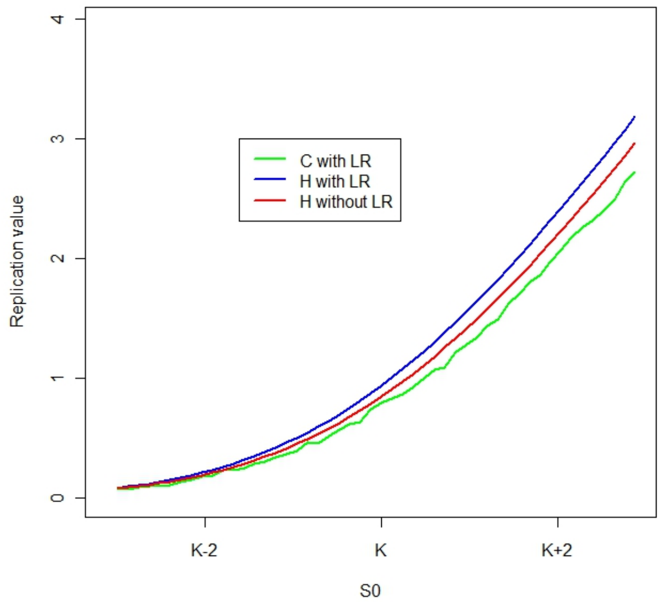

Figure 6.4.: Replication value: $S_0 \approx K$

6.2. A Black-Scholes model with deterministic liquidity

We assume that the unaffected stock price is the Black-Scholes price and the liquidity process is again a linear function with constant, positive slope

$$
\begin{aligned}
dS_t &= \sigma S_t dB_t, & S_0 > 0, \\
dM_t &= c dt, & M_0 > 0,
\end{aligned}
$$

where $\sigma > 0$ and $c > 0$. Then $S_t = S_0 \exp(\sigma B_t - \frac{\sigma^2}{2} t) > 0$. Fitting this model in the general model of Chapter 4 means

$$\psi_1^0 = c$$
$$\psi_1^1 = \sigma S_t$$

and $\psi_j^i = 0$ otherwise. $\Psi_u = \psi_{1,u}^1 > 0$ but the process $\Theta_u = \frac{c}{\sigma^2 S_u^2}$ is not bounded. Hence, for a claim H we consider

$$H_L^N = \mathbb{E}^{\mathbb{Q}} \Big[\max\left(\min(H, N), -N\right) \Big| \Big],$$

with the stopping time $\tau_L \leqslant T$ defined in (4.13). Then, $\mathbb{1}_{\{t \leqslant \tau_L\}} \Theta_t \leqslant L^2$ and the solution $(Z^{N,L}, V^{N,L})$ of the stochastic integral equation

$$H_L^N = V_t - \lambda \int_t^T \Theta_u^L Z_u^2 du + \int_t^T Z_u dB_u, \qquad 0 \leqslant t \leqslant T, \tag{6.3}$$

is unique. By Theorem 4.13 we get $\pi_0(H_L^N) = V_0^{N,L}$. Furthermore, $\tau_L \to T$ a.s. and $H_L^N \to H$ in L^1.

Since we do not know a closed form solution of this BSDE, we look for a numerical approximation. To apply numerical methods we have to get rid of the dependency of the claim H_L^N on S_{τ_L}. Instead of equation (6.3) we consider

$$H = V_t - \frac{\lambda c}{\sigma^2} \int_t^T \frac{Z_u^2}{S_u^2} du + \int_t^T Z_u dB_u, \qquad 0 \leqslant t \leqslant T, \tag{6.4}$$

where H is a function of the unaffected quoted price S_T. We assume that if there exists a solution (Z, V) of (6.4), then $V_0^{N,L} \to V_0$.

Let us define the value process

$$V_t = F(t, S_t)$$

with $F(T, S_T) = H$ and $F(0, S_0) = Y_0$. If F is sufficiently smooth, then by Ito´s formula

$$\begin{aligned} dV_t &= \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial S_t} dS_t + \frac{1}{2} \frac{\partial F^2}{\partial S_t^2} d[S,S]_t \\ &= \left(\frac{\partial F}{\partial t} + \frac{\partial F^2}{\partial S_t^2} \frac{\sigma^2}{2} S_t^2 \right) dt + \frac{\partial F}{\partial S_t} \sigma S_t dB_t. \end{aligned}$$

On the other hand, by (6.4) V_t satisfies

$$dV_t = -\frac{\lambda c}{\sigma^2 S_t^2} Z_t^2 dt + Z_t dB_t.$$

The BSDE satisfies the Markov property and a comparison of the coefficients yields

(i)

$$Z_t = \frac{\partial F}{\partial S_t} \sigma S_t,$$

(ii)

$$\frac{\partial F}{\partial t} + \frac{\partial F^2}{\partial S_t^2} \frac{\sigma^2}{2} S_t^2 = -\frac{\lambda c}{\sigma^2 S_t^2} Z_t^2.$$

Condition (ii) with Z_t by (i) results in the following partial differential equation (PDE).

If the value process $F(t, x)$ is two-times differentiable then it satisfies

$$\frac{\partial F}{\partial t} + \lambda c \left(\frac{\partial F}{\partial x} \right)^2 + \frac{\sigma^2}{2} x^2 \frac{\partial F^2}{\partial x^2} = 0 \tag{6.5}$$

for $t \in [0, T]$ and $x \in (0, \infty)$. Conversely, if (6.5) has a classical solution then $V_t = F(t, S_t)$ satisfies (6.4).

Remark. A. Roch [3] proved that $F(t, x)$ is a viscosity solution of (6.5).

Remark. If H depends on the stock price $S_T^{\hat{X}} = S_T - \lambda \int_0^T \hat{X}_{u-}^2 dM_u$, where $\hat{X}$ is the hedge in the Black-Scholes model without illiquidtiy, the claim depends on $(S_t)_{0 \leqslant t \leqslant T}$ and $(M_t)_{0 \leqslant t \leqslant T}$. To get rid of this path dependency, Roch [3] defines a process

$$L_t = 2\lambda \int_0^t \hat{X}_{u-}^2 dM_u.$$

Then H is a function of the quoted stock price S_T and L_T. The resulting PDE has 3 dimensions and is therefore more complicated than equation (6.5), where $H = F(T, S_T)$.

To obtain the numerical replication costs of a claim $H = F(T, S_T)$ we use two different numerical methods. We first solve the corresponding PDE (6.5) in a finite difference scheme. In addition, we solve the BSDE (6.4) in a binomial tree. From now on we assume that the solution is correct if the results of these two methods match each other. In that case we denote the numerical replication costs by $\pi_0^N(H)$. See Appendix B for further information on the numerical methods.

Let us have a look at the replication costs $\pi_0^N(H)$ of a European put option $H = (K - S_T)_+$. Remember that the price of this option in classical theory is given by the Black-Scholes formula

$$P_0^{BS} = K\Phi(-d^{BS} + \sigma\sqrt{T}) - S_0\Phi(-d^{BS})$$

where

$$d^{BS} = \frac{\log(\frac{S_0}{K} + \frac{1}{2}\sigma^2 T)}{\sigma\sqrt{T}}.$$

We define $K = 100$ and $T = 0.5$ and choose the following model parameters: $\lambda = 1$, $\sigma = 0.3$ and $c = 0.5$. The finite difference method calculates the prices of options for $S_0 \in [0, U]$, for an upper bound $U = 200$. Figure 6.5 compares the price of the put option in the BS model without illiquidity (red) and the (numerical) replication value in the model with liquidity risk (blue). A close-up (Figure 6.6) for initial values around

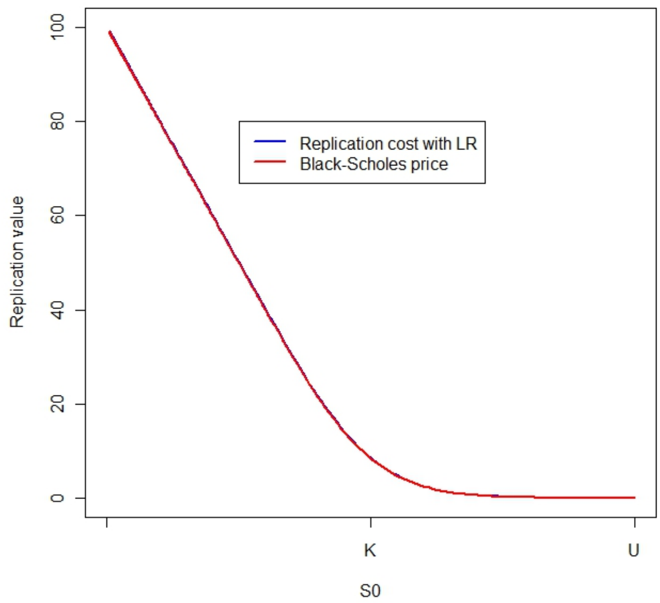

Figure 6.5.: Replication value: European put option

the strike price K shows that the numerical replication value in the model with liquidity risk is bigger than the Black-Scholes price. This is due to the positive liquidity costs

$$L_T(X^n) \to \lambda c \int_0^T X_{u-}^2 du$$

where (X^n, Y^n) is the smooth approximating replication strategy and $X_u = \frac{Z_{1,u}}{\sigma S_u}$, where (Z, V) is the solution of the BSDE (6.4). See Proposition 3.15.

We are interested in the average price per share if x shares are purchased. We therefore consider the value

$$H(x) = \frac{1}{x} \pi_0^N \left(x(K - S_T)_+ \right)$$

where x is the order size. Figure 6.7 represents $H(x)$ with initial underlying value $S_0 = 105$ dependent on the size of the transaction x. Recall that the supply curve of the

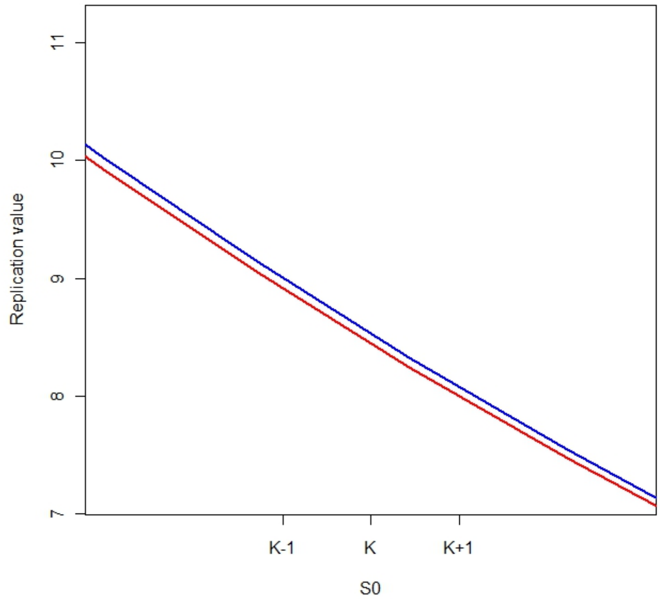

Figure 6.6.: Replication value: $S_0 \approx K$

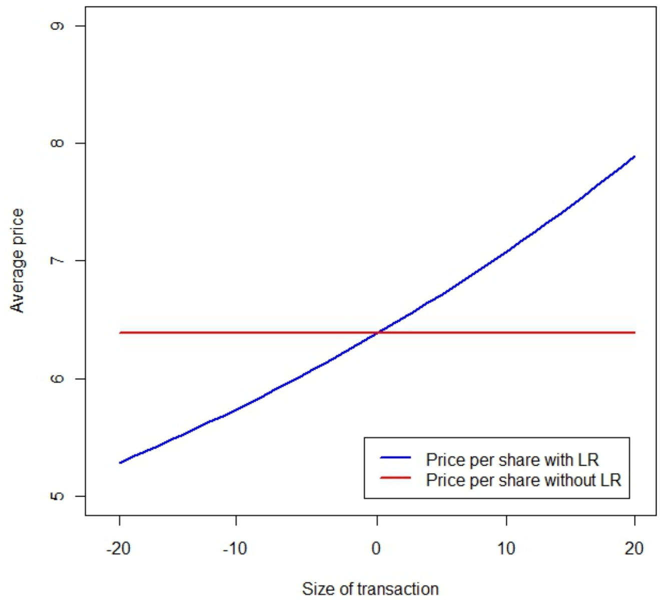

Figure 6.7.: The supply curve of the put option

stock price is linear with a positive slope. Figure 6.7 suggests that similar to the stock price the average price per share of the claim H is an increasing linear function in the order size. Since $H = h(S_T)$ is bounded and h Lipschitz continuous, we get the slope of the supply curve in $x = 0$ by (4.31). We then use the approximation

$$H(x) \approx H(0) + H^{'}(0) \cdot x,$$

where

$$H^{'}(0) = \lambda E\left[\int_0^T \Theta_u(\widehat{Z}_{1,u})^2 du\right] = \lambda E\left[\int_0^T c(\widehat{X}_u)^2 du\right].$$

The Δ-Hedge $\widehat{X}$ in the classical theory is

$$\widehat{X} = \Phi\left(\frac{\log(\frac{S_t}{K}) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) - 1.$$

A Monte-Carlo simulation yields

$$H^{'} \approx 0.0658$$

and $H(0) = 6.385$. Hence by (4.29)

$$H(x) \approx 6.385 + 0.0658x. \tag{6.6}$$

Figure 6.8 compares this approximation (red) of the price per share with the supply curve $H(x)$ of Figure 6.7. We notice that the impact of liquidity is more than linear. For big values x the replication value is significantly higher than the linear approximation. This is expected since $L(xX) = x^2 L(X)$ for $x \in \mathbb{R}$. Nevertheless, if the number of purchased claims is small, (6.6) serves as a good approximation of the exact (numerical) results.

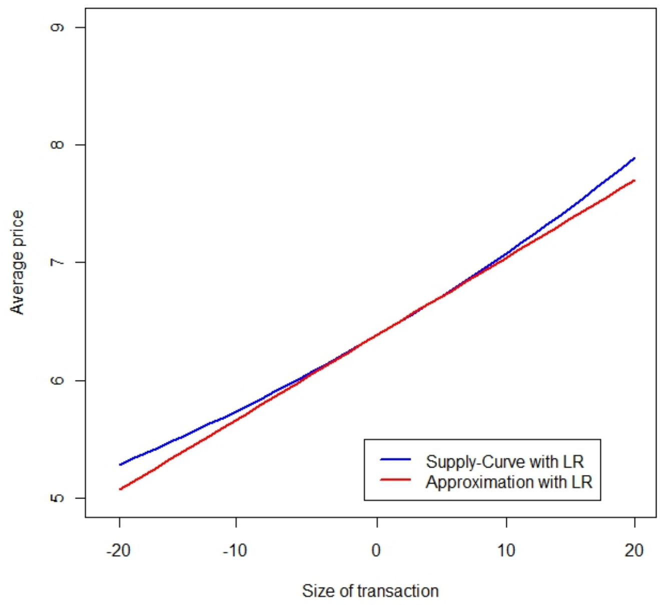

Figure 6.8.: The approximation of the supply curve

A. Smoothing of stochastic integrals

Definition A.1. *Let X be a semimartingale. X is called a* ***special semimartingale*** *if it has a decomposition*

$$X = \overline{N} + \overline{A}$$

where $\overline{N}$, $\overline{N}_0 = X_0$, is a local martingale and $\overline{A}$, $\overline{A}_0 = 0$, a predictable process which has finite variation.

Remark. It can be shown that a semimartingale X is special if and only if X is locally integrable. Furthermore, note that in case X is continuous, the processes $\overline{A}$ and $\overline{N}$ are continuous.

Definition A.2. *The $\mathcal{H}_T^2$* ***norm*** *of a special semimartingale X, with canonical decomposition $X = \overline{N} + \overline{A}$, is*

$$\|X\|_{\mathcal{H}_T^2} = \mathbb{E}\left[\int_0^T d[\overline{N}, \overline{N}]\right] + \mathbb{E}\left[\left(\int_0^T |d\overline{A}|\right)^2\right].$$

The space of all special semimartingales with finite $\mathcal{H}_T^2$ norm is denoted by $\mathcal{H}^2(0,T)$. For $X \in \mathcal{H}^2(0,T)$ with canonical decomposition $X = \overline{N} + \overline{A}$ and H and J predictable processes we define a metric

$$d_X(H,J) = \|(H-J)\circ X\|_{\mathcal{H}_T^2} = \mathbb{E}\left[\int_0^T (H_u - J_u)^2 d[\overline{N}, \overline{N}]_u\right] + \mathbb{E}\left[\left(\int_0^T |H_u - J_u| \left|d\overline{A}\right|_u\right)^2\right].$$

Lemma A.3. *Let X be a semimartingale in $\mathcal{H}_T^2$. Then*

$$\mathbb{E}\left[\left(\sup_{0\leqslant t\leqslant T} | X_t |\right)^2\right] \leqslant 8 \|X\|_{\mathcal{H}_T^2}^2 .$$

Proof. See Theorem 5 of Chapter *IV* in Protter [16].

♦

Theorem A.4. *Let $X \in \mathcal{H}^2$ be continuous with decomposition $X = \overline{N} + \overline{A}$ and $k \in \{1,2\}$. Z is a predictable caglad process which is integrable with respect to X and $\|Z^k \circ X\|_{\mathcal{H}_T^2} < \infty$. Then Z is the a.s. limit of the sequence $(Z^n)_{n \geqslant 1}$, with*

$$Z_t^n = n \int_{(t-\frac{1}{n}) \vee 0}^{t} \overline{Z}_u^n du \tag{A.1}$$

and $\overline{Z}_u^n = \max(\min(Z_u, n), -n)$. For every n the process Z^n is bounded, continuous and of finite variation. Furthermore,

$$\int_0^T (Z_u^n)^k dX_u \xrightarrow{L^2} \int_0^T (Z_u)^k dX_u, \qquad (n \to \infty).$$

Proof. By definition of Z^n and because Z is caglad $Z^n \to Z$ almost surely for $n \to \infty$. Z^n can be represented as the difference of two increasing, continuous processes

$$\begin{aligned} Z^n = {} & n \int_0^t | \overline{Z}_u^n | \, du + n \int_0^{t-\frac{1}{n}} | \overline{Z}_u^n | - \overline{Z}_u^n du \\ & - \left(n \int_0^{t-\frac{1}{n}} | \overline{Z}_u^n | \, du + n \int_0^t | \overline{Z}_u^n | - \overline{Z}_u^n du \right). \end{aligned}$$

Consequently, Z^n is bounded, continuous and of finite variation. The process X is continuous and so are the processes $\overline{N}$ and $\overline{A}$. Since $\|Z^k \circ X\|_{\mathcal{H}_T^2} < \infty$ and Z^n is bounded, we get $\int_0^T ((Z_u^n)^k - (Z_u)^k)^2 d[\overline{N}, \overline{N}]_u < \infty$. Lebesgue's dominated convergence theorem yields

$$\int_0^T (Z_u^n - Z_u)^2 d[\overline{N}, \overline{N}]_u \xrightarrow{ucp} 0.$$

Moreover for every n,

$$\mathbb{E}\left[\int_0^T (Z_u^n - Z_u)^2 d[\overline{N}, \overline{N}]_u\right] \leqslant 2n^{2k} T \mathbb{E}\left[[\overline{N}, \overline{N}]_T\right] + 2\mathbb{E}\left[\int_0^T Z_u^{2k} d[\overline{N}, \overline{N}]_u\right] < \infty.$$

By the dominated convergence theorem

$$\mathbb{E}\left[\int_0^T ((Z_u^n)^k - (Z_u)^k)^2 d[\overline{N}, \overline{N}]_u\right] \to 0.$$

Analogously we show that

$$\mathbb{E}\left[\left(\int_0^T \left|(Z_u^n)^k - (Z_u)^k\right| \left|d\overline{A}\right|_u\right)^2\right] \to 0.$$

The convergence in L^2 results from Lemma A.3. ♦

The next theorem is similar to Theorem A.4 in [22]. While Cetin, Jarrow and Protter construct processes (trading strategies) with $Z_T^n = 0$, we construct processes where Z_T^n is equal to a bounded random variable W.

Theorem A.5. *Let $X \in \mathcal{H}^2$ be continuous with decomposition $X = \overline{N} + \overline{A}$. The process Z is caglad, bounded and integrable with respect to X. In addition let $W : \Omega \to \mathbb{R}$ be a $\mathcal{F}_T$-measureable bounded random variable and $k \in \{1, 2\}$. For every $m \geqslant 1$ we consider the process $Z^m = (Z_t^m)_{0 \leqslant t \leqslant T}$ with*

$$Z_t^m = L_t \mathbb{1}_{[0,T_m]} + \left(\mathbb{E}[W \mid \mathcal{F}_t] + (L_{T_m} - \mathbb{E}[W \mid \mathcal{F}_t]) \frac{T-t}{T-T_m} \right) \mathbb{1}_{(T_m,T]}$$

and $T_m = T - \frac{1}{m}$. Then $Z^m \to Z$ almost surely for $m \to \infty$ and

$$\int_0^T (Z_u^m)^k dX \xrightarrow{L^2} \int_0^T Z_u^k dX, \qquad (n \to \infty).$$

Proof. Consider

$$(Z_t^m)^k = \underbrace{Z_t^k \mathbb{1}_{[0,T_m]}}_{(1)} + \underbrace{\left(\mathbb{E}[W \mid \mathcal{F}_t] + (Z_{T_m} - \mathbb{E}[W \mid \mathcal{F}_t]) \frac{T-t}{T-T_m} \right)^k \mathbb{1}_{(T_m,T]}}_{(2)}. \tag{A.2}$$

We first show that (1) converges to Z^k in $\mathcal{H}_T^2$. Since Z is bounded

$$\int_0^T (Z_u^k - Z_u^k \mathbb{1}_{[0,T_m]})^2 d\left[\overline{N}, \overline{N}\right]_u \leqslant \int_0^T Z_u^{2k} d\left[\overline{N}, \overline{N}\right]_u < \infty, \qquad a.s..$$

Since $Z^m \to Z$ almost surely by definition, the Lebesgue's dominated convergence theorem yields

$$\int_0^T (Z_u^k - Z_u^k \mathbb{1}_{[0,T_m]})^2 d\left\langle \overline{N}, \overline{N} \right\rangle_u \to 0, \qquad a.s..$$

Because $X \in \mathcal{H}^2$ we get $\mathbb{E}^{\mathbb{Q}}\left[\int_0^T (Z_u^k - Z_u^k \mathbb{1}_{[0,T_m]})^2 d\left[\overline{N}, \overline{N}\right]_u\right] < \infty$. Hence by the dominated convergence theorem

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T (Z_u^k - Z_u^k \mathbb{1}_{[0,T_m]})^2 d\left[\overline{N}, \overline{N}\right]_u\right] \to 0.$$

Similarly we get that

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \left| Z_u^k - Z_u^k \mathbb{1}_{[0,T_m]} \right| \left| d\overline{A}_u \right| \right] \to 0.$$

Next we show that the part (2) in (A.2) converges to 0 in $\mathcal{H}^2_T$. Because Z and W are bounded

$$\int_0^T \left(\mathbb{E}[W \mid \mathcal{F}_t] + (Z_{T_m} - \mathbb{E}[W \mid \mathcal{F}_t]) \frac{T-u}{T-T_m} \right)^{2k} \mathbb{1}_{(T_m,T]} d\left[\overline{N}, \overline{N}\right]_u$$
$$\leqslant \int_0^T K \mathbb{1}_{(T_m,T]} d\left[\overline{N}, \overline{N}\right]_u = K \left(\left[\overline{N}, \overline{N}\right]_T - \left[\overline{N}, \overline{N}\right]_{T_m} \right) \to 0,$$

where $K = \left\| \max_{0 \leqslant t \leqslant T} (|Z_t| + |W|)^{2k} \right\|_\infty$. The dominated convergence theorem yields

$$\mathbb{E}\left[\int_0^T \left(\mathbb{E}[W \mid \mathcal{F}_t] + (Z_{T_m} - \mathbb{E}[W \mid \mathcal{F}_t]) \frac{T-u}{T-T_m} \right)^{2k} \mathbb{1}_{(T_m,T]} d\left[\overline{N}, \overline{N}\right]_u \right] \to 0.$$

We use a similar argument to show that

$$\mathbb{E}\left[\int_0^T \left| \mathbb{E}[W \mid \mathcal{F}_t] + (Z_{T_m} - \mathbb{E}[W \mid \mathcal{F}_t]) \frac{T-u}{T-T_m} \right|^{k} \mathbb{1}_{(T_m,T]} \mid d\overline{A}_u \mid \right] \to 0.$$

♦

Theorem A.6. *The process $X \in \mathcal{H}^2$ is continuous with decomposition $X = \overline{N} + \overline{A}$ and Z, with $\|Z \circ X\|_{\mathcal{H}^2_T} < \infty$, is caglad and integrable with respect to X. Then there exists a sequence of continuous processes $(Z^j)_{j \geqslant 1}$, where $Z^j = (Z^j_t)_{0 \leqslant t \leqslant T}$ is bounded, of finite variation with $Z^j_0 = Z^j_T = 0$, such that $Z^j \to Z$ almost surely and*

$$\int_0^T Z^j dX \xrightarrow{L^2} \int_0^T Z dX, \qquad (j \to \infty).$$

Proof. By Theorem A.4 with $k = 1$ there exists a sequence $(Z^n)_{n \geqslant 1}$ of bounded, continuous processes of finite variation with $Z^n_0 = 0$. By Theorem A.5 with $W = 0$ and $k = 1$, we create a sequence $(Z^{n,m})_{m \geqslant 1}$ with $Z^{m,n} = Z^n$ on $[0, T - \frac{1}{m}]$ and $Z^{m,n} = nZ^n(T-t)$ linear on $[T - \frac{1}{m}, T]$. Hence, for every n the process $Z^{n,m}$ is bounded, continuous and of finite variation with $Z^{n,m}_0 = Z^{n,m}_T = 0$. Furthermore,

$$\int_0^T Z^{n,m} dX \xrightarrow{L^2} \int_0^T Z dX \qquad (n, m \to \infty).$$

♦

B. Approximation of the replications costs

To calculate the minimal replication costs of contingent claims in the liquidity risk models used in Chapter 6 we have to solve the corresponding BSDE. In this chapter we introduce two different methods how the liquidity costs can be approximated in case no closed form solution is known.

B.1. Time discrete approximation in a binomial tree

For a random variable H, let us consider the following BSDE

$$V_t = H + \int_t^T f(s, S_s, Z_s)ds - \int_t^T Z_s dW_s \tag{B.1}$$

where W is a Brownian motion and $f(t, \omega, z)$

$$f : [0,T] \times \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$$

the driver of the BSDE. For simplicity we supress the dependence on $\omega \in \Omega$. The driver satisfies

$$f(t, , Z_t) = \begin{cases} \frac{\lambda c}{\sigma^2} Z_t^2 & \text{in the Bachelier model,} \\ \frac{\lambda c}{\sigma^2} \frac{Z_t^2}{S_t^2} & \text{in the Black-Scholes model.} \end{cases}$$

Instead of solving this BSDE we transform the equation to a backward stochastic difference equation ($BS\Delta E$) and find a solution of a discrete time approximation of this stochastic differential equation.

For every N let us consider a partition of the time intervall $[0, T]$ tending to identity, such that $0 = t_0^N < ... < t_{i_N}^N = T$ and

$$t_i^N = i \cdot \frac{T}{N}.$$

We approximate the Brownian motion by a Bernoulli random walk

$$W_{t_i^N}^N = \sqrt{\frac{T}{N}} \sum_{j=1}^{i} X_j^N$$

for i.i.d random variables X_j^N on a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{Q}})$ with distribution $\widetilde{\mathbb{Q}}[X_j^N = 1] = \widetilde{\mathbb{Q}}[X_j^N = -1] = \frac{1}{2}$. Let us extend $W_{t_i^N}^N$ to the time continuous process W_t^N which is constant on the intervals $[t_i^N, t_{i+1}^N)$. Furthermore, we denote by $\mathcal{F}_t^N$ the augmented filtration of the process W_t^N. The approximating $BS\Delta E$ is driven by the function

$$f^N : [0,T] \times \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$$

such that for $t \in (t_i^N, t_{i+1}^N]$

$$f^N(t, Z_t) = f(t_i^N, Z_{t_i}^N).$$

Instead of a solution (Z, V) to the BSDE (B.1) we are looking for a pair of processes (Z_t^N, V_t^N), with V_t^N constant on the intervals $[t_i^N, t_{i+1}^N)$ and Z_t^N constant on $(t_i^N, t_{i+1}^N]$, such that,

$$V_t^N = H^N + \int_t^T f(s, S_s, Z_s) d[W^N, W^N]_s - \int_t^T Z_s^N dW_s^N.$$

See Cheridito and M. Stadje [15]. Since $\Delta[W^N, W^N]_{t_i^N} = \mathbb{E}\left[(\Delta W_{t_i^N}^N)^2\right] = \frac{T}{N}$, this is equivalent to

$$V_{t_i^N}^N = V_{t_{i+1}^N}^N + f(t_i^N, Z_{t_i}^N)\frac{T}{N} - Z_{t_i^N}^N \left(\Delta W_{t_{i+1}^N}^N\right) \tag{B.2}$$

$$V_T^N = H^N \tag{B.3}$$

where $\left(\Delta W_{t_{i+1}^N}^N\right) = W_{t_{i+1}^N}^N - W_{t_i^N}^N$ and $H^N = h(W_T^N)$. Let us take the conditional expectation $E[V_{t_i^N}^N \mid \mathcal{F}_{t_i^N}^N]$,

$$V_{t_i^N}^N = \mathbb{E}\left[V_{t_{i+1}^N}^N \middle| \mathcal{F}_{t_i^N}^N\right] + f(t_i^N, Z_{t_i}^N)\left(t_{i+1}^N - t_i^N\right) \tag{B.4}$$

When multiplying equation (B.2) with $(\Delta W_{t_{i+1}^N}^N)$ on both sides, the conditional expectation yields

$$Z_{t_i^N}^N = \frac{N}{T}\mathbb{E}\left[V_{t_{i+1}^N}^N (\Delta W_{t_{i+1}^N}^N) \middle| \mathcal{F}_{t_i^N}^N\right]. \tag{B.5}$$

Note that the information $\mathcal{F}_{t_i^N}^N$ is equal to knowing the process $W_{t_i^N}^N$ at time T_i^N. Therefore

$$V_{t_i^N}^N\Big|_{\mathcal{F}_{t_i^N}^N} = V_{t_i^N}^N\Big|_{N_{t_i^N}^N}$$

where $N_{t_i^N}^N = \#\left\{j \leqslant i \middle| X_j^N = 1\right\}$. From now on we use the following notation,

$$V_{t_i^N}^N(k) = V_{t_i^N}^N\Big|_{N_{t_i^N}^N = k},$$

$$Z_{t_i^N}^N(k) = Z_{t_i^N}^N\Big|_{N_{t_i^N}^N = k}.$$

By equation (B.4) and (B.5) we obtain a discrete time approximation, which can be solved going backward in time. At time T,

$$V_T = H^N \tag{B.6}$$

and

$$Z^N_{t^N_i}(k) = \frac{1}{2}\sqrt{\frac{N}{T}}\left(V^N_{t^N_{i+1}}(k+1) - V^N_{t^N_{i+1}}(k)\right) \tag{B.7}$$

$$V^N_{t^N_i}(k) = \frac{1}{2}\left(V^N_{t^N_{i+1}}(k) + V^N_{t^N_{i+1}}(k+1)\right) + f(t^N_i, Z^N_{t^N_i}(k))\frac{T}{N} \tag{B.8}$$

for $k = 0, ..., i$ and $t^N_i = t^N_{(i_N-1)}, ..., t^N_0$.

The convergence of this method is shown for drivers with subquadratic growth in Z, see P. Cheridito and M. Stadje [15]. The resulting BSDEs in the liquidity risk model of A. Roch are of quadratic growth, but calculations for different claims in models used in Chapter 6 have shown that this method converges to the exact solution.

See Cheridito and M. Stadje [15] for further information on $BS\Delta E$s and Ma, Protter, San Martin, Torres [8], such as Bouchard, Touzi [4] for information on the discrete time approximation of BSDEs.

B.2. The numerical solution of the PDE

In Chapter 6 we transform the BSDE to a partial differential equation. In this section we use a finite difference scheme to solve this PDE. For different claims in the Black-Scholes and Bachelier model, the numerical solutions obtained with this method are identical to the values that result from the discrete time approximation (B.6) - (B.8) .

When the unaffected price process is a geometric Brownian motion and the illiquidity process deterministic (see Section 6.2), we consider the PDE

$$\frac{\partial F(t,x)}{\partial t} + \lambda c\left(\frac{\partial F(t,x)}{\partial x}\right)^2 + \frac{\sigma^2}{2}x^2\frac{\partial F(t,x)^2}{\partial x^2} = 0 \tag{B.9}$$

for $t \in [0,T]$ and $x \in [1,U]$. Depending on the claim H we define boundary conditions. For instance, a European put option satisfies

$$\begin{aligned} F(T,x) &= (K-x)_+, \\ F(t,1) &= K-1, \\ F(t,U) &= 0, \end{aligned}$$

where $U >> K$. The resulting PDE is quadratic in the first derivative with respect to x. We show how this equation can be approximated using a finite difference method.

The factor x^2 in the Black-Scholes PDE, causes oscillations when we try to solve the PDE directly. Instead of solving the PDE (B.9) we apply an exponential transformation

$$G(t,z) = F(t,\exp(z)) \quad \Leftrightarrow \quad G(t,\log(x)) = F(t,x).$$

Then for $z = \log(x)$

$$\begin{aligned}
\frac{\partial F(t,x)}{\partial t} &= \frac{\partial G(t,z)}{\partial t}, \\
\frac{\partial F(t,x)}{\partial x} &= \frac{\partial G(t,z)}{\partial z}\frac{1}{\exp z}, \\
\frac{\partial F(t,x)^2}{\partial x^2} &= \frac{\partial G(t,z)^2}{\partial z^2}\frac{1}{\exp 2z} - \frac{\partial G(t,z)}{\partial z}\frac{1}{\exp 2z}.
\end{aligned}$$

By (B.9) this leads to

$$\frac{\partial G(t,z)}{\partial t} + \lambda c \exp(-2z)\left(\frac{\partial G(t,z)}{\partial z}\right)^2 - \frac{\sigma^2}{2}\frac{\partial G(t,z)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial G(t,z)^2}{\partial z^2} = 0 \tag{B.10}$$

for $t \in [0,T]$ and $z \in [0, \log U]$. The boundaries of a European put option are transformed to

$$\begin{aligned}
&G(T,z) = (K - \exp(z))_+ \\
&G(t,0) = K - 1 \\
&G(t,\log(U)) = 0.
\end{aligned}$$

We use a finite difference method to solve (B.10). For $N_z \in \mathbb{N}_+$ and $N_t \in \mathbb{N}_+$ we use gird points $z_i = i\Delta z$ with mesh size $\Delta z = \frac{\log(U)}{N_z}$ and discrete time points $t_j = j\Delta t$ where $\Delta t = \frac{T}{N_t}$ for $i = 0, ..., N_z$ and $j = 0, ..., N_t$. Let $G_{j,i}$ be the numerical approximation of $G(t_j, z_i)$ and

$$\begin{aligned}
\frac{\partial G(t_j,z_i)}{\partial t} &\approx \left(\frac{\partial G}{\partial t}\right)_{j,i} = \frac{G_{j+1,i} - G_{j,i}}{\Delta t} \\
\frac{\partial G(t_j,z_i)}{\partial z} &\approx \left(\frac{\partial G}{\partial z}\right)_{j,i} = \frac{G_{j,i+1} - G_{j,i-1}}{2\Delta z} \\
\frac{\partial G^2(t_j,z_i)}{\partial z^2} &\approx \left(\frac{\partial G^2}{\partial z^2}\right)_{j,i} = \frac{G_{j,i+1} - 2G_{j,i} + G_{j,i-1}}{(\Delta z)^2}.
\end{aligned}$$

We use a the first order forward difference approximation with respect to t and central difference approximation with respect to z, such as the central difference approximation for the second derivative with respect to z. To approximate

$$\left(\frac{\partial G(t,z)}{\partial z}\right)^2$$

we assume that

$$\left(\frac{\partial G(t,z)}{\partial z}\right)^2 \approx \left(\frac{\partial G(t,z)}{\partial z}\right) \cdot \left(\frac{\partial G(t+\Delta t,z)}{\partial z}\right).$$

Then

$$\left(\frac{\partial G}{\partial z}\right)^2_{j,i} \approx \left(\frac{\partial G}{\partial z}\right)_{j+1,i} \left(\frac{\partial G}{\partial z}\right)_{j,i}.$$

Given the numerical solution $G_{j+1,i}$ for $j = 0, ..., N_t$ we solve the implicit equation

$$\left(\frac{\partial G}{\partial t}\right)_{j,i} + \lambda c \exp(-2z_i) \left(\frac{\partial G}{\partial z}\right)^2_{j,i} - \frac{\sigma^2}{2} \left(\frac{\partial G}{\partial z}\right)_{j,i} + \frac{\sigma^2}{2} \left(\frac{\partial G^2}{\partial z^2}\right)_{j,i} = 0, \tag{B.11}$$

or equivalently

$$G(t_j, z_i) = G(t_{j+1}, z_i) + \Delta t \lambda c \exp(-2z_i) \left(\frac{\partial G}{\partial z}\right)^2_{j,i} - \frac{\Delta t \sigma^2}{2} \left(\frac{\partial G}{\partial z}\right)_{j,i} + \frac{\Delta t \sigma^2}{2} \left(\frac{\partial G^2}{\partial z^2}\right)_{j,i},$$

with respect to $G_{j,i}$. We therefore use a simple iteration with n steps. For every time step j and $i = 0, ..., N_z$ consider $G^k_{j,i}$ $(k = 1, ..., n)$ with $G^1_{j,i} = G_{j+1,i}$ and

$$\begin{aligned} G^{k+1}_{j,i} &= G_{j+1,i} + G^k_{j,i+1} \left(-\frac{\Delta t \sigma^2}{(\Delta z)^2}\right) \\ &+ G^k_{j,i+1} \left(\frac{\Delta t \sigma^2}{2(\Delta z)^2} - \frac{\Delta t \sigma^2}{4 \Delta z} + \frac{\Delta t \lambda c \exp(-2z_i)}{4(\Delta z)^2} \left(G_{j+1,i+1} - G_{j+1,i-1}\right)\right) \\ &+ G^k_{j,i-1} \left(\frac{\Delta t \sigma^2}{2(\Delta z)^2} + \frac{\Delta t \sigma^2}{4 \Delta z} - \frac{\Delta t \lambda c \exp(-2z_i)}{4(\Delta z)^2} \left(G_{j+1,i+1} - G_{j+1,i-1}\right)\right). \end{aligned}$$

At maturity the numerical solution is given due to the boundary condition, e.g. for a European put option $G_{N_t,i} = (K - \exp(z_i))_+$. By solving (B.11) we step backward in time and this way obtain a numerical solution $G_{0,i}$ at time 0. Then,

$$F(0, \exp(z_i)) \approx G_{0,i}.$$

For a small time grid ($\Delta t \approx \frac{1}{200}$) the numerical results show that this method converges to the same values as the discrete time approximation of Section B.1.

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