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Investigations on Edge Intersection Graphs of Paths on a Grid with Focus on Monotonic Representations

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Abstract

A graph is called edge intersection graph of paths on a grid if there is a grid and there are paths on this grid, such that the vertices correspond to the paths and two vertices are adjacent if and only if the corresponding paths share a grid edge. Such a representation is called EPG representation. A graph is in B_k if and only if there is an EPG representation where every path has at most k bends. Furthermore a graph is in B_k^m if it is in B_k and every path is monotonic, that is it is only ascending in both columns and rows.

In the thesis we first present an overview of the existing results on edge intersection graphs of paths on a grid. Then we show that every outerplanar graph is in B_2^m. Moreover, we give an exact characterization of the graphs contained in B_0, B_1^m, B_1, and B_2^m for both maximal outerplanar graphs and cacti. Then we proceed by proving, that $B_k^m \subsetneqq B_k$ for $k = 2$, $k = 5$, and $k \geqslant 7$, give a condition on when a $K_{m,n}$ is in B_k^m and prove that $B_1 \subseteq B_3^m$. In the end we present a mixed integer linear programming (MILP) formulation of the problem to find the minimum k such that a given graph is in B_k^m and a MILP formulation of the problem whether a given graph is in B_k^m for a fixed k. The latter we generalize to a MILP formulation of the problem, whether a given graph is in B_k for a fixed k.

Kurzfassung

Ein Graph wird Kanten-Überschneidungsgraph von Pfaden auf einem Gitter (edge intersection graph of paths on a grid, EPG) genannt, wenn es ein Gitter und Pfade im Gitter gibt, so dass zwei Knoten im Graphen adjazent sind, genau dann, wenn die zugehörigen Pfade im Gitter eine Gitterkante gemeinsam haben. So eine Repräsentation eines Graphen wird EPG Repräsentation genannt. Ein Graph ist in B_k, wenn es eine EPG Repräsentation gibt, so dass jeder Pfad maximal k Knicke hat. Außerdem ist ein Graph in B_k^m, wenn er in B_k ist und zusätzlich jeder Pfad der EPG Repräsentation vom Startpunkt immer nur nach oben oder nach rechts geht.

In der Masterarbeit präsentieren wir zuerst einen Überblick über die existierenden Resultate über Kanten-Überschneidungsgraphen von Pfaden auf einem Gitter. Dann zeigen wir, dass jeder außenplanare Graph in B_2^m ist. Darüber hinaus geben wir genaue Kriterien an, wann Kaktusgraphen und außenplanare Triangulierungen in B_0, B_1^m, B_1 und B_2^m sind. Dann beweisen wir, dass $B_k^m \subsetneq B_k$ für $k = 2$, $k = 5$ und $k \geqslant 7$ gilt. Zusätzlich geben wir eine Bedingung an, die erfüllt sein muss, wenn $K_{m,n}$ in B_k^m ist, und beweisen außerdem, dass $B_1 \subseteq B_3^m$. Am Ende der Masterarbeit geben wir eine lineare gemischt-ganzzahlige (mixed integer linear programming, MILP) Formulierung des Problems an, das minimale k zu bestimmen, so dass ein gegebener Graph in B_k^m ist. Außerdem präsentieren wir eine zweite MILP Formulierung um herauszufinden, ob ein gegebener Graph für ein fixes k in B_k^m ist. Letztere verallgemeinern wir zu einer MILP Formulierung um zu bestimmen, ob ein gegebener Graph für ein fixes k in B_k ist.

Contents

1 Introduction

A graph is called edge intersection graph of paths on a grid (EPG) if there is a grid and there are paths on this grid such that the vertices correspond to the paths and two vertices are adjacent in the graph if and only if the corresponding paths share a grid edge. Such a representation of the graph is called an EPG representation.

Closely related to EPGs are edge intersection graphs of paths on a tree (EPT). They were introduced already in 1985 by Golumbic and Jamison in [15] and [16]. In an EPT, the paths are not paths on a grid but paths on a tree. A first generalization of EPTs was made in [18] where k-edge intersection graphs of paths on a tree were studied. Here the vertices are adjacent if and only if the paths have at least k edges in common. In [17] and [19] it turned out, that EPTs in trees with degree 4 are of certain interest. Finally, edge intersection graphs of paths on a grid were introduced in 2009 by Golumbic, Lipshteyn, and Stern in [20]. They are a generalization of EPTs in trees with degree 4 because in every grid point there are 4 outgoing grid edges.

In all the above mentioned graph classes, an intersection of two paths means, that they share an edge. Nevertheless also the case where intersection only means sharing a point was considered. A graph is called vertex intersection graph of paths on a tree (VPT) if there is a tree and there are paths on this tree, such that two vertices are adjacent if and only if the paths intersect in at least one vertex of the tree. Vertex intersection graphs of paths on a tree are also called path graphs and were studied for example in [13]. Also VPTs were generalized analogously to EPTs, namely vertex intersection graphs of paths on a grid were introduced in [1].

There has been done a lot of research on EPG graphs recently [2–4, 8, 10, 20, 21, 25, 26], especially restricted classes of EPG are considered. A graph is in B_k if there is an EPG representation of the graph, where every path has at most k bends. Another considered class is B_k^m. Here all the paths are only allowed to have at most k bends and furthermore are only allowed to be ascending in both columns and rows, so the paths look like stairs going upwards from the left to the right. The bend number and the monotonic bend number are defined as the minimum k and k', such that a graph is in B_k and $B_{k'}^m$ respectively.

Edge intersection graphs of paths on a grid initially were introduced because of two applications. The first one comes from circuit layout setting. In this setting the wires correspond to the paths on the grid. In the knock-knee layout model we want to place the wires on the grid in multiple layers, such that the wires of each layer to not share a grid edge, but crossing and bending of wires is allowed. In our notation that corresponds to finding a coloring of the vertices of the graph, such that two adjacent vertices are not colored with the same color. For more information see [7, 32]. Another application comes from chip manufacturing. There a transition whole is required, whenever a wire bends. Many transition wholes may enlarge the area and furthermore increase the cost of the chip, hence we want to minimize the number of bends or equivalently find the minimum k such that the corresponding graph is in B_k. Further information can be found in [20].

The rest of the thesis is organized as follows. In Chapter 2 we will introduce all the definitions needed throughout the whole thesis. Then we will give an overview of the known results concerning edge intersection graphs of paths on a grid in Chapter 3. That includes for example structural properties, as well as results on complexity, the hierarchy of B_k and B_k^m, and upper and lower bounds on the bend number and the monotonic bend number of some graph classes and with respect to some graph properties respectively.

We will proceed by deriving new results. In Chapter 4 we will show, that the monotonic bend number of outerplanar graphs is 2. This implies that the monotonic bend number and the bend number coincide for outerplanar graphs. Additional to that we will consider two subclasses of outerplanar graphs, namely outerplanar triangulations and cacti, and derive a full characterization of the graphs of these classes which are in B_0, B_1^m, B_1, and B_2^m. These characterizations are done by stating forbidden induced subgraphs.

We will close the discussion of the graph S_n by giving the bend number as well as the monotonic bend number of it in the beginning of Chapter 5. Then we proceed by showing, that $B_k^m \subsetneqq B_k$ for $k = 2$, $k = 5$, and $k \geqslant 7$, which answers an open question of [20] for almost all values of k. In order to prove that, we derive an inequality on m, n, and k which has to be fulfilled if a $K_{m,n}$ is in B_k^m. With this inequality we will also show, that for even $k \geqslant 6$ there is a graph in B_k which is not in B_{2k-8}^m and for odd $k \geqslant 6$ there is a graph in B_k which is not in B_{2k-9}^m respectively. Additional to that we will show, that $B_1 \subseteq B_3^m$, giving the first result of this kind.

In Chapter 6 we will present a mixed integer linear programming (MILP) formulation of the problem of finding the bend number of a given graph. It is based on introducing a binary variable for every path and for every grid edge which determines, whether a path uses this grid edge or not. Also another MILP formulation is given. With this formulation we can determine whether a given graph is in B_k^m for a fixed k. The formulation is based on introducing integer variables which represent the bend points of the paths. The latter is also generalized to a MILP model which determines, whether a graph is in B_k for a fixed k in Chapter 7. In the end we present our conclusions and open questions in Chapter 8.

2 Basic Definitions

The purpose of this section is to point out, which terms of graph theory are used in the thesis and where to find their definitions. Furthermore we define edge intersection graphs of paths on a grid and related terms.

2.1 Definitions from Graph Theory

Throughout the whole thesis we assume familiarity with terms of graph theory like graph, directed graph, vertex, edge, adjacent, incident, neighbor, neighborhood, degree, maximum degree, connected, bipartite, isolated vertex, clique, maximal clique, maximum clique, independent set, path, cycle, tree, perfect graph, plane, planar, outerplanar, inner and outer face, subgraph, and induced subgraph. For mathematically rigorous definitions of all above mentioned terms see for example Diestel [9].

In the whole thesis we will stick to the notation of [9], except for the notation of the edges. For a graph $G = (V, E)$ we will denote the elements of E with (u, v) for $u, v \in V$. Note, that this implies that $(u, v) = (v, u)$ in our notation.

Additionally let us mention, that whenever we refer to an induced subgraph, that means that we take a subset of the vertices and all the edges inbetween the vertices of the subset. If take a subset of the vertices and only some of the edges inbetween them, we will refer to a subgraph. This will make a major difference in the proceeding.

Moreover we will find out, that the following graph plays a major role in investigating on edge intersection graphs of paths on a grid.

Definition. The graph $K_{m,n} = (V, E)$ is defined in the following way. It has vertex set $V = A \cup B$ with $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$. Furthermore it has edge set $E = \{(a_i, b_j) \mid 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$. The graph $K_{m,n}$ is called *complete bipartite graph on m and n vertices.*

2.2 Definitions related to Edge Intersection Graphs of Paths on a Grid

In this subsection, we give the basic definitions related to edge intersection graphs of paths on a grid, which are used throughout the thesis.

We start with the definition of a grid and a path.

Definition. The term *grid* is used to denote a rectangular grid in the plane. A grid consists of *horizontal* and *vertical grid lines* which are also called *grid rows* and *grid* *columns*, respectively. The crossings of two grid lines are called *grid points*. The segment on a grid line between two consecutive grid points is called a *grid edge*. If a vertical grid line is called x and a horizontal grid line is called y then we denote by (x, y) the grid point which is the crossing of the grid line x and the grid line y.

Definition. A *path* on a grid consists of a *start point* and an *end point*, which are both grid points, and of consecutive grid edges joining the start point with the end point. Hence a path on a grid goes only along the grid lines. A turn of a path in the grid is called *bend* and the grid point, in which the path turns, is called a *bend point*. The part of a path between two consecutive bend points is called a *segment*. Also the part of the path from the start point to the first bend point and the part of the path from the last bend point to the end point are called *segments*.

Now we define the term intersection of paths.

Definition. We say, that two paths on a grid *intersect*, if they both have at least one common grid edge. If two paths share a grid point, but no grid edge which is adjacent to the grid point, we say, that the paths have a *crossing* in this grid point.

Finally we are able to define edge intersection graphs of paths on a grid according to the definition of Golumbic, Lipshteyn, and Stern [20].

Definition. Let $\mathcal{P}$ be a collection of paths on a grid $\mathcal{G}$. Then the *edge intersection graph* $EPG(\mathcal{P})$ is a graph, in which the vertices correspond to the paths in $\mathcal{P}$ and there is an edge between two vertices, if and only if the corresponding paths intersect in the grid $\mathcal{G}$.

An undirected graph G is an *edge intersection graph of paths on a grid* (EPG), if there exist a grid $\mathcal{G}$ and a collection of paths $\mathcal{P}$ such that $G = EPG(\mathcal{P})$. In this case we say that G *is EPG* and we call $\langle \mathcal{P}, \mathcal{G} \rangle$ an *EPG representation* of G.

Additional to that we want to consider two restrictions to the class of edge intersection graphs of paths on a grid.

Definition. An EPG representation is called *B_k-EPG representation*, if every path has at most k bends. A graph G is called *B_k-EPG*, if there exists a B_k-EPG representation of G. We let B_k be the class of all graphs, which are B_k-EPG.

Definition. A path on a grid is called *monotonic*, if it is ascending in both columns and rows. An EPG representation is called *monotonic*, if every path is monotonic. A graph G is called *monotonic EPG*, if there exists a monotonic EPG representation of G.

Definition. A monotonic B_k-EPG representation is called *B_k^m-EPG representation*. A graph G is called *B_k^m-EPG*, if there exists a B_k^m-EPG representation of G. We let B_k^m be the class of all graphs, which are B_k^m-EPG.

The following facts are a direct consequence of the above definitions.

Observation 2.2.1. *$B_0 \subseteq B_1 \subseteq B_2 \subseteq \dots$ and $B_0^m \subseteq B_1^m \subseteq B_2^m \subseteq \dots$ hold. Furthermore $B_k^m \subseteq B_k$ holds for every k. Additional to that $B_0 = B_0^m$ and $B_0 \subseteq B_1^m$.*

Moreover we define the bend number according to the definition of Heldt, Knauer, and Ueckerdt [26] and analogously define the monotonic bend number.

Definition. The bend number of a graph G is the minimum $k \in \mathbb{N}^* = \{0, 1, 2, \dots\}$, such that $G \in B_k$. The monotonic bend number of a graph G is the minimum $k \in \mathbb{N}^*$, such that $G \in B_k^m$.

Note, that according to these definitions the bend number of a graph G is always less or equal to the monotonic bend number of G.

3 Known Results

The aim of this section is to present an overview of the existing results on edge intersection graphs of paths on a grid. We will start by giving the proof, that every graph is EPG and monotonic EPG in Section 3.1. Then we will give more detailed results on graphs in B_1 and B_1^m including results on complexity and approximation algorithms for certain combinatorial optimization problems in Section 3.2. After that, we will present known upper and lower bounds on the bend number of graphs of certain classes and graphs with certain properties in Section 3.3 and Section 3.4 respectively. In the end we will present existing results on the graph $K_{m,n}$ in Section 3.5.

3.1 First Results

The very first result on edge intersection graphs of paths on a grid comes from Golumbic, Lipshteyn, and Stern [20]. The proved the following result.

Theorem 3.1.1 (Golumbic, Lipshteyn, Stern [20])**.** *Every graph is EPG.*

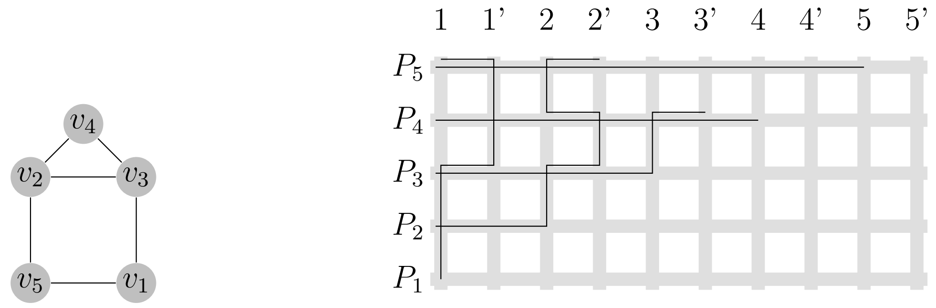
Proof. Let $G = (V, E)$ be a graph with $V = \{v_1, \dots, v_n\}$. We consider the grid $\mathcal{G}$ which has grid rows 1, 2, ..., n from bottom to top and grid columns 1, $1'$, 2, $2'$, ..., n, n' from left to right. Before we construct the paths, we define the forward neighborhood $N^+(v_i) = \{v_j \mid (v_i, v_j) \in E, i < j\}$. Now we can define the paths. For every $1 \leqslant i \leqslant n$ the path P_i starts at the grid point $(1, i)$ and goes horizontally to the grid point (i, i). Then the path P_i goes up vertically in the columns i and i'. It changes the column in every row j with $v_j \in N^+(v_i)$. Let $j^* = \max\{j \mid ((v_i, v_j) \in E \wedge i < j) \vee j = i\}$. Then P_i changes the column in row j^* one last time and then ends in either the grid point (i, j^*) or the grid point (i', j^*). Let us denote this collection by $\mathcal{P} = \{P_i \mid 1 \leqslant i \leqslant n\}$.

An example of the construction can be seen in Figure 3.1.

What is left to show is, that $\langle \mathcal{P}, \mathcal{G} \rangle$ is an EPG representation of G. Clearly path P_i corresponds to v_i, so we only have to show, that two paths intersect in $\mathcal{G}$ if and only if the corresponding vertices are adjacent in G.

Assume two vertices v_i and v_j are adjacent in G with $i < j$. Then $v_j \in N^+(v_i)$ and hence both the paths P_i and P_j use the grid edge from (i, j) to (i', j) and the paths intersect.

Assume two paths P_i and P_j with $i < j$ intersect. An intersection can only be on the grid line from (i, j) to (i', j), since every path P_ℓ uses only the parts of the grid which are in grid row ℓ before column ℓ' or between the grid columns ℓ and ℓ' above the grid row ℓ. Nevertheless, path P_i uses this grid line only if $v_j \in N^+(v_i)$. So v_j and v_i are adjacent in G.



(a) (b)

Figure 3.1: (a) A graph G. (b) An EPG representation of G.

This implies that the above construction yields an EPG representation and hence G is EPG. $\square$

This result does not only reveal, that every graph is EPG, but it also points out, that restricting the number of allowed bends is a natural further question. Actually, the construction of the proof of Theorem 3.1.1 already gives an upper bound on the number of bends needed in an EPG representation in terms of the maximum degree of a graph.

Corollary 3.1.2 (Golumbic, Lipshteyn, Stern [20])**.** *Let G be a graph with maximum degree Δ. Then $G \in B_{2\Delta}$.*

Proof. Every path in the construction of Theorem 3.1.1 bends one time in order to get into the corresponding column. Then it bends at most two additional times for every vertex in the forward neighborhood except for the last one, for which it only bends once. Every vertex has at most Δ neighbors and hence also at most Δ vertices in the forward neighborhood, therefore every path bends at most $1 + 2(\Delta - 1) + 1 = 2\Delta$ times. $\square$

This is the first result giving an upper bound on the bend number in terms of a graph property. Many more upper bounds of this kind can be found in Section 3.4. There also the upper bound on the bend number with respect to the maximum degree Δ is improved in Corollary 3.4.11.

Another result given in the same paper is, that every graph is monotonic EPG.

Theorem 3.1.3 (Golumbic, Lipshteyn, Stern [20])**.** *Every graph is monotonic EPG.*

Proof. Let $G = (V, E)$ be a graph with $V = \{v_1, \dots, v_n\}$. We define the grid $\mathcal{G}$ in the following way. It has grid rows 1, 2, ..., n from bottom to top. Furthermore it has grid columns $(1, 1)$, $(2, 2)$, ..., (n, n) and additional to that a grid column (i, j) for every $(v_i, v_j) \in E$ with $i < j$. All these grid columns are ordered lexicographically from left to right, hence $(1, 1)$ is the grid column farthest to the left and (n, n) is the grid column farthest to the right. Let N^+ be the forward neighborhood as defined in the proof of Theorem 3.1.1 and let $N^+(v_i) = \{v_{j_1}, v_{j_2}, \dots, v_{j_k}\}$ with $j_\ell \leqslant j_{\ell+1}$ for all $1 \leqslant \ell \leqslant k - 1$.

Then we define a path P_i for every $1 \leqslant i \leqslant n$. For simplicity let $j_0 = i$. The path P_i starts at grid point $((1, 1), i)$ and goes horizontally to the grid point $((i, i), i)$. Then for every $0 \leqslant \ell \leqslant k-1$ the path goes from the grid point $((i, j_\ell), j_\ell)$ vertically to the grid point $((i, j_\ell), j_{\ell+1})$ and then horizontally to the grid point $((i, j_{\ell+1}), j_{\ell+1})$.

An example of the construction is depicted in Figure 3.2.

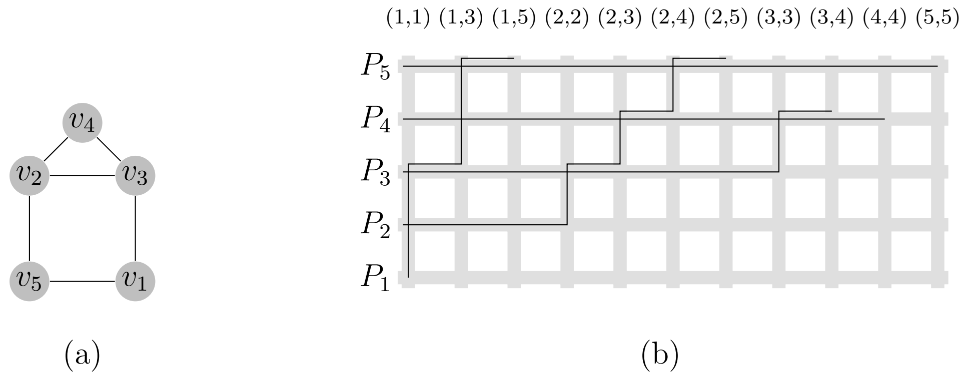


Figure 3.2: (a) A graph G. (b) A monotonic EPG representation of G.

It is easy to see, that every path is monotonic, so in order to show, that the obtained construction is a monotonic EPG representation, we only have to prove, that two paths intersect in $\mathcal{G}$ if and only if the corresponding vertices are adjacent in G.

Assume two vertices v_i and v_j with $i < j$ are adjacent. Then $v_j \in N^+(v_i)$. If v_j is the first vertex in the forward neighborhood of v_i, then let $\ell = i$. If there is a vertex with a lower index than j in the forward neighborhood of v_i, then let ℓ be the index of the vertex with the highest index which is lower than j. Then both paths P_i and P_j use the grid edge from $((i, \ell), j)$ to $((i, j), j)$ and hence intersect.

Assume two paths P_i and P_j with $i < j$ intersect. It follows from the definition of the construction, that a path P_ℓ only uses the grid row ℓ before the column (ℓ, ℓ) and then only uses the grid columns (ℓ, k) for some k which are above the grid row ℓ. So P_i and P_j can only have a grid edge from $((i, \ell_1), j)$ to $((i, \ell_2), j)$ for some ℓ_1 and ℓ_2 in common. But P_i uses this grid edges only, if j is in the forward neighborhood of i. This implies that if the paths intersect, v_j is adjacent to v_i. $\square$

A drawback of the monotonic EPG representation is, that it needs a larger grid size. Namely the representation of Theorem 3.1.1 needs a grid size of $n \times 2n$ whereas the monotonic representation requires a grid size of $n \times (n+m)$. Nevertheless also the construction of the proof of Theorem 3.1.3 gives an upper bound on the monotonic bend number in terms of the maximum degree.

Corollary 3.1.4 (Golumbic, Lipshteyn, Stern [20])**.** *Let G be a graph with maximum degree Δ. Then $G \in B^m_{2\Delta}$.*

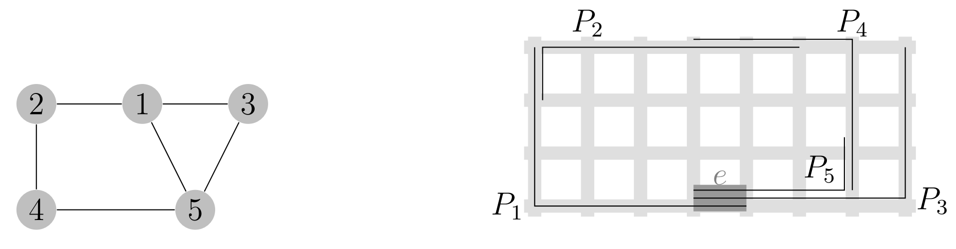
Proof. The paths of the construction of Theorem 3.1.3 use two bends for every vertex in the forward neighborhood. A vertex has at most Δ vertices in its forward neighborhood, therefore every path has at most 2Δ bends. $\square$

Like the bound on the bend number, the bound on the monotonic bend number with respect to the maximum degree Δ is improved in Corollary 3.4.11.

3.2 Basic Known Results on B_1 and B_1^m

3.2.1 Structural Results

In this subsection we want to derive properties of B_1-EPG representations which we will need throughout the thesis. We first consider the following definition.



(a) (b)

Figure 3.3: (a) A graph G with the clique $\{1, 3, 5\}$. (b) A B_1-EPG representation of G with the edge clique $\{P_1, P_3, P_5\}$.

(a) (b)

Figure 3.4: (a) A graph G with the clique $\{1, 2, 3\}$. (b) A B_1-EPG representation of G with the claw clique $\{P_1, P_2, P_3\}$.

Definition. Let $\langle \mathcal{P}, \mathcal{G} \rangle$ be a B_1-EPG representation of G. For every grid edge e, the collection $\{P \in \mathcal{P} \mid e \in P\}$ is called *edge clique*. For every copy of the claw graph $K_{1,3}$ in the grid, the collection $\{P \in \mathcal{P} | P \text{ uses 2 edges of the claw}\}$ is called *claw clique*.

The following observation follows from the definition of the claw clique.

Observation 3.2.1. *A claw clique can not be represented in neither B_0 nor B_1^m.*

It is easy to see, that the vertices corresponding to the paths of an edge clique form a clique in G. Also the vertices corresponding to the paths of a claw clique form a clique in G. The following result reveals, that also the converse is true.

Lemma 3.2.2 (Golumbic, Lipshteyn, Stern [20])**.** *Let G be a graph. Then in every B_1-EPG representation of G, every maximal clique of G corresponds either to an edge clique or to a claw clique.*

In order to derive the next result, we will need two more definitions.

Definition. Let $G = (V, E)$ be a graph and C a subset of the set of vertices of G. Then the *branch graph* $B(G/C) = (V_B, E_B)$ is defined in the following way. V_B contains all the vertices of G, which are not in C but adjacent to a vertex of C. Let $u, v \in V_B$. Then the edge $(u, v) \in E_B$ if and only if

- $(u, v) \notin E$,
- there exists a vertex $w \in C$ such that $(w, u) \in E$ and $(w, v) \in E$,
- there exists a vertex $w_1 \in C$ such that $(w_1, u) \in E$ but $(w_1, v) \notin E$, and
- there exists a vertex $w_2 \in C$ such that $(w_2, v) \in E$ but $(w_2, u) \notin E$.

Definition. A graph $G = (V, E)$ is called *k-colorable* if there exists a coloring of the vertices, such that every vertex is colored with one of k colors and whenever two vertices are adjacent, they are colored with different colors. In other words there exists a mapping $f : V \to \{1, 2, \ldots, k\}$ such that $(u, v) \in E$ implies $f(u) \neq f(v)$ for every $u, v \in V$.

Now we are able to prove another result of [20].

Lemma 3.2.3 (Golumbic, Lipshteyn, Stern [20])**.** *Let $G = (V, E)$ be a graph in B_1 and let $\langle \mathcal{P}, \mathcal{G} \rangle$ be a B_1-EPG representation of G. Let furthermore C be a maximal clique in G. Then the branch graph $B(G/C)$ is 2-colorable if C corresponds to an edge clique in $\langle \mathcal{P}, \mathcal{G} \rangle$ and $B(G/C)$ is 3-colorable if C corresponds to a claw clique in $\langle \mathcal{P}, \mathcal{G} \rangle$.*

Proof. We start by proving, that $B(G/C)$ is 3-colorable if C corresponds to a claw clique. Let q be the grid point in the center of the claw and let q_1, q_2, and q_3 be the other grid points of the claw, such that the grid edges from q_1 to q and from q_3 to q are either both horizontal or both vertical. Let $\mathcal{Q} = \{P_v \mid v \in C\}$. We partition the grid edges which are used by the paths of $\mathcal{Q}$ into 4 parts. The first part consists of the 3 grid edges which form the claw. The remaining grid edges are separated into $\mathcal{Q}_1$, $\mathcal{Q}_2$, and $\mathcal{Q}_3$ such that a grid edge e is contained in $\mathcal{Q}_i$ if there is a path $P \in \mathcal{Q}$ that uses e and the grid point q_i is contained in P in the part from e to q.

Then we color the grid edges with 3 colors, such that all the grid edges from $\mathcal{Q}_i$ are colored with color i. The coloring of the grid edges is well-defined, since every grid edge is contained in only one set of $\mathcal{Q}_1$, $\mathcal{Q}_2$, and $\mathcal{Q}_3$.

Afterwards we color the vertices of the branch graph. A vertex of the branch graph is colored with color i, if it uses a grid edge which is colored with color i.

Let v be a vertex of the branch graph. Then v is colored with a color because v is, as vertex of the branch graph, adjacent to a vertex w of the maximal clique and hence the paths P_v and P_w share a grid edge. This grid edge was colored, and hence P_v uses a colored grid edge.

Assume v is colored with at least 2 colors. Then P_v has to use 2 differently colored grid edges, one of Q_{i_1} and one of Q_{i_2} for $i_1 \neq i_2$ and $i_1, i_2 \in \{1, 2, 3\}$. All the grid edges of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are either in the grid line from q to q_1 and q_2 on the side of q_1 and q_2

respectively, or on a grid line parallel to the grid edge from q to q_3. All the grid edges of $\mathcal{Q}_3$ are in the grid line with the grid edge from q to q_3 on the side of q_3. That implies, that it is not possible that a path with only one bend contains a grid edge of Q_{i_1} and one of Q_{i_2} without using two grid edges of the claw. Nevertheless, the vertex v is not allowed to use two grid edges of the claw, because otherwise it would be in the clique, a contradiction to its maximality. Therefore every vertex is colored with at most 1 color and hence also the coloring of the vertices of the branch graph is well-defined.

What is left to show is, that the coloring is valid, that is two adjacent vertices are not colored with the same color. Assume there are two vertices x and y, which are adjacent in the branch graph but colored with the same color. By the definition of the branch graph, there are vertices $v, u, w \in C$ such that $(x, u), (y, u), (x, v), (y, w) \in E$ but $(x, y), (x, w), (y, v) \notin E$. Let P_x, P_y, P_u, P_v, P_w be the corresponding paths. The vertices x and y are colored with the same color i and both are adjacent to u, therefore the grid edges of $P_x \cap P_u$ and of $P_y \cap P_u$ are also colored with i and hence are all contained in $\mathcal{Q}_i$. Furthermore P_x and P_y do not share a grid edge and hence also $P_x \cap P_u$ and $P_y \cap P_u$ do not share a grid edge. Suppose without loss of generality that $P_x \cap P_u$ is closer to q_i than $P_y \cap P_u$ on the path P_u. The vertex w is contained in the maximal clique and hence P_w uses two grid edges of the claw. Furthermore w is adjacent to y and therefore the path P_w has to use the grid edges of $P_x \cap P_u$ in order to share a grid edge with P_y and have only one bend. This is a contradiction because x and w are not adjacent. That means, that two neighbors in the branch graph have always a different color, hence the coloring is valid.

The statement for a clique that comes from an edge clique is a special case of the above proof. We have to define q_1 and q_2 as the end points of the grid edge from which the edge clique comes from and omit q_3. Then $\mathcal{Q}_3$ is the empty set and therefore only 2 colors are necessary. □

This result can be used in order to show in a very simple way, that there are graphs with arbitrary many vertices which are not in B_1. Namely we consider the following graph.

Definition. The graph $A_d = (V, E)$ is defined as follows. It has vertices $V = P \cup Q$ with $P = \{p_{i,j} \mid 1 \leqslant i < j \leqslant d\}$ and $Q = \{q_i \mid 1 \leqslant i \leqslant d\}$. Furthermore A_d has the edge set $E = E_P \cup E_Q^1 \cup E_Q^2$ with $E_P = \{(p_{i,j}, p_{k,\ell}) \mid 1 \leqslant i < j \leqslant d, 1 \leqslant k < \ell \leqslant d, p_{i,j} \neq p_{k,\ell}\}$, $E_Q^1 = \{(p_{i,j}, q_i) \mid 1 \leqslant i < j \leqslant d\}$, and $E_Q^2 = \{(p_{i,j}, q_j) \mid 1 \leqslant i < j \leqslant d\}$.

The graph A_4 can be found in Figure 3.5 (a). It was shown in [20] that the graph A_d is not in B_1 for arbitrary large d.

Lemma 3.2.4 (Golumbic, Lipshteyn, Stern [20])**.** *The graph $A_d \notin B_1$ for any $d \geqslant 4$.*

Proof. Assume A_d has a B_1-EPG representation. By the definition of A_d, P forms a maximal clique in A_d and hence by Lemma 3.2.2 the collection of paths corresponding to vertices of P corresponds to either an edge clique or a claw clique in the B_1-EPG representation. If we consider the branch graph $B(A_d/P) = (V_B, E_B)$, it is easy to see, that $V_B = Q$ and $E_B = \{(q_i, q_j) \mid 1 \leqslant i < j \leqslant d\}$. Note, that $B(A_d/P)$ is a clique on d

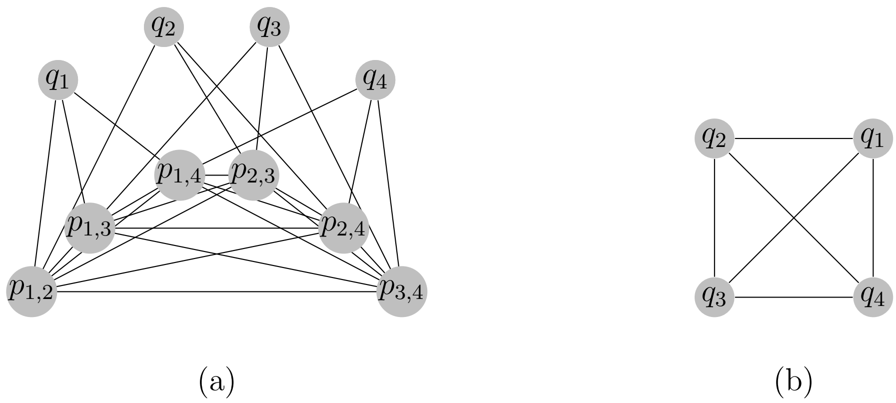


Figure 3.5: (a) The graph A_4 with $P = \{p_{1,2}, p_{1,3}, p_{1,4}, p_{2,3}, p_{2,4}, p_{3,4}\}$. (b) The branch graph $B(A_4/P)$.

vertices. As soon as $d \geqslant 4$ that means, that it is not possible to color $B(A_d/P)$ with 3 colors, a contradiction to Lemma 3.2.3, so A_d is not in B_1. □

Now we want to consider another graph called S_n.

Definition. Let $n \geqslant 3$. Then the n-sun graph $S_n = (V, E)$ is the graph with the vertices $V = \{x_1, \dots, x_n, y_1, \dots, y_n\}$ and edges $E = E_1 \cup E_2$ with $E_1 = \{(x_i, x_j) \mid 1 \leqslant i < j \leqslant n\}$ and $E_2 = \{(x_i, y_i), (x_{i+1}, y_i) \mid 1 \leqslant i < n\} \cup \{(x_1, y_n), (x_n, y_n)\}$.

A picture of S_3 can be found in Figure 3.6 (a) and S_n is depicted in Figure 5.1 (a). Golumbic, Lipshteyn, Morgenstern, and Stern showed in [21] that S_n is another example of a graph with arbitrary many vertices which is not in B_1. Additional to that, they gave a B_1-EPG representation of S_3 in [20].

Lemma 3.2.5 (Golumbic, Lipshteyn, Stern [20])**.** *The graph S_3 is in B_1.*

Proof. To prove, that S_3 is in B_1 it is enough to give a B_1-EPG representation. Such a representation can be seen in Figure 3.6 (b). □

Lemma 3.2.6 (Golumbic, Lipshteyn, Morgenstern, Stern [21])**.** *For $n \geqslant 4$ the graph S_n is not in B_1.*

Now we want to further investigate on S_3 because it will turn out, that S_3 is a very important graph in Section 4.2.

Definition. The vertices $\{x_1, x_2, x_3\}$ of the graph S_3 depicted in Figure 3.6 (a) are called *center vertices* and the edges between them are called *center edges*.

We proceed by making an easy but nevertheless very useful observation which was already mentioned by Biedl and Stern in [4].

Observation 3.2.7 (Biedl, Stern [4])**.** *In every B_1-EPG representation of the graph S_3 the clique $\{x_1, x_2, x_3\}$ corresponds to a claw clique.*

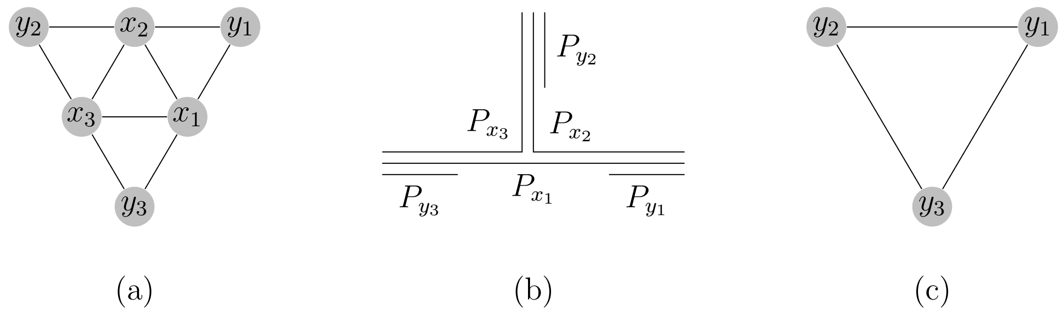


Figure 3.6: (a) The graph S_3. (b) A B_1-EPG representation of the graph S_3. (c) The branch graph of S_3 over the maximal clique $\{x_1, x_2, x_3\}$.

Proof. Consider the branch graph over the maximal clique $\{x_1, x_2, x_3\}$, that is the graph on the vertices y_1, y_2, and y_3 with edge set $\{(y_1, y_2), (y_2, y_3), (y_1, y_3)\}$ as depicted in Figure 3.6 (c). By Lemma 3.2.2 $\{x_1, x_2, x_3\}$ comes from an edge clique or a claw clique. If the clique $\{x_1, x_2, x_3\}$ would come from an edge clique, it follows from Lemma 3.2.3 that the branch graph would be 2-colorable. Nevertheless it takes 3 colors to color the branch graph, so the clique comes from a claw clique. □

Now we are able to prove the following lemma, which is a direct consequence of a result of Cameron, Chaplick, and Hoàng [8].

Lemma 3.2.8 (Cameron, Chaplick, Hoàng [8]). *The graph S_3 is not in B_0 and not in B_1^m.*

Proof. Assume S_3 has a B_0-EPG or a B_1^m-EPG representation, which are B_1-EPG representations as well. The clique $\{x_1, x_2, x_3\}$ comes from a claw clique in this B_1-EPG representation by Observation 3.2.7, and hence it comes from a claw clique also in the B_0-EPG and in the B_1^m-EPG representation. This is a contradiction, since a claw clique can not be represented in neither B_0 nor B_1^m by Observation 3.2.1. □

This result is also the first step towards determining the relationship between B_k and B_k^m, which is an open question of [20]. Note, that in the case $k = 0$ the relationship $B_0^m = B_0$ holds by Observation 2.2.1. In [20] it was conjectured, that $B_1 \subsetneq B_1^m$ holds. This conjecture was confirmed in [8] with the following result.

Corollary 3.2.9 (Cameron, Chaplick, Hoàng [8]). *It holds that $B_1^m \subsetneq B_1$.*

Proof. It is clear, that $B_1^m \subseteq B_1$. We know that $S_3 \in B_1$ due to Lemma 3.2.5. Moreover Lemma 3.2.8 yields, that $S_3 \notin B_1^m$. □

Eventually we want to present another result on the structure of the graphs in B_1 which comes from Asinowski and Ries [2].

Theorem 3.2.10 (Asinowski, Ries [2]). *Let $G = (V, E)$ be a graph in B_1 with n vertices. Then G contains either a clique or an independent set of size $n^{\frac{1}{3}}$.*

3.2.2 Results on Complexity and Approximation Algorithms

Throughout the thesis we will assume basic knowledge on complexity theory and approximation algorithms. Rigorous definitions and introductions into both topics can be found in Korte and Vygen [31] in Chapter 15 and Chapter 16.

As we will see in Subsection 3.3.2, it follows from a result of Booth and Lueker [6] that the graphs in B_0 can be recognized in $\mathcal{O}(m+n)$ time. Nevertheless the problem becomes much harder, if we want to know, whether a given graph is in B_k for higher values of k. The first result on the complexity of deciding, whether a given graph is in B_k for a fixed k, was proved by Heldt, Knauer, and Ueckerdt [26].

Definition. The problem *SINGLE-BEND-RECOGNITION* is the problem of deciding, whether a given graph is in B_1.

Theorem 3.2.11 (Heldt, Knauer, Ueckerdt [26])**.** *SINGLE-BEND-RECOGNITION is NP-complete.*

Basic Idea of the Proof. It is easy to see, that a B_1-EPG representation can be verified in polynomial time, so it is clear, that SINGLE-BEND-RECOGNITION is in NP. In order to prove, that the problem is also NP-hard, the authors of [26] used a reduction from ONE-IN-THREE-3-SAT. For more information about ONE-IN-THREE-3-SAT see for example Garey and Johnson [11]. $\square$

In [8] the authors considered the class $[\llcorner]$, which is another restriction to B_1 and also to B_1^m.

Definition. The class $[\llcorner]$ is the class of all graphs, which have a B_1-EPG representation, in which every path either has only a horizontal segment, or has only a vertical segment, or has a horizontal segment and a vertical segment which starts at the lower end point of the horizontal segment and goes to the right.

The problem $[\llcorner]$-RECOGNITION is the problem of deciding, whether a given graph is in $[\llcorner]$.

In other words $[\llcorner]$ does only contain graphs, which have a B_1-EPG representation where every path has the shape of $\llcorner$. The classes $[\urcorner]$, $[\ulcorner]$, $[\lrcorner]$ are defined analogously. By rotating the grid it is easy to see, that $[\llcorner] = [\urcorner] = [\ulcorner] = [\lrcorner]$. Furthermore the classes $[\llcorner, \lrcorner]$, $[\lrcorner, \ulcorner]$, and $[\ulcorner, \urcorner, \llcorner]$ are defined analogously. By rotating the grid it is again easy to see, that $[\llcorner]$, $[\llcorner, \lrcorner]$, $[\lrcorner, \ulcorner]$, and $[\ulcorner, \urcorner, \llcorner]$ are the only strict subclasses of B_1 obtained by allowing only certain directions of bends. In [8] they showed the following.

Theorem 3.2.12 (Cameron, Chaplick, Hoàng [8])**.** *The problems* $[\llcorner]$*-RECOGNITION,* $[\llcorner, \lrcorner]$*-RECOGNITION and* $[\lrcorner, \ulcorner]$*-RECOGNITION are all NP-complete.*

Basic Idea of the Proof. Again showing, that the problems are in NP is not hard, because representations can be verified easily. For the proof of NP-hardness the authors of [8] used a reduction from 3-SAT. For more information about 3-SAT see [11]. $\square$

That means, that also deciding, whether a given graph is in B_1^m is NP-complete since $B_1^m = [\lrcorner, \ulcorner]$. It was also conjectured in [8] that $[\ulcorner, \urcorner, \llcorner]$-RECOGNITION is NP-complete and that deciding, whether a given graph is in B_k for $k \geqslant 2$ is NP-complete.

After knowing, that deciding whether a given graph is in B_1 is NP-complete, it is a natural further question whether problems, which are NP-complete for general graphs, remain NP-complete for graphs in B_1.

The first problem we consider is MINIMUM-COLORING. In this problem we want to find the minimum k, such that a given graph is k-colorable. This minimum k is called chromatic number and denoted by $\chi(G)$ for a given graph G. It is well known that MINIMUM-COLORING is NP-complete for general graphs, see for example Karp [29]. In [14] it was shown by Golumbic, that there is a polynomial algorithm for MINIMUM-COLORING on interval graphs, that is on graphs in B_0. Epstein, Golumbic, and Morgenstern proved in [10], that that is not the case for graphs in B_1.

Theorem 3.2.13 (Epstein, Golumbic, Morgenstern [10])**.** *MINIMUM-COLORING is NP-complete on graphs in B_1, even if a B_1-EPG representation is given. Furthermore MINIMUM-COLORING is NP-complete on graphs in $[\llcorner]$, even if a $[\llcorner]$-EPG representation is given.*

Basic Idea of the Proof. It is easy to see that coloring a graph with $\chi(G)$ colors is in NP for all graphs. In order to show NP-hardness in [10] they used a reduction from coloring circle graphs. For more information on that problem and the proof, that it is NP-complete see Garey, Johnson, Miller, and Papadimitriou [12]. □

Now, that we know that coloring a graph in B_1 is hard, a natural further question is, whether there are approximation algorithms that run in polynomial time. In [10] the following result was proved.

Theorem 3.2.14 (Epstein, Golumbic, Morgenstern [10])**.** *There is an algorithm that runs in polynomial time and determines a coloring of the vertices of a graph in B_1 with at most $4\chi(G)$ colors. For a graph in $[\llcorner, \lrcorner]$ it needs at most $2\chi(G)$ colors.*

This means, that there is a 4-approximation algorithm for the coloring problem for graphs in B_1. This is in fact a big improvement, because for a general graph the best known approximation ratio is $\mathcal{O}\left(n\frac{\log(\log(n))^2}{\log(n)^3}\right)$, where n is the number of vertices in the graph, see Halldórsson [24].

Now we want to investigate on another problem, the MAXIMUM-INDEPENDENT-SET problem. In this problem we want to find an independent set of maximum cardinality in a given graph G. Let $\alpha(G)$ denote the cardinality of the maximum independent set. It is well known, that the problem is NP-complete for general graphs, a proof can be found in [31]. Nevertheless that is not the case for graphs in B_0. In [14] is was proved, that interval graphs are perfect graphs, and hence the graphs in B_0 are perfect. Moreover it was shown by Grötschel, Lovász, and Schrijver [22], that the MAXIMUM-INDEPENDENT-SET problem can be solved in polynomial time for perfect graphs. Hence for graphs in B_0 the problem MAXIMUM-INDEPENDENT-SET is in P. Nevertheless that is not the case for the graphs in B_1.

Theorem 3.2.15 (Epstein, Golumbic, Morgenstern [10]). *The problem MAXIMUM-INDEPENDENT-SET is NP-complete on graphs in B_1. Furthermore it is NP-complete on graphs in $[\lrcorner, \ulcorner]$ and on graphs in $[\llcorner, \lrcorner]$.*

Basic Idea of the Proof. It is easy to see that the problem is in NP because an independent set can be verified very easily. The NP-hardness result is shown by a reduction from MAXIMUM-INDEPENDENT-SET on planar graphs with maximum degree 4. This problem is known to be NP-complete, see [11]. □

Theorem 3.2.15 also implies, that MAXIMUM-INDEPENDENT-SET is NP-complete for graphs in B_1^m.

Again there are some positive approximation results.

Theorem 3.2.16 (Epstein, Golumbic, Morgenstern [10]). *There is a polynomial time algorithm that finds an independent set of size at least $\frac{1}{4}\alpha(G)$ for every graph in B_1 and an independent set of size at least $\frac{1}{2}\alpha(G)$ for every graph in $[\llcorner, \lrcorner]$.*

The last problem we consider is MAXIMUM-CLIQUE. In this problem we want to find a clique of maximum cardinality in a given graph. Again, the problem is NP-complete for general graphs [29]. Nevertheless that is not the case for the graphs in B_1.

Theorem 3.2.17 (Epstein, Golumbic, Morgenstern [10]). *Let G be a graph in B_1. There is an algorithm which determines the maximum clique of G in $\mathcal{O}(n^3)$ if a B_1-EPG representation is given. Additional to that the maximum clique can be determined in $\mathcal{O}(n^5)$, even if no B_1-EPG representation is given.*

This is an interesting result, considering that the recognition of graphs from B_1 is NP-hard.

3.3 Known Results with Respect to Graph Classes

3.3.1 Overview

At the beginning of this section we want to give a short summary of the known results with respect to graph classes listed in tabular form. Every graph of the graph class is contained in the class of the upper bound. Furthermore there is a graph in the graph class, which is not contained in the class of the lower bound.

Graph Class	Upper Bound		Lower Bound	Reference	Section
Interval Graph	$\subseteq B_0$				3.3.2
Tree	$\subseteq B_1$	$\subseteq B_1^m$	$\not\subseteq B_0$	[20]	3.3.3
Outerplanar	$\subseteq B_2$		$\not\subseteq B_1$	[25], [4]	3.3.4
Planar	$\subseteq B_4$		$\not\subseteq B_2$	[25]	3.3.5
Planar & Bipartite	$\subseteq B_2$		$\not\subseteq B_1$	[4]	3.3.5
Planar & Treewidth $\leqslant 3$	$\subseteq B_3$		$\not\subseteq B_2$	[25]	3.3.5
Line Graph	$\subseteq B_2$			[4]	3.3.6
Line Graph of Bipartite Graph	$\subseteq B_1$	$\subseteq B_1^m$		[26]	3.3.6

3.3.2 Interval Graphs

The first class we consider is the well-known class of interval graphs. They were first introduced by Hajós in [23].

Definition. Let S_1, …, S_n be intervals on the real line. The *interval graph* $G = (V, E)$ has vertex set $V = \{1, \ldots, n\}$ and edge set $E = \{(v_i, v_j) \mid S_i \cap S_j \neq \emptyset\}$.

It is easy to see, that the graphs in B_0 are exactly the interval graphs. In [6] Booth and Lueker presented an algorithm to determine in $\mathcal{O}(n + m)$ time, whether a given graph G with n vertices and m edges is an interval graph or not. Hence also the graphs in B_0 can be determined in $\mathcal{O}(n + m)$ time.

3.3.3 Trees

The first graph class, for which an upper bound on the bend number was newly derived, was the class of trees. In [20] the following result was given.

Theorem 3.3.1 (Golumbic, Lipshteyn, Stern [20])**.** *Every tree is in B_1^m.*

Proof. Let $T = (V, E)$ be a tree. We choose an arbitrary vertex $v_0 \in V$ to be the root and define level $i \geqslant 1$ as the vertices, that have exactly i edges on their unique path to v_0. Furthermore we let v_0 be level 0. Then every vertex of level i is adjacent to exactly one vertex of level $i - 1$. Let k denote the number of levels.

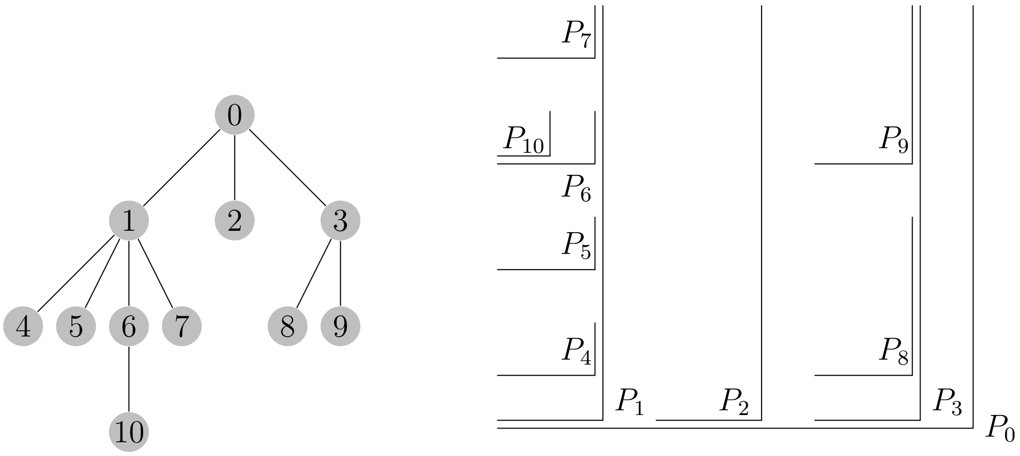
We give a recursive procedure for embedding the vertices of each level. We define the path P_0 corresponding to vertex v_0 as a path that uses the bottom grid line and the grid line farthest to the right.

Let $1 \leqslant i \leqslant k$. When we have already constructed the paths of all levels from 0 to $i - 1$ we construct the paths corresponding to level i in the following way. Let v be a vertex of level $i - 1$ and let v_1, v_2, …, v_ℓ be the vertices of level i which are adjacent to v. We denote the rectangular part of the grid from the bottom left to the top right grid point of the path P_v by $\mathcal{G}'$ and partition $\mathcal{G}'$ into ℓ distinct pieces. If i is odd, every piece consists of all the grid rows of $\mathcal{G}'$ and the grid columns of $\mathcal{G}'$ are split among the pieces such that there is always one free grid column between two pieces and there is a free grid column between the column in which P_v is and the piece closest to P_v. If i is even, every piece consists of all the grid columns of $\mathcal{G}'$ and the grid rows of $\mathcal{G}'$ are split among the pieces such that there is always one free grid row between two pieces and there is a free grid row between the row in which P_v is and the piece closest to P_v. Then for every $1 \leqslant j \leqslant \ell$ the path P_j corresponding to v_j is constructed in such a way, that it uses the bottom grid line and the grid line farthest to the right of the j-th piece.

An example of the construction can be seen in Figure 3.7.

We only have to show, that two vertices are adjacent if and only if the corresponding paths share a grid edge, since it is obvious that every path is monotonic and there is a vertex for every path.

If two vertices v and u are adjacent, then they are in neighbored levels. Let P_v and P_u be the paths corresponding to v and u respectively. Assume without loss of generality,



(a) (b)

Figure 3.7: (a) A tree T. (b) A B_1^m-EPG representation of T.

that v is in level $i-1$ and u is in level i. That means, that we have constructed the path P_u after the path P_v. When we partition the grid from the bottom left to the top right grid point of P_v into pieces, then, depending on the parity of i, either the grid line farthest to the right or the bottom grid line is contained in every piece of the partition. Furthermore the path P_v uses this grid line in every piece. Due to the fact, that also the newly introduced path P_u uses this grid line, this means that P_v and P_u intersect.

Before we prove the second direction of the equivalence, we observe, that the part of the grid, that a vertex uses, is always completely contained in the part of the grid that the vertex, which is one level above it and adjacent to it, uses. Furthermore the parts of the grid, that two vertices on the same level use, do not share any grid edges or grid points.

Assume two vertices v and u are not adjacent, u is on level i and v is on level j with $i \geqslant j$. We distinct two cases.

In the first case v lies on the unique path from u to v_0. Let v' be the vertex which is directly before v and v'' be the vertex which is directly before v' on the unique path from u to v_0. Let $P_{v'}$ and $P_{v''}$ be the paths corresponding to them. The vertices u and v are not adjacent, therefore $u \neq v'$ holds, but u could probably be equal to v''. Then in the construction, depending on the parity of j, either the horizontal or the vertical segment of P_v is completely not contained in the part of the grid, that $P_{v'}$ uses. Furthermore also the other segment of P_v is completely not contained in the part of the grid that $P_{v''}$ uses. That means, that P_u does not intersect with P_v, since P_u only uses the part of the grid, that $P_{v''}$ uses.

If v does not lie on the unique path from u to v_0, then there is a vertex w, such that w is on the unique path from u to v_0, w is on the unique path from v to v_0 and w is on the level with the highest number of all the vertices which fulfil the first two conditions. Note, that $w = v_0$ could probably hold. Let ℓ be the level of w. The choice of w implies, that there are vertices u' and v' on level $\ell+1$, such that u' and v' are adjacent to w and

u' and v' lie on the unique path from u and v to v_0 respectively. The observation from above implies, that the parts of the grid that $P_{u'}$ and $P_{v'}$ use are completely distinct and that furthermore P_u uses only the part of the grid that $P_{u'}$ uses and P_v uses only the part of the grid that $P_{v'}$ uses. Hence also the parts of the grid that P_u and P_v use are distinct and therefore they do not intersect also in this case.

□

This upper bound is tight, because there is a tree which is not in B_0. An example of such a tree is depicted in Figure 4.9 (a), the proof that it is not in B_1 can be found in Lemma 4.3.2.

3.3.4 Outerplanar Graphs

In [4] it was shown, that every outerplanar graph is in B_3. Furthermore is was conjectured, that every outerplanar graph is in B_2. This conjecture was confirmed of Heldt, Knauer, and Ueckerdt in [25].

Corollary 3.3.2 (Heldt, Knauer, Ueckerdt [25])**.** *Every outerplanar graph is in B_2.*

Proof. In Bodlaender [5] the proof, that every outerplanar graph has treewidth at most 2 can be found. Therefore it follows from Theorem 3.4.6 that every outerplanar graph is in B_2. □

This result is best possible with respect to the bend number, because there is a graph which is outerplanar and not in B_1. Biedl and Stern gave an example of such a graph in [4]. See Lemma 4.1.2 for details.

3.3.5 Planar Graphs

In [4] it was shown, that every planar graph is in B_5. This result was improved in [25] in the following way.

Theorem 3.3.3 (Heldt, Knauer, Ueckerdt [25])**.** *Every planar graph is in B_4. Moreover, a B_4-EPG representation can be determined in linear time with respect to the number of vertices of the graph.*

Additional to that, assuming further properties on planar graphs reduces the maximum bend number. The following result was proved in [4].

Theorem 3.3.4 (Biedl, Stern [4])**.** *Every planar and bipartite graph is in B_2.*

This result can not be further improved, because of the following.

Lemma 3.3.5 (Biedl, Stern [4])**.** *The graph $K_{2,5}$ is planar, bipartite and not in B_1.*

Another possibility of reducing the bend number for planar graphs is to require the treewidth to be less than 4. The definition of treewidth can be found in Subsection 3.4.5.

Theorem 3.3.6 (Heldt, Knauer, Ueckerdt [25]). *Let G be a planar graph with treewidth less or equal to 3. Then $G \in B_3$.*

Also this result can not be further improved.

Lemma 3.3.7 (Heldt, Knauer, Ueckerdt [25]). *There is a planar graph G with treewidth at most 3, such that $G \notin B_2$.*

The last result also gives a lower bound on the bend number needed for planar graphs in general. Eventually we conclude, that it is known, that every planar graph is in B_4 and there is a planar graph which is not in B_2. Hence there is still a gap to close.

3.3.6 Line Graphs

We first define line graphs.

Definition. Let G be a graph. G is called *line graph*, if there exists a graph H such that every vertex of G corresponds to an edge in H and two vertices are adjacent in G, if and only if the corresponding edges are incident in H. The graph H is called *root graph*.

The investigation on line graphs started in [4].

Theorem 3.3.8 (Biedl, Stern [4]). *Every line graph is in B_2.*

This result was improved for bipartite graphs in [26] by using Theorem 3.4.9.

Theorem 3.3.9 (Heldt, Knauer, Ueckerdt [26]). *Let G be a line graph with a bipartite root graph. Then $G \in B_1^m$.*

3.4 Known Results with Respect to Graph Properties

3.4.1 Overview

In the beginning we present a short overview of known results with respect to graph properties in tabular form.

Property	Upper Bound		Lower Bound	Reference	Section
pathwidth $\leqslant k$	$\subseteq B_{2k-2}$			[4]	3.4.2
κ-regular orientable	$\subseteq B_{2\kappa+1}$			[4]	3.4.3
κ-regular orientable & bipartite	$\subseteq B_{2\kappa}$			[4]	3.4.3
degeneracy k	$\subseteq B_{2k-1}$		$\nsubseteq B_{2k-2}$	[26]	3.4.4
treewidth $\leqslant k$, $k \geqslant 2$	$\subseteq B_{2k-2}$		$\nsubseteq B_{2k-3}$	[26], [25]	3.4.5
global clique covering number k	$\subseteq B_{k-1}$	$\subseteq B_{k-1}^m$		[26]	3.4.6
edge chromatic number χ'	$\subseteq B_{\chi'-1}$	$\subseteq B_{\chi'-1}^m$		[26]	3.4.7
maximum degree Δ	$\subseteq B_{\Delta}$	$\subseteq B_{\Delta}^m$	$\nsubseteq B_{\lceil \frac{\Delta}{2} \rceil -1}$	[26]	3.4.8
maximum degree Δ & bipartite	$\subseteq B_{\Delta-1}$	$\subseteq B_{\Delta-1}^m$	$\nsubseteq B_{\lceil \frac{\Delta}{2} \rceil -1}$	[26]	3.4.8
local clique covering number k	$\subseteq B_{2k-2}$			[26]	3.4.9

3.4.2 Pathwidth

We start by defining the pathwidth of a graph.

Definition. A graph has *pathwidth* k, if there is an ordering of the vertices $v_1, \ldots, v_n$ such that for any j with $2 \leqslant j \leqslant n$, at most k vertices of $v_1, \ldots, v_{j-1}$ have a neighbor in $v_j, \ldots, v_n$.

In [4] an upper bound on the required bend number for a graph with pathwidth at most k was given.

Theorem 3.4.1 (Biedl, Stern [4]). *Every graph with pathwidth at most k is in B_{2k-2}.*

3.4.3 κ-regular Orientable

In order to give upper bounds in terms of κ-regular orientable graphs, we need the following definition.

Definition. An *edge orientation* determines a head and a tail for every edge. So there are two mappings $h, t : E \to V$ such that $h(e) \in \{v, u\}$, $t(e) \in \{v, u\}$, and $h(e) \neq t(e)$ hold for every edge $e = (v, u) \in E$.

The *indegree* of a vertex v in an edge orientation is defined as the cardinality of the set $\{u \in V \mid (v, u) \in E, h((v, u)) = v\}$.

A graph is called *κ-regular orientable*, if there is an edge orientation, such that every vertex has at most indegree κ.

Now we are able to present the next result.

Theorem 3.4.2 (Biedl, Stern [4]). *Every κ-regular orientable graph G is in $B_{2\kappa+1}$. If G is bipartite, then it is even in $B_{2\kappa}$.*

3.4.4 Degeneracy

The next graph property we consider is degeneracy. In order to define degeneracy, we restrict the class of κ-regular orientable graphs.

Definition. An edge orientation is called *acyclic*, if there is no cycle contained in the directed graph obtained by the edge orientation.

A graph is called *κ-regular acyclic orientable*, if there exists an acyclic edge orientation, such that every vertex has at most indegree κ.

The *degeneracy* of a graph is the minimum κ, such that the graph is κ-regular acyclic orientable.

The first upper bound concerning degeneracy was implicitly given in [20]. There the construction of Theorem 3.1.1 reveals, that every graph with degeneracy k is in B_{2k}. Then in [4] the bound was improved for $k = 2$.

Theorem 3.4.3 (Biedl, Stern [4]). *Every graph with degeneracy 2 is in B_3.*

The authors of [4] suspected, that the bound could be improved to $2k-1$ in general. This conjecture was finally proved in [26].

Theorem 3.4.4 (Heldt, Knauer, Ueckerdt [26]). *Every graph with degeneracy k is in B_{2k-1}.*

They also proved that this bound is tight.

Lemma 3.4.5 (Heldt, Knauer, Ueckerdt [26]). *There is a graph G with degeneracy at most k which is not in B_{2k-2}.*

3.4.5 Treewidth

In order to investigate on an upper bound for the bend number with respect to treewidth, we start by giving the definition of it.

Definition. Let $k \geqslant 1$. A *k-tree* is a graph, that can be constructed by starting with a clique consisting of $k+1$ vertices. Then iteratively a vertex and the edges from this vertex to k other vertices that form a clique are added.

A graph has *treewidth k* if it is the subgraph of a k-tree.

It is easy to see, that connected graphs with treewidth 1 are trees, so by Theorem 3.3.1 we know, that graphs with treewidth at most 1 are in B_1. The first result concerning higher values of treewidth was given in [25].

Theorem 3.4.6 (Heldt, Knauer, Ueckerdt [25]). *Every graph with treewidth at most 2 is in B_2.*

They later generalized their result.

Theorem 3.4.7 (Heldt, Knauer, Ueckerdt [26]). *Every graph with treewidth at most k is in B_{2k-2}.*

Additional to that, they showed that this bound is tight.

Theorem 3.4.8 (Heldt, Knauer, Ueckerdt [26]). *For every $k \geqslant 2$ there is a graph G with treewidth at most k which is not in B_{2k-3}.*

Proof. We start by considering $k \geqslant 3$. According to Theorem 3.5.12 $K_{k,n} \notin B_{2k-3}$ for $n = k^4 - 2k^3 + 5k^2 - 4k + 1$. $K_{k,n}$ has treewidth less or equal to k because we can construct a k-tree by introducing a clique with $k+1$ vertices. Then we iteratively attach $k^4 - 2k^3 + 5k^2 - 4k + 1$ vertices and connect all of them to the same k vertices of the starting clique. It is easy to see, that $K_{k,n}$ is a subgraph of this k-tree and hence has treewidth at most k.

In the case $k = 2$ by Lemma 3.5.2 we know that $K_{2,5}$ is not in B_1, but the graph has treewidth at most 2 because of the same construction as in the first case. $\square$

The upper bound 1 for $k = 1$ is tight as well, because there is a tree which is not in B_0 by Lemma 4.3.2.

3.4.6 Global Clique Covering Number

In the beginning we give the definition of the global clique covering number according to [26].

Definition. Let $\mathcal{C}$ be the class of all cliques and their disjoint unions and let $G = (V, E)$ be a graph. Let furthermore $C_1, C_2, \ldots, C_n \in \mathcal{C}$ with $C_i = (V_i, E_i)$ for every $1 \leqslant i \leqslant n$. Then the collection $C_1, C_2, \ldots, C_n$ is called *edge clique cover of size n* if there exist mappings $f_1, f_2, \ldots, f_n$ such that

- $f_i : V_i \cup E_i \to V \cup E$ with $f_i(E_i) \subseteq E$ and $f_i(V_i) \subseteq V$,
- an edge e and a vertex v are incident in C_i if and only if the edge $f_i(e)$ and the vertex $f_i(v)$ are incident in G,
- for every $e \in E$ and for every $1 \leqslant i \leqslant n$ there is at most one $e' \in E_i$ with $e = f_i(e')$, and
- for every $e \in E$ there is a $1 \leqslant i \leqslant n$ and there is an $e' \in E_i$ with $e = f_i(e')$.

The *global clique covering number* is the minimum n, such that there is an edge clique cover of size n.

In [26] it turns out, that the following holds.

Theorem 3.4.9 (Heldt, Knauer, Ueckerdt [26]). *Every graph with global clique covering number k is in B^m_{k-1}.*

Proof. Let $G = (V, E)$ be a graph with global clique covering number k and let furthermore $C_1, C_2, \ldots, C_k$ be an edge clique cover of size k and denote the corresponding mappings by $f_1, f_2, \ldots, f_k$. Whenever for a vertex v there is no vertex v_i such that $f_i(v_i) = v$ for some $1 \leqslant i \leqslant k$, then we add an additional isolated vertex v_i^* to V_i and define $f_i(v_i^*) = v$. In other words we do nothing else than adding cliques of size 1 to the existing cliques in C_i, such that $f_i(V_i) = V$.

Let the disjoint cliques of C_i be $C_i^1, C_i^2, \ldots, C_i^{k_i}$. Note, that $k_i \leqslant n$ for every $1 \leqslant i \leqslant k$ because the maximum number of cliques contained in C_i is n. This bound is obtained if there are n isolated vertices in C_i.

For every $v \in V$ we define $x_i^v = j$ where j is such that $v \in f_i(C_i^j)$. Note, that $1 \leqslant x_i^v \leqslant k_i$ holds.

Then we define the paths corresponding to the vertices as staircases with $k-1$ bends. Let $v \in V$ and P_v be the path corresponding to v. Then P_v starts in the grid point p_0, has bend points $p_1, p_2, \ldots, p_{k-1}$ and ends in the grid point p_k. For simplicity let $k_0 = 0$, $x_0^v = 1$, and $x_{k+1}^v = 1$ for every v. For every $0 \leqslant i \leqslant k$ we define the point $p_i = (p_i^x, p_i^y)$

with

$$p_i^x = x_{i+1}^v + \sum_{j=1}^{\frac{i}{2}} (k_{2j-1} + 1) \quad \text{and} \quad p_i^y = x_i^v + \sum_{j=1}^{\frac{i}{2}} (k_{2j-2} + 1) \qquad \text{if } i \text{ is even and}$$

$$p_i^x = x_i^v + \sum_{j=1}^{\frac{i-1}{2}} (k_{2j-1} + 1) \quad \text{and} \quad p_i^y = x_{i+1}^v + \sum_{j=1}^{\frac{i+1}{2}} (k_{2j-2} + 1) \qquad \text{if } i \text{ is odd.}$$

It is easy to see, that the above construction yields a path on a grid for every v, because if we go from an even i to an odd $i+1$ the x-coordinate stays the same and if we go from an odd i to an even $i+1$ the y-coordinate stays the same. Furthermore every path has only $k-1$ bends because it consists of k segments.

What is left to show is, that two vertices v and u are adjacent in G if and only if the corresponding paths intersect in $\mathcal{G}$. It follows from the construction, that an intersection of the paths must be located on the segment with the same number for both P_u and P_v. Moreover the paths P_u and P_v share a grid edge in segment i if and only if the vertices u and v are in the image of same clique contained in C_i. Hence if u and v are adjacent, then there are i^* and j^* such that u and v are both in the image of $C_{i^*}^{j^*}$ and therefore intersect. If u and v are not adjacent, then they are never in the same image and hence their paths do not intersect.

In conclusion the above construction yields a B_{k-1}^m-EPG representation. □

3.4.7 Edge Chromatic Number

For investigating on further graph properties we next consider the edge chromatic number defined as follows.

Definition. A graph is called *k-edge colorable* if there exists a coloring of the edges, such that every edge is colored with one of k colors and two incident edges are not colored with the same color. In other words there exists a mapping $f : E \to \{1, \dots, k\}$ such that $(u_1, v), (u_2, v) \in E$ implies that $f((u_1, v)) \neq f((u_2, v))$ for all $u_1, u_2, v \in V$.

The *edge chromatic number* $\chi'(G)$ of a graph G is the minimum value of k, such that the graph is k-edge colorable.

In [26] it was shown as a direct consequence of Theorem 3.4.9, that every graph with edge chromatic number χ' can be represented with at most $\chi' - 1$ bends.

Corollary 3.4.10 (Heldt, Knauer, Ueckerdt [26]). *Let G be a graph with edge chromatic number χ'. Then G is in $B_{\chi'-1}^m$.*

Proof. By the definition of χ' there exists a coloring of the edges with χ' colors, such that two edges which are incident to the same vertex are not colored with the same color. It is easy to see, that a coloring of the edges is in fact an edge clique cover of size χ', where for every color i the corresponding set C_i consists of a clique with two vertices for every edge that is colored with color i. Hence the global edge clique cover number is at most χ' and hence the graph is in $B_{\chi'-1}^m$ by Theorem 3.4.9. □

3.4.8 Maximum Degree

The first bound on the bend number with respect to the maximum degree Δ was derived in [20] as already mentioned in Corollary 3.1.2. They showed that every graph is in $B_{2\Delta}$. This bound was improved in [4] by using Theorem 3.4.2. It implies that every graph is in $B_{2\lceil\frac{\Delta+1}{2}\rceil+1}$ and every bipartite graph is in $B_{2\lceil\frac{\Delta}{2}\rceil}$. The currently best known bound comes from [26] and is a consequence of Corollary 3.4.10.

Corollary 3.4.11 (Heldt, Knauer, Ueckerdt [26])**.** *Let G be a graph with maximum degree Δ. Then G is in B^m_Δ. If G is bipartite, then G is in $B^m_{\Delta-1}$.*

Proof. Let χ' be the edge chromatic number of G. We know, that every graph is in $B^m_{\chi'-1}$ because of Corollary 3.4.10. By Vizing's theorem [33] either $\chi' = \Delta$ or $\chi' = \Delta+1$ holds. Hence every graph is in B^m_Δ. Furthermore for bipartite graphs König [30] showed that $\chi' = \Delta$ holds and hence bipartite graphs are in $B^m_{\Delta-1}$. ☐

It is not known yet, whether this bound is tight. The best known lower bound mentioned in [26] is the following.

Observation 3.4.12 (Heldt, Knauer, Ueckerdt [26])**.** *There is a graph with maximum degree Δ which is not in $B_{\lceil\frac{\Delta}{2}\rceil-1}$.*

Proof. The graph $K_{m,m}$ is by Theorem 3.5.7 not in $B_{\lceil\frac{m}{2}\rceil-1}$. The statement follows due to the fact, that $\Delta = m$ holds for $K_{m,m}$. ☐

3.4.9 Local Clique Covering Number

Another graph parameter considered in [26] is the local clique covering number, which is closely related to the global clique covering number.

Definition. Let $G = (V, E)$ be a graph, $C_1, C_2, \ldots, C_n$ be an edge clique cover of size n with corresponding mappings $f_1, f_2, \ldots, f_n$. Furthermore let $C_i = (V_i, E_i)$ for every $1 \leqslant i \leqslant n$. We say that a vertex $v \in V$ is *contained in the image of f_i* if there exists an $v_i \in V_i$ such that $f_i(v_i) = v$. We say that a vertex $v \in V$ is *contained in k images*, if there exist $i_1, \ldots, i_k$ such that v is contained in the image of f_{i_j} for every $1 \leqslant j \leqslant k$.

The *local clique covering number* of a graph is the minimum k, such that there exists an edge clique cover in which every vertex is contained in at most k images.

In [26] there was given an upper bound on the bend number with respect to the local clique covering number.

Theorem 3.4.13 (Heldt, Knauer, Ueckerdt [26])**.** *Every graph with local clique covering number k is in B^m_{2k-2}.*

3.5 Known Results on $K_{m,n}$

The probably best studied graph class in terms of the bend number is the complete bipartite graph on m and n vertices. For instance all the examples for lower bounds on the required bend number with respect to certain graph properties are proved by using a $K_{m,n}$ for certain values of m and n.

The first step into the direction of proving, that with increasing m and n also the bend number of $K_{m,n}$ increases, was done in [20] with the following result.

Lemma 3.5.1 (Golumbic, Lipshteyn, Stern [20])**.** *The graph* $K_{3,3} \notin B_1$.

Before we give the next result, we mention that it is easy to see, that $K_{1,n}$ is in B_0 for every $n \geqslant 1$, so the characterization of $K_{1,n}$ is done. Asinowski and Suk [3] considered the graph $K_{2,n}$ in more detail.

Lemma 3.5.2 (Asinowski, Suk [3])**.** *The graph* $K_{2,n} \in B_1$ *if and only if* $n \leqslant 4$.

Additional to that, they gave a bound on the maximum number of bends needed to represent a $K_{m,n}$.

Theorem 3.5.3 (Asinowski, Suk [3])**.** $K_{m,n} \in B_{\max\{\lceil \frac{m}{2} \rceil, \lceil \frac{n}{2} \rceil\}}$.

A direct consequence of this is the following.

Corollary 3.5.4 (Asinowski, Suk [3])**.** *The graph* $K_{m,m} \in B_{\lceil \frac{m}{2} \rceil}$.

Another upper bound on the number of bends needed in order to represent $K_{m,n}$ comes from [4].

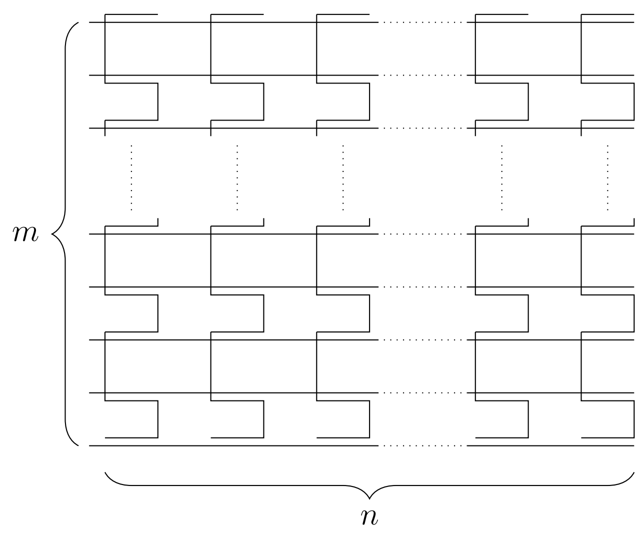


Figure 3.8: A B_{2m-2}-EPG representation of $K_{m,n}$.

Lemma 3.5.5 (Biedl, Stern [4]). *The graph $K_{m,n} \in B_{2m-2}$.*

Proof. This result follows from slightly modifying the construction which is used in the proof of Theorem 3.1.1. The construction can be seen in Figure 3.8. □

As it turned out, this bound is tight for large values of n. Two results were derived independently. The first result was given in [4] by Biedl and Stern. They showed, that $K_{m,N}$ is not in B_{2m-3} for some $N \in \mathcal{O}(m^4)$. The second result comes from Asinowski and Suk [3]. They also proved that $K_{m,N}$ is not in B_{2m-3} for some N sufficiently large. Both results were finally slightly improved by Heldt, Knauer, and Ueckerdt as we will see in Theorem 3.5.12. In order to prove their result, they first derived two inequalities that hold, if a $K_{m,n}$ is in B_k for certain values of m, n, and k. The first result gives a lower bound on k if we know, that $K_{m,n}$ is in B_k for fixed m and n ranging from $\frac{m}{2}$ to $m-1$.

Lemma 3.5.6 (Heldt, Knauer, Ueckerdt [26]). *Let $3 \leqslant m \leqslant n$. Then for every B_k-EPG representation of $K_{m,n}$*

$$(k+1)(m+n) \geqslant mn + \sqrt{2k(m+n)}$$

holds.

They used Lemma 3.5.6 in order to prove the next result.

Theorem 3.5.7 (Heldt, Knauer, Ueckerdt [26]). *Let $m \geqslant 3$. Then $K_{m,m}$ is not in $B_{\lceil \frac{m}{2} \rceil -1}$.*

Proof. Assume $K_{m,m}$ is in $B_{\lceil \frac{m}{2} \rceil -1}$. By applying Lemma 3.5.6 for $n = m$ and $k = \lceil \frac{m}{2} \rceil - 1$ we get that

$$\begin{aligned}
& & \left\lceil \frac{m}{2} \right\rceil (m+m) &\geqslant m^2 + \sqrt{2\left(\left\lceil \frac{m}{2} \right\rceil - 1\right)(m+m)} \\
&\Leftrightarrow & 2m \left\lceil \frac{m}{2} \right\rceil &\geqslant m^2 + 2\sqrt{m\left(\left\lceil \frac{m}{2} \right\rceil - 1\right)} \\
&\Rightarrow & 2m\left(\frac{m+1}{2}\right) &\geqslant m^2 + 2\sqrt{m\left(\frac{m-2}{2}\right)} \\
&\Leftrightarrow & m^2 + m &\geqslant m^2 + \sqrt{2}\sqrt{m(m-2)} \\
&\Rightarrow & m &\geqslant \sqrt{2}(m-2) \\
&\Leftrightarrow & m &\leqslant \frac{2\sqrt{2}}{\sqrt{2}-1} \approx 6.83
\end{aligned}$$

has to hold, which is a contradiction for $m \geqslant 3$. So our assumption was wrong and hence $K_{m,m}$ is not in $B_{\lceil \frac{m}{2} \rceil -1}$ for $m \geqslant 3$. □

Theorem 3.5.7 enabled Heldt, Knauer, and Ueckerdt to answer the question of [20], which relationship B_k and B_{k+1} have. It is easy to see, that $B_0 \subsetneqq B_1$ with for example the graph S_3. In [20] Golumbic, Lipshteyn, and Stern showed with Lemma 3.5.1 that $B_1 \subsetneqq B_2$ and asked, whether $B_{k-1} \subsetneqq B_k$ holds in general. The first partial answer to that question was given in [3] by Asinowski and Suk. They proved $B_{k-1} \subsetneqq B_k$ for every even k by using their result, that $K_{m,N}$ is not in B_{2m-3} for some N sufficiently large. Finally, in [26] the question was fully answered.

Theorem 3.5.8 (Heldt, Knauer, Ueckerdt [26]; Asinowski, Suk [3])**.** *For every $k \geqslant 1$ it holds that $B_{k-1} \subsetneqq B_k$.*

Proof. For $k = 1$ we know that the graph S_3 is in B_1 by Lemma 3.2.5 but not in B_0 due to Lemma 3.2.8. For $k \geqslant 2$ we know that $K_{2k,2k} \in B_k$ by Corollary 3.5.4 but $K_{2k,2k} \notin B_{k-1}$ because of Theorem 3.5.7. □

This means, that for every k there is a graph which has bend number exactly k. Another consequence of Lemma 3.5.6 is the following.

Theorem 3.5.9 (Heldt, Knauer, Ueckerdt [26])**.** *If m is even, then $K_{m,\frac{1}{4}m^3-\frac{1}{2}m^2-m+4}$ is in B_{m-1} but not in B_{m-2}. If m is odd, then $K_{m,\frac{1}{4}m^3-m^2+\frac{3}{4}m}$ is in B_{m-1} but not in B_{m-2}.*

One more result of [26] is as follows. It is used in order to derive another bound on the bend number which is needed in order to represent $K_{m,n}$.

Lemma 3.5.10 (Heldt, Knauer, Ueckerdt [26])**.** *Let $3 \leqslant m \leqslant n$. In a B_k-EPG representation of $K_{m,n}$ let c denote the total number of crossings between the paths corresponding to the vertices of the component which has m vertices. Then*

$$n(2m-k-2) \leqslant 2c + 2(k+1)m$$

holds.

Proof. We consider an arbitrary but fixed B_k-EPG representation of $K_{m,n}$. Let A be the component of $K_{m,n}$ with m vertices and B be the vertices of the other component. In order to prove the above inequality, we will assign end points and crossings of the paths corresponding to A to the paths corresponding to B.

Let $b \in B$ and denote the corresponding path with P_b. Let ℓ be the number of segments of P_b on which there is no intersection with any path corresponding to a vertex of A. The vertex b is adjacent to all vertices of A, so it has to intersect all the corresponding paths. Let the vertices a_1, a_2, $\ldots$, $a_{m'}$ be such that P_b intersects the corresponding paths in this order. Note, that $m' \geqslant m$ because P_b could probably intersect a vertex from A more than once. Let a and a' be two consecutive vertices in this ordering and denote the segment of P_a and $P_{a'}$, on which the intersection with P_b is located, with s and s' respectively. Now we consider two possible configurations. Note, that there also could be others than that.

In the first possible configuration, s and s' intersect P_b on the same segment which we will denote by t. That means, that there is an end point of both s and s' completely contained in t. We assign this two end points to P_b. The path P_b consists of $k+1$ segments, but only $k+1-\ell$ of them have intersections with paths corresponding to the vertices of A on them. That implies, that there are at least $m-(k+1-\ell) = m-k+\ell-1$ pairs of s and s' such that they intersect P_b on the same segment. That means that the total number of assignments to P_b in this configuration is at least $2(m-k+\ell-1)$ because for every pair of s and s' there are two assignments.

In the second possible configuration, s and s' intersect P_b on two consecutive segments t and t'. If s and s' cross in the bend point of P_b that both t and t' use, then we assign this crossing to P_b. If s and s' do not cross in this bend point, then one of them has to end before it reaches that bend point and hence at least one end point of one of the segments s or s' is completely contained in one of the segments t or t'. We assign this end point to P_b. There are $k+1$ segments of P_b and hence P_b has k pairs of consecutive segments. This second configuration occurs at least $k-2\ell$ times, since there are ℓ segments that are not used. So there are at least $k-2\ell$ assignments to P_b in this configuration.

In total, if we sum up over both considered configurations, we have assigned at least $2(m-k+\ell-1)+k-2\ell = 2m-k-2$ crossings and end points to P_b. If we do that for every vertex in B, that yields at least $n(2m-k-2)$ assignments to paths corresponding to vertices of B.

Now we consider the vertices of A. It is easy to see, that every crossing of A can only be assigned twice, because whenever a crossing is assigned to a path P_b, the path P_b bends at the crossing point and hence uses 2 grid edges attached to the crossing point. All the paths corresponding to vertices of B are not allowed to share a grid edge and every grid point does only have 4 attached grid edges, hence there can be at most 2 assignments of one crossing of paths corresponding to vertices of A. Furthermore every end point of a segment can only be assigned at most twice, since whenever it is assigned, one of the two attached segments shares a grid edge with another path and no 3 paths can share a grid edge because the graph is bipartite. Every path has $k+1$ segments and there are m paths corresponding to the vertices of A. That means, that at most $2c+2(k+1)m$ assignments can be made.

Putting together, that at least $n(2m-k-2)$ and at most $2c+2(k+1)m$ assignments are made, yields that

$$n(2m-k-2) \leqslant 2c+2(k+1)m$$

holds. □

This result and another result, which gives an upper bound on the possible number of crossings of two paths with at most k bends, imply the next result.

Lemma 3.5.11 (Heldt, Knauer, Ueckerdt [26]). *Let $3 \leqslant m \leqslant n$. Then for every B_k-EPG representation of $K_{m,n}$*

$$n(2m-k-2) \leqslant m(m-1)\left\lceil \frac{k+1}{2} \right\rceil \left\lceil \frac{k+3}{2} \right\rceil + 2(k+1)m$$

holds.

In [26] the authors were finally able to improve the result for which n the graph $K_{m,n}$ is not in B_{2m-3}. Remember, that Lemma 3.5.5 showed, that $K_{m,n} \in B_{2m-2}$. The next result gives the currently best known bound on n, such that $2m-2$ bends are required for representing $K_{m,n}$.

Theorem 3.5.12 (Heldt, Knauer, Ueckerdt [26]). *Let* $3 \leqslant m \leqslant n$.
If $n \geqslant m^4 - 2m^3 + 5m^2 - 4m + 1$ *then the graph* $K_{m,n} \notin B_{2m-3}$.

Proof. Assume $K_{m,n} \in B_{2m-3}$. Then by Lemma 3.5.11 with $k = 2m-3$ we know that

$$\begin{aligned} & n(2m-(2m-3)-2) \leqslant m(m-1)\left\lceil \frac{2m-2}{2} \right\rceil \left\lceil \frac{2m}{2} \right\rceil + 2((2m-3)+1)m \\ \Leftrightarrow \quad & n \leqslant m^2(m-1)^2 + 2(2m-2)m \\ \Leftrightarrow \quad & n \leqslant m^4 - 2m^3 + 5m^2 - 4m \end{aligned}$$

has to hold. Nevertheless this contradicts for $n \geqslant m^4 - 2m^3 + 5m^2 - 4m + 1$. Hence $K_{m,n} \notin B_{2m-3}$ for $n \geqslant m^4 - 2m^3 + 5m^2 - 4m + 1$. $\square$

Additional to that, Heldt, Knauer, and Ueckerdt [26] gave a lower bound on n such that the bend number of $K_{m,n}$ is $2m-2$.

Theorem 3.5.13 (Heldt, Knauer, Ueckerdt [26]). *If* $n \leqslant m^4 - 2m^3 + \frac{5}{2}m^2 - 2m - 4$ *then the graph* $K_{m,n} \in B_{2m-3}$.

Note, that this leaves only a quadratic gap on n for which it is not known, whether $2m-3$ bends are sufficient or $2m-2$ bends are necessary in order to represent $K_{m,n}$.

4 Outerplanar Graphs

The purpose of this chapter is to determine the minimum k, such that every outerplanar graph is in B_k^m. Additional to that we will find out more about the characterization of special graph subclasses of outerplanar graphs in terms of bend number and monotonic bend number. We will consider two graph classes, namely outerplanar triangulations and cacti, in more detail.

4.1 Outerplanar Graphs are in B_2^m

In [4] it was shown, that every outerplanar graph is in B_3, but also conjectured, that this bound is not tight and all outerplanar graphs are in B_2. This conjecture was confirmed by [25]. We will improve the result by showing, that all outerplanar graphs are not only in B_2, but even in B_2^m.

Theorem 4.1.1. *Every outerplanar graph is in B_2^m.*

Proof. The proof is done by construction. Let G be an outerplanar graph. Note, that if we refer to G, we actually refer to the drawing of G as plane graph where all vertices lie on the boundary of the outer face. Furthermore, we will present the construction only for connected outerplanar graphs. If a graph is not connected, the construction can be applied on every connected component separately.

The construction starts by taking an arbitrary vertex, giving it the number 1, and then going along the boundary of the outer face and numbering the vertices consecutively. Let v_1, v_2, …, v_n be the vertices of the graph in this order.

The path corresponding to the vertex v_1 is constructed as straight horizontal line in the grid and the vertex v_1 is colored green. The rest of the graph is considered as one part.

Then we do the following, until there are no vertices left. We take the green vertex with the lowest number v_i. Let v_{i_1}, …, v_{i_ℓ} be the neighbors of v_i, such that $i_j < i_{j+1}$ for all $1 \leqslant j < \ell$. Furthermore, let v_{i^*} be the green vertex with the second lowest number, if there exists one.

Then we construct the paths corresponding to v_{i_1}, …, v_{i_ℓ} in the following way on the horizontal free line corresponding to v_i. We construct the path corresponding to v_{i_1} as depicted in Figure 4.1 (a). Then we construct the paths corresponding to the vertices v_{i_2}, …, $v_{i_{\ell-1}}$ in this order one by one. Let $1 < j < \ell$. If we have already constructed the path corresponding to $v_{i_{j-1}}$ and the vertices $v_{i_{j-1}}$ and v_{i_j} are not adjacent, then we construct the path corresponding to v_{i_j} as depicted in Figure 4.1 (b). If we have already constructed the path corresponding to $v_{i_{j-1}}$ and the vertices $v_{i_{j-1}}$ and v_{i_j} are adjacent,

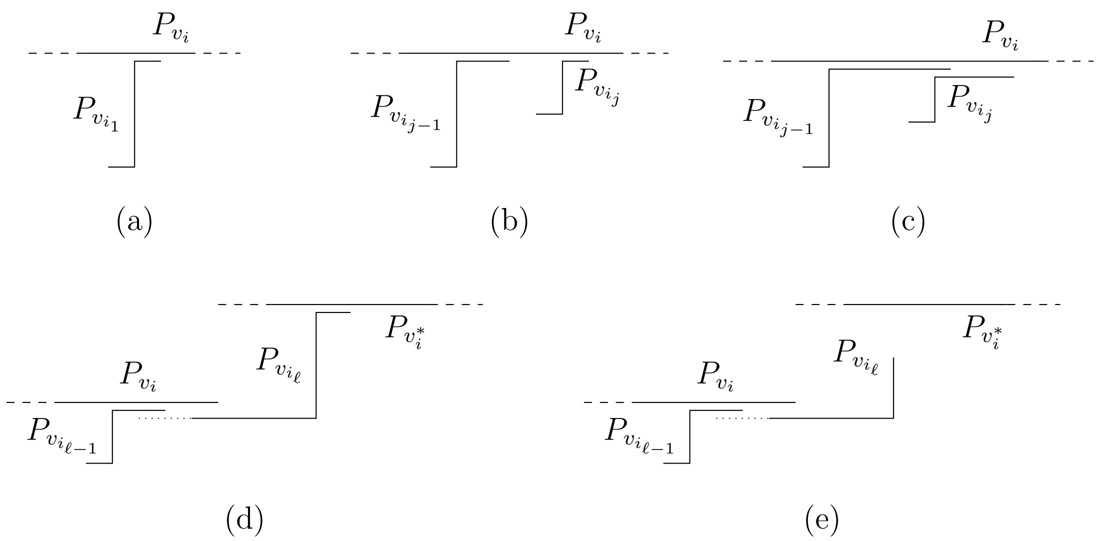


Figure 4.1: Parts of the construction of the proof of Theorem 4.1.1.

then we construct the path corresponding to v_{i_j} as in Figure 4.1 (c). The construction of the path corresponding to v_{i_ℓ} depends on the vertex v_{i^*}. If v_{i_ℓ} is adjacent to v_{i^*}, then we construct the path corresponding to v_{i_ℓ} like depicted in Figure 4.1 (d). If v_{i_ℓ} is not adjacent to v_{i^*} or there exists no vertex v_{i^*}, then the path is constructed as it is done in Figure 4.1 (e). In both cases, the dotted line belongs to the path corresponding to v_{i_ℓ} if $v_{i_{\ell-1}}$ and v_{i_ℓ} are adjacent and the dotted line does not belong to the path corresponding to v_{i_ℓ} if $v_{i_{\ell-1}}$ and v_{i_ℓ} are not adjacent.

Then we delete all the edges (v_i, v_{i_j}), $1 \leqslant j \leqslant \ell$ and the vertex v_i. Furthermore, we delete all the edges $(v_{i_j}, v_{i_{j+1}})$, $1 \leqslant j < \ell$ and the edge (v_{i_ℓ}, v_i^*) if they exist. Then we color the vertices $v_{i_1}, \ldots, v_{i_\ell}$ green. In the end, we adjust the parts. We delete the part between v_i and v_{i^*} and instead introduce new parts between v_{i_j} and $v_{i_{j+1}}$ for every $1 \leqslant j < \ell$ and furthermore we introduce a new part between v_{i_ℓ} and v_{i^*}. The part between two vertices v_{k_1} and v_{k_2} is always the induced subgraph of the vertices $\{v_i \mid k_1 \leqslant i \leqslant k_2\}$. Then we start again by taking the next green vertex with the lowest number.

Now we have to prove, that this construction yields a B_2^m-EPG representation of the graph G. It is clear from the construction, that every path has at most 2 bends and is monotonic, so we only have to show, that the algorithm is well-defined and the resulting representation is an EPG representation.

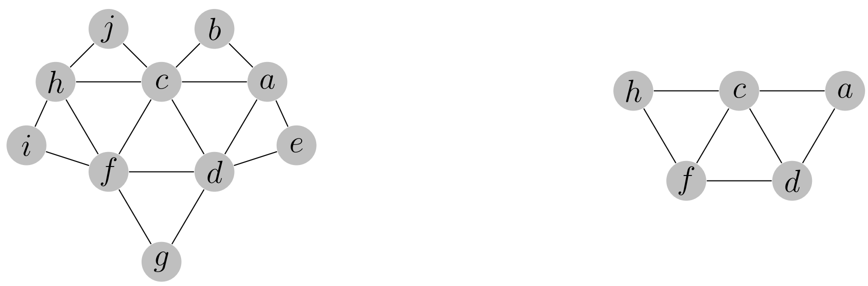
We start by showing that the algorithm is well-defined. In the beginning of the algorithm, a vertex is green. Whenever a green vertex is chosen, all its neighbors which are still in the graph are colored green. The neighbors which are not in the graph anymore have been deleted and since we only delete green vertices, they have been colored green before. Hence whenever a vertex has been colored green, all its neighbors have been colored green during the algorithm as well. That means that every vertex has been colored green at some point since the graph is connected. That implies that every vertex is chosen at some point and that there always is a vertex we can choose until the algorithm stops after finitely many steps. Furthermore, this implies that there is a path corresponding to every vertex, since we only color a vertex green when the path corresponding to it has already been constructed.

Before we will show, that it is always possible to embed the paths in the way the construction does, we consider some invariants throughout the whole algorithm. The following property holds for all the vertices on the outer face since the graph is outerplanar. If we take two vertices v_i and v_j with $i < j$, then all the vertices which are between them on the outer face can not be adjacent to any vertex which is not between them. In other words there can not be an edge between two vertices v_k and v_ℓ with $i < k < j$ and $\ell < i$ or $j < \ell$. As a consequence, throughout the whole algorithm there can only be edges between two vertices which are in the same part and there can never be edges between two vertices which are not in the same part. Another consequence holding the whole algorithm is, that if we start with the green vertex with the lowest number and go along the parts, then the numbers of the green vertices are always increasing.

Now we are able to prove, that the construction can actually be done in the way we described it. With the invariants it is easy to see, that throughout the whole algorithm, for two green vertices which have a part between them, the paths corresponding to them both have a free horizontal line, where the one corresponding to the vertex with the higher number is located to the top left of the one corresponding to the vertex with the lower number. So there always is the free line of the vertex v_i which is needed in the constructions of Figure 4.1 (a) - (e) and furthermore the position of v_{i^*} is always like needed in Figure 4.1 (d) and (e).

The only thing left to show is, that the corresponding paths of two vertices share a grid edge if and only if the vertices are adjacent. Nevertheless that is easy to see, because whenever we perform a step and have chosen a vertex, we construct the paths in such a way, that the paths of two vertices have a grid edge in common if and only if the edge between the vertices is deleted in the end of the step. □

In order to point out, that the theorem can not be further improved, we consider the following result. Note, that this result has already been proved by Biedl and Stern in [4] in a different way.



(a) (b)

Figure 4.2: (a) The graph H_1. (b) The reduced graph of H_1.

Lemma 4.1.2 (Biedl, Stern)**.** *The graph H_1 of Figure 4.2 (a) is not in B_1.*

Proof. We consider the reduced graph of H_1 as defined in the beginning of Subsection 4.2.2. The reduced graph of H_1 is depicted in Figure 4.2 (b). Clearly H_1 is an outerplanar triangulation and the reduced graph is the graph M_1, which is also defined in Subsection 4.2.2. So by Theorem 4.2.7 the graph H_1 is not in B_1. □

So eventually we know, that every outerplanar graph is in B_2^m and furthermore there is an outerplanar graph which is not in B_1, so Theorem 4.1.1 can not be improved.

4.2 Outerplanar Triangulations

In order to understand the definition of outerplanar triangulations better, we also introduce triangulations.

Definition. A graph is called *triangulation* if it is planar and every face is a triangle.

Definition. A graph is called *outerplanar triangulation* if it is outerplanar, the boundary of the outer face is a cycle containing all vertices and every inner face is a triangle.

Note, that an outerplanar triangulation is, except for the graph C_3, not a triangulation, because the outer face is not a triangle. Nevertheless triangulations are planar graphs with as much edges as possible, that is adding one single arbitrary edge to a triangulation would destroy planarity. Exactly the analogous is true for outerplanar triangulations, because adding one single arbitrary edge would yield a graph which is not outerplanar anymore. Due to this fact, outerplanar triangulations are sometimes also called maximal outerplanar graphs.

4.2.1 Outerplanar Triangulations in B_0

Our first step towards determining the minimum k and k^* such that G is in B_k and $B_{k^*}^m$ respectively for every outerplanar graph G, is to investigate which outerplanar triangulations are in B_0. This is not too difficult, so we can immediately give our first result.

Theorem 4.2.1. *Let G be an outerplanar triangulation. Then G is in B_0 if and only if G does not have S_3 as induced subgraph.*

Proof. Let G be an outerplanar triangulation which has a B_0-EPG representation. It follows from Lemma 3.2.8 that G does not contain S_3 as induced subgraph.

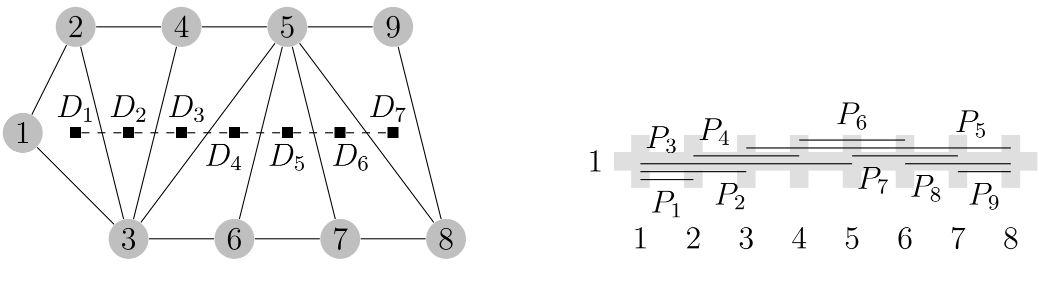
The proof of the other direction of the equivalence is done by construction. Let G be an outerplanar triangulation which does not have S_3 as induced subgraph. We consider the graph $\widehat{G}$ defined as follows. For every inner face of G there is vertex in $\widehat{G}$. Two vertices of $\widehat{G}$ are adjacent if and only if the corresponding inner faces of G share an edge in G. Every vertex of $\widehat{G}$ has degree at most 3 since G is a triangulation. Due to the fact, that S_3 is not an induced subgraph of G, every vertex of $\widehat{G}$ has at most degree 2. Furthermore, because G is connected, $\widehat{G}$ is connected too. In conclusion $\widehat{G}$ is a path.

Let $D_1, \dots, D_\ell$ be the consecutive vertices on this path in $\widehat{G}$ or equivalently the corresponding triangles in G. The B_0-EPG representation is defined as follows. One horizontal line of the grid and $\ell + 1$ columns of the grid are used, hence the used grid points are $(1,1), \dots, (\ell + 1, 1)$. From grid point $(i,1)$ to grid point $(i+1,1)$ we draw a part of the path P_v for every vertex v which is part of the triangle D_i. All the according paths are connected since every vertex v can only be part of consecutive triangles.

For an example of this construction see Figure 4.3. □

Note, that the graph $\widehat{G}$ of the proof is an induced subgraph of the dual graph of G. For information about the dual graph see for example [9].

Additional to that, the proof reveals, that it is possible to either give a B_0-EPG representation of an outerplanar triangulation or determine, that such a representation does not exist in linear time with respect to the number of vertices. Given an outerplanar triangulation, an outerplanar embedding can be found in linear time, because a graph is outerplanar if and only if the graph obtained by adding a vertex and an edge from this vertex to every other vertex is planar. A planar embedding can be found in linear time for example with an algorithm of Hopcroft and Tarjan [27]. By deleting the additional vertex of the planar embedding, one gets an outerplanar embedding of the original graph. Hence an outerplanar embedding can be found in linear time. Furthermore the graph $\widehat{G}$ can, as induced subgraph of the dual graph, be determined in linear time in the outerplanar embedding. Then it is possible to find out in linear time, whether there is a vertex in $\widehat{G}$ having degree at least 3, in which case the graph is not in B_0. If every vertex in $\widehat{G}$ has degree at most 2, a B_0-EPG representation can be found in linear time in the way it is done in the proof of Theorem 4.2.1.



(a) (b)

Figure 4.3: (a) A graph G with $\widehat{G}$. (b) A B_0-EPG representation of the graph in (a).

Theorem 4.2.1 can also be reformulated in the following way.

Corollary 4.2.2. *Let G be an outerplanar triangulation. Then G is in B_0 if and only if the boundary of every inner face of G contains at least two consecutive vertices of the outer face.*

Proof. In order to prove this result, because of Theorem 4.2.1 it is enough to prove that an outerplanar triangulation does not contain S_3 as induced subgraph if and only if the boundary of every inner face contains at least two consecutive vertices of the outer face.

Equivalently it is enough to prove that an outerplanar triangulation contains S_3 as induced subgraph if and only if there is at least one inner face with a boundary which does not contain two consecutive vertices of the outer face.

On the one hand, if S_3 is contained as induced subgraph, the triangle $\{x_1, x_2, x_3\}$ of S_3 as in Figure 3.6 (a) is an inner face in which no two consecutive vertices of the outer face are in, because y_1, y_2, and y_3 are between x_1 and x_2, x_2 and x_3, and x_1 and x_3 on the outer face respectively.

On the other hand, if there is at least one inner face in which no two consecutive vertices of the outer face are in, one can easily construct a S_3 as induced subgraph by defining the vertices of that triangle as $\{x_1, x_2, x_3\}$ and the third vertices of the three neighbored triangles as $\{y_1, y_2, y_3\}$. □

4.2.2 Outerplanar Triangulations in B_1

Now we know exactly which outerplanar triangulations are in B_0, therefore the next aim is to find out which of them are in B_1^m. The following result is a direct consequence of Theorem 4.2.1.

Corollary 4.2.3. *Let G be an outerplanar triangulation. Then G is in B_1^m if and only if G does not have S_3 as induced subgraph.*

Proof. On the one hand, since by Lemma 3.2.8 the graph S_3 is not in B_1^m, it is clear, that a graph containing S_3 as induced subgraph is not in B_1^m.

On the other hand by Theorem 4.2.1 every outerplanar triangulation which does not have S_3 as induced subgraph is in B_0 and hence in B_1^m by Observation 2.2.1. □

So, surprisingly, the outerplanar triangulations in B_1^m are just the same as the ones in B_0. This means that allowing two more shapes of paths does not increase the number of graphs which can be represented. Nevertheless that is not the case of B_1, as we will see in the proceeding. First we need a definition.

Definition. Let G be an outerplanar triangulation. Then the *reduced graph* $\tilde{G}$ is defined in the following way. For every copy of S_3 in G, we color the center vertices and the center edges of S_3 green. Then we remove every vertex and every edge which is not colored from G. The resulting graph is the reduced graph $\tilde{G}$.

Note, that the resulting object is indeed a graph, since whenever an edge is colored also the end vertices are colored, so it can not happen that an edges is in $\tilde{G}$ but a vertex, to which the edge is incident to, is not in $\tilde{G}$.

The following result is easy to see.

Lemma 4.2.4. *Let G be an outerplanar triangulation and $\tilde{G}$ its reduced graph. Then every triangle in $\tilde{G}$ corresponds to the center vertices of a copy of S_3 in G.*

Proof. Let $\{v_1, v_2, v_3\}$ be the vertices of an arbitrary triangle T of $\tilde{G}$. If all the edges of T were colored at once, the triangle corresponds to the center vertices of a S_3 in G by definition.

If only two edges of T were colored at once, then the last edge of the triangle which was colored in G has to be the last edge of T, because there can be only one edge incident to two vertices. So the third edge was colored at the same time as the first two edges and so the vertices of T correspond to the center vertices of a S_3 by definition again.

If all three edges were colored at a different time, then consider the edge (v_1, v_2). It was colored, hence there is a vertex $v_4 \neq v_3$ such that $\{v_1, v_2, v_4\}$ form the center vertices of a S_3. Analogously to the vertex v_4 for the edge (v_1, v_2) there are vertices v_5 and v_6 such that $\{v_2, v_3, v_5\}$ and $\{v_1, v_3, v_6\}$ are the center vertices of a S_3 in G. None of the vertices v_4, v_5, and v_6 can be adjacent to any of the others since the graph is outerplanar. Therefore, the induced subgraph of the vertices $\{v_i | 1 \leqslant i \leqslant 6\}$ forms a S_3 in G and hence the vertices of T are the center vertices of a S_3 in G. □

This result is useful in order to prove the following Corollary.

Corollary 4.2.5. *Let G be an outerplanar triangulation and $\tilde{G}$ its reduced graph. Then the paths corresponding to the vertices of any triangle in $\tilde{G}$ form a claw clique in every B_1-EPG representation.*

Proof. The vertices of every triangle in $\tilde{G}$ are the center vertices of a S_3 in G by Lemma 4.2.4. Then their corresponding paths have to form a claw clique because of Observation 3.2.7. □

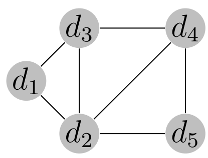


Figure 4.4: The graph M_1.

Lemma 4.2.6. *Let G be an outerplanar triangulation and $\tilde{G}$ its reduced graph. If the graph M_1 depicted in Figure 4.4 or the graph M_1^n depicted in Figure 4.5 is an induced subgraph of $\tilde{G}$ for some $n \geqslant 0$, then G is not in B_1.*

Proof. Assume that M_1^n is contained as induced subgraph in $\tilde{G}$ for some $n \geqslant 0$ and there is a B_1-EPG representation of G.

One of the edges (a_1, a_3) and (a_2, a_4) is contained in the copy of M_1^n in $\tilde{G}$, therefore the vertices $\{a_1, a_2, a_3, a_4\}$ form two triangles. The corresponding paths form a claw clique for each triangle, due to Corollary 4.2.5. In both of these claw cliques 2 paths are bended and there cannot be a path which is bended in both claw cliques, so it follows that all paths corresponding to $\{a_1, a_2, a_3, a_4\}$ are bended in the claw cliques of the two

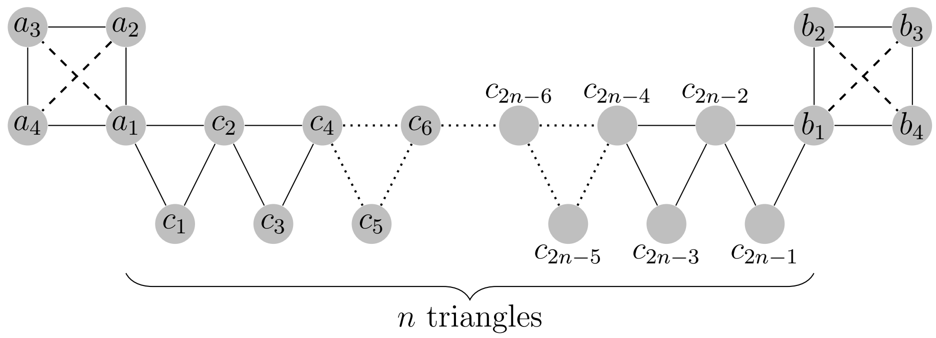


Figure 4.5: The graph M_1^n with $n \geqslant 0$ consecutive triangles between the vertices a_1 and b_1. Furthermore either the edge (a_1, a_3) or the edge (a_2, a_4) is contained and either the edge (b_1, b_3) or the edge (b_2, b_4) is contained. Note that $a_1 = b_1$ for $n = 0$.

triangles. Analogously all the paths corresponding to $\{b_1, b_2, b_3, b_4\}$ are bended in the two triangles formed by these vertices.

So for $n = 0$ the vertex $a_1 = b_1$ has to be bended in two different claw cliques, a contradiction. So assume $n \geqslant 1$ from now on.

The first triangle $\{a_1, c_1, c_2\}$ comes from a claw clique due to Corollary 4.2.5 and a_1 is bended in another claw clique, hence it follows that c_1 and c_2 are bended in the claw clique corresponding to this triangle.

By induction and setting $b_1 = c_{2n}$ we get that in the i-th triangle, the paths corresponding to the vertices c_{2i-1} and c_{2i} are bended in the claw clique corresponding to the i-th triangle for $1 \leqslant i \leqslant n$. So $c_{2n} = b_1$ has to be bended in the n-th triangle, as well as in one of the triangles formed by $\{b_1, b_2, b_3, b_4\}$. This is a contradiction, hence if M_1^n is contained as induced subgraph in $\tilde{G}$ for some $n \geqslant 0$, then G is not in B_1.

Assume that M_1 is an induced subgraph of $\tilde{G}$ and G is in B_1. Then analogously to $\{a_1, a_2, a_3, a_4\}$ it follows that $\{d_1, d_2, d_3, d_4\}$ are all bended in the claw cliques corresponding to the two triangles formed by the corresponding vertices. But then the paths corresponding to $\{d_2, d_4, d_5\}$ have to form a claw clique while two of them already have to be bended in another claw clique, so they can not be bended in this claw clique, again a contradiction. Hence M_1 can not be contained as induced subgraph of $\tilde{G}$ if G has a B_1-EPG representation. □

So now we have a criterion, when an outerplanar triangulation is not in B_1. In fact, this is the only possibility for an outerplanar triangulation not to have a B_1-EPG representation as the following result reveals.

Theorem 4.2.7. *Let G be an outerplanar triangulation and $\tilde{G}$ its reduced graph. Then G is in B_1 if and only if neither the graph M_1 nor the graph M_1^n is an induced subgraph of $\tilde{G}$ for any $n \geqslant 0$.*

Proof. As soon as M_1 or M_1^n is an induced subgraph of $\tilde{G}$ the graph G is not in B_1 by Lemma 4.2.6.

The proof for the other side of the equivalence is done by construction. Assume $\tilde{G}$ does neither have M_1 nor M_1^n for any n as induced subgraph. We first construct the graph $\widehat{G}$ like in the proof of Theorem 4.2.1. Again, because of the fact that G is an outerplanar triangulation, every vertex in $\widehat{G}$ has at most degree 3, $\widehat{G}$ is connected and does not contain any cycles. In conclusion $\widehat{G}$ is a tree.

If 3 consecutive vertices of $\widehat{G}$ would all have degree 3, then M_1 would be contained as induced subgraph in $\tilde{G}$, so that is not possible.

Furthermore, if a vertex of $\widehat{G}$ has degree 2, the corresponding paths can be represented as paths in B_0 as we have already seen in the proof of Theorem 4.2.1, so especially the vertices of degree 3 in $\widehat{G}$ are of interest.

Prior to the presentation of the construction we show the following property.

Claim 4.2.8. *For every vertex of degree 3 in $\widehat{G}$, or equivalently for every triangle in $\tilde{G}$, or equivalently for every center triangle of a S_3 in G, it is possible to determine 2 vertices which are bended in the claw clique corresponding to the triangle according to Corollary 4.2.5 and are not bended in any other claw clique.*

Proof of Claim. We start by calling two triangles of $\tilde{G}$ which have an edge in common *neighbored*. Due to the fact, that M_1 is not an induced subgraph of $\tilde{G}$, it follows that a triangle is neighbored to at most one other triangle. Moreover all the vertices of two neighbored triangles can not be in any other neighbored triangles since M_1^0 is not an induced subgraph of $\tilde{G}$. So if two triangles T_1 and T_2 are neighbored, we choose the vertex which is only in T_1 and one of the two vertices which are in both triangles to be bended in the claw clique corresponding to T_1. The other vertex which is in both triangles and the vertex which is only in T_2 are supposed to be bended in the claw clique corresponding to T_2. So now we have chosen two vertices which are supposed to be bended for every triangle which is neighbored to any other in such a way, that every vertex is chosen at most once.

Now we have to choose two vertices for every triangle which is not neighbored to any other triangle in $\tilde{G}$. We do that in the following way:

1 As long as there is a triangle which shares a vertex v with a triangle, in which two vertices have already been chosen, we choose the 2 vertices which are not v for this triangle.

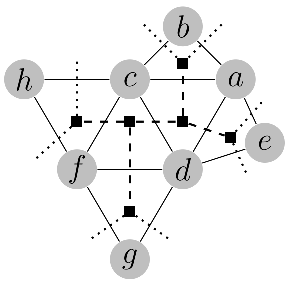
2 If there is no triangle which has a vertex in common with a triangle where the vertices are already chosen, but there is still a triangle where no vertices are chosen yet, we choose two arbitrary vertices of the triangle and proceed with 1.

It can never happen, that a triangle where no vertices are chosen yet shares 2 vertices with triangles where the vertices have already been chosen, because in this case we either would have a M_1^n as induced subgraph of $\tilde{G}$ or a cycle in $\tilde{G}$ which is not a triangle or two neighbored triangles, but every edge of the cycle is part of a triangle. In this case all the triangles lie outside the cycle because G is outerplanar and because G is an outerplanar triangulation, in the interior of this cycle in G there are only triangles. Hence all of these triangles are center triangles of a S_3 in G and all of these triangles are in $\tilde{G}$. So M_1 is contained as induced subgraph in $\tilde{G}$.

There are only finitely many triangles and in every step we choose two vertices of a triangle where we have not chosen the vertices before, therefore in the end we have chosen two vertices of every triangle of $\tilde{G}$ as desired in such a way, that every vertex is chosen only once. □

Now we are finally able to present the construction. There is at least one vertex of degree 1, since $\widehat{G}$ is a tree. Let $\widehat{v}$ be a vertex of degree 1 in $\widehat{G}$ and let $\{a, b, c\}$ be the vertices of the triangle corresponding to $\widehat{v}$. We will start the construction at $\widehat{v}$ by constructing the paths P_a, P_b, and P_c as parallel paths using one grid line. No cycles are contained in $\widehat{G}$, therefore we can give a construction for every neighbor of $\widehat{v}$ in $\widehat{G}$ separately and will go on by always giving the construction for the neighbors of already constructed vertices which are not constructed yet. In this way we will finally have a construction for the whole graph. In the construction we will distinct 3 cases.

In all of these cases we have already constructed parts of the paths of a triangle $\{a, b, c\}$ and the corresponding vertex $\widehat{v}$ of $\widehat{G}$ has degree at most 2 and we want to proceed the construction for the not constructed neighbor $\widehat{u}$ of $\widehat{v}$ in $\widehat{G}$. If a neighbor $\widehat{u}$ has degree 1, then all the corresponding paths stop, so all the dotted and dashed lines end. We distinct the following 3 cases.



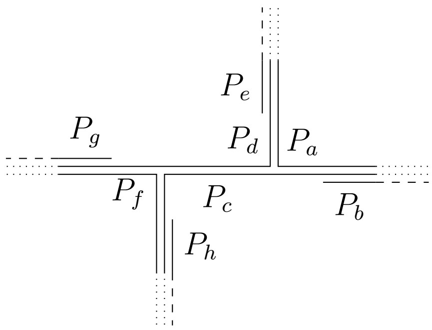
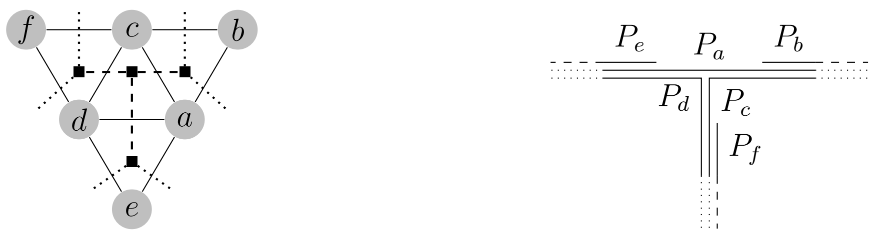


Figure 4.6: (a) A part of a graph G with $\widehat{G}$. (b) The B_1-EPG representation of the graph in (a) where at most one of two dotted edges incident to one vertex in $\widehat{G}$ can exist.

Case 1. If $\widehat{u}$ has degree 3 in $\widehat{G}$ and $\widehat{u}$ has a neighbor $\widehat{w}$ with degree 3, the construction goes on as depicted in Figure 4.6, where $\{a, c, d\}$ are the vertices corresponding to $\widehat{u}$ and a and d are chosen in $\widehat{u}$ and furthermore $\{c, d, f\}$ are the vertices corresponding to $\widehat{w}$ and d and f are chosen in $\widehat{w}$. Note, that in this case all the other neighbors of $\widehat{u}$ and $\widehat{w}$ have to have at most degree 2 in $\widehat{G}$ because otherwise M_1 would be an induced subgraph of $\tilde{G}$. Then the construction proceeds for at most 3 vertices in $\widehat{G}$, namely the ones corresponding to the triangles $\{a, d, e\}$, $\{c, f, h\}$, and $\{d, f, g\}$ in G.



(a) (b)

Figure 4.7: (a) A part of a graph G with $\widehat{G}$. (b) The B_1-EPG representation of the graph in (a) where at most one of two dotted edges incident to one vertex in $\widehat{G}$ can exist.

Case 2. If $\widehat{u}$ has degree 3 in $\widehat{G}$ but all of its neighbors have degree 2 or less, we have to distinct, which vertices corresponding to $\widehat{u}$ are chosen. If only one vertex which corresponds to both $\widehat{v}$ and $\widehat{u}$ is chosen and additional to that the vertex which only corresponds to $\widehat{u}$ and not to $\widehat{v}$ is chosen, then let d and c be the chosen vertices of $\widehat{u}$. Then the construction proceeds as in Figure 4.7. If both vertices which correspond to both $\widehat{v}$ and $\widehat{u}$ are chosen in $\widehat{u}$, namely c and a, then in the construction of Figure 4.7 we exchange the paths P_a and P_d and additional to that P_b and P_f in order to make sure, that the chosen vertices of $\widehat{u}$ are bended in the claw clique corresponding to $\widehat{u}$. The construction proceeds in the vertices in $\widehat{G}$ corresponding to the triangles $\{a, d, e\}$ and $\{c, d, f\}$ in G.

(a) (b)

Figure 4.8: (a) A part of a graph G with $\widehat{G}$. (b) The B_1-EPG representation of the graph in (a) where at most one of two dotted edges incident to one vertex in $\widehat{G}$ can exist.

Case 3. If $\widehat{u}$ has degree at most 2 in $\widehat{G}$ and $\{a, c, d\}$ are the vertices of $\widehat{u}$, then the construction goes on as depicted in Figure 4.8. In the end the construction proceeds with the vertex in $\widehat{G}$ corresponding to the triangle $\{a, c, d\}$ in G.

$\widehat{G}$ is a tree and hence connected, so it is easy to see, that with this construction every path is constructed. Moreover, for every vertex the corresponding path is only bended in the claw clique corresponding to the triangle in which it is chosen. Every path has at most one bend since every vertex is chosen at most once. Hence this construction yields a B_1-EPG representation of G. □

The proof of Theorem 4.2.7 also gives a linear time algorithm to find a B_1-EPG representation of an outerplanar triangulation if it exists. The graph $\widehat{G}$ can, as induced subgraph of the dual graph of G, be determined in linear time with respect to n, if n denotes the number of vertices of G. Moreover the graph $\tilde{G}$ can be determined in linear time, since for every vertex of degree 3 in $\widehat{G}$ there are 3 vertices and 3 edges between them in $\tilde{G}$. Furthermore, the chosen vertices can be determined in $\tilde{G}$ according to the proof in linear time. Then the paths can be constructed in the way it is done in the proof in linear time.

In Theorem 4.1.1 we have seen that all outerplanar graphs are in B_2^m and hence all outerplanar triangulations are in B_2^m. Summarizing we get a full characterization of which outerplanar triangulations belong to B_0, B_1^m, B_1, and B_2^m.

4.3 Cacti

We now want to derive such a full characterization for another subclass of outerplanar graphs, namely the cacti. We first need the following definitions.

Definition. The graph C_n is a graph with vertices $\{v_1, \ldots, v_n\}$ and with the edge set $\{(v_i, v_{i+1}) \mid 1 \leqslant i \leqslant n-1\} \cup \{(v_n, v_1)\}$, that is a cycle on n vertices.

Definition. A *simple cycle* of a graph G is a subgraph C_n for some n where all the vertices are pairwise disjoint.

Definition. A graph is called *cactus* if it is connected and any two simple cycles have at most one vertex in common.

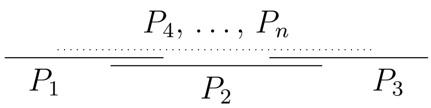
It is easy to see that every cactus is outerplanar. Moreover cacti are in some sense the opposite to outerplanar triangulations, because in a cactus there are no edges allowed inside a cycle, whereas in outerplanar triangulations there have to be as many edges as possible in every cycle.

4.3.1 Cacti in B_0

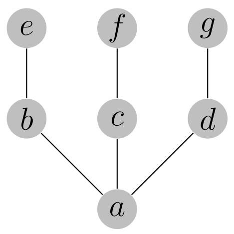
Again, we start by figuring out which cacti are in B_0. In order to do so, we need the following results.

Observation 4.3.1. *The cycle C_n is not in B_0 for $n \geqslant 4$.*

Proof. Let v_1, v_2, ..., v_n be the consecutive vertices of C_n with $n \geqslant 4$. Assume C_n has a B_0-EPG representation and let P_i be the path corresponding to v_i. Both P_1 and P_3 have to share a grid edge with P_2 since v_1 and v_3 are adjacent to v_2. Nevertheless P_1 and P_3 do not share a grid edge with each other because v_1 and v_3 are not adjacent. So P_1 is completely on one side of P_3 with P_2 using all of the grid edges between them.



Furthermore v_1 and v_3 are not only connected via v_2, but also using $v_4, \ldots, v_n$, so at least one of the paths $P_4, \ldots, P_n$ has to have a common grid edge with P_2, a contradiction since all of the corresponding vertices are not adjacent to v_2. □



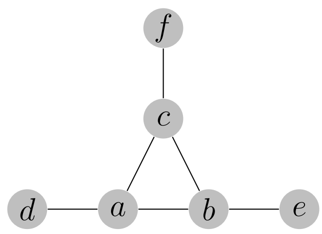


Figure 4.9: (a) The graph M_2. (b) The graph M_3.

Lemma 4.3.2. *The graphs M_2 and M_3 shown in Figure 4.9 are not in B_0.*

Proof. Assume M_2 has a B_0-EPG representation. There is no edge joining any two vertices in $\{b, c, d\}$, therefore their corresponding paths are disjoint lines and hence one path has to be in the middle of the other two paths. Due to the fact, that b, c, and d are all adjacent to a, the path corresponding to a has to have a grid edge in common with all 3 corresponding paths and hence the one path in the middle is completely contained in the path corresponding to a. That is a contradiction, since all 3 vertices b, c, and d have a neighbor which is not adjacent to a. Hence M_2 is not in B_0.

Assume M_3 has a B_0-EPG representation. The vertices d, e, and f are pairwise disjoint, so also their corresponding paths are pairwise disjoint. Furthermore a, c, and b form a clique, so the corresponding paths have to form an edge clique by Lemma 3.2.2, which can not share a grid edge with any of the paths d, e, and f. Hence on one side of the edge clique there are at least two paths corresponding to the vertices of d, e, and f, one being closer to the edge clique than the other. But then the corresponding path of the vertex of the clique $\{a, b, c\}$ which is adjacent to the vertex which is further away contains the path of the vertex which is closer, a contradiction since every vertex of $\{a, b, c\}$ is only adjacent to exactly one vertex of $\{d, e, f\}$. □

Now we can give a characterization of the cacti in B_0.

Theorem 4.3.3. *Let G be a cactus. Then G is in B_0 if and only if G does not have M_2, M_3, and C_n, $n \geqslant 4$ as induced subgraph.*

Proof. Every graph which has at least one of the above graphs as induced subgraph is not in B_0 since M_2, M_3, and C_n, $n \geqslant 4$ are not in B_0 by Lemma 4.3.2 and Observation 4.3.1.

In order to prove the other direction, assume that G is a cactus, which does not contain any of the graphs M_2, M_3, and C_n, $n \geqslant 4$ as induced subgraph. Hence the only cycles that G probably contains are triangles.

If there is a triangle contained in G, let v_1, v_2, v_3 be the vertices of an arbitrary triangle. At least one vertex among v_1, v_2, and v_3 has degree 2 since M_3 is not contained as induced subgraph. Without loss of generality we can assume, that v_3 has degree 2. If there would be another vertex $v_4 \neq v_3$ which is adjacent to both v_1 and v_2, then the simple cycles $\{v_1, v_2, v_3\}$ and $\{v_1, v_2, v_4\}$ would share an edge, which is a contradiction to G being a cactus. Let G' be the graph we get by deleting the vertex v_3 and the edges (v_1, v_3) and (v_2, v_3) of G. If G' has a B_0-EPG representation, we can insert a new grid edge such that only the paths corresponding to v_1 and v_2 use this grid edge, because the vertices v_1 and v_2 are adjacent and there is no other vertex being adjacent to both of them. If we add the path corresponding to v_3 as just this grid edge, we obtain a B_0-EPG representation for G. Hence it is enough, to find a B_0-EPG representation for G'. We can delete a vertex of degree 2 of every triangle since the triangle was chosen arbitrarily, and it is enough to find a B_0-EPG representation for the remaining graph. This graph does not contain any triangles, but is still a cactus and does not contain any of the graphs M_2, M_3, and C_n with $n \geqslant 4$.

Hence we can assume, that G does not contain any triangles. G is connected because it is a cactus and does not contain any C_n, $n \geqslant 3$, hence G is in fact a tree. Let G'' be the graph we obtain by deleting all vertices with degree 1 of G. Assume, that there is a vertex v in G'' with degree at least 3 in G''. Let v_1, v_2, and v_3 three of its neighbors in G''. They are of degree at least 2 in G since they have not been deleted, and hence all of them have another neighbor different to v in G. That implies that the graph M_2 is contained in G because there are no cycles in G. Nevertheless that is a contradiction. So in G'', every vertex has at most degree 2 and furthermore G'' is connected because G was connected. So G'' is in fact a path. Let $w_1, \ldots, w_k$ be the vertices in this order on the path. Now we can construct the paths corresponding to $w_1, \ldots, w_k$ as depicted in Figure 4.10.

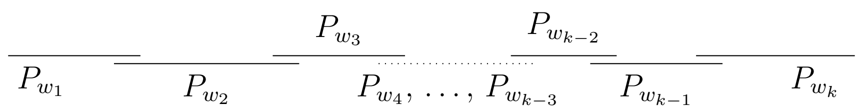


Figure 4.10: The B_0-EPG representation of the path G''.

After that, we can obtain a B_0-EPG representation of G in the following way. Let v be a vertex in G which is missing in G'', so it has degree 1 in G. Let w_ℓ be its neighbor in G, which is in G'' as well. Then we construct the path P_v as a part of the path P_{w_ℓ}, where P_{w_ℓ} does not intersect any other path. This can be done for all the vertices of G which are not in G'' and in the end we get a B_0-EPG representation of G. □

Note, that by following the steps of the proof, one can determine a B_0-EPG representation for a cactus in B_0 with n vertices in $\mathcal{O}(n \log(n))$ time. In order to do so, we assume that the graph has vertices $1, 2, \ldots, n$ and we use a data structure, in which we can access a vertex in constant time and all the neighbors of a vertex are stored in ascending order. Note, that a cactus has only $\mathcal{O}(n)$ edges, hence such a data structure can be build up in $\mathcal{O}(n\log(n))$ time.

First of all in every triangle one vertex with degree 2 has to be deleted. This can be done by going through all the vertices. Whenever a vertex has degree 2, we determine, whether its two neighbors i and j are adjacent. This can be done in $\mathcal{O}(\log(n))$ by applying logarithmic search on the list of the neighbors of i. If they are adjacent, then the vertex and the two edges are deleted. Also this can be performed in $\mathcal{O}(\log(n))$ time. So in total we need $\mathcal{O}(n\log(n))$ time to delete one vertex with degree 2 of every triangle.

Then all the vertices with degree 1 have to be deleted in the remaining graph. Also this can be performed in $\mathcal{O}(n\log(n))$. In the end, a B_0-EPG representation can be constructed of the resulting graph according to the proof in linear time. Eventually the paths corresponding to the deleted vertices are added to the B_0-EPG representation like it is done in the proof.

Probably the running time of the algorithm could be improved.

4.3.2 Cacti in B_1

Next we want to know which cacti are in B_1^m. It turns out that the following holds.

Theorem 4.3.4. *Every cactus is in B_1^m.*

Proof. In order to prove this result, we present a recursive procedure. We start by constructing the corresponding path of an arbitrary vertex as straight horizontal path and mark the whole path and the whole grid. If we have already constructed a path corresponding to a vertex v and marked a straight part of the path and its surrounding, we construct the paths of all neighbors of v and all vertices which lie on a cycle with v. All the corresponding paths are constructed inside the marked part of the path corresponding to v and its surrounding. Furthermore we mark a straight part of every new constructed path and its surrounding, in which the construction goes on in a recursive way.

We now focus on the details of the construction. Let v be a vertex of which the path has already been constructed and a straight part of the path and a part of its surrounding has already been marked. We want to construct the paths of all vertices which are adjacent to v or lie on a cycle with v. We can split the neighborhood of v in such a way, that every part corresponds to one of the following cases since the graph is a cactus. All the paths corresponding to vertices of one part can be constructed separately and then the constructions corresponding to different parts can be aligned consecutively on the marked part of the path corresponding to v. The cases are

(a) $\exists u$, such that v and u share an edge but are not on a cycle with each other,

(b) $\exists u_1, u_2$, such that v, u_1, and u_2 form a triangle,

(c) $\exists u_1, u_2, u_3$ such that v, u_1, u_2, and u_3 form a quadrangle, or

(d) $\exists u_1, u_2, \ldots, u_{n-1}$ such that v, u_1, u_2, $\ldots$, u_{n-1} form a C_n for $n \geqslant 5$.

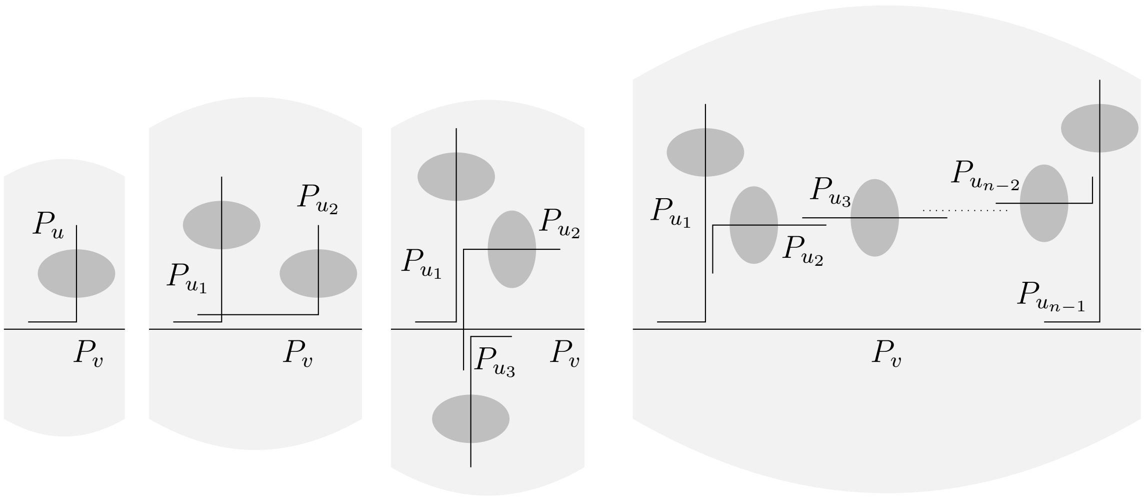


Figure 4.11: The recursive construction for a B_1^m-EPG representation of a cactus with marked parts of the paths and their surroundings.

How the paths corresponding to u in case (a), u_1 and u_2 in case (b), u_1, u_2, and u_3 in case (c), and $u_1, \ldots, u_{n-1}$ in case (d) are constructed, is depicted in Figure 4.11.

Note, that Figure 4.11 presents a construction if the marked part of the path corresponding to the starting vertex v is a horizontal straight line. Nevertheless, in all the cases there are paths, for which the marked part is a vertical line. If the marked part of a path is a vertical line, the depicted construction is turned 90° counterclockwise and then flipped vertically.

Every cycle is represented at once in the construction and by contracting all the cycles in a cactus one gets a tree, so it is easy to see, that the construction is indeed an EPG representation. This is because there are no intersections and there also should not be any intersections of paths corresponding to different marked parts of paths.

Due to the fact, that the starting vertex has a path with no bends, all the paths in the construction in Figure 4.11 are in B_1^m. Also by turning the paths of Figure 4.11 by 90° counterclockwise and then flipping them vertically one gets paths in B_1^m, so the construction yields a B_1^m-EPG representation.

□

The proof is done by construction, therefore it is easy to see that a B_1-EPG representation for a cactus can be obtained in polynomial time with respect to the number of vertices of G.

Finally we can say, that in contrast to outerplanar triangulations, where only graphs which are also in B_0 are in B_1^m, the converse is true for cacti. Namely all cacti are in B_1^m. This means that we have different behavior in terms of bend number and monotonic bend number of outerplanar triangulations and cacti. However, the graph classes itself have major structural differences, so this behavior is not too surprising.

5 Monotonic EPG

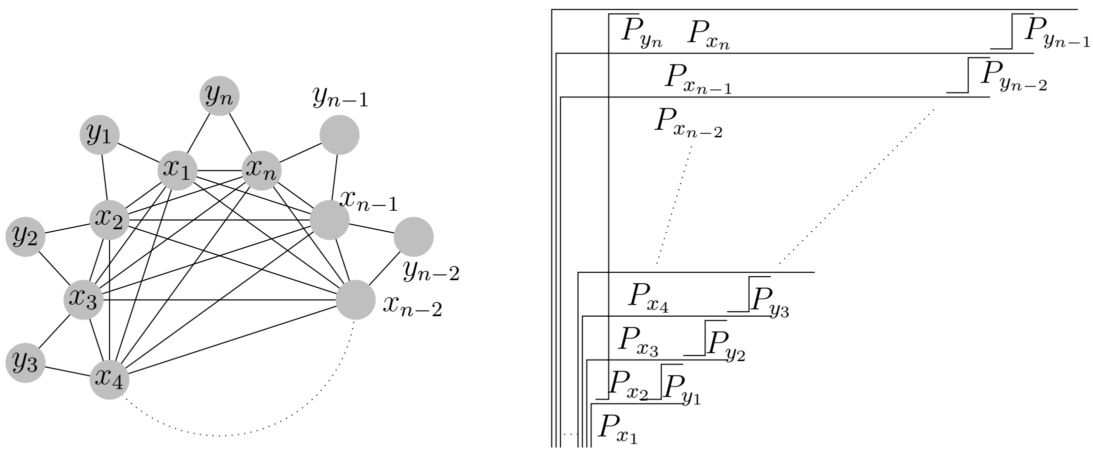
The purpose of this chapter is to investigate in more detail on the graph classes B_k^m. First of all we consider the graph S_n and determine the minimum k, such that S_n is in B_k^m. Then we determine the relationship between B_2 and B_2^m and generalize this result by investigating on B_k^m-EPG representations of $K_{m,n}$. In the end we determine the relationship between B_1 and B_3^m, giving the first result of this kind.

5.1 Characterization of the n-Sun

The graph S_n was one of the first graphs, for which it was shown, that it is not in B_1 for arbitrary large n, as long as $n \geqslant 4$. This result comes from Golumbic, Lipshteyn, and Stern in [21]. Furthermore they proved in [20], that $S_3 \in B_1$. Additional to that Cameron, Chaplick, and Hoàng proved in [8] that the graph $S_3 \notin B_1^m$. Hence with the next result, the characterization of S_n is fully determined.

Theorem 5.1.1. *The n-sun S_n depicted in Figure 5.1 (a) is in B_2^m for all $n \geqslant 3$.*

Proof. In order to prove, that S_n is in B_2^m, it is enough to give a B_2^m-EPG representation, which can be seen in Figure 5.1 (b). □



(a) (b)

Figure 5.1: (a) The graph S_n. (b) A B_2^m-EPG representation of S_n.

Note, that the construction of Theorem 5.1.1 implies, that it is possible to get a B_2^m-EPG representation of S_n in linear time with respect to n.

5.2 Relationship between B_2^m and B_2

It is an open question of [20] to determine the relationship between B_k^m and B_k for $k \geqslant 1$. Obviously $B_k^m \subseteq B_k$ holds for every k. In [20] Golumbic, Lipshteyn, and Stern conjectured that $B_1^m \subsetneq B_1$. This conjecture was confirmed by Cameron, Chaplick, and Hoàng in [8] by showing, that the graph S_3 is not in B_1^m. From [20] it is known, that S_3 is in B_1. The aim of this section is to prove that also $B_2^m \subsetneq B_2$ holds. For this purpose we will show, that there is a graph which is in B_2 but not in B_2^m.

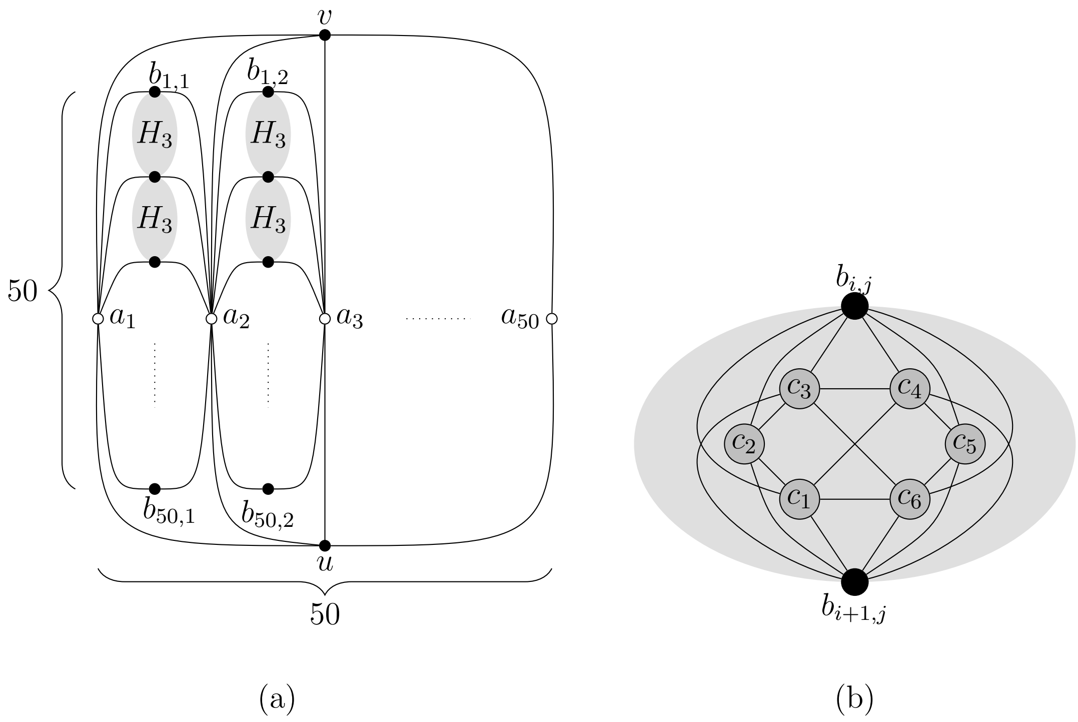


Figure 5.2: (a) The graph H_2. (b) The graph H_3 contained in every gray part of H_2.

The graph we will consider is the graph H_2 from Figure 5.2. Its definition is as follows.

Definition. The graph H_2 depicted in Figure 5.2 is constructed in the following way. The vertices $\{u, v\}$ and $\{a_1, \ldots, a_{50}\}$ form a $K_{2,50}$. Furthermore, for every $1 \leqslant j < 50$ the vertices $\{a_j, a_{j+1}\}$ and $\{b_{1,j}, \ldots, b_{50,j}\}$ form a $K_{2,50}$. Additional to that for every $1 \leqslant j < 50$ and for every $1 \leqslant i < 50$ there is the graph H_3 of Figure 5.2 (b) placed between the vertices $b_{i,j}$ and $b_{i+1,j}$.

The next result follows from a proof in [25]. Heldt, Knauer, and Ueckerdt there used a similar construction in order to prove, that there is a planar graph with treewidth at most 3 which is not in B_2.

Lemma 5.2.1 (Heldt, Knauer, Ueckerdt). *In any B_2-EPG representation of the graph H_2 depicted in Figure 5.2 there are i and j such that one end segment of each of the paths corresponding to $b_{i,j}$ and $b_{i+1,j}$ is completely contained in one segment of the path* *corresponding to a_j. Additional to that, the other end segment of each of the paths corresponding to $b_{i,j}$ and $b_{i+1,j}$ is completely contained in one segment of the path corresponding to a_{j+1}.*

With this auxiliary result we are able to prove the following result.

Lemma 5.2.2. *The graph H_2 is not in B_2^m.*

Proof. Assume that H_2 is in B_2^m. Every B_2^m-EPG representation is a B_2-EPG representation as well, therefore Lemma 5.2.1 holds also for any B_2^m-EPG representation. Assume without loss of generality, that the center segment of the path corresponding to $b_{i,j}$ is a horizontal segment, that it is above the center segment of the path corresponding to $b_{i+1,j}$ and that the segment of the path corresponding to a_j is on the right side of the segment of the path corresponding to a_{j+1}. Then the positioning of the segments of the paths has to look like in Figure 5.3.

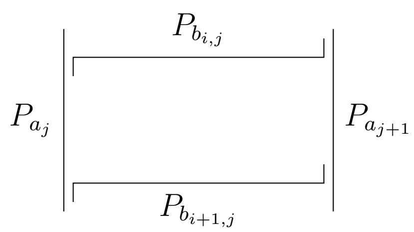


Figure 5.3: A part of the hypothetical B_2^m-EPG representation of H_2.

All of the vertices c_1, c_2, c_3, c_4, c_5, and c_6 of the graph H_3 between $b_{i,j}$ and $b_{i+1,j}$ are adjacent to both $b_{i,j}$ and $b_{i+1,j}$, but neither to a_j nor to a_{j+1}, therefore all of the corresponding paths have to share a grid edge with both center segments of the paths corresponding to $b_{i,j}$ and $b_{i+1,j}$. So all the paths corresponding to $c_1, \ldots, c_6$ have a horizontal part in common with the center segment of $b_{i+1,j}$, go up vertically and then share a horizontal part with the center segment of $b_{i,j}$.

The vertices c_1, c_3, and c_5 form an independent set, therefore all their corresponding paths and hence their vertical segments have to be disjoint, so they are in a unique order from left to right. We will refer to the leftmost, middle and rightmost vertical segment of those 3 segments as L, M, and R respectively.

Now we take a closer look at the paths corresponding to c_4 and c_6. The corresponding paths of c_4 and c_6 have to intersect all the corresponding paths of c_1, c_3, and c_5 since both vertices are adjacent to c_1, c_3, and c_5.

If the vertical segment of the path corresponding to c_4 would be between, or directly on one of the segments L and M, then the third segment of the path corresponding to c_4 would have to use the entire third segment of the path corresponding to M in order to be able to share a grid edge with the path corresponding to R. But then the path corresponding to c_6 would not be allowed to use the third segment of the path corresponding to M because c_6 is not adjacent to c_4. So c_6 would have to use the first segment of the path corresponding to M and hence also have to use the first segment of the path corresponding to L. Nevertheless this first segment of the path corresponding to L is already occupied by the path corresponding to c_4, a contradiction.

Analogously the path corresponding to c_6 is not between or directly on the segments L and M. Just in the same way it can also be shown, that the paths corresponding to c_4 and c_6 are also not between or on the segments M and R. So both vertical segments corresponding to c_4 and c_6 either have to be on the left side of L or on the right side of R. If both of them would be on the same side, then both of the corresponding paths would only be able to use the same segments (first or last) of the paths corresponding to L, M, and R, a contradiction.

So one of the vertical segments of the paths corresponding to c_4 and c_6 is on the left side of L and uses all the third segments of the paths corresponding to L, M, and R and the other vertical segment is on the right side of R and uses all the first segments of the paths corresponding to L, M, and R. For an illustration of this configuration see Figure 5.4.

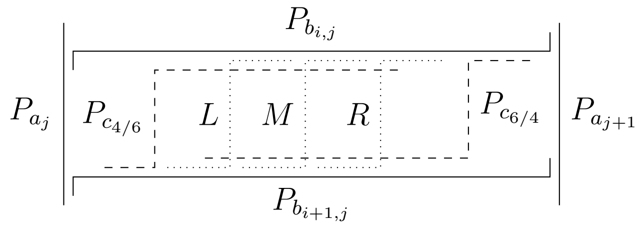


Figure 5.4: The only possible placement of paths corresponding to c_1, c_3, c_5, c_4, and c_6 in the hypothetical B_2^m-EPG representation of H_2.

But no matter, how the order of c_1, c_3, and c_5 is within L, M, and R, it is not possible to place the path corresponding to c_2 in such a way, that is does not intersect the paths corresponding to c_4 and c_6 but does intersect the paths corresponding to c_1 and c_5, because no two parts of L, M, and R can be connected by using a monotonic path which does not intersect the paths corresponding to c_4 and c_6. So the path corresponding to c_2 can not be positioned and hence we have a contradiction.

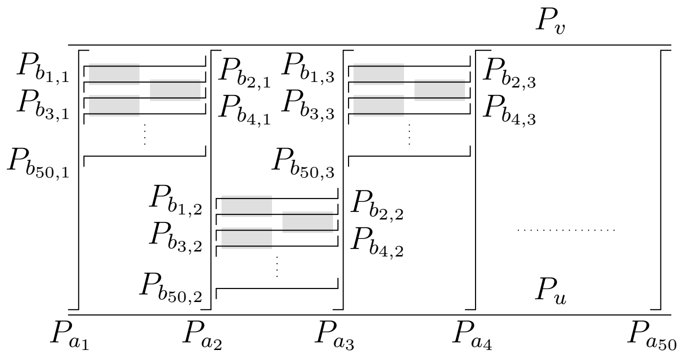
$\square$

After knowing, that there is a graph which is not in B_2^m we can formulate the next result.

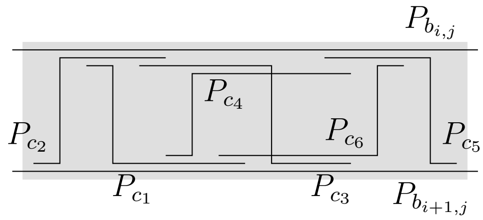
Lemma 5.2.3. *It holds that $B_2^m \subsetneq B_2$.*

Proof. The fact that $B_2^m \subseteq B_2$ follows by definition. In order to see, that strict inclusion holds, we consider the graph H_2 depicted in Figure 5.2. We have already seen in Lemma 5.2.2 that the graph H_2 is not in B_2^m. So it is enough to show, that H_2 is in B_2. In order to prove that, it is sufficient to give a B_2-EPG representation of H_2. Such a representation can be seen in Figure 5.5. $\square$

So in the end we proved that $B_k^m \subsetneq B_k$ does not only hold for $k = 1$, but also for $k = 2$.



(a)



(b)

Figure 5.5: (a) A B_2-EPG representation of the graph H_2 of Figure 5.2, where in every gray part there is the B_2-EPG representation depicted in (b).

5.3 Relationship between B_k^m and B_k for $k \geqslant 3$

In order to investigate the relationship between B_k^m and B_k for bigger k, we first derive a Lower-Bound-Lemma for B_k^m-EPG representations similarly like it is done in [26] for B_k-EPG representations. We start with the following result.

Lemma 5.3.1. *Consider two paths P_1, P_2 in a B_k^m-EPG representation. If one path starts horizontally and the other one starts vertically, the paths can cross in at most $k+1$ points. If either both paths start horizontally or both paths start vertically, the paths can cross in at most k points.*

Proof. Consider an arbitrary but fixed segment of P_1. It is easy to see, that there can be at most one crossing with the monotonic path P_2 on this segment. Hence there are at most $k+1$ crossings in any case because P_1 has at most $k+1$ segments.

Assume, that there are $k+1$ crossings between P_1 and P_2. Without loss of generality let P_1 start horizontally. There is exactly one crossing in every segment of P_1 since at most one crossing can be in every segment of P_1. The same is true for P_2. Furthermore, a crossing can only occur between two segments, if one segment is horizontal and the other is vertical. Hence the segment of P_2 which intersects the first segment of P_1 has to be vertical. Furthermore this segment is the first of P_2 because the path is monotonic. So P_2 starts vertically.

This implies, that if two paths start in the same direction, there can be at most k crossings between them. □

Now we are able to prove the following result.

Lemma 5.3.2. *Let $3 \leqslant m \leqslant n$. In every B_k^m-EPG representation of $K_{m,n}$*

$$n(2m-k-2) \leqslant k(m-1)m + \frac{1}{2}m^2 + 2(k+1)m$$

holds.

Proof. Let c denote the number of crossings of the paths corresponding to the vertices of the component with m vertices. Every B_k^m-EPG representation is a B_k-EPG representation too, therefore it follows from Lemma 3.5.10 that

$$n(2m-k-2) \leqslant 2c + 2(k+1)m \tag{5.1}$$

holds for every B_k^m-EPG representation of $K_{m,n}$. Now we want to find an upper bound on c. Let ℓ be the number of paths among the paths corresponding to the vertices of the component with m vertices, which start horizontally. It follows that $m-\ell$ paths start vertically since there are m paths in total. The number of total crossings between the paths can be calculated as the sum of all crossings between any two paths. If two paths both start in the same direction, there can be at most k crossings by Lemma 5.3.1. There are $\binom{\ell}{2} + \binom{m-\ell}{2}$ possibilities of choosing two paths that start in the same direction. If the paths start in different directions, by Lemma 5.3.1 there can be at most $k+1$ crossings. There are exactly $\ell(m-\ell)$ pairs of paths which start in different directions. In total there can be at most

$$\begin{aligned}
&k\left(\binom{\ell}{2} + \binom{m-\ell}{2}\right) + (k+1)\ell(m-\ell)\\
&= k\left(\binom{\ell}{2} + \ell(m-\ell) + \binom{m-\ell}{2}\right) + \ell(m-\ell)\\
&= k\binom{m}{2} + \ell(m-\ell)
\end{aligned}$$

crossings. Furthermore $\ell(m-\ell) \leqslant \left(\frac{m}{2}\right)^2$ for all $0 \leqslant \ell \leqslant m$. So

$$c \leqslant k\binom{m}{2} + \left(\frac{m}{2}\right)^2 = \frac{1}{2}\left(k(m-1)m + \frac{1}{2}m^2\right)$$

holds. In the end it follows from (5.1) that

$$n(2m-k-2) \leqslant k(m-1)m + \frac{1}{2}m^2 + 2(k+1)m$$

holds. □

With this result we will first of all consider the relationship between B_5^m and B_5.

Lemma 5.3.3. *It holds that* $B_5^m \subsetneq B_5$.

Proof. It is obvious that $B_5^m \subseteq B_5$, so it is enough to show that $B_5^m \neq B_5$.

For $m = 4$ Theorem 3.5.13 yields that $K_{4,156} \in B_5$. Assume, that $K_{4,156} \in B_5^m$. Then by Lemma 5.3.2 it holds that

$$\begin{aligned} & 156(2 \cdot 4 - 5 - 2) \leqslant 5 \cdot 3 \cdot 4 + \frac{1}{2} 4^2 + 2 \cdot 6 \cdot 4 \\ \Leftrightarrow \quad & 156 \leqslant 116, \end{aligned}$$

which is a contradiction. So $K_{4,156} \notin B_5^m$ but $K_{4,156} \in B_5$, hence $B_5^m \neq B_5$. $\square$

Additional to that, we can determine the relationship between B_k^m and B_k for $k \geqslant 7$.

Lemma 5.3.4. *It holds that* $B_k^m \subsetneq B_k$ *for* $k \geqslant 7$.

Proof. We start by proving the statement for odd k. By Theorem 3.5.9 it follows that $K_{k+1, \frac{1}{4}(k+1)^3 - \frac{1}{2}(k+1)^2 - (k+1) + 4} = K_{k+1, \frac{1}{4}k^3 + \frac{1}{4}k^2 - \frac{5}{4}k + \frac{11}{4}} \in B_k$. Assume that the graph is in B_k^m, hence by Lemma 5.3.2 with $m = k + 1$ and $n = \frac{1}{4}k^3 + \frac{1}{4}k^2 - \frac{5}{4}k + \frac{11}{4}$ it follows that

$$\begin{aligned} & \left(\frac{1}{4}k^3 + \frac{1}{4}k^2 - \frac{5}{4}k + \frac{11}{4}\right)(2(k+1) - k - 2) \leqslant k^2(k+1) + \frac{1}{2}(k+1)^2 + 2(k+1)^2 \\ \Leftrightarrow \quad & k\left(\frac{1}{4}k^3 + \frac{1}{4}k^2 - \frac{5}{4}k + \frac{11}{4}\right) \leqslant k^3 + \frac{7}{2}k^2 + 5k + \frac{5}{2} \\ \Leftrightarrow \quad & k^4 - 3k^3 - 19k^2 - 9k - 10 \leqslant 0, \end{aligned}$$

which is a contradiction for $k \geqslant 7$. Hence for odd $k \geqslant 7$ there is a graph in B_k which is not in B_k^m and therefore $B_k^m \subsetneq B_k$ for odd $k \geqslant 7$.

Now we consider the case for even k. In this case Theorem 3.5.9 yields that the graph $K_{k+1, \frac{1}{4}(k+1)^3 - (k+1)^2 + \frac{3}{4}(k+1)} = K_{k+1, \frac{1}{4}k^3 - \frac{1}{4}k^2 - \frac{1}{2}k} \in B_k$. Assume again that the graph is in B_k^m, hence by Lemma 5.3.2 with $m = k + 1$ and $n = \frac{1}{4}k^3 - \frac{1}{4}k^2 - \frac{1}{2}k$ it follows that

$$\begin{aligned} & \left(\frac{1}{4}k^3 - \frac{1}{4}k^2 - \frac{1}{2}k\right)(2(k+1) - k - 2) \leqslant k^2(k+1) + \frac{1}{2}(k+1)^2 + 2(k+1)^2 \\ \Leftrightarrow \quad & k\left(\frac{1}{4}k^3 - \frac{1}{4}k^2 - \frac{1}{2}k\right) \leqslant k^3 + \frac{7}{2}k^2 + 5k + \frac{5}{2} \\ \Leftrightarrow \quad & k^4 - 5k^3 - 16k^2 - 20k - 10 \leqslant 0, \end{aligned}$$

which contradicts for $k \geqslant 8$. Hence for even $k \geqslant 8$ there is a graph in B_k which is not in B_k^m and therefore $B_k^m \subsetneq B_k$ for even $k \geqslant 8$. $\square$

Finally we can sum up the previous results and combine them in the following.

Theorem 5.3.5. *It holds that* $B_k^m \subsetneq B_k$ *for* $k = 2$, $k = 5$, *and* $k \geqslant 7$.

Proof. This is a direct consequence of Lemma 5.2.3, Lemma 5.3.3 and Lemma 5.3.4. □

That means, that the question concerning the relationship between B_k^m and B_k raised in [20] is answered for almost all k. The relationship for the cases $k = 3$, $k = 4$, and $k = 6$ remains an open question. Nevertheless we have the following conjecture.

Conjecture 5.3.6. *$B_k^m \subsetneq B_k$ holds also for $k = 3$, $k = 4$, and $k = 6$.*

5.4 Relationship between B_k and B_ℓ^m for $\ell > k$

After finding out, that $B_k^m \subsetneq B_k$ holds for almost all k a natural further question is, what the relationship between B_k and B_{k+1}^m is. Neither $B_k \subseteq B_{k+1}^m$ nor $B_{k+1}^m \subseteq B_k$ is known. The following result partially answers this question and also points out, that another question is of interest.

Theorem 5.4.1. *Let $k \geqslant 6$. If k is odd, then there is a graph which is in B_k but not in B_{2k-8}^m. If k is even, there is a graph which is in B_k but not in B_{2k-9}^m.*

Proof. At first we will consider the case if k is odd. In this case because of Theorem 3.5.9 we know that $K_{k+1,\frac{1}{4}(k+1)^3-\frac{1}{2}(k+1)^2-(k+1)+4} = K_{k+1,\frac{1}{4}k^3+\frac{1}{4}k^2-\frac{5}{4}k+\frac{11}{4}} \in B_k$. Assume that this graph is also in B_{2k-8}^m. Then it follows from Lemma 5.3.2 that

$$\begin{aligned}
\left(\frac{1}{4}k^3+\frac{1}{4}k^2-\frac{5}{4}k+\frac{11}{4}\right)(2(k+1)-2k+8-2) &\leqslant (2k-8)(k+1)k+\frac{1}{2}(k+1)^2 \\
&\quad + 2(2k-7)(k+1) \\
\Leftrightarrow \qquad 8\left(\frac{1}{4}k^3+\frac{1}{4}k^2-\frac{5}{4}k+\frac{11}{4}\right) &\leqslant 2k^3-\frac{3}{2}k^2-17k-\frac{27}{2} \\
\Leftrightarrow \qquad 0 &\leqslant -\frac{7}{2}k^2-7k-\frac{71}{2}
\end{aligned}$$

has to hold. Nevertheless this is an obvious contradiction for $k \geqslant 0$. So the graph is not in B_{2m-8}^m. Hence for odd $k \geqslant 6$, there is a graph in B_k which is not in B_{2k-8}^m.

It remains to consider the case that k is even. By Theorem 3.5.9 we know that in this case $K_{k+1,\frac{1}{4}(k+1)^3-(k+1)^2+\frac{3}{4}(k+1)} = K_{k+1,\frac{1}{4}k^3-\frac{1}{4}k^2-\frac{1}{2}k} \in B_k$. If we assume, that the graph is also in B_{2m-9}^m, due to Lemma 5.3.2

$$\begin{aligned}
\left(\frac{1}{4}k^3-\frac{1}{4}k^2-\frac{1}{2}k\right)(2(k+1)-2k+9-2) &\leqslant (2k-9)(k+1)k+\frac{1}{2}(k+1)^2 \\
&\quad + 2(2k-8)(k+1) \\
\Leftrightarrow \qquad 9\left(\frac{1}{4}k^3-\frac{1}{4}k^2-\frac{1}{2}k\right) &\leqslant 2k^3-\frac{5}{2}k^2-20k-\frac{31}{2} \\
\Leftrightarrow \qquad 0 &\leqslant -\frac{1}{4}k^3-\frac{1}{4}k^2-\frac{31}{2}k-\frac{31}{2}
\end{aligned}$$

has to hold. But this is again an obvious contradiction for $k \geqslant 0$, so for even $k \geqslant 6$ there is a graph which is in B_k but not in B_{2k-9}^m. □

Theorem 5.4.1 shows, what the relationship between the classes B_k and B_{k+1}^m is not, namely $B_k \not\subseteq B_{k+1}^m$. In fact, it even reveals that $B_k \not\subseteq B_{2k-8}^m$ for odd k and that $B_k \not\subseteq B_{2k-9}^m$ for even k. This fact points out, that restricting the direction of bends really is a big limitation. Nevertheless it is still unknown, whether there is a graph which is in B_{k+1}^m but is not in B_k. Another question of interest raised by Theorem 5.4.1 can be found in the next section.

5.5 Relationship between B_1 and B_3^m

In the previous sections we have seen, that $B_k^m \subsetneqq B_k$ for all $k \geqslant 1$ is already known except for $k = 3$, $k = 4$, and $k = 6$. In other words allowing the same number of bends but restricting the possible directions of the bends does decrease the number of representable graphs. A natural question that arises is the reverse, hence to find out how much more bends are needed if we want to represent a graph with fewer possibilities of directions of the bends. Equivalently we want to find the minimum ℓ such that $B_k \subseteq B_\ell^m$. In Theorem 5.4.1 we have already seen that $\ell \geqslant 2k - 9$ for $k \geqslant 6$. We now want to investigate the relationship for small values of k. It follows from Observation 2.2.1, that $B_0 = B_0^m$, so for $k = 0$ the question can be answered easily. We go a step further and give an upper bound on the minimum ℓ for $k = 1$.

Theorem 5.5.1. *It holds that $B_1 \subseteq B_3^m$.*

Proof. We have to prove, that every graph in B_1 is also in B_3^m. Let G be a graph in B_1. We consider a B_1-EPG representation of G and transform it into a B_3^m-EPG representation in the following way. Let R_1 be an arbitrary B_1-EPG representation of G. We place another copy of the same B_1-EPG representation to the top right of R_1 and call this second copy R_2.

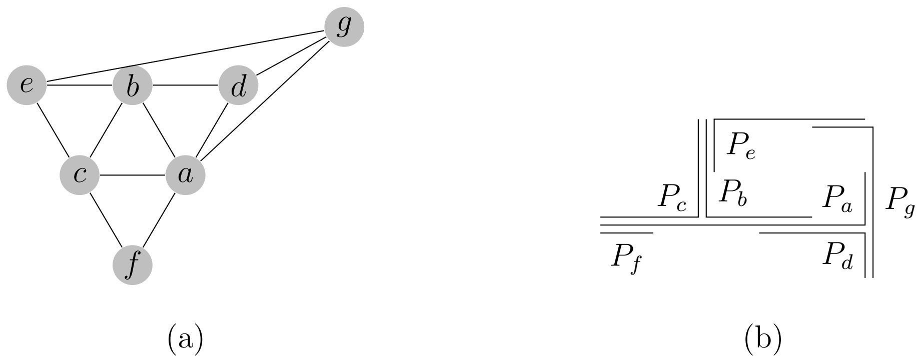


Figure 5.6: (a) A graph G. (b) A B_1-EPG representation of the graph G.

An example how to modify the B_1-EPG representation in Figure 5.6 (b) of the graph G in Figure 5.6 (a) is depicted in Figure 5.7.

Let v be a vertex and P_v the path corresponding to it in the B_1-EPG representation. If P_v has a horizontal segment, we split the vertical grid line of R_1, in which the right

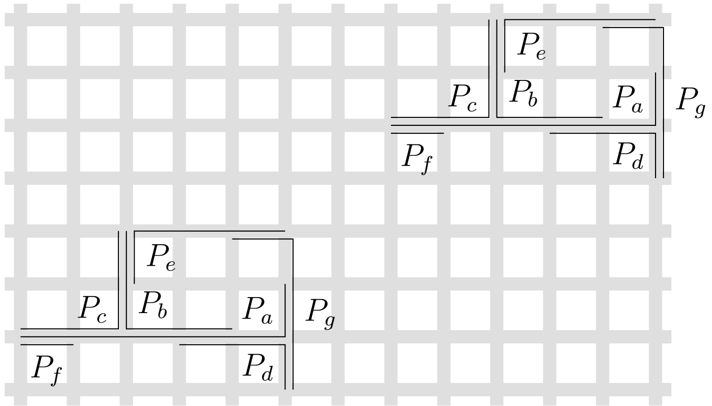


Figure 5.7: The grid with components R_1 and R_2.

end point of the horizontal segment of P_v is, into two grid lines. For this purpose, we introduce a new vertical grid line $L_v^|$ immediately left to it and then we modify the path P_v such that it does not use the original vertical grid line, but uses the vertical grid line $L_v^|$ instead. All the other paths remain in the original vertical grid line. If P_v has a vertical segment, then in R_2 we introduce a new horizontal grid line L_v^- directly beneath the grid line in which the lower end point of the vertical segment of P_v is. Additional to that we modify the path P_v such that it uses L_v^- instead of the original grid line, all the other paths are not modified.

An example of the modified grid and paths of the graph in Figure 5.6 (a) can be seen in Figure 5.8.

Now we define the B_3^m-EPG representation of G with a path Q_v for every vertex v in the following way. If the path P_v is only a horizontal segment, we define Q_v as the horizontal segment of P_v in R_1. If the path P_v is only a vertical segment, we define Q_v as the vertical segment of P_v in R_2. If the path P_v has a horizontal and a vertical segment, then the path Q_v uses the horizontal segment of P_v in R_1. Then the path Q_v uses the vertical grid line $L_v^|$ until it intersects the grid line L_v^- and proceeds in this horizontal grid line until it reaches the vertical grid line of P_v in R_2. Finally it uses this grid line and ends in the upper end point of the vertical segment of P_v in R_2.

An example of the B_3^m-EPG representation of the graph in Figure 5.6 (a) is depicted in Figure 5.9.

It is easy to see, that every path is monotonic and bends at most 3 times. What is left to show is, that two paths Q_{v_1} and Q_{v_2} intersect if and only if the vertices v_1 and v_2 are adjacent in G. For that it is enough to show that two paths Q_{v_1} and Q_{v_2} intersect, if and only if the paths P_{v_1} and P_{v_2} intersect in the original B_1-EPG representation.

Assume Q_{v_1} and Q_{v_2} intersect. There are no intersections of the paths in any new introduced grid lines because we introduced different grid lines for Q_{v_1} and Q_{v_2}. Additional to that, for every v the first segment of Q_v is contained entirely in R_1, the last segment of Q_v is contained entirely in R_2 and R_1 and R_2 are separated. So the intersection of Q_{v_1} and Q_{v_2} is either for both paths located at the first or for both paths located

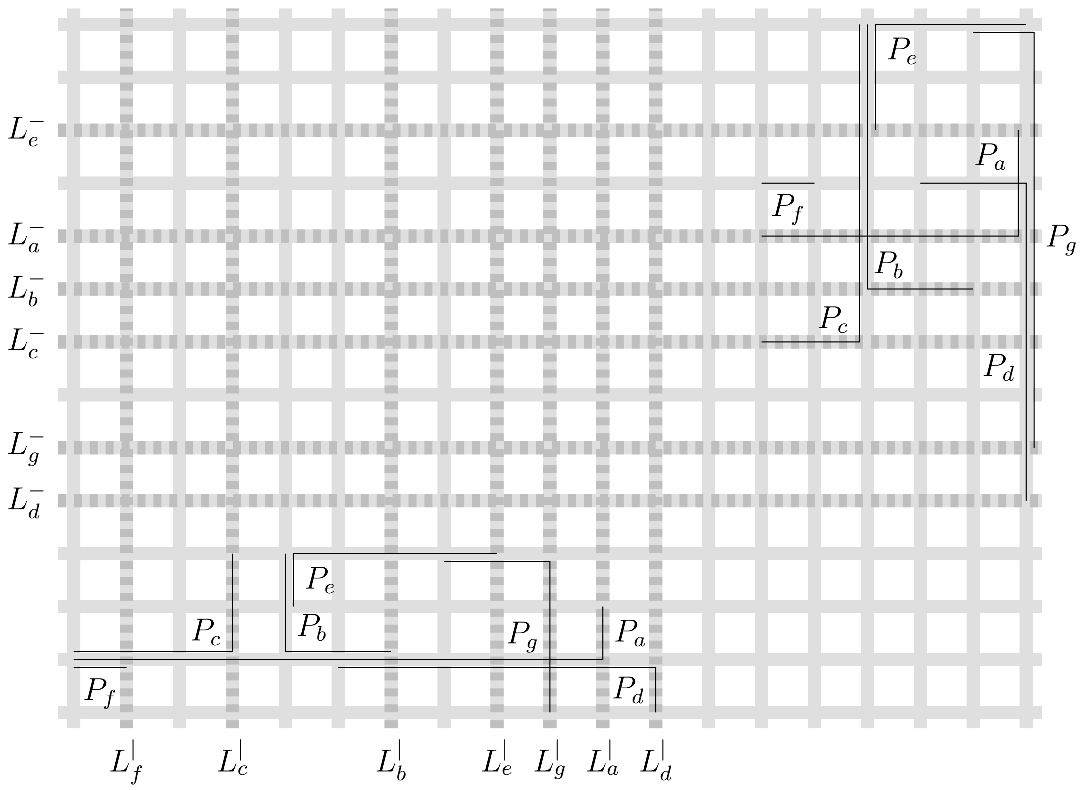


Figure 5.8: The grid with newly introduced grid lines and modified paths.

at the last segment. Assume without loss of generality that the intersection is located at the first segment of both paths. It follows that also P_{v_1} and P_{v_2} intersect since the first segments of Q_{v_1} and Q_{v_2} are completely contained in the original horizontal segments of P_{v_1} and P_{v_2} respectively.

What is left to show is the other direction of the equivalence, that is that if P_{v_1} and P_{v_2} intersect, also Q_{v_1} and Q_{v_2} intersect. It follows from the definition of the modified paths, that if two original paths P_{v_1} and P_{v_2} intersect horizontally in R_1 or vertically in R_2, also the modified paths P_{v_1} and P_{v_2} intersect horizontally in R_1 or vertically in R_2 respectively. But then by the definition of the path Q_v, for every v the first segment of Q_v contains the whole horizontal part of the modified path P_v in R_1 and the last segment of Q_v contains the whole vertical part of the modified path P_v in R_2. Hence if the paths P_{v_1} and P_{v_2} intersect horizontally, the paths Q_{v_1} and Q_{v_2} intersect horizontally on the first segment of each path. If the paths P_{v_1} and P_{v_2} share a vertical grid edge, the paths Q_{v_1} and Q_{v_2} intersect vertically on the last segment of each path.

□

Due to the fact, that the proof is done by construction, an immediate consequence of Theorem 5.5.1 is, that with a given B_1-EPG representation of a graph, a B_3^m-EPG representation can be obtained in linear time with respect to the number of vertices of the graph.

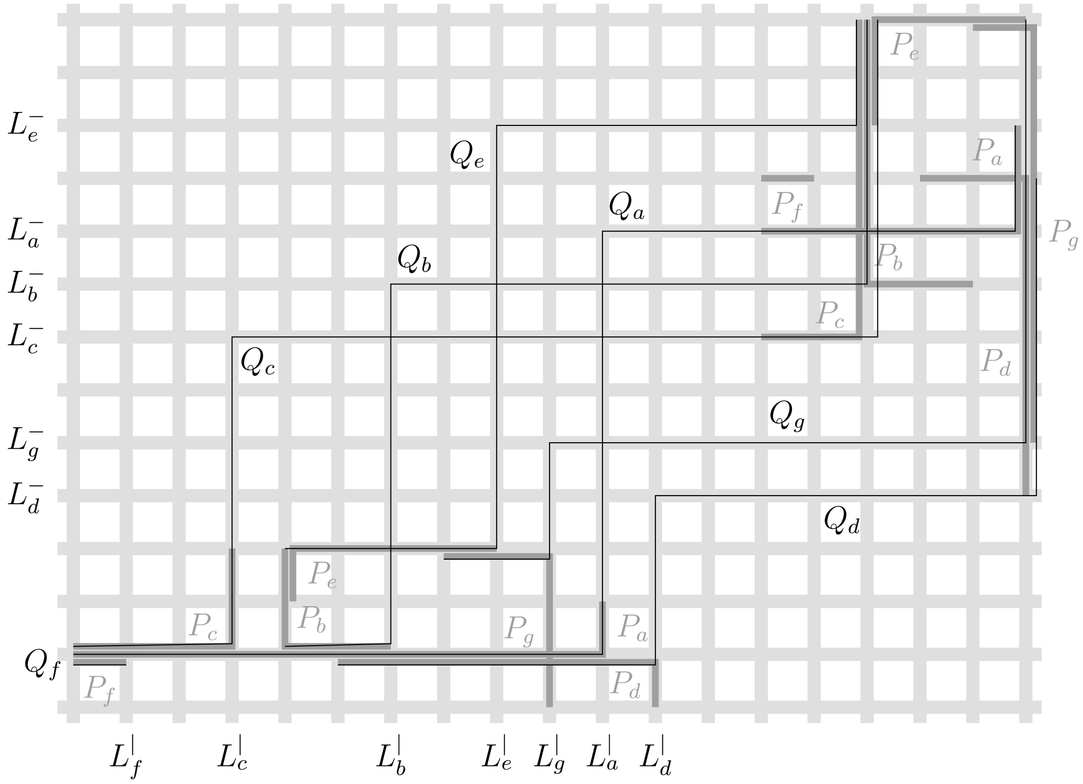


Figure 5.9: The obtained B_3^m-EPG representation of the graph of Figure 5.6 (a).

It is still an open question, whether this result is tight, so whether $\ell = 3$ is really the minimum ℓ such that $B_1 \subseteq B_\ell^m$ or even $B_1 \subseteq B_2^m$ holds.

5.6 Further Investigation on $K_{m,n} \in B_k^m$

In [26] there has been done many work on determining upper and lower bounds on k such that $K_{m,n}$ is in and is not in B_k respectively. One result there was, that for $n \geqslant m^4 - 2m^3 + 5m^2 - 4m + 1$ the graph $K_{m,n} \notin B_{2m-3}$ but $K_{m,n} \in B_{2m-2}$. The aim of this section is to deduce a similar result for the monotonic case.

We first generalize a result of [4]. There it was shown by slightly modifying a construction of [20], that $K_{m,n} \in B_{2m-2}$ for all n. By again modifying the construction of [4], we can give the same result for the monotonic case.

Theorem 5.6.1. *It holds that $K_{m,n} \in B_{2m-2}^m$.*

Proof. In order to prove this, it is enough to give a B_{2m-2}^m-EPG representation, which can be found in Figure 5.10. If we denote the vertices of $K_{m,n}$ which are in the component of size m with $A = \{a_1, \dots, a_m\}$ and the vertices of the other component with $B = \{b_1, \dots, b_n\}$, then the paths corresponding to vertices of A have only a horizontal segment. The paths corresponding to vertices of B are staircases with $2m-2$ bends. $\square$

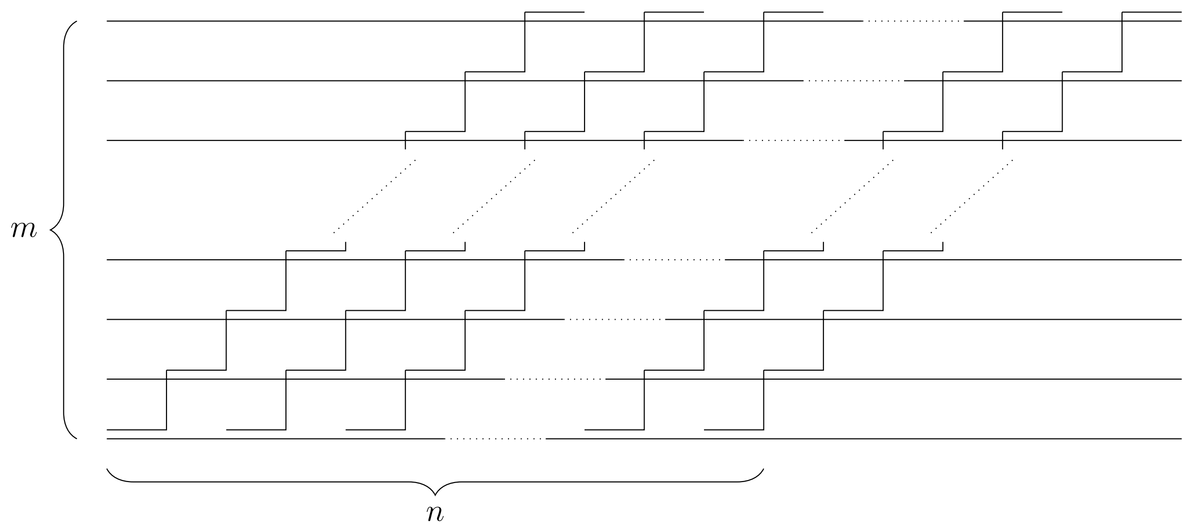


Figure 5.10: A B_{2m-2}^m-EPG representation for $K_{m,n}$.

This means, that the upper bound on bends needed for an EPG representation of $K_{m,n}$ with $m \leqslant n$ is the same, no matter whether only monotonic bends are allowed or all kind of bends are allowed. This fact is even more surprising, if we take into account Theorem 5.4.1, which tells us, that the gap between the number of minimum bends needed in a monotonic EPG representation and the number of minimum bends needed in an EPG representation can be arbitrary large. However, it turns out, that in the monotonic case the maximum number of bends is needed earlier than in the non monotonic case. That is, $K_{m,n} \notin B_{2m-3}$ for $n \geqslant N_1$ for some $N_1 \in \mathcal{O}(m^4)$, but $K_{m,n} \notin B_{2m-3}^m$ for $n \geqslant N_2$ already for some $N_2 \in \mathcal{O}(m^3)$. This is a consequence of the following result.

Theorem 5.6.2. *Let $3 \leqslant m$. If $n \geqslant 2m^3 - \frac{1}{2}m^2 - m + 1$ then the graph $K_{m,n} \notin B_{2m-3}^m$.*

Proof. Assume that $K_{m,n} \in B_{2m-3}^m$. Then by applying Lemma 5.3.2 for $k = 2m-3$ we get that

$$\begin{aligned} n(2m-(2m-3)-2) &\leqslant (2m-3)(m-1)m + \frac{1}{2}m^2 + 2(2m-2)m \\ \Leftrightarrow \qquad n &\leqslant 2m^3 - \frac{1}{2}m^2 - m \end{aligned}$$

has to hold. This is a contradiction for $n \geqslant 2m^3 - \frac{1}{2}m^2 - m + 1$. $\square$

6 Monotonic MILP Formulation

In combinatorial optimization it is very common to formulate a problem as mixed integer linear program (MILP). The aim of this chapter is to provide a MILP formulation of the problem, whether a given graph G is in B_k^m or not. We will derive two distinct formulations with different advantages. One version will be based on the grid edges, the other version will be based on the grid points.

6.1 Grid Edge Based Monotonic MILP Formulation

At first we will present a MILP called (EB_k^m) which determines the minimum k, such that a given graph G is in B_k^m. The basic idea is to introduce a binary variable for each grid edge and for each path so as to determine whether the path uses this grid edge or not.

Definition. Let $G = (V, E)$ be a graph with vertex set $V = \{1, \dots, n\}$ and let $\mathcal{G}$ be a grid with m_x vertical and m_y horizontal lines. Let

$$\begin{aligned} M_x &:= \{1, \dots, m_x\}, \\ M_x^0 &:= \{0, 1, \dots, m_x\}, \\ M_x^+ &:= \{1, \dots, m_x, m_x + 1\}, \\ M_y &:= \{1, \dots, m_y\}, \\ M_y^0 &:= \{0, 1, \dots, m_y\}, \text{ and} \\ M_y^+ &:= \{1, \dots, m_y, m_y + 1\}. \end{aligned}$$

The meaning of the variables is the following. The variable $u_{x,y}^v = 1$ if the path corresponding to vertex v uses the edge of the grid which goes up at the grid point (x, y), $u_{x,y}^v = 0$ otherwise. In the same way $r_{x,y}^v = 1$ if the path corresponding to vertex v uses the edge of the grid which goes to the right at the grid point (x, y), $r_{x,y}^v = 0$ otherwise. The variable $u_{x,y}^{v,w} = 1$ if both the paths corresponding to v and w use the edge of the grid which goes up at the grid point (x, y), $u_{x,y}^{v,w} = 0$ otherwise. So $u_{x,y}^{v,w} = 1$ if and only if the paths corresponding to v and u intersect at the edge that goes up at the grid point (x, y). Analogously $r_{x,y}^{v,w} = 1$ if both the paths corresponding to v and w use the edge of the grid which goes to the right at the grid point (x, y), $r_{x,y}^{v,w} = 0$ otherwise. We set $s_{x,y}^v = 1$ if the path corresponding to vertex v starts at the grid point (x, y) and $s_{x,y}^v = 0$ otherwise. Furthermore we let $e_{x,y}^v = 1$ if the path corresponding to vertex v ends at the grid point (x, y) and $e_{x,y}^v = 0$ otherwise. In the end, we set $b_{x,y}^v = 1$ if the path corresponding to v bends at the grid point (x, y) and $b_{x,y}^v = 0$ otherwise.

The grid edge based monotonic MILP (EB_k^m) is defined in the following way.

$$
\begin{aligned}
(\mathbf{EB}_k^m) \quad & \min \quad k \\
\text{s.t.} \quad & u_{x,y}^v + u_{x,y}^w \leqslant 1 \qquad \forall v,w \in V,\, v<w,\, (v,w) \notin E,\, \forall x \in M_x,\, \forall y \in M_y^0 && (6.1)\\
& r_{x,y}^v + r_{x,y}^w \leqslant 1 \qquad \forall v,w \in V,\, v<w,\, (v,w) \notin E,\, \forall x \in M_x^0,\, \forall y \in M_y && (6.2)\\
& u_{x,y}^v + u_{x,y}^w \geqslant 2u_{x,y}^{v,w} \qquad \forall (v,w) \in E,\, \forall x \in M_x,\, \forall y \in M_y^0 && (6.3)\\
& r_{x,y}^v + r_{x,y}^w \geqslant 2r_{x,y}^{v,w} \qquad \forall (v,w) \in E,\, \forall x \in M_x^0,\, \forall y \in M_y && (6.4)\\
& \sum_{\substack{y \in M_y^0 \\ x \in M_x}} u_{x,y}^{v,w} + \sum_{\substack{y \in M_y \\ x \in M_x^0}} r_{x,y}^{v,w} \geqslant 1 \qquad \forall (v,w) \in E && (6.5)\\
& r_{x-1,y}^v + u_{x,y}^v \leqslant 1 + b_{x,y}^v \qquad \forall v \in V,\, \forall x \in M_x,\, \forall y \in M_y && (6.6)\\
& r_{x,y}^v + u_{x,y-1}^v \leqslant 1 + b_{x,y}^v \qquad \forall v \in V,\, \forall x \in M_x,\, \forall y \in M_y && (6.7)\\
& \sum_{\substack{y \in M_y \\ x \in M_x}} b_{x,y}^v \leqslant k \qquad \forall v \in V && (6.8)\\
& r_{x-1,y}^v + u_{x,y-1}^v + s_{x,y}^v = r_{x,y}^v + u_{x,y}^v + e_{x,y}^v \qquad \forall v \in V,\, \forall x \in M_x,\, \forall y \in M_y && (6.9)\\
& s_{0,y}^v = r_{0,y}^v \qquad \forall v \in V,\, \forall y \in M_y && (6.10)\\
& r_{m_x,y}^v = e_{m_x+1,y}^v \qquad \forall v \in V,\, \forall y \in M_y && (6.11)\\
& s_{x,0}^v = u_{x,0}^v \qquad \forall v \in V,\, \forall x \in M_x && (6.12)\\
& u_{x,m_y}^v = e_{x,m_y+1}^v \qquad \forall v \in V,\, \forall x \in M_x && (6.13)\\
& \sum_{\substack{y \in M_y^+ \\ x \in M_x^+}} e_{x,y}^v \leqslant 1 \qquad \forall v \in V && (6.14)\\
& \sum_{\substack{y \in M_y^0 \\ x \in M_x^0}} s_{x,y}^v \geqslant 1 \qquad \forall v \in V && (6.15)\\
& u_{x,y}^v \in \{0,1\} \qquad \forall v \in V,\, \forall x \in M_x,\, \forall y \in M_y^0 && (6.16)\\
& r_{x,y}^v \in \{0,1\} \qquad \forall v \in V,\, \forall x \in M_x^0,\, \forall y \in M_y && (6.17)\\
& u_{x,y}^{v,w} \in \{0,1\} \qquad \forall (v,w) \in E,\, \forall x \in M_x,\, \forall y \in M_y^0 && (6.18)\\
& r_{x,y}^{v,w} \in \{0,1\} \qquad \forall (v,w) \in E,\, \forall x \in M_x^0,\, \forall y \in M_y && (6.19)\\
& s_{x,y}^v \in \{0,1\} \qquad \forall v \in V,\, \forall x \in M_x^0,\, \forall y \in M_y^0 && (6.20)\\
& e_{x,y}^v \in \{0,1\} \qquad \forall v \in V,\, \forall x \in M_x^+,\, \forall y \in M_y^+ && (6.21)\\
& b_{x,y}^v \in \{0,1\} \qquad \forall v \in V,\, \forall x \in M_x,\, \forall y \in M_y && (6.22)\\
& k \in \mathbb{N}^* = \{0,1,2,\dots\} && (6.23)
\end{aligned}
$$

Note, that if m is the number of edges of the graph G, then the MILP (EB_k^m) has $n(4m_xm_y+3m_x+3m_y+3)+m(2m_xm_y+m_x+m_y+1)$ binary variables, 1 integer variable and there are $\left(\frac{n^2}{2}+m\right)(2m_xm_y+m_x+m_y+1)+m+n(2m_xm_y+\frac{3}{2}m_x+\frac{3}{2}m_y+\frac{5}{2})$ constraints.

Theorem 6.1.1. *For a fixed k, every feasible solution of (EB_k^m) corresponds to a B_k^m-EPG representation of G on the grid $\mathcal{G}$ and vice versa.*

Proof. Throughout the whole proof, we let k be fixed.

We start by showing, that every B_k^m-EPG representation on the grid $\mathcal{G}$ corresponds to a feasible solution. In order to do so, we assign the values of the decision variables as it is described above.

Now we will show, that this assignment yields a feasible solution of (EB_k^m). We have assigned only binary values to all variables, so (6.16) - (6.22) hold. If two vertices v and w are not adjacent, their corresponding paths do not share a grid edge, so every grid edge can be used by at most one of the paths, hence (6.1) and (6.2) are fulfilled. If the vertices v and w are adjacent, then it follows from the definition of our assignment, that (6.3) and (6.4) hold. Furthermore for vertices which are adjacent there has to be an intersection of the paths anywhere in the grid, and hence there has to be a grid edge, which both paths use, so (6.5) is true. If we consider (6.6), then it is easy to see, that this inequality is always fulfilled, if at most one variable on the left hand side is 1. If both variables are 1, then the path bends at this point and hence also $b_{x,y}^v = 1$ and the inequality is fulfilled. The same holds for (6.7). Additional to that it is easy to see, that (6.8) holds because every path has at most k bends. If we consider (6.9), it is easy to see, that the equation is true if the path corresponding to vertex v does not use the grid point (x, y), because then all the variables are 0. If the path uses the grid point (x, y), then the path has to come to the grid point (either it starts there, or it uses the edge from the left or it uses the edge from below) and has to leave the grid point (either it ends there or it uses the grid edge to the right or it uses the grid edge up). In all of these cases exactly one of the variables on both the left and the right hand side is 1 and the equality is true and hence (6.9) is always fulfilled. Equation (6.10) - (6.13) are just special cases of (6.9) at the border of the grid. (6.14) is true since every path ends only in one grid point and (6.15) is true because every path starts in only one grid point. Hence all constraints are fulfilled.

What is left so show is, that every feasible solution corresponds to a B_k^m-EPG representation. In order to do so, we consider a feasible solution and construct a B_k^m-EPG representation. We construct it in the following way. We first define a pseudo path for every vertex v, then we will show, that every pseudo path is actually a path. A grid edge going up at grid point (x, y) is part of the pseudo path corresponding to vertex v if and only if $u_{x,y}^v = 1$. In the same way the grid edge going to the right at grid point (x, y) belongs to the pseudo path corresponding to vertex v if and only if $r_{x,y}^v = 1$.

Now we will show, that every pseudo path is a monotonic path. In order to do so, we start by making a claim. Beforehand let us mention, that it is easy to see, that equation (6.10) - (6.13) are just special cases of (6.9). So if we refer to (6.9) in the proceeding, we refer either to exactly this equation or to the special cases if we consider a grid point at the border of the grid.

Claim 6.1.2. *Whenever a grid edge is contained in a pseudo path, there is a connected monotonic path from the left or upper end of the grid edge to a grid point (x^*, y^*) with $e_{x^*,y^*}^v = 1$.*

Proof of Claim. If we consider (6.9) for the grid point of the right or upper end of the grid edge, the left hand side is at least 1. Hence also the right hand side has to be at least 1. That means that either we are in a point (x, y) in with $e^v_{x,y} = 1$ or the pseudo path proceeds up or to the right. Both possibilities lead to a monotonic proceeding of the path. If the path proceeds, we can iteratively apply (6.9) for the grid point in which the right or upper end point of the new grid edge is. The path we get by applying this iteratively is monotonic and the grid size is restricted, hence we come to a grid point (x^*, y^*) with $e^v_{x^*,y^*} = 1$ if we reach the border of the grid at the latest. □

With this claim we are able to prove the following fact.

Claim 6.1.3. *For every two grid edges contained in the pseudo path, one of them has to lie on a monotonic path from the upper or right end point of the other one to a grid point (x^*, y^*) with $e^v_{x^*,y^*} = 1$.*

Proof of Claim. Assume that there are two grid edges e_1 and e_2 contained in the pseudo path, such that neither of them lies on a monotonic path from the right or upper end point of the other grid edge to a grid point (x^*, y^*) with $e^v_{x^*,y^*} = 1$. In other words we assume that there is a monotonic path from e_1, e_2 to grid points $(x_1^*, y_1^*), (x_2^*, y_2^*)$ with $e^v_{x_1^*,y_1^*} = e^v_{x_2^*,y_2^*} = 1$ respectively. Furthermore neither e_1 is contained in the path from e_2 to (x_2^*, y_2^*), nor e_2 is contained in the path from e_1 to (x_1^*, y_1^*). Inequality (6.14) implies that $(x_1^*, y_1^*) = (x_2^*, y_2^*)$. We distinct two cases.

If both the grid edges from below and from the left to the grid point (x_1^*, y_1^*) are contained in the pseudo path, then the left hand side of equation (6.9) is at least 2, hence also the right hand side has to be at least 2. We already know that $e^v_{x_1^*,y_1^*} = 1$, but at least one of the grid edges going up or to the right at grid point (x_1^*, y_1^*) has to be contained in the pseudo path as well. But then we can apply Claim 6.1.2 and know, that from the endpoint of that grid edge there has to be a monotonic path to another point (x_3^*, y_3^*) with $e^v_{x_3^*,y_3^*} = 1$, which is again a contradiction to (6.9).

In the second case, only one grid edge of the grid edges from below and from left to the grid point (x_1^*, y_1^*) is contained in the pseudo path and hence both monotonic paths use the same grid edge to reach the grid point (x_1^*, y_1^*). In this case we go back in the monotonic paths until we reach the first grid point, for which both the grid edges from below and left are contained in the pseudo path. Let this grid point be (x_4^*, y_4^*). The left hand side of (6.9) for this grid point is at least 2, hence also the right hand side of the equation has to be at least 2. We already know, that $e^v_{x_1^*,y_1^*} = 1$, therefore it follows from (6.14) that $e^v_{x_4^*,y_4^*} = 0$ and hence both the edges from (x_4^*, y_4^*) to the right and to the top are used in the pseudo path. Only one of them is contained in the monotonic paths, hence from the upper or right end point of the other grid edge e_5 Claim 6.1.2 can be applied again. That means that also from that end point of the grid edge there has to be a monotonic path to a grid point (x_5^*, y_5^*) with $e^v_{x_5^*,y_5^*} = 1$. Again, (6.14) implies that $(x_5^*, y_5^*) = (x_1^*, y_1^*)$. For the same reason as in the last case, it is not possible, that the monotonic path from e_5 to (x_1^*, y_1^*) and the monotonic paths from e_1 and e_2 over (x_4^*, y_4^*) to (x_1^*, y_1^*) use different grid edges in order to reach the grid point (x_1^*, y_1^*). But since the beginning at grid point (x_4^*, y_4^*) is different but the ending at grid point (x_1^*, y_1^*) is the same, there has to be a grind point on the monotonic path from (x_4^*, y_4^*) to (x_1^*, y_1^*) in which the path from e_1 and the path from e_5 enter differently but leave in the same way. Hence there has to be a point (x_6^*, y_6^*) between (x_4^*, y_4^*) and (x_1^*, y_1^*) such that both the edges from below and from left are contained in the pseudo path. But this is a contradiction to the definition of (x_4^*, y_4^*), so our assumption was wrong. Hence there can not be two grid edges contained in the pseudo path, such that neither of them lies on a monotonic path from the right or upper end point of the other grid edge to a grid point (x^*, y^*) with $e^v_{x^*,y^*} = 1$. □

Hence by Claim 6.1.3 every pseudo path is a monotonic path.

What is still left to show is, that every path has at most k bends and that two paths intersect if and only if the corresponding vertices are adjacent in G. We start by showing, that every path has at most k bends. Every path is monotonic, so we know, that there can only be two different types of bends at a grid point (x, y). The first possibility is, that the path uses the edge coming from the left hand side at the grid point and then the grid edge going up at the grid point. In this case $r^v_{x-1,y} = u^v_{x,y} = 1$ and hence because (6.6) is fulfilled, also $b^v_{x,y} = 1$. The second possibility is, that the path uses the grid edge going down at the grid point and then proceeds in the grid edge going to the right. For this case $u^v_{x,y-1} = r^v_{x,y} = 1$ holds and by (6.7) $b^v_{x,y} = 1$ again. Hence whenever a path bends at a grid point (x, y), the variable $b^v_{x,y} = 1$. Eventually it follows from (6.8), that every path has at most k bends.

In the end we will prove, that two paths intersect if and only if the corresponding vertices share an edge in G. If the vertices are not adjacent in G, then if follows from (6.1) and (6.2) that at every grid edge there can be at most one path. Hence whenever two vertices are not adjacent, their corresponding paths do not share a grid edge. If two vertices v and w are adjacent, then it follows from (6.5) that there are x^* and y^* such that $u^{v,x}_{x^*,y^*} = 1$ or $r^{v,x}_{x^*,y^*} = 1$. In the first case, it follows from (6.3) that both paths use the grid edge going up at the grid point (x^*, y^*) and hence the paths intersect there. In the second case, because of (6.4) the paths intersect on the grid edge going to the right at grid point (x^*, y^*). Hence in any case the paths intersect. □

Note, that in the MILP (EB^m_k) the constraints (6.9) - (6.15) are actually the constraints of a multicommodity flow problem. Every vertex of G corresponds to one commodity and the network can be obtained by introducing a vertex for every grid point and additional to that a source and a sink. The directed edges of the network are on the one hand the left-to-right and bottom-to-top directed grid edges respectively and on the other hand an edge from the source to every grid point and one edge from every grid point to the sink. The capacities are all 1. In this formulation, a feasible solution for the mulitcommodity flow problem corresponds to a monotonic path in the grid. For more information on the mulitcommodity flow problem see for example Hu [28].

A direct consequence of Theorem 6.1.1 is the next result.

Corollary 6.1.4. *For every graph G the MILP (EB^m_k) determines the minimum k, such that G is in B^m_k.*

The obvious advantage of (EB_k^m) is, that k is part of the optimization process, hence one does not have to determine beforehand, for which k he wants to know, whether a given graph is in B_k^m.

Nevertheless, a disadvantage of (EB_k^m) is, that the number of variables and constraints strongly depends on the grid size and there is no bound known on the grid size for a B_k^m-EPG representation for every k. Such upper bounds are only known for some values of k. Namely Corollary 3.1.4 implies, that for every graph with n vertices, m edges and maximum degree Δ, there is a B_k^m-EPG representation on a grid of size $n \times (n+m)$ for $k \geqslant 2\Delta$.

6.2 Grid Point Based Monotonic MILP Formulation

The second MILP we present is called (PB_k^m). The basic idea is to introduce integer variables representing the x- and y-coordinate of the bend points for every path. Furthermore for every path one auxiliary segment is introduced, such that it is known, which segments of the paths are horizontal and which are vertical. This turns out to be very useful in determining whether two paths intersect or not.

Definition. Let $G = (V, E)$ be a graph with $V = \{1, \ldots, n\}$ and let $\mathcal{G}$ be a rectangular grid from the bottom left grid point $(1, 1)$ to the top right grid point (m_x, m_y). Let $L := \{0, 1, \ldots, k+1\}$ and $L^+ := \{0, 1, \ldots, k+2\}$.

The meaning of the variables is the following. If the first segment of the path corresponding to v is horizontal, we let $h_v = 1$, if it is vertical we let $h_v = 0$. Then we assign consecutive numbers to consecutive segments for every path. If the path starts horizontally, we assign 0 to the first segment, if the path starts vertically, we assign 1 to the first segment and assign 0 to the segment which is just the grid point where segment 1 starts. Every assigned number is at most $k+1$ since every path has at most $k+1$ segments. If a path does not reach $k+1$ in the numbering, that is it uses less than $k+1$ segments or starts horizontally, then we let the remaining segments all be just the grid point in which the last segment ends. Then we let (x_v^ℓ, y_v^ℓ) be the coordinates of the grid point, at which segment ℓ of the path corresponding to v starts for every $0 \leqslant \ell \leqslant k+1$. Furthermore, we let (x_v^{k+2}, y_v^{k+2}) be the grid point, in which segment $k+1$ ends. Note, that if in the path corresponding to v segment ℓ is just a grid point, then $(x_v^\ell, y_v^\ell) = (x_v^{\ell+1}, y_v^{\ell+1})$. For every $v, w \in V$ with $v < w$ and for every $\ell, j \in L$ with $\ell \equiv j \bmod 2$ we let $i_{v,\ell,w,j} = 1$ if segment ℓ of the path corresponding to v intersects segment j of the path corresponding to w and $i_{v,\ell,w,j} = 0$ otherwise. Furthermore, for every $v, w \in V$ and for every $\ell, j \in L$, where either $v \neq w$ or $v = w$ and $j = \ell + 1$, we let $g^x_{v,\ell,w,j} = 1$ if $x_v^\ell \geqslant x_w^j$ and $g^x_{v,\ell,w,j} = 0$ otherwise. In the same way, we set $g^y_{v,\ell,w,j} = 1$ if $y_v^\ell \geqslant y_w^j$ and $g^y_{v,\ell,w,j} = 0$ otherwise.

The grid point based monotonic MILP (PB_k^m) is defined in the following way.

$$(\mathbf{PB}_k^m) \quad \text{s.t.} \quad x_v^\ell \leqslant x_v^{\ell+1} \qquad \forall v \in V, \forall \ell \in L \tag{6.24}$$

$$y_v^\ell \leqslant y_v^{\ell+1} \qquad \forall v \in V, \forall \ell \in L \tag{6.25}$$

$$x_v^\ell = x_v^{\ell+1} \qquad \forall v \in V,\, \forall \ell \in L,\, \ell \equiv 1 \bmod 2 \tag{6.26}$$

$$y_v^\ell = y_v^{\ell+1} \qquad \forall v \in V,\, \forall \ell \in L,\, \ell \equiv 0 \bmod 2 \tag{6.27}$$

$$x_v^{k+2} - x_v^{k+1} \leqslant m_x(1 - h_v) \qquad \forall v \in V \tag{6.28}$$

$$y_v^{k+2} - y_v^{k+1} \leqslant m_y(1 - h_v) \qquad \forall v \in V \tag{6.29}$$

$$x_v^1 - x_v^0 \leqslant m_x h_v \qquad \forall v \in V \tag{6.30}$$

$$x_v^\ell - x_w^j \geqslant m_x(g^x_{v,\ell,w,j} - 1) \qquad \forall v, w \in V,\, \forall \ell, j \in L^+,\; v \neq w \vee (v = w \wedge j = \ell + 1) \tag{6.31}$$

$$x_v^\ell - x_w^j \leqslant m_x g^x_{v,\ell,w,j} - 1 \qquad \forall v, w \in V,\, \forall \ell, j \in L^+,\; v \neq w \vee (v = w \wedge j = \ell + 1) \tag{6.32}$$

$$y_v^\ell - y_w^j \geqslant m_y(g^y_{v,\ell,w,j} - 1) \qquad \forall v, w \in V,\, \forall \ell, j \in L^+,\; v \neq w \vee (v = w \wedge j = \ell + 1) \tag{6.33}$$

$$y_v^\ell - y_w^j \leqslant m_y g^y_{v,\ell,w,j} - 1 \qquad \forall v, w \in V,\, \forall \ell, j \in L^+,\; v \neq w \vee (v = w \wedge j = \ell + 1) \tag{6.34}$$

$$5(1 - i_{v,\ell,w,j}) \geqslant g^x_{v,\ell,v,\ell+1} + g^x_{w,j,w,j+1} + (2 - g^y_{v,\ell,w,j} - g^y_{w,j,v,\ell}) + g^x_{w,j,v,\ell+1} + g^x_{v,\ell,w,j+1} \qquad \forall v, w \in V,\, \forall \ell, j \in L,\; v < w,\, \ell \equiv j \equiv 0 \bmod 2 \tag{6.35}$$

$$1 - i_{v,\ell,w,j} \leqslant g^x_{v,\ell,v,\ell+1} + g^x_{w,j,w,j+1} + (2 - g^y_{v,\ell,w,j} - g^y_{w,j,v,\ell}) + g^x_{w,j,v,\ell+1} + g^x_{v,\ell,w,j+1} \qquad \forall v, w \in V,\, \forall \ell, j \in L,\; v < w,\, \ell \equiv j \equiv 0 \bmod 2 \tag{6.36}$$

$$5(1 - i_{v,\ell,w,j}) \geqslant g^y_{v,\ell,v,\ell+1} + g^y_{w,j,w,j+1} + (2 - g^x_{v,\ell,w,j} - g^x_{w,j,v,\ell}) + g^y_{w,j,v,\ell+1} + g^y_{v,\ell,w,j+1} \qquad \forall v, w \in V,\, \forall \ell, j \in L,\; v < w,\, \ell \equiv j \equiv 1 \bmod 2 \tag{6.37}$$

$$1 - i_{v,\ell,w,j} \leqslant g^y_{v,\ell,v,\ell+1} + g^y_{w,j,w,j+1} + (2 - g^x_{v,\ell,w,j} - g^x_{w,j,v,\ell}) + g^y_{w,j,v,\ell+1} + g^y_{v,\ell,w,j+1} \qquad \forall v, w \in V,\, \forall \ell, j \in L,\; v < w,\, \ell \equiv j \equiv 1 \bmod 2 \tag{6.38}$$

$$1 \leqslant \sum_{\substack{\ell, j \in L \\ \ell \equiv j \bmod 2}} i_{v,\ell,w,j} \qquad \forall (v, w) \in E \tag{6.39}$$

$$0 \geqslant \sum_{\substack{\ell, j \in L \\ \ell \equiv j \bmod 2}} i_{v,\ell,w,j} \qquad \forall v, w \in V,\, v < w,\, (v, w) \notin E \tag{6.40}$$

$$x_v^\ell \in \{1, \ldots, m_x\} \quad \forall v \in V,\, \forall \ell \in L^+ \tag{6.41}$$

$$y_v^\ell \in \{1, \ldots, m_y\} \quad \forall v \in V,\, \forall \ell \in L^+ \tag{6.42}$$

$$h_v \in \{0, 1\} \quad \forall v \in V \tag{6.43}$$

$$i_{v,\ell,w,j} \in \{0, 1\} \quad \forall v, w \in V,\, v < w,\, \forall \ell, j \in L,\, \ell \equiv j \bmod 2 \tag{6.44}$$

$$g^x_{v,\ell,w,j} \in \{0,1\} \quad \forall v,w \in V, \forall \ell, j \in L^+, v \neq w \vee (v = w \wedge j = \ell+1) \tag{6.45}$$
$$g^y_{v,\ell,w,j} \in \{0,1\} \quad \forall v,w \in V, \forall \ell, j \in L^+, v \neq w \vee (v = w \wedge j = \ell+1) \tag{6.46}$$

In this MILP we have $2n(k+3)$ integer variables and additional to that there are $(n^2-n)\left(\frac{1}{2}\left\lceil\frac{(k+2)^2}{2}\right\rceil + 2(k+3)^2\right) + n(2k+5)$ binary variables. Furthermore the MILP consists of $(n^2-n)\left(4(k+3)^2 + \left\lceil\frac{(k+2)^2}{2}\right\rceil + \frac{1}{2}\right) + n(7k+17)$ constraints.

Theorem 6.2.1. *Every feasible solution of (PB^m_k) corresponds to a B^m_k-EPG representation of G on the grid $\mathcal{G}$ and vice versa.*

Proof. At first we will show, that every B^m_k-EPG representation corresponds to a feasible solution of (PB^m_k). In order to do so, for a B^m_k-EPG representation we consider the above described assignment and show, that this assignment is a feasible solution for (PB^m_k) indeed.

It follows from our assignment, that the variables are in the right scope, so (6.41) - (6.46) are fulfilled. Every path is only ascending in both columns and rows because we have a monotonic EPG representation, so (6.24) and (6.25) hold. Furthermore our numbering was chosen in such a way, that every horizontal segment has an even number, every vertical segment has an odd number and the number of every segment is also the number of the grid point in which the segment starts. So if ℓ is even, the segment is horizontal and hence remains in the same row of the grid. That means that $y^\ell_v = y^{\ell+1}_v$ for every even ℓ. Analogously $x^\ell_v = x^{\ell+1}_v$ for every odd ℓ, so (6.26) and (6.27) are fulfilled. For (6.28) and (6.29) it is easy to see, that they do not impose a restriction if $h_v = 0$. If $h_v = 1$, the path starts horizontally and hence uses the first segment and does not use the last segment. That means that segment $k+1$ is only a point and hence $x^{k+1}_v = x^{k+2}_v$ and $y^{k+1}_v = y^{k+2}_v$ hold, implying that both inequalities hold in any case. Also (6.30) does not restrict any variables, if $h_v = 1$. If $h_v = 0$, then the path starts vertically and hence segment 0 is a grid point, so $x^0_v = x^1_v$ holds and the inequality is true. If $x^\ell_v \geqslant x^j_w$, then $g^x_{v,\ell,w,j} = 1$ by our assignment and both (6.31) and (6.32) hold. Also if $x^\ell_v < x^j_w$, then $g^x_{v,\ell,w,j} = 0$ by our assignment and because of the integrality of both x^ℓ_v and x^j_w, even $x^\ell_v - x^j_w \leqslant -1$ holds, which implies that both inequalities are fulfilled in this case as well. That (6.33) and (6.34) hold, follows analogously. In order to consider (6.35) and (6.36), we first determine, when two horizontal segments of different paths do not intersect each other. Consider the segment ℓ of the path corresponding to v and segment j of the path corresponding to w, with $v \neq w$. If both segments are horizontal, then $\ell \equiv j \equiv 0 \bmod 2$ holds. The segments do not intersect if one of them consists of only a grid point ($g^x_{v,\ell,v,\ell+1} = 1$ or $g^x_{w,j,w,j+1} = 1$), or if they do not lie on the same grid row ($2 - g^y_{v,\ell,w,j} - g^y_{w,j,v,\ell} = 1$), or if the start point of one segment lies on the right side of the end point of the other segment ($g^x_{v,\ell,w,j+1} = 1$ or $g^x_{w,j,v,\ell+1} = 1$). In the case, that they do not intersect, $i_{v,\ell,w,j} = 0$ and (6.35) is not a restriction. Furthermore at least one of the above conditions is fulfilled and hence the right side of (6.36) is at least 1. The constraint is fulfilled since the left side in this case is 1 too. In the case, that they intersect, $i_{v,\ell,w,j} = 1$ and furthermore none of the above conditions is true, so all of the mentioned terms are 0. Hence both constraints hold in this case as well. In the same way, both (6.37) and (6.38) are true. It is easy to see, that (6.39) holds since in every B_k^m-EPG representation for every edge (v, w) there has to be an intersection of the corresponding paths and two segments can only intersect, if both of them are horizontal or vertical. Furthermore in every B_k^m-EPG representation the corresponding paths of two vertices v and w which are not adjacent are not allowed to intersect. So the sum on the right hand side of (6.40) is 0 and hence this inequality holds. Now we have shown, that every constraint of the MILP is fulfilled, and therefore we have proved, that every B_k^m-EPG representation of G on $\mathcal{G}$ corresponds to a feasible solution.

What is left to show is, that every feasible solution of (PB_k^m) corresponds to a B_k^m-EPG representation of G on $\mathcal{G}$. Consider a feasible solution. We let segment ℓ of the path corresponding to vertex v go from the grid point (x_v^ℓ, y_v^ℓ) to the grid point $(x_v^{\ell+1}, y_v^{\ell+1})$. This way we get a connected path. It goes along the grid lines because for every segment either the x-coordinate or the y-coordinate stays the same due to (6.26) and (6.27). Furthermore the path is monotonic, because of (6.24) and (6.25). Ever path has at most $k+2$ segments since we have $k+3$ points. But in every feasible solution h_v is either 0 or 1. If it is 0, it follows from (6.24) and (6.30) that $x_v^0 = x_v^1$. Moreover $y_v^0 = y_v^1$ holds because of (6.27). So the segment between (x_v^0, y_v^0) and (x_v^1, y_v^1) is only a point. Hence for $h_v = 0$, the path corresponding to v has at most $k+1$ segments. If $h_v = 1$, then it follows from (6.28), (6.24), (6.29) and (6.25) that $(x_v^{k+1}, y_v^{k+1}) = (x_v^{k+2}, y_v^{k+2})$ and hence the last segment is only a point. So also in the case of $h_v = 1$ the path consists of at most $k+1$ segments. This makes sure, that in any case every path has at most k bends.

It remains to show, that the paths corresponding to two vertices intersect in the grid if and only if the vertices are adjacent in G. In order to prove that, we first will show, that $g^x_{v,\ell,w,j} = 1$ if and only if $x_v^\ell \geqslant x_w^j$. It follows from (6.31) that if $g^x_{v,\ell,w,j} = 1$ it holds that $x_v^\ell \geqslant x_w^j$. If $g^x_{v,\ell,w,j} = 0$, it follows from (6.32) that $x_v^\ell \leqslant x_w^j - 1$. That implies that $x_v^\ell < x_w^j$. In the same way it can be shown that $g^y_{v,\ell,w,j} = 1$ if and only if $y_v^\ell \geqslant y_w^j$ by using (6.33) and (6.34). Next we will show that $i_{v,\ell,w,j} = 1$ if and only if $g^x_{v,\ell,v,\ell+1} = 0$ and $g^x_{w,j,w,j+1} = 0$ and $(2 - g^y_{v,\ell,w,j} - g^y_{w,j,v,\ell}) = 0$ and $g^x_{w,j,v,\ell+1} = 0$ and $g^x_{v,\ell,w,j+1} = 0$. The one side of the equivalence follows from (6.36), the other from (6.35). Applying the observation, that $g^x_{v,\ell,w,j} = 1$ if and only if $x_v^\ell \geqslant x_w^j$ yields the following equivalence. It holds that $i_{v,\ell,w,j} = 1$ if and only if $x_v^\ell < x_v^{\ell+1}$ and $x_w^j < x_w^{j+1}$ and $y_v^\ell = y_w^j$ and $x_v^\ell < x_w^{j+1}$ and $x_j^w < x_v^{\ell+1}$. Putting all these conditions together we get that $i_{v,\ell,w,j} = 1$ if and only if the horizontal segment ℓ of the path corresponding to v intersects the horizontal segment j of the path corresponding to w. By using the equivalent constraints for vertical segments, we obtain that $i_{v,\ell,w,j} = 1$ if and only if the vertical segment ℓ of the path corresponding to v intersects the vertical segment j of the path corresponding to w. In the end it follows from (6.39) that the paths corresponding to vertices which are adjacent intersect each other. Furthermore it follows from (6.40) that there is no intersection between two vertical or between two horizontal segments of paths, where there is no edge between the corresponding vertices. It follows that there is no intersection at all since paths can only intersect each other if both segments are horizontal or both segments are vertical.

□

The advantage of (PB_k^m) over (EB_k^m) is, that the number of variables and constraints is not influenced by the grid size, hence solving an instance on a bigger grid does not make the MILP larger. Another advantage is, that (PB_k^m) can also be generalized for the non monotonic case, like it is done in Chapter 7. A drawback of (PB_k^m) is, that k is a crucial part in the modelling, hence with solving one MILP one can only decide whether a given graph G is in B_k^m or not for one single k. Nevertheless it is possible to determine the minimum k such that G is in B_k^m by solving at most $\lceil \log_2(\Delta) \rceil$ instances of (PB_k^m) for different values of k. In order to do so, one has to perform a binary search on the values $\{0, 1, \ldots, \Delta - 1\}$ in order to find a k^* such that $G \in B_{k^*}$ but $G \notin B_{k^*-1}$.

7 MILP Formulation

In this chapter we want to present a MILP formulation of the problem, whether a given graph G is in B_k for a fixed value of k.

7.1 Grid Point Based Formulation

The formulation we consider is a generalization of (PB_k^m) in Section 6.2. The basic idea is still to introduce integer variables which represent the x- and y- coordinates of the bend points of the paths corresponding to the vertices. However, it is more challenging than in the monotonic case to find out, if two paths intersect or not. This is because we can not assume anymore, that a bend point with a lower number of a path is to the left of a bend point with higher number of that path. We first define the MILP (PB_k).

Definition. Let $G = (V, E)$ be a graph with vertex set $V = \{1, \ldots, n\}$ and let $\mathcal{G}$ be a grid from the bottom left grid point $(1, 1)$ to the top right grid point (m_x, m_y). Let $L := \{0, 1, \ldots, k + 1\}$ and furthermore $L^+ := \{0, 1, \ldots, k + 2\}$.

The meaning of the variables is, analogous to the meaning of the variables of the MILP (PB_k^m), the following. If the first segment of the path corresponding to v is horizontal, we let $h_v = 1$, if it is vertical we let $h_v = 0$. Then we assign consecutive numbers to consecutive segments for every path. If the path starts horizontally, we assign 0 to the first segment, if the path starts vertically, we assign 1 to the first segment and assign 0 to the segment which is just the grid point where segment 1 starts. Every assigned number is at most $k + 1$ since every path has at most $k + 1$ segments. If a path does not reach $k + 1$ in the numbering, that is it uses less than $k + 1$ segments or starts horizontally, then we let the remaining segments all be just the grid point in which the last segment ends. Then we let (x_v^ℓ, y_v^ℓ) be the coordinates of the grid point, at which segment ℓ of the path corresponding to v starts for every $0 \leqslant \ell \leqslant k + 1$. Furthermore, we let (x_v^{k+2}, y_v^{k+2}) be the grid point, in which segment $k + 1$ ends. Note, that if in the path corresponding to v segment ℓ is just a grid point, then $(x_v^\ell, y_v^\ell) = (x_v^{\ell+1}, y_v^{\ell+1})$. For every $v, w \in V$ with $v < w$ and for every $\ell, j \in L$ with $\ell \equiv j \bmod 2$ we let $i_{v,\ell,w,j} = 1$ if segment ℓ of the path corresponding to v intersects segment j of the path corresponding to w and $i_{v,\ell,w,j} = 0$ otherwise. Furthermore, for every $v, w \in V$ and for every $\ell, j \in L$, where either $v \neq w$ or $v = w$ and $j = \ell \pm 1$, we let $g^x_{v,\ell,w,j} = 1$ if $x_v^\ell \geqslant x_w^j$ and $g^x_{v,\ell,w,j} = 0$ otherwise. In the same way, we set $g^y_{v,\ell,w,j} = 1$ if $y_v^\ell \geqslant y_w^j$ and $g^y_{v,\ell,w,j} = 0$ otherwise.

The grid point based MILP (PB_k) is defined in the following way.

$$(\mathbf{PB}_k) \quad \text{s.t.} \quad x_v^\ell = x_v^{\ell+1} \qquad \forall v \in V,\ \forall \ell \in L,\ \ell \equiv 1 \bmod 2 \tag{7.1}$$

$$y_v^\ell = y_v^{\ell+1} \qquad \forall v \in V,\, \forall \ell \in L,\, \ell \equiv 0 \bmod 2 \tag{7.2}$$
$$x_v^{k+2} - x_v^{k+1} \leqslant m_x(1-h_v) \qquad \forall v \in V \tag{7.3}$$
$$x_v^{k+1} - x_v^{k+2} \leqslant m_x(1-h_v) \qquad \forall v \in V \tag{7.4}$$
$$y_v^{k+2} - y_v^{k+1} \leqslant m_y(1-h_v) \qquad \forall v \in V \tag{7.5}$$
$$y_v^{k+1} - y_v^{k+2} \leqslant m_y(1-h_v) \qquad \forall v \in V \tag{7.6}$$
$$x_v^1 - x_v^0 \leqslant m_x h_v \qquad \forall v \in V \tag{7.7}$$
$$x_v^0 - x_v^1 \leqslant m_x h_v \qquad \forall v \in V \tag{7.8}$$
$$x_v^\ell - x_w^j \geqslant m_x(g^x_{v,\ell,w,j} - 1) \qquad \forall v,w \in V,\, \forall \ell, j \in L^+,\, v \neq w \vee (v = w \wedge j = \ell \pm 1) \tag{7.9}$$
$$x_v^\ell - x_w^j \leqslant m_x g^x_{v,\ell,w,j} - 1 \qquad \forall v,w \in V,\, \forall \ell, j \in L^+,\, v \neq w \vee (v = w \wedge j = \ell \pm 1) \tag{7.10}$$
$$y_v^\ell - y_w^j \geqslant m_y(g^y_{v,\ell,w,j} - 1) \qquad \forall v,w \in V,\, \forall \ell, j \in L^+,\, v \neq w \vee (v = w \wedge j = \ell \pm 1) \tag{7.11}$$
$$y_v^\ell - y_w^j \leqslant m_y g^y_{v,\ell,w,j} - 1 \qquad \forall v,w \in V,\, \forall \ell, j \in L^+,\, v \neq w \vee (v = w \wedge j = \ell \pm 1) \tag{7.12}$$
$$3(1 - i_{v,\ell,w,j}) \geqslant (g^x_{v,\ell,v,\ell+1} + g^x_{v,\ell+1,v,\ell} - 1) + (g^x_{w,j,w,j+1} + g^x_{w,j+1,w,j} - 1) + (2 - g^y_{v,\ell,w,j} - g^y_{w,j,v,\ell}) \qquad \forall v,w \in V,\, \forall \ell, j \in L,\, v < w,\, \ell \equiv j \equiv 0 \bmod 2 \tag{7.13}$$
$$4 - i_{v,\ell,w,j} \geqslant g^x_{v,\ell,w,j} + g^x_{v,\ell,w,j+1} + g^x_{v,\ell+1,w,j} + g^x_{v,\ell+1,w,j+1} \qquad \forall v,w \in V,\, \forall \ell, j \in L,\, v < w,\, \ell \equiv j \equiv 0 \bmod 2 \tag{7.14}$$
$$4 - i_{v,\ell,w,j} \geqslant g^x_{w,j,v,\ell} + g^x_{w,j,v,\ell+1} + g^x_{w,j+1,v,\ell} + g^x_{w,j+1,v,\ell+1} \qquad \forall v,w \in V,\, \forall \ell, j \in L,\, v < w,\, \ell \equiv j \equiv 0 \bmod 2 \tag{7.15}$$
$$2 + i_{v,\ell,w,j} \geqslant g^x_{v,\ell,w,j} + (1 - g^x_{v,\ell,w,j+1}) + (1 - g^x_{v,\ell,v,\ell+1}) - (2 - g^y_{v,\ell,w,j} - g^y_{w,j,v,\ell}) \qquad \forall v,w \in V,\, \forall \ell, j \in L,\, v < w,\, \ell \equiv j \equiv 0 \bmod 2 \tag{7.16}$$
$$2 + i_{v,\ell,w,j} \geqslant g^x_{w,j,v,\ell} + (1 - g^x_{w,j,v,\ell+1}) + (1 - g^x_{w,j,w,j+1}) - (2 - g^y_{v,\ell,w,j} - g^y_{w,j,v,\ell}) \qquad \forall v,w \in V,\, \forall \ell, j \in L,\, v < w,\, \ell \equiv j \equiv 0 \bmod 2 \tag{7.17}$$
$$2 + i_{v,\ell,w,j} \geqslant g^x_{v,\ell+1,w,j} + (1 - g^x_{v,\ell+1,w,j+1}) + (1 - g^x_{v,\ell+1,v,\ell}) - (2 - g^y_{v,\ell,w,j} - g^y_{w,j,v,\ell}) \qquad \forall v,w \in V,\, \forall \ell, j \in L,\, v < w,\, \ell \equiv j \equiv 0 \bmod 2 \tag{7.18}$$
$$2 + i_{v,\ell,w,j} \geqslant g^x_{w,j,v,\ell+1} + (1 - g^x_{w,j,v,\ell}) + (1 - g^x_{w,j,w,j+1}) - (2 - g^y_{v,\ell,w,j} - g^y_{w,j,v,\ell}) \qquad \forall v,w \in V,\, \forall \ell, j \in L,\, v < w,\, \ell \equiv j \equiv 0 \bmod 2 \tag{7.19}$$

$$2 + i_{v,\ell,w,j} \geqslant g^x_{v,\ell,w,j+1} + (1 - g^x_{v,\ell,w,j}) + (1 - g^x_{v,\ell,v,\ell+1}) - (2 - g^y_{v,\ell,w,j} - g^y_{w,j,v,\ell}) \qquad \forall v, w \in V,\, \forall \ell, j \in L,\ v < w,\ \ell \equiv j \equiv 0 \bmod 2 \tag{7.20}$$

$$2 + i_{v,\ell,w,j} \geqslant g^x_{w,j+1,v,\ell} + (1 - g^x_{w,j+1,v,\ell+1}) + (1 - g^x_{w,j+1,w,j}) - (2 - g^y_{v,\ell,w,j} - g^y_{w,j,v,\ell}) \qquad \forall v, w \in V,\, \forall \ell, j \in L,\ v < w,\ \ell \equiv j \equiv 0 \bmod 2 \tag{7.21}$$

$$2 + i_{v,\ell,w,j} \geqslant g^x_{v,\ell+1,w,j+1} + (1 - g^x_{v,\ell+1,w,j}) + (1 - g^x_{v,\ell+1,v,\ell}) - (2 - g^y_{v,\ell,w,j} - g^y_{w,j,v,\ell}) \qquad \forall v, w \in V,\, \forall \ell, j \in L,\ v < w,\ \ell \equiv j \equiv 0 \bmod 2 \tag{7.22}$$

$$2 + i_{v,\ell,w,j} \geqslant g^x_{w,j+1,v,\ell+1} + (1 - g^x_{w,j+1,v,\ell}) + (1 - g^x_{w,j+1,w,j}) - (2 - g^y_{v,\ell,w,j} - g^y_{w,j,v,\ell}) \qquad \forall v, w \in V,\, \forall \ell, j \in L,\ v < w,\ \ell \equiv j \equiv 0 \bmod 2 \tag{7.23}$$

$$3(1 - i_{v,\ell,w,j}) \geqslant (g^y_{v,\ell,v,\ell+1} + g^y_{v,\ell+1,v,\ell} - 1) + (g^y_{w,j,w,j+1} + g^y_{w,j+1,w,j} - 1) + (2 - g^x_{v,\ell,w,j} - g^x_{w,j,v,\ell}) \qquad \forall v, w \in V,\, \forall \ell, j \in L,\ v < w,\ \ell \equiv j \equiv 1 \bmod 2 \tag{7.24}$$

$$4 - i_{v,\ell,w,j} \geqslant g^y_{v,\ell,w,j} + g^y_{v,\ell,w,j+1} + g^y_{v,\ell+1,w,j} + g^y_{v,\ell+1,w,j+1} \qquad \forall v, w \in V,\, \forall \ell, j \in L,\ v < w,\ \ell \equiv j \equiv 1 \bmod 2 \tag{7.25}$$

$$4 - i_{v,\ell,w,j} \geqslant g^y_{w,j,v,\ell} + g^y_{w,j,v,\ell+1} + g^y_{w,j+1,v,\ell} + g^y_{w,j+1,v,\ell+1} \qquad \forall v, w \in V,\, \forall \ell, j \in L,\ v < w,\ \ell \equiv j \equiv 1 \bmod 2 \tag{7.26}$$

$$2 + i_{v,\ell,w,j} \geqslant g^y_{v,\ell,w,j} + (1 - g^y_{v,\ell,w,j+1}) + (1 - g^y_{v,\ell,v,\ell+1}) - (2 - g^x_{v,\ell,w,j} - g^x_{w,j,v,\ell}) \qquad \forall v, w \in V,\, \forall \ell, j \in L,\ v < w,\ \ell \equiv j \equiv 1 \bmod 2 \tag{7.27}$$

$$2 + i_{v,\ell,w,j} \geqslant g^y_{w,j,v,\ell} + (1 - g^y_{w,j,v,\ell+1}) + (1 - g^y_{w,j,w,j+1}) - (2 - g^x_{v,\ell,w,j} - g^x_{w,j,v,\ell}) \qquad \forall v, w \in V,\, \forall \ell, j \in L,\ v < w,\ \ell \equiv j \equiv 1 \bmod 2 \tag{7.28}$$

$$2 + i_{v,\ell,w,j} \geqslant g^y_{v,\ell+1,w,j} + (1 - g^y_{v,\ell+1,w,j+1}) + (1 - g^y_{v,\ell+1,v,\ell}) - (2 - g^x_{v,\ell,w,j} - g^x_{w,j,v,\ell}) \qquad \forall v, w \in V,\, \forall \ell, j \in L,\ v < w,\ \ell \equiv j \equiv 1 \bmod 2 \tag{7.29}$$

$$2 + i_{v,\ell,w,j} \geqslant g^y_{w,j,v,\ell+1} + (1 - g^y_{w,j,v,\ell}) + (1 - g^y_{w,j,w,j+1}) - (2 - g^x_{v,\ell,w,j} - g^x_{w,j,v,\ell}) \qquad \forall v, w \in V,\, \forall \ell, j \in L,\ v < w,\ \ell \equiv j \equiv 1 \bmod 2 \tag{7.30}$$

$$2 + i_{v,\ell,w,j} \geqslant g^y_{v,\ell,w,j+1} + (1 - g^y_{v,\ell,w,j}) + (1 - g^y_{v,\ell,v,\ell+1}) - (2 - g^x_{v,\ell,w,j} - g^x_{w,j,v,\ell}) \qquad \forall v, w \in V,\, \forall \ell, j \in L,\ v < w,\ \ell \equiv j \equiv 1 \bmod 2 \tag{7.31}$$

$$2 + i_{v,\ell,w,j} \geqslant g^y_{w,j+1,v,\ell} + (1 - g^y_{w,j+1,v,\ell+1}) + (1 - g^y_{w,j+1,w,j}) - (2 - g^x_{v,\ell,w,j} - g^x_{w,j,v,\ell}) \quad \forall v, w \in V,\, \forall \ell, j \in L,\, v < w,\, \ell \equiv j \equiv 1 \bmod 2 \tag{7.32}$$

$$2 + i_{v,\ell,w,j} \geqslant g^y_{v,\ell+1,w,j+1} + (1 - g^y_{v,\ell+1,w,j}) + (1 - g^y_{v,\ell+1,v,\ell}) - (2 - g^x_{v,\ell,w,j} - g^x_{w,j,v,\ell}) \quad \forall v, w \in V,\, \forall \ell, j \in L,\, v < w,\, \ell \equiv j \equiv 1 \bmod 2 \tag{7.33}$$

$$2 + i_{v,\ell,w,j} \geqslant g^y_{w,j+1,v,\ell+1} + (1 - g^y_{w,j+1,v,\ell}) + (1 - g^y_{w,j+1,w,j}) - (2 - g^x_{v,\ell,w,j} - g^x_{w,j,v,\ell}) \quad \forall v, w \in V,\, \forall \ell, j \in L,\, v < w,\, \ell \equiv j \equiv 1 \bmod 2 \tag{7.34}$$

$$1 \leqslant \sum_{\substack{\ell, j \in L \\ \ell \equiv j \bmod 2}} i_{v,\ell,w,j} \quad \forall (v,w) \in E \tag{7.35}$$

$$0 \geqslant \sum_{\substack{\ell, j \in L \\ \ell \equiv j \bmod 2}} i_{v,\ell,w,j} \quad \forall v, w \in V,\, v < w,\, (v,w) \notin E \tag{7.36}$$

$$x^\ell_v \in \{1, \ldots, m_x\} \quad \forall v \in V,\, \forall \ell \in L^+ \tag{7.37}$$

$$y^\ell_v \in \{1, \ldots, m_y\} \quad \forall v \in V,\, \forall \ell \in L^+ \tag{7.38}$$

$$h_v \in \{0, 1\} \quad \forall v \in V \tag{7.39}$$

$$i_{v,\ell,w,j} \in \{0, 1\} \quad \forall v, w \in V,\, v < w,\, \forall \ell, j \in L,\, \ell \equiv j \bmod 2 \tag{7.40}$$

$$g^x_{v,\ell,w,j} \in \{0, 1\} \quad \forall v, w \in V,\, \forall \ell, j \in L^+,\, v \neq w \vee (v = w \wedge j = \ell \pm 1) \tag{7.41}$$

$$g^y_{v,\ell,w,j} \in \{0, 1\} \quad \forall v, w \in V,\, \forall \ell, j \in L^+,\, v \neq w \vee (v = w \wedge j = \ell \pm 1) \tag{7.42}$$

Note, that the MILP (PB_k) has $2n(k+3)$ integer variables and additional to that $(n^2 - n)\left(\frac{1}{2}\left\lceil \frac{(k+2)^2}{2} \right\rceil + 2(k+3)^2\right) + n(4k+9)$ binary variables. Moreover, in total there are $(n^2 - n)\left(4(k+3)^2 + 11\left\lceil \frac{(k+2)^2}{2} \right\rceil + \frac{1}{2}\right) + n(9k+24)$ constraints.

It turns out, that the following holds for (PB_k).

Theorem 7.1.1. *Every feasible solution of (PB_k) corresponds to a B_k-EPG representation of G on the grid $\mathcal{G}$ and vice versa.*

Proof. The proof is very similar to the one of Theorem 6.2.1, especially the assignment of the variables is done in just the same way in both directions.

We start by showing, that every B_k-EPG representation of G corresponds to a feasible solution of (PB_k). The assignment of numbers to the segments and bend points of the paths, and also the assignment of the variables is done like in the MILP (PB^m_k). So analogously to the proof of Theorem 6.2.1 it follows that the constraints (7.1) - (7.12) and (7.35) - (7.42) are fulfilled. What is left to show is, that the constraints (7.13) - (7.34) are true.

It is easy to see, that the right hand side of (7.13) does not impose a restriction if $i_{v,\ell,w,j} = 0$, because all 3 parenthesized expressions on the right hand side can be at most

1. In the case that $i_{v,\ell,w,j} = 1$ there is an intersection between segment ℓ of the path corresponding to v and segment j of the path corresponding to w. Hence neither of the both segments can be only a point, nor can the grid rows, in which the segments are, be different. That means that all 3 parenthesized expressions are 0 and (7.13) is true. Also (7.14) and (7.15) fulfilled, if $i_{v,\ell,w,j} = 0$. If segment ℓ of the path corresponding to v and segment j of the path corresponding to w intersect, it is not possible, that all 2 end points of one of the segments lie to the right of all 2 end points of the other segment. Hence both right hand sides of (7.14) and (7.15) are at most 3 if $i_{v,\ell,w,j} = 1$ and hence the two constrains are fulfilled also in this case. If we consider (7.16) - (7.23) it is obvious, that the inequalities are always fulfilled if the right hand sides are at most 2. If a right hand side is 3, then all 3 terms in the first line have to be 1 and $(2 - g^y_{v,\ell,w,j} - g^y_{w,j,v,\ell}) = 0$. The latter implies that $y^\ell_v = y^j_w$, hence segment ℓ of the path corresponding to v and segment j of the path corresponding to w are in the same grid row. If we take for example (7.16), that also implies that $x^\ell_v \geqslant x^j_w$, $x^\ell_v < x^{j+1}_w$, and $x^\ell_v < x^{\ell+1}_v$. In other words $x^j_w \leqslant x^\ell_v < x^{j+1}_w$ and $x^\ell_v < x^{\ell+1}_v$, hence the two corresponding segments intersect, because both segments use the grid edge to the right at x^ℓ_v. So also $i_{v,\ell,w,j} = 1$ and the inequality (7.16) holds. That the constraints (7.17) - (7.23) are fulfilled, can be shown in just the same way. Also that the inequalities (7.24) - (7.34) hold is shown analogously.

Now we have seen, that every B_k-EPG representation corresponds to a feasible solution of (PB_k). What remains to show is, that every feasible solution corresponds to a B_k-EPG representation. In order to prove this, we use the same construction of the paths as in the proof of Theorem 6.2.1. Hence it follows in the same way, that every vertex is represented as a path in the grid $\mathcal{G}$ with at most k bends from constraints (7.1) - (7.8) and (7.37) - (7.42). Furthermore we know from (7.9) - (7.12) that $g^x_{v,\ell,w,j} = 1$ if and only if $x^\ell_v \geqslant x^j_w$, and $g^y_{v,\ell,w,j} = 1$ if and only if $y^\ell_v \geqslant y^j_w$. What is left to show is, that two paths intersect if and only if the corresponding vertices are adjacent in G.

Assume two vertices v and w are adjacent in G. Then it follows from (7.35) that there are ℓ^* and j^* such that $i_{v,\ell^*,w,j^*} = 1$. If both $j^* \equiv \ell^* \equiv 0 \bmod 2$ then it follows from (7.13) that $g^x_{v,\ell^*,v,\ell^*+1} + g^x_{v,\ell^*+1,v,\ell^*} - 1 = 0$ and hence segment ℓ^* of the path corresponding to v is not only a point. Furthermore it follows that $g^x_{w,j^*,w,j^*+1} + g^x_{w,j^*+1,w,j^*} - 1 = 0$ so also segment j^* of the path corresponding to w is not only a segment. Moreover $2 - g^y_{v,\ell^*,w,j^*} - g^y_{w,j^*,v,\ell^*} = 0$ has to hold, hence $y^{\ell^*}_v = y^{j^*}_w$. So the two segments are in the same grid row. Moreover it follows from (7.14) and (7.15) that neither both end points of segment ℓ^* of the path corresponding to v lie on the right side of both end points of segment j^* of the path corresponding to w nor vice versa. But that means, that the both segments intersect. Hence also the paths intersect. The same follows from (7.24) - (7.26) in the case $j^* \equiv \ell^* \equiv 1 \bmod 2$.

It remains to prove, that the corresponding paths do not intersect if two vertices v and w are not adjacent in G. If the vertices do not share an edge in G, then it follows from (7.36) that $i_{v,\ell,w,j} = 0$ holds for every ℓ and j. Then for horizontal segments (7.16) implies that $2 - g^y_{v,\ell,w,j} - g^y_{w,j,v,\ell} = 1$, hence the two corresponding segments are not on the same gird row and hence do not intersect, or $g^x_{v,\ell,w,j} = 0$ or $g^x_{v,\ell,w,j+1} = 1$ or $g^x_{v,\ell,v,\ell+1} = 1$. The latter are equivalent to $x^\ell_v < x^j_w$ or $x^\ell_v \geqslant x^{j+1}_w$ or $x^\ell_v \geqslant x^{\ell+1}_v$. This means, that it can not happen, that $x_w^j \leqslant x_v^\ell < x_w^{j+1}$ and $x_v^\ell < x_v^{\ell+1}$ hold. This implies, that is can not happen that $x_w^j < x_w^{j+1}$ and $x_v^j < x_v^{j+1}$ and both paths share the grid edge going to the right from x_v^j. In the same way (7.17) - (7.23) imply the same statement for different configurations of the segments. Taking all the implications together, this implies that two horizontal segments can not intersect. Analogously, (7.27) - (7.34) make sure, that two vertical segments of the corresponding paths do not intersect. This implies, that the paths do not intersect, because intersections can only occur on two segments, if both are directed in the same way.

□

So we are able to solve a MILP in order to find out, whether a given graph G is in B_k or not. If we want to determine the bend number of G, we can do that like the determination of the monotonic bend number is done at the end of Section 6.2. So we have to perform a binary search on $\{0, 1, 2, \ldots, \Delta - 1\}$ in order to find the minimum k, such that G is in B_k but not in B_{k-1}. This can be done by solving at most $\lceil \log_2(\Delta) \rceil$ instances of (PB_k) for different values of k.

8 Conclusions and Open Problems

In this thesis we gave an overview of the existing results on edge intersection graphs of paths on a grid. Furthermore we proved, that all outerplanar graphs are in B_2^m and that there is an outerplanar graph which is not in B_1^m. Due to the fact, that it is known, that all outerplanar graphs are in B_2 and there is an outerplanar graph which is not in B_1, this means, that the bend number and the monotonic bend number of outerplanar graphs coincide. It would be of interest to know, whether this is also true for planar graphs. The first step into this direction would be to give an upper bound on the monotonic bend number of planar graphs, because such a bound is not known yet.

Furthermore we gave an exact characterization which outerplanar triangulations and cacti are in B_0, B_1^m, B_1 and B_2^m. It would be interesting to get such characterizations for more subclasses of outerplanar graphs or even for all outerplanar graphs.

Then we derived an inequality that has to hold whenever a $K_{m,n}$ is in B_k^m. This inequality imposes a strong restriction only for high values of k, that is close to $2m-2$. It would be of interest to also derive another inequality for low values of k like it is done in [26] for B_k.

We used the inequality to prove, that $B_k^m \subsetneq B_k$ for $k = 2$, $k = 5$, and $k \geqslant 7$, which answers an open question of [20] for almost all k. Of course it is a pressing question to prove the statement also for the remaining cases $k = 3$, $k = 4$, and $k = 6$.

Another question of interest which is not answered yet is, whether $B_k^m \subsetneq B_{k+1}^m$. We conjecture, that that is the case.

Additional to that we showed, that for every $k \geqslant 6$ there is a graph in B_k which is not in B_{2k-9}^m. A natural question that arises is, whether also the converse is true, hence whether there is a graph in B_{2k-9}^m that is not in B_k.

We showed, that $B_1 \subseteq B_3^m$. It would be of interest to answer a more general question. If $b(G)$ and $b^m(G)$ denote the bend number and the monotonic bend number of a graph respectively, is there a function f such that $b(G) \leqslant f(b^m(G))$ for every graph G?

Finally we formulated the problem of whether a given graph is in B_k and B_k^m for a fixed k as MILP. Here it would, especially in the case of B_k, be of interest whether there is a formulation with fewer variables and constraints.

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