Michael MOSSHAMMER

Phase transitions in series-parallel graphs

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Betreuerin:
Univ.-Prof. Ph.D. Mihyun KANG

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1 Introduction

Since the paper of Erdős and Rényi [9] on random graphs in 1960, random graphs have been widely studied. The idea is to draw a graph out of a certain set of graphs according to some probability distribution, for example a uniform distribution on the set of all graphs with a given number of vertices and edges, and to study its typical properties, like for example connectivity or the emergence of the giant component. Since the seminal work of Erdős and Rényi [9], it has been tried to get similar results for various random graph models, for example planar graphs.

The classes of planar graphs and random planar graphs have recieved considerable attention during the last decades. The first results on this topic were asymptotic formulas on the number of such graphs which got increasingly more exact. Let $pl(n)$ be the number of planar graphs with n vertices. Then McDiarmid, Steger, and Welsh [25] showed, that $(pl(n)/n!)^{1/n}$ converges to a limit $0 < \gamma < \infty$ as $n \to \infty$. A number of upper and lower bounds was given for γ, for example by Osthus, Prömel, and Taraz [28] or Bender, Gao, and Wormald [1], until Giménez and Noy [13] proved $pl(n) \sim cn^{-7/2}\gamma^n n!$ with c and γ analytically given with $\gamma \sim 27.2$. At the second international seminar on random graphs [19] the same question about $pl(n, M)$, where M is the number of edges, was asked. This question was answered in multiple steps. A first observation showed that the typical random graph $G(n, an)$ with $a < \frac{1}{2}$ is planar [24]. Gerke et al. [11] proved that $(pl(n, an)/n!)^{1/n}$ converges to a limit dependent only on a. This was further specified by Gimenez and Noy [14] that $pl(n, an) = c_a n^{-4}\gamma_a^n n!$, if $1 < a < 3$. Finally, Kang and Łuczak [18] proved the asymptotics of $pl(n, an)$ in the case of $\frac{1}{2} < a < 1$. Similarly, the number of forests $F(n, M)$ with n vertices and M edges was given by Cayley [7]. As for series-parallel graphs, there are not many results in this direction.

A graph is series-parallel, if it does not contain the graph K_4 as a minor. As K_5 and $K_{3,3}$ do contain K_4 as a minor, all series-parallel graphs are planar [20]. In contrast to that, all forests are series-parallel. Comparisons of these graph classes will be made throughout this thesis. Let $sp(n)$ be the number of series-parallel graphs with n vertices. Bodirsky, Giménez, Kang, and Noy [2] showed that $sp(n) \approx gn^{-5/2}\gamma^n n!$. Also, it was proven quite recently by Uno, Uehara, and Nakano [32] that the number of series-parallel graphs with M edges is bounded from above by $2^{\lceil 2.5285M-2 \rceil}$. About the number $sp(n, M)$ it is only known that, as above, the general random graph $G(n, an)$ is series-parallel, if $a < \frac{1}{2}$.

Another point of interest besides the asymptotic number of these graph classes is its structure, and in particular the size of the largest component. It follows from results of Erdős and Rény [9], Bollobás [3], Łuczak [21], Łuczak, Pitel and Wierman [24], Janson et al. [16] and Janson [15] that in the case of $G(n,M)$, the giant component (which is the unique largest component) suddenly emerges at $M = \frac{n}{2} + O(n^{2/3})$. This phenomenon is called a phase transition and will be defined more exactly in Chapter 2.3. Nowadays this phenomenon is widely studied and similar results have been found for a couple of graph classes, including planar graphs [18] and random forests [23]. In both of these cases the giant component emerges at $M = \frac{n}{2} + O(n^{2/3})$ with the same size estimate, which does differ from the estimates of $G(n,M)$. Again there are no such theorems for series-parallel graphs. Also, it was shown by Kang and Łuczak [18] that there is a second point, where the asymptotics of the size estimates change, at $M = n + O(n^{3/5})$. This second phase transition exists neither for $G(n,M)$ nor for $F(n,M)$.

In this thesis, we will give asymptotic formulas for the number $sp(n,M)$ of series-parallel graphs. We will also show that the giant component will emerge at $M = \frac{n}{2} + O\left(n^{2/3}\right)$ and the size estimates will be the same as for planar graphs. Furthermore we will show that the second phase transition of planar graphs does also exist for series-parallel graphs with the same size estimates as in the planar case with a size of $n-(2+o(1))\,|t|$, if $M = n+t$ and $t \ll -n^{3/5}$ and of size $n-(\alpha+o(1))\left(\frac{n}{t}\right)^{3/2}$ for $M = n+t$ and $n^{3/5} \ll t \ll n^{2/3}$.

To do this, we will use a similar approach as in [18]. In Chapter 2, we will establish some general definitions and formulas, which will be needed throughout the thesis. Then we will give some general results abut series-parallel graphs and random graphs. Furthermore we will state some properties of phase transition in random graphs. Additionally we will give some explanations on the symbolic methods and on singularity analysis, as we will need these techniques to find the number of series-parallel graphs. In Chapter 3, we will calculate the number $sp(n,M)$ for $M = an$ with $a \leq 1$. To do this, we will first restate some results on trees and unicyclic components in Chapter 3.1. Then we will count the number of 3-regular series-parallel multigraphs in Chapter 3.2. We will then give a method to get an estimate on the number of complex series-parallel graphs (graphs are complex, if they have at least 2 cycles) in chapters 3.3 and 3.4. Finally, by merging these results, we will get the number of series-parallel graphs $sp(n,M)$ in Chapter 3.5. In Chapter 4.3, we will then calculate the size of the largest component in the different ranges. To do this, we will again use the structure we looked at for calculating the number of series-parallel graphs. Thus we will not only get the size of the largest component but also estimates for some other parts of the graph emerging from the way of counting, the deficiency and the excess in Chapter 4.1 and kernel and core in Chapter 4.2. In the last chapter we will summarize all results and compare them to other graph classes.

2 Basics

In this chapter, we will first establish some basic definitions and formulas we will need in this thesis. Then, in Section 2.2, we will provide an introduction on series-parallel graphs and in Section 2.3 an introduction on random graphs and exact definitions on phase transition. Finally, in Section 2.4 we will describe the symbolic method and singularity analysis.

2.1 Definitions and formulas

In this section, we will give some definitions and discuss some formulas occurring frequently throughout this thesis.

At first, to discuss asymptotic formulas, we need the following notations.

Definition 2.1. *Let $f, g : \mathbb{N} \to \mathbb{R}$ be functions. Then we write*

- $f(n) = O(g(n))$, *if* $\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$.
- $f(n) = o(g(n)$, *if* $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.
- $f(n) = \Theta(g(n))$, *if* $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$ *for some constant* $0 < c < \infty$.
- $f(n) \ll g(n)$, *if* $f(n) = o(g(n))$ *for* $f(n)$ *and* $g(n)$ *positive and* $g(n) = o(f(n))$ *for* $f(n)$ *and* $g(n)$ *negative.*
- $f(n) \gg g(n)$, *if* $g(n) = o(f(n))$ *for* $f(n)$ *and* $g(n)$ *positive and* $f(n) = o(g(n))$ *for* $f(n)$ *and* $g(n)$ *negative.*
- $f(n) \approx g(n)$, *if* $f(n) = \Theta(g(n))$.

This notation will be used mainly for approximating error terms, for example in the following theorems.

Theorem 2.1 (Stirling's Formula)**.** *For all $n \in \mathbb{N}$ we have:*

$$n! = \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} \left(1 + O\left(\frac{1}{n}\right)\right). \tag{1}$$

Theorem 2.2. *Let $x \in \mathbb{R}$. Then:*

$$1 + x = \exp\left(x - \frac{x^2}{2} + \frac{x^3}{3} + O\left(x^4\right)\right). \tag{2}$$

From this, we get the following approximation for the falling factorial $(k)_i$, which is then used to give different approximations for the binomial coefficient.

Theorem 2.3. *Let $i, k \in \mathbb{N}$ and $\frac{i^4}{k^3} = o(1)$. Then:*

$$(k)_i = k^i \exp\left(-\frac{i^2}{2k} - \frac{i^3}{6k^2} + O\left(\frac{i}{k} + \frac{i^4}{k^3}\right)\right). \tag{3}$$

Proof. We have:

$$\begin{aligned}
(k)_i &= \prod_{j=0}^{i-1} (k-j) \\
&= k^i \prod_{j=0}^{i-1} \left(1 + \left(-\frac{j}{k}\right)\right) \\
&\overset{(2)}{=} k^i \prod_{j=0}^{i-1} \exp\left(-\frac{j}{k} - \frac{j^2}{2k^2} - \frac{j^3}{3k^3} + O\left(\frac{j^4}{k^4}\right)\right) \\
&= k^i \exp\left(\sum_{j=0}^{i-1} \left(-\frac{j}{k} - \frac{j^2}{2k^2} - \frac{j^3}{3k^3}\right) + O\left(\frac{j^4}{k^4}\right)\right).
\end{aligned}$$

Concluding in the result described above, after using summation formulas and collecting corresponding terms. □

For binomial coefficients we will need different approximations depending on the precision. A very rough approximation is the following:

Theorem 2.4. *For all $k \leq n \in \mathbb{N}$ we have:*

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!} \leq \left(\frac{en}{k}\right)^k.$$

Proof. We have:

$$\binom{n}{k} = \frac{(n)_k}{k!} = \prod_{i=0}^{k-1} \frac{n-i}{k-i}.$$

The middle inequality follows directly from the first equation with $(n)_k \leq n^k$. The leftmost inequality follows from the last equation by noting that for all $0 \leq i \leq k-1$: $\frac{n-i}{k-i} \geq \frac{n}{k}$. This is true because the following is valid.

$$\begin{aligned}
&\frac{n-i}{k-i} \geq \frac{n}{k} \\
\Leftrightarrow\ &(n-i)\,k \geq (k-i)\,n \\
\Leftrightarrow\ &ni \geq ki
\end{aligned}$$

The last part follows from Stirling's formula (1):

$$\begin{aligned} k! &= \sqrt{2\pi} k^{k+\frac{1}{2}} e^{-k} \left(1 + O\left(\frac{1}{k}\right)\right) \\ &= \left(\frac{k}{e}\right)^k \left(\sqrt{2\pi k}\left(1 + O\left(\frac{1}{k}\right)\right)\right) \\ &\geq \left(\frac{k}{e}\right)^k . \end{aligned}$$

□

By being more precise with $(n)_k$ and $k!$, using (1) and (3), we get the following lemma.

Lemma 2.5. *Let $n, k \in \mathbb{N}$. Then we have:*

$$\binom{n}{k} = \left(1 + O\left(\frac{1}{k}\right)\right) \frac{n^k}{\sqrt{2\pi k} k^k} \exp\left(k - \frac{k^2}{2n} - \frac{k^3}{6n^2} + O\left(\frac{k}{n} + \frac{k^4}{n^3}\right)\right).$$

Another approximation for the binomial coefficient is the following.

Lemma 2.6. *Let $n, k \in \mathbb{N}$. Then the following equation holds:*

$$\binom{n}{k} = \frac{n^{n+\frac{1}{2}}}{\sqrt{2\pi} k^{k+\frac{1}{2}} (n-k)^{n-k+\frac{1}{2}}} \left(1 + O\left(\frac{1}{n}\right)\right).$$

Proof. This formula is obtained immediately by using Stirling's formula three times in $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. □

From these approximations we get results for special binomial coefficients occurring throughout the thesis.

Corollary 2.7. *Let $n, k \in \mathbb{N}$. Then we have:*

$$\binom{\binom{n}{2}}{k} = \frac{n^{2k}}{\sqrt{2\pi k} (2k)^k} \exp\left(k - \frac{k}{n} - \frac{k^2}{n^2} + O\left(\frac{k}{n^2} + \frac{1}{k}\right)\right). \tag{4}$$

Proof. By using $\binom{n}{2} = \frac{n(n-1)}{2}$ in the previous lemma, we get:

$$\binom{\binom{n}{2}}{k} = \left(1 - O\left(k^{-1}\right)\right) \frac{n^k (n-1)^k}{\sqrt{2\pi k} (2k)^k} \exp\left(k - \frac{k^2}{n(n-1)} + O\left(\frac{k}{n(n-1)}\right)\right).$$

Using

$$\begin{aligned}(n-1)^k &= n^k \left(1-\frac{1}{n}\right)^k \\ &= n^k \exp\left(k\left(-\frac{1}{n}+O\left(\frac{1}{n^2}\right)\right)\right),\end{aligned}$$

and

$$\begin{aligned}\frac{k^2}{n(n-1)} &= \frac{k^2}{n^2}+\frac{k^2}{n^2(n-1)} \\ &= \frac{k^2}{n^2}+O\left(\frac{k^2}{n^3}\right),\end{aligned}$$

yields the claimed result. □

Another special case of a binomial coefficient is the following.

Corollary 2.8. *Let $n \in \mathbb{N}$. Then we have:*

$$\binom{2n}{n} = \left(1+O\left(n^{-1}\right)\right)\frac{4^n}{\sqrt{\pi n}}. \tag{5}$$

Proof. Using Stirling's formula (1), we have:

$$\begin{aligned}\binom{2n}{n} &= \frac{(2n)!}{(n!)^2} \\ &= \frac{\sqrt{2\pi}\,(2n)^{(2n)+\frac{1}{2}}\,e^{-2n}\left(1+O\left(n^{-1}\right)\right)}{\left(\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}\right)^2} \\ &= \left(1+O\left(n^{-1}\right)\right)\frac{4^n}{\sqrt{\pi n}}.\end{aligned}$$

□

2.2 Series-parallel graphs

In this section we will introduce series-parallel graphs, which will be our main object of interest throughout this thesis. We will first give several ways of defining series-parallel graphs. We will then define some substructures of series-parallel graphs which will play essential roles in counting series-parallel graphs.

The first way of defining series-parallel graphs is as follows.

Definition 2.2. *A terminated series-parallel graph is a graph with two special vertices* u *and* v*, called terminals defined inductively as follows:*

- $(\{u, v\}, \{(u, v)\})$ *is a terminated series-parallel graph with terminals* u*,* v*.*
- *If* $G_1 = (V_1 \cup \{u_1, v_1\}, E_1)$ *with terminals* u_1*,* v_1 *and* $G_2 = (V_2 \cup \{u_2, v_2\}, E_2)$ *with terminals* u_2*,* v_2 *are series-parallel graphs, then so is* $S = (V_1 \cup V_2 \cup \{u_1, v_1 = u_2, v_2\}, E_1 \cup E_2)$ *with terminals* u_1*,* v_2*, merging the two graphs on one of the terminals. (Series composition)*
- *If* $G_1 = (V_1 \cup \{u_1, v_1\}, E_1)$ *with terminals* u_1*,* v_1 *and* $G_2 = (V_2 \cup \{u_2, v_2\}, E_2)$ *with terminals* u_2*,* v_2 *are series-parallel graphs, then so is* $P = (V_1 \cup V_2 \cup \{u_1 = v_1, u_2 = v_2\}, E_1 \cup E_2)$ *with terminals* u_1*,* u_2*, merging the two graphs on both terminals. (Parallel composition)*
- *If* $G_1 = (V_1 \cup \{u_1, v_1\}, E_1)$ *with terminals* u_1*,* v_1 *and* $G_2 = (V_2 \cup \{u_2, v_2\}, E_2)$ *with terminals* u_2*,* v_2 *are series-parallel graphs, then so is* $S = (V_1 \cup V_2 \cup \{u_1, v_1 = u_2, v_2\}, E_1 \cup E_2)$ *with terminals* u_1*,* u_2*, merging the two graphs on one of the terminals. (Source merge)*

A graph is series-parallel, if it is a terminated series parallel graph for some two of its vertices as terminals.

This definition is also the reason for the name of this class of graphs. The last property is needed, if one wants to include trees to this class. It is also needed in the second characterization.

Theorem 2.9. *A graph* G *is series-parallel iff* G *does not contain* K_4 *as a minor (e.g. by Oxley in [29]).*

There is also a third way of defining series-parallel graphs, which gives a good way of proving properties of series-parallel graphs inductively.

Theorem 2.10. *A graph* G *is series-parallel iff starting from* G*,* K_2 *can be reached by iteratively applying one of the following three actions:*

- *replacing a multiple edge by a single edge between the same vertices,*
- *deleting a vertex of degree* 2 *with an edge connecting its two neighbours,*
- *deleting a vertex of degree* 1*.*

This follows directly from the minor characterizations of this graph class.

As we have said before, our aim is to count series-parallel graphs with n vertices and M edges, where M will be dependent on n. For this we will use the following definitions.

Definition 2.3. *Let $G = (V, E)$ be a series-parallel graph. Let $G' = (V', E')$ be a graph arising from G by recursively cutting away all vertices of degree one and the corresponding edges. Then G' is called the core of G, $G' = \mathrm{cr}(G)$.*

As can easily be seen, the minimum degree of the core is at least two.

Definition 2.4. *Let $G = (V, E)$ be a series-parallel graph. Let $G' = (V', E')$ be a multigraph arising from the core of G by replacing all vertices of degree two and its adjacent edges by one edge connecting its neighbours. Then G' is called the kernel of G, $G' = \ker(G)$.*

Obviously the kernel can have loops and multiple edges and has a minimum degree of at least three.

We will use this in the following way: At first we will calculate the number of possible kernels. Then we will put some additional vertices on its edges to get the core and finally we will add a rooted forest to the vertices to get all series-parallel graphs.

In order to get the number of kernels, we will need the following:

Definition 2.5. *Let $G = (V, E)$ be a series-parallel graph with $|V| = n$, $|E| = M$.*

- *If the kernel of G is 3-regular, G is called clean.*
- *The excess $\mathrm{ex}\,(G)$ is defined as $\mathrm{ex}\,(G) = M - n$.*
- *Suppose the kernel of G has k vertices and l edges. Then the deficiency $\mathrm{def}\,(G)$ of G is defined as $2l - 3k$.*

The deficiency is a measure of some sort of how far G is away from being clean, as seen in the next lemma.

Lemma 2.11. *Let G be as above. Then $\mathrm{def}\,(G) \geq 0$ and $\mathrm{def}\,(G) = 0 \Leftrightarrow G$ is clean.*

Proof. As the minimum degree in the kernel is 3, we have $2l = \sum_v d\,(v) \geq 3k$ with equality iff all vertices have degree 3 and hence G is clean. □

The next lemma gives a reason, as to why the excess will be needed in getting a connection between the kernel of a graph and the graph itself.

Lemma 2.12. *Let G be as above a graph where all connected components have positive excess. Then $\mathrm{ex}(G) = \mathrm{ex}(\ker(G))$. Furthermore a connected component with excess ≤ 0 has an empty kernel and as a consequence its kernel has excess zero. Such components are either trees or unicyclic components.*

Proof. Suppose a connected component has negative excess. Then it has fewer edges than vertices and is therefore a tree. As such it has no kernel, as iteratively deleting leaves results in an empty graph. If a connected component has excess zero, it has exactly one circle. Iteratively deleting vertices yields exactly this circle and again the kernel of this is the empty graph.

Let G be a connected graph with excess ≥ 1. Then G has at least two circles and its kernel is not empty. Then $ker(G)$ is constructed from G by iteratively deleting vertices with degree one and its adjacent edge, which does not change the excess, or by deleting a vertex with degree two, its two neighbouring edges and inserting another edge instead. This does not change the excess either. As a consequence the excess stays the same in all connected components and therefore in the whole graph. □

2.3 Random graphs

In this section we will introduce random graphs and some concepts and properties occurring with random graphs. To do this we will first formally define random graphs and show some basic properties. After that we will introduce the concept of phase transition in random graphs and will give some examples. For a more detailed overview, see for example [4].

In recent history there have been two different models of random graphs. In this section we will see some basic properties of these graphs and look at the differences in the concept. These models are:

- Suppose $n, M \in \mathbb{N}$. Then $G(n, M)$ is a graph with n vertices and M edges chosen uniformly at random from the set of all graphs with n vertices and M edges.
- Suppose $n \in \mathbb{N}$ and $0 < p < 1$. Then $G(n, p)$ is a graph on n vertices, such that for each pair u, v of vertices $\{u, v\}$ is an edge with probability p. These probabilities are independent for all possible edges.

The first of these was introduced by Erdős and Rényi in their paper [9] in 1960. The second is a well-known alternative model defined by Gilbert [12]. If $p \approx M\binom{n}{2}^{-1}$, meaning that the expected number of edges in $G(n, p)$ is M, then these two classes are expected to have similar properties. The exact result for this is for example given by Bollobás [4].

Theorem 2.13. *Let P be a property such that for all $A \subset B \subset C$ where A and C have P also B has P. Let also $p(1-p)\binom{n}{2} \to \infty$. Then then a.a.s. every graph in $G(n,p)$ has property P iff a.a.s. every graph in $G(n,M)$ has P where $M = p\binom{n}{2} + c\sqrt{p(1-p)\binom{n}{2}}$ with any fixed constant c.*

There are two different kinds of questions one might ask for such graphs. The first is about properties of $G(n, M)$ or $G(n, p)$ for fixed n and M or p. Questions of this kind are: Given n and M what is the probability of some property the graph might have, like the probability that the graph is connected. The other type of question reverses the roles in these questions. They are of the sort: Given some property and some value of n, what are the values of M as a function of n such that almost all or almost no graph in $G(n, M(n))$ do have this property as n tends to ∞? In this second category are for example the questions about the giant component given in the introduction to this thesis, as they are questions about the change of the largest component, if one changes the value of $M(n)$. Such questions lead to the definition of phase transitions and threshold functions.

Definition 2.6. *Let $\wp$ be a property, $f = f(n)$, $M = M(n)$ some functions of n and $P(n)$ the probability that $G(n, M(n))$ satisfies $\wp$. Furthermore suppose that the following two properties hold.*

- *If $\lim_{n\to\infty} \frac{M(n)}{f(n)} = 0$, then $lim_{n\to\infty} P(n) = 0$.*
- *If $\lim_{n\to\infty} \frac{M(n)}{f(n)} = \infty$, then $lim_{n\to\infty} P(n) = 1$.*

Then the function f is called a threshold function for property $\wp$.

There are many different questions concerning threshold functions. Some of them are shown for example in [31].

In this thesis we will work especially with phase transitions. The phase transition is a phenomenon observed in many fundamental problems in graph theory like, for example, graph coloring. The phase transition observed in different random graph models refers to a phenomenon that there is a critical value of edge density such that adding a small number of edges around the critical value results in a dramatic change in the size of the largest components. Usually one measures the edge density as the asymptotics of $M(n)$ when changing the number of edges in dependance of the vertices.

Probably one of the most well known phase transitions on random graphs is the occurance of the giant component. The exact result for this, as stated below, follows from papers of Erdős and Rény [9], Bollobás [3], Łuczak [21], Łuczak, Pitel and Wierman [24], Janson et al. [16] and Janson [15]

Theorem 2.14. *Let $G(n, M)$ be the class of all graphs with n vertices and M edges. If $M = \frac{n}{2} + s$ with $-n \ll s \ll -n^{2/3}$, then $G(n, M)$ consists of trees and unicyclic component with probability tending to 1 as n appraches ∞. The largest of these components is a tree of size $(1 + o(1))\frac{n^2}{2s^2} \log\left(\frac{|s^3|}{n^2}\right)$.*

In contrast, if $n^{2/3} \ll s \ll n$, then $G(n, M)$ contains exactly one component with more edges than vertices, which has a size of $(4 + o(1))s$. Furthermore for such a value of s this is the unique component with this size and all other components have a size of at most $O(n^{2/3})$.

This is a prime example of a phase transition. For $s \ll -n^{2/3}$ there are many small components, all of which have at most one cycle and no component is considerably bigger than the others. For $s \gg O(n^{2/3})$ this changes dramatically. After this point there is exactly one component which is asymptotically bigger than all other components and it has more edges than vertices. There have also been theorems about what happens for $s = cn^{2/3}$, especially by Łuczak [21]. Such a region where the properties change dramatically is called the critical phase or the critical point.

Similar results have been proven for a variety of graph classes. Kang and Łuczak [18] showed a similar result for planar graphs only differing in the size of the largest component after the phase transition, which they showed to be $(2 + o(1))s$. The same result as this has also been shown for random forests by Łuczak and Pittel [23]. In an other direction, Bollobás, Janson and Riordan [5] showed that there does also exist such a phase transition for inhmogenious graphs. Inhomogenious graphs are a class of random graphs where the degree distribution of all vertices is not uniform, but follows a power-law degree distribution. If one uses the model $G(n, p)$ for these graphs, one has to relax the condition of independence of edges. In this thesis, we will add another graph class with similar properties to this list.

2.4 Applied methods

In this section we will discuss two different methods used to get the number of series-parallel graphs in this thesis. First we will describe the symbolic method. This is a way to get algebraic equations for the number of combinatorial objects. After that, we will use singularity analysis to get asymptotic formulas for the combinatorial objects out of this algebraic equation.

The symbolic method is a way of finding (systems of) equations for generating functions of combinatorial objects. To do this, the symbolic method gives a set of rules, by which the combinatorial objects can be manipulated and how this manipulation can be translated into equations. In this thesis there will only be a short introduction to this method. For a more exact approach, see for example [10].

To do this, we need the following:

Definition 2.7. *Suppose $\mathcal{A}$ is a class of labelled combinatorial objects and $s : \mathcal{A} \to \mathbb{Z}_{\geq 0}$ is a function on these objects with $\forall n \in \mathbb{Z}_{\geq 0} : s^{-1}(n) < \infty$.*

Such a function is called a size function on $\mathcal{A}$. Let a_n be the number of objects of size n. Then

$$A(z) = \sum_{n \geq 0} \frac{a_n}{n!} z^n$$

is the exponential generating function of $\mathcal{A}$. Furthermore, let $[z^n]A(z) := \frac{a_n}{n!}$ be the coefficient of z^n in $A(z)$.

With this, we have the following concept.

Definition 2.8. *Let $\mathcal{P} \subset \mathcal{A}$ be two combinatorial classes. Then $\mathcal{P} = \mathcal{A}$ asymptotically almost surely (a.a.s.), if $\lim_{n \to \infty} \frac{p_n}{a_n} = 1$.*

This notion is also used in probability theory in the same way, as one can interpret $\frac{p_n}{a_n}$ as the probability that an object in $\mathcal{A}$ has property $\mathcal{P}$.
Easy classes of combinatorial objects, are the following.

Definition 2.9. *Let $\mathcal{E}$ be a set with one element of size zero. Then $\mathcal{E}$ is a combinatorial class called the neutral class. Let $\mathcal{Z}$ be a set with one element of size one. Then $\mathcal{E}$ is a combinatorial class called the atomic class.*

The symbolic method is a set of instructions of how to combine combinatorial classes and how the generating functions of such combinations look like. The most important of those are the following.

Definition 2.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be combinatorial classes. We have the following constructions:*

- *The disjoint union $\mathcal{A} + \mathcal{B}$.*
- *The labelled product $\mathcal{A} * \mathcal{B} = \sum_{\alpha \in \mathcal{A}, \beta \in \mathcal{B}} \alpha * \beta$. For this, let $\alpha * \beta$ be the set of all pairs (α', β') where its atoms get distinct labels from 1 to $n = s(\alpha) + s(\beta)$, such that the order of the labellings of α and β is preserved.*
- *The sequence $\operatorname{Seq}(\mathcal{A}) = \sum_{k \geq 0} \operatorname{Seq}_k(\mathcal{A})$ of elements of $\mathcal{A}$ where $\operatorname{Seq}_k(\mathcal{A}) = \mathcal{A} * \cdots * \mathcal{A}$ is the labelled product of k copies of $\mathcal{A}$.*
- *The set $\operatorname{Set}(\mathcal{A}) = \operatorname{Seq}(\mathcal{A}) / R$ where R is the set of all permutations. Therefore this is a collection of a finite number of copies of $\mathcal{A}$ without regarding any order.*

For these we get the following equations:

Theorem 2.15. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be combinatorial classes. Then we have the following rules.*

1. *If $\mathcal{C} = \mathcal{E}$, then $C(z) = 1$.*
2. *If $\mathcal{C} = \mathcal{Z}$, then $C(z) = z$.*
3. *If $\mathcal{C} = \mathcal{A} + \mathcal{B}$, then $C(z) = A(z) + B(z)$.*
4. *If $\mathcal{C} = \mathcal{A} * \mathcal{B}$, then $C(z) = A(z) \cdot B(z)$.*
5. *If $\mathcal{C} = \mathrm{Seq}(\mathcal{A})$, then $C(z) = \sum_{n\geq 0} A(z)^n = \frac{1}{1-A(z)}$.*
6. *If $\mathcal{C} = \mathrm{Set}(\mathcal{A})$, then $C(z) = \sum_{n\geq 0} \frac{A(z)^n}{n!} = \exp(A(z))$.*

There can be modifiers for Set and Seq of the form Set_{Rc}, where R is some order relation, usually "$\leq$", "$\geq$"or "$=$". These relate to a set of n copies of the class, where nRc. These modifiers translate directly to bounds of the corresponding summation in the following form.

Corollary 2.16. *With the same conditions as above, we have:*

- *If $\mathcal{C} = \mathrm{Seq}_{Ra}(\mathcal{A})$, then $C(z) = \sum_{nRa} A(z)^n$,*
- *if $\mathcal{C} = \mathrm{Set}_{Ra}(\mathcal{A})$, then $C(z) = \sum_{nRa} \frac{A(z)^n}{n!}$.*

To show the use of this construction, consider the following example.

One wants to find the number of all labelled binary trees with exactly n vertices. Let $\mathcal{B}$ be the combinatorial class and $B(z)$ it's generating function. A vertex can either be a binary tree in itself (the atomic class) or (disjoint union) it has two sorted neighbours 'left' and 'right' (labelled product or a sequence of exactly two elements) of binary trees. Using the symbolic method, we get:

$$\mathcal{B} = \mathcal{Z} + \mathcal{Z} * \mathcal{B} * \mathcal{B}.$$

So, for the generating function, we have:

$$B(z) = z + z \cdot B(z)^2 . \tag{6}$$

Now one can use the binomial theorem to get exact formulas for this function. If the equation obtained from this method is not as easy as in this case, one can use the method described in the next chapter to obtain at least asymptotic bounds for the coefficients of the generating function. Using this we get an algebraic equation for the generating function. We want to find an asymptotic value of coefficients for implicitly given functions. For this, we can use singularity analysis.

Singularity analysis is a way to find the asymptotics of the coefficients of a power series $s(z) = \sum_{n=0}^{\infty} \frac{s_n}{n!} z^n$, if the power series is implicitly given. Here it will be shown how an expression of the form $s_n = g\gamma^n n^\alpha (1 + o(1))$ can be derived given some polynomial $p(z, s)$ in z and $s(z) = \sum_{n \geq 0} \frac{s_n}{n!} z^n$. We will use this to get the asymptotics of $B(z)$ from equation (6). To obtain the desired results, we will follow the book Analytic Combinatorics from Flajolet and Sedgewick [10] and in particular chapter VII.7.1. These calculations will be done in different phases. First we will determine the value of γ. From this we will get some conditions for the value α and simultaneously get an algebraic equation for g. As a last step, we will conduct some calculations to derive the explicit values of α and g from these conditions. To get the exponential factor γ for $s(x)$, we will need the notion of an analytic function.

Definition 2.11. *Let $s(z)$ be a function over a region Ω and $z_0 \in \Omega$. Then s is called analytic in z_0, if $s(z)$ is representable by a convergent power series expansion:*

$$s(z) = \sum_{n \geq 0} c_n (z - z_0)^n .$$

If $s(z)$ is analytic for all $z \in \Omega$, s is called analytic.

With this, there is the following well known theorem [10]:

Theorem 2.17. *Suppose $f(z)$ is analytic at the origin and*

$$R = \sup\{r \geq 0 : f \text{ is analytic for } |z| < r\}. \tag{7}$$

Then the coefficients $f_n = [z^n]f(z)$ of f satisfy $f_n = R^{-n}\theta(n)$, where

$$\limsup_{n \to \infty} |\theta|^{\frac{1}{n}} = 1. \tag{8}$$

If in addition $f_n \geq 0$ for all n, then

$$R = \sup\{r \geq 0 : f \text{ is analytic in } [0, r)\}. \tag{9}$$

From this theorem we can conclude, that we have to find the smallest positive value for which our function is not analytic. Such points are the so-called singularities.

Now we want to get all the singularities of the polynomial $p(x, s)$. Suppose the degree of p with respect to s is d. So:

$$p(z, s) = \sum_{i=0}^{d} s^i p_i(z),$$

where $p_i(z)$ are polynomials in z.

In general $p(z, s)$ has d different values for some fixed z. Values for z in which there are not d different values of $p(z, s)$ in $\mathbb{C}$ can give rise to singularities. Consequently we have the following lemma.

Lemma 2.18. *Suppose for some z_0 that $p(z_0, s) = 0$ has d different solutions $(s_1, \ldots, s_d)$. Then there exists some disc around (z_0, s_i) in which $p(z, s)$ is analytic.*

Proof. For this we use the following statement: Suppose $q(y)$ is a polynomial and y_0 is a root. Then $q'(y) = 0$, iff y_0 is a root with multiplicity at least 2. From this statement and as the polynomial $p(z_0, s)$ has d different roots, we can conclude that $\frac{\partial p}{\partial s}(z_0, s)$ is not zero at $s = s_i$. Hence we can use the implicit function theorem and get $y_i = y_i(z)$ is an analytic function in some neighbourhood of (z_0, y_i). □

Let us consider the points where there are less than d roots. Such points z_0 satisfy one of the following two properties. There can be $p_d(z_0) = 0$. In this case the degree of the polynomial would be reduced and we would have fewer roots. The other possibility would be that two roots are the same, therefore there is a root with multiplicity ≥ 2. In this case the derivative of the polynomial at that point is also zero, as seen in the previous proof. Accordingly, we have to look for points, where either $p_d(z)$ is zero or $p(z, s)$ and $\frac{\partial p}{\partial s}(z, s)$ are zero simultaneously. Values of z satisfying at least one of these conditions are exactly the zeros of the discriminant of $p(z, s)$ as a polynomial in s [10]. Such values are called exceptional points.

Proposition 2.19. *The exceptional values for the polynomial* (6) *are $z \in \{0, 0.5, -0.5\}$.*

Proof. The discriminant of the polynomial is equal to $1 - 4z^2$. The roots of this polynomial are ± 0.5. Additionally, the coefficient of B^2 is zero, if $z = 0$. □

The corresponding values of s are either points at infinity, if $p_d(x) = 0$ or else the value that is a root of multiplicity at least two of $p(x_0, s)$. By construction such a value of s does exist.

Proposition 2.20. *The corresponding values for c to the values of z above are $\{\infty, 1, -1\}$ respectively.*

Proof. If $z = 0$ the corresponding polynomial is B resulting in the root zero. As a consequence, we have just $p_d(z_0, s) = 0$ and the corresponding value of B lies at infinity. For the other values of z, the degree of the polynomial stays two. Calculating all the solutions does yield one solution with multiplicity two in each case with the values stated as above. □

As was first discovered by Newton in his book "De Methodis Serierum et Fluxionum"[26], and later further developed by Puiseux [30], a polynomial at such a singular point does admit a series expansion of the following form.

Theorem 2.21 (Newton-Puiseux)**.** *Let $p(x, s)$ be a polynomial and $(0, 0)$ a point as above. Then there exist some values $\kappa \in \mathbb{Z}^+$, $k_0 \in \mathbb{Z}$ and $c_k \in \mathbb{C}$ such that*

$$s(x) = \sum_{k \geq k_0} c_k x^{\frac{k}{\kappa}}, \tag{10}$$

and $s(0) = 0$. This series expansion is locally convergent.

In this expansion κ is called the *branching type*. If the point of interest is not $(0, 0)$, one can use the translation $z = Z + z_0$ and $s = S + s_0$, if $s_0 < \infty$ or $S = \frac{1}{s}$, if $s_0 = \infty$. With this some arbitrary singularity z_0, s_0 translates to the origin and the series expansion from above can also easily be transposed back.

The next step is to find the Puiseux-series expansion in the origin. For this we suppose $s(z) = az^{\alpha}(1 + o(1))$. This will correspond to the first term in the Puiseux-expansion. Replacing this in the polynomial, we get some set of exponents for z depending on α. As we are near the origin the main contribution will be from terms with small exponents. Because all coefficients are not zero, we have to have cancellation. We want to find some value for α such that any two of the exponents have the same value and all other exponents are bigger or equal. Such a value can be found for example by calculating α for each pair and then check which value yields the desired result. Now the coefficients of the terms where cancellation has to take place have to add up to zero. This gives an algebraic equation for a. Each of the different solutions corresponds to a value s with $p(0, s) = 0$. Finally one can subtract this leading term from s and iterate to obtain the next terms in the expansion.

Proposition 2.22. *At $(0.5, 1)$ the Puiseux-expansion of the polynomial (6) has a main term of $\pm 2(x - 0.5)^{\frac{1}{2}}$.*

We will later use the fact that c is the generating function of some combinatorial object and as such its coefficients are positive to determine whether the sign in this expression should be $+$ or $-$.

Proof. At first, we have to transform the polynomial by setting $B = B_1 + 1$ and $z = Z + 0.5$. From this we get the following polynomial:

$$p(Z, B_1) = \frac{B_1^2}{2} + 2Z + 2B_1 Z + B_1^2 Z.$$

By replacing B_1 by aZ^α, one can easily see that the main contribution is set by the terms B_1^2 and Z and we get $\alpha = \frac{1}{2}$ as branching type. Furthermore for cancellation, we need that the two coefficients sum up to zero, therefore $\frac{a^2}{2} - 2 = 0$ or $a = \pm 2$.

□

In this thesis we will only work with power series emerging from generating functions of combinatorial objects. Hence we can use the following theorem.

Theorem 2.23. *If s is the generating function of some combinatorial objects, then the main contribution to the asymptotics of its coefficients s_n is attained by the corresponding coefficients from expanding the leading term of the Puiseux-expansion around the smallest positive singularity.*

Proof. As s is counting combinatorial objects, $s_n \geq 0$ for all n, we know from Theorem 2.17 that

$$s_n = R^{-n}\theta(n),$$

where R is the smallest positive singularity. The main term of $\theta(n)$ is obtained at the main term of the Puiseux-expansion in this singularity. □

From this we get:

Theorem 2.24. *The coefficient c_n for c satisfying the polynomial* (6) *is*

$$c_n = a\gamma^n n^{-\frac{3}{2}} \tag{11}$$

where $a = 0.434$ and $\gamma = \frac{1}{0.418} = 2.392$.

Proof. As we have stated in the previous theorem, we will get the main contribution from the main term in the Puiseux-expansion at the smallest positive singularity. We have seen that this singularity is at $\rho = 0.5$. At this point, the main term of the Puiseux-expansion is $\pm 2\sqrt{\rho - z} = \pm 2 (\rho - z)^{\frac{1}{2}}$. From this, using the binomial theorem, we get:

$$\begin{aligned}
[z^n]B(z) &= [z^n] \pm 2\sqrt{\rho - z} \\
&= [z^n] \pm 2 \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} (-z)^i (\rho)^{\frac{1}{2}-i} \\
&= \pm 2 \binom{\frac{1}{2}}{n} (-1)^n \rho^{\frac{1}{2}-n} \\
&= \pm 2 (-1)^{n+1} \frac{1}{4^n (2n-1)} \binom{2n}{n} (-1)^n \rho^{\frac{1}{2}-n} \\
&= - \pm 2\rho^{\frac{1}{2}} \frac{1}{4^n (2n-1) \rho^n} \binom{2n}{n}.
\end{aligned}$$

As this coefficient has to be positive, we see that the sign of the main term in the Puiseux-expansion has to be negative. Also, using equation (5) for this binomial coefficient, we get:

$$\begin{aligned}
b_n &= \frac{2}{\sqrt{2}} \frac{1}{4^n (2n-1) 2^{-n}} \frac{4^n}{\sqrt{\pi n}} (1 + o(1)) \\
&= \sqrt{\frac{2}{\pi}} \frac{1}{2n^{\frac{3}{2}}} 2^n (1 + o(1)) \\
&= \frac{1}{\sqrt{2\pi}} n^{-\frac{3}{2}} 2^n (1 + o(1)).
\end{aligned}$$

□

3 Counting series-parallel graphs

In this chapter we will get the asymptotic number of series-parallel graphs. To do this, we will use the structure we saw in Chapter 2.2. The following steps will be made to get to the final result. First we will recall the number of trees and unicyclic components in Section 3.1. Then we will count the number of complex graphs. To do this, we will first count all 3-regular graphs in Section 3.2. From them we will derive an estimate for the kernel in Section 3.3. By using combinatorial arguments, we will derive the number

$C(n, M)$ of complex series-parallel graphs from this number in Section 3.4. Finally we will calculate the asymptotics for $sp(n, M)$ to get the following result.

Theorem 3.1. *Suppose $n \in \mathbb{N}$, $M = \alpha n + s$ with $\alpha \leq 1$ and $s \ll n$. Then the following holds.*

- *If $\alpha < \frac{1}{2}$ or if $\alpha = \frac{1}{2}$ and $s \ll -n^{2/3}$, then*

$$sp(n, M) = (1 + o(1))\binom{\binom{n}{2}}{M} = (1 + o(1))\frac{n^{n+2s}e^{\frac{n-1}{2}+s}}{\sqrt{\pi}(n + 2s)^{\frac{n+1}{2}+s}}. \quad (12)$$

- *If $\alpha = \frac{1}{2}$ and $s = cn^{2/3}$ for some constant c, then*

$$sp(n, M) = (\tau(c) + o(1))\binom{\binom{n}{2}}{M} = (\tau(c) + o(1))\frac{n^{n+2s}e^{\frac{n-1}{2}+s}}{\sqrt{\pi}(n + 2s)^{\frac{n+1}{2}+s}}, \quad (13)$$

  where $\tau(c)$ is analytically given and can be explicitly calculated for any given c.

- *If $\alpha = \frac{1}{2}$ and $s \gg n^{2/3}$, then*

$$\begin{aligned} sp(n, M) = &(1 + o(1))\, 2^{-\frac{5}{2}} 3^{\frac{3}{2}} \pi^{-\frac{1}{2}} \gamma^{-2} g I n^{n+\frac{7}{6}} (n - 2s)^{-\frac{n}{2}+s} s^{-\frac{5}{2}} \\ &\times \exp\left(\frac{n}{2} - s - \frac{3}{4} + \gamma^{\frac{4}{3}} s n^{-\frac{2}{3}}\right), \end{aligned} \quad (14)$$

  where γ, g and I are explicitly given constants.

- *If $\frac{1}{2} < \alpha < 1$, then*

$$sp(n, M) = \theta(1)\, n^{\alpha n - \frac{4}{3}} \left(\frac{e}{n(1 - 2\alpha)}\right)^{\left(\frac{1}{2}-\alpha\right)n} \exp\left(\gamma^{\frac{4}{3}} \alpha n^{\frac{1}{3}}\right). \quad (15)$$

- *If $\alpha = 1$ and $s \ll -n^{3/5}$, then*

$$\begin{aligned} sp(n,M) = &\, C n^{n-\frac{1}{2}} \frac{(2(l_0 - s))^{s-1/6}}{\sqrt{5l_0 - 3s}} l_0^{-3/2} s^{l_0} (l_0 - s)^{-l_0} \\ &\times \exp\left(\frac{5l_0}{2} - s - \frac{3l_0^2}{n + 2s} + B\sqrt{\frac{l_0^3}{n - 2(l_0 - s)}} + O\left(\frac{l_0^2}{n}\right)\right), \end{aligned} \quad (16)$$

  where $l_0 = D\frac{n+2s}{s^{2/3}}$ and B, C and D are given constants.

- *If $\alpha = 1$ and $s = cn^{3/5}$ for some constant c, then*

$$sp\left(n, M\right) = C_0 n^{n-\frac{1}{2}} s^{s-11/6} \exp\left(C_1 s - C_2 \frac{s^2}{n} + B\sqrt{\frac{s^3}{n}} + O\left(n^{-1/5}\right)\right), \tag{17}$$

for some values C_0, C_1 and C_2 depending only on c and some constant B.

- *If $\alpha = 1$ and $n^{3/5} \ll s \ll n^{2/3}$, then*

$$\begin{aligned} sp(n,M) = {} & C_0 n^{n-\frac{1}{2}} \frac{(2z)^{s+1/6}}{\sqrt{5z+2s}} \left(z+s\right)^{-3/2} s^{3(z+s)/2} \left(z+s\right)^{-3(z+s)/2} \\ & \times \exp\left(\frac{5z+3s}{2} - \frac{3\left(z+s\right)z}{n} + B\sqrt{\frac{(z+s)^3}{n-2z}} + O\left(\frac{(z+s)^2}{n-2z}\right)\right), \end{aligned} \tag{18}$$

where $z = D\left(\frac{n}{s}\right)^{3/2}$ and B, C_0 and D are given constants.

For getting these results, we will consider the following graph classes.

Definition 3.1. *Let $\mathcal{U}$ be the class of graphs with at most one cycle and $\mathcal{S}_P$ the class of all series-parallel graphs. Let $U\left(n, M\right)$ be the number of graphs in $\mathcal{U}$ with exactly n vertices and M edges and $sp(n, M)$ be the number of series-parallel graphs with n vertices and M edges. Furthermore let $C\left(n, M\right)$ be the number of graphs in $\mathcal{S}_P$ with n vertices and M edges, where no connected component is in $\mathcal{U}$. Such graphs are called complex.*

These classes can be used to calculate the number $sp(n, M)$ in the following way.

Lemma 3.2. *Let $n, M \in \mathbb{N}$. Then:*

$$sp\left(n, M\right) = \sum_{k,l} \binom{n}{k} C\left(k, k+l\right) U\left(n-k, M-k-l\right). \tag{19}$$

Proof. Suppose k vertices and $k + l$ edges of a series-parallel graph are in its complex components. As each complex component has at least 1 more edges than vertices, l is positive. Furthermore, as the vertices are labelled, we have to choose the k vertices in one of $\binom{n}{k}$ ways. Furthermore, we get the complex components in one of $C\left(k, k+l\right)$ ways and the simple component in $S\left(n-k, M-k-l\right)$ ways, as all simple components are series-parallel.

Now summing up over all k and l yields the result. □

In the next section we will state some results on the number $U(n,M)$ and in chapter 3.4 we will give the following estimate for $C(k,k+l)$:

$$C(k,k+l) = 2^{-3} g\gamma^{2l} l^{-\frac{5}{2}} k^{k+\frac{3l-1}{2}} e^{\frac{3l}{2}} \exp\left(O\left(\frac{l^2}{k}+\frac{1}{l}\right)\right) \times \exp\left(\beta\sqrt{\frac{l}{k}} + O\left(\frac{l^2}{k}\right)\right).$$

3.1 Series-parallel graphs with at most one cycle

In this section, we will calculate the number $U(n,M)$ as follows.

Theorem 3.3. *[6] Let $n, M \in \mathbb{N}$. Then:*

$$U(n,M) = \frac{\rho(n,M) n^{2M}}{\sqrt{2\pi M}\,(2M)^M} \exp\left(M - \frac{M}{n} - \frac{M^2}{n^2} + O\left(\frac{M}{n^2}+\frac{1}{M}\right)\right), \quad (20)$$

where $0 < \rho(n,M) < 1$ is an explicitly given function.

The number $U(n,M)$ has been studied for example by Britikov in [6]. There he showed the following:

Theorem 3.4. *Let $\rho(n,M)$ be the probability that a random graph with n vertices and M edges is simple and let $M = \frac{n}{2} + s$, then*

- *if $s^3 n^{-2} \to -\infty$, then $\rho(n,M) = 1 + O(n^2 |s^{-3}|)$,*
- *if $s^3 n^{-2} \to c$ for some constant $c \in \mathbb{R}$, then $\rho(n,M) = (1+o(1))\,\nu(c)$,*
- *if $s^3 n^{-2} \to \infty$, then $\rho(n,M) \le \exp(-s^3 n^{-2})$,*

where

$$\nu(c) = \sqrt{\frac{2}{3\pi}} e^{-\frac{4}{3}c} \sum_{r=0}^{\infty} \frac{(-9c)^{\frac{r}{3}}}{r!} \Gamma\left(\frac{2r}{3}+\frac{1}{2}\right) \cos\frac{r\pi}{3}. \quad (21)$$

This probability can also be written as

$$\rho(n,M) = \frac{U(n,M)}{G(n,M)}.$$

This is the number of simple graphs divided by the number of all graphs, which is also given by $\binom{\binom{n}{2}}{M}$, which is choosing M pairs from the possible $\binom{n}{2}$. Then we can write

$$U(n,M) = \rho(n,M)\binom{\binom{n}{2}}{M}. \tag{22}$$

By using equation (4), we get the result claimed in Theorem 3.3.

The important facts about the function (21) are the following.

Proposition 3.5. *$\nu(c)$ is monotonously decreasing with* $\lim_{c\to-\infty}\nu(c) = 1$ *and* $\nu(c) \le \exp\left(-\frac{(4+o(1))}{3}c\right)$.

This means, we have a phase transition at $M = \frac{n}{2} + cn^{\frac{2}{3}}$, as for $c \to \infty$ $G(n,M)$ is almost surely simple and $G(n,M)$ almost surely has a complex component, if $c \to \infty$. In the next sections we will see that this is also true for series-parallel graphs.

3.2 3-regular series-parallel graphs

From now on we will try to count all complex series-parallel graphs. To this end, we will follow the steps taken in [18] to count planar graphs. As a first step we will count the number of 3-regular weighted complex labelled series-parallel multigraphs with n vertices and $M = \frac{3}{2}n$ edges. Based on the definitions in the first chapter this is exactly the number of clean kernels. Additionally we will have to use weights for the graphs. As we want to put vertices on edges to obtain the core of a graph, we would have to distinguish between all edges in the kernel. But as the kernel is a multigraph, we cannot distinguish parallel edges or the direction of loops. To compensate this, we have to give weights to such graphs.

Definition 3.2. *Suppose a clean kernel has f_1 loops, f_2 pairs of double edges and f_3 triple edges. Then the weight w of the graph is $2^{-f_1-f_2}6^{-f_3}$.*

These weights will exactly cancel out the different possibilities to select one edge or one direction for each of these cases. We will get the following result.

Theorem 3.6. *Let $r(n)$ be the number of 3-regular weighted complex labelled series-parallel multigraphs with n vertices. Then*

$$r(n) = (1 + O(n^{-1}))gn^{-\frac{5}{2}}\gamma^n n!,$$

where g and γ are analytically given. This asymptotics does also hold for r_c, the number of connected graphs with these properties for some other constant g_c.

To compute the asymptotics of the number of weighted series-parallel multigraphs, we will use the symbolic method for finding a system of equations for the generating function. By solving this, we will get a polynomial equation in the generating function, from which we will derive its asymptotics.

First we will look at the number of 3-regular connected, weighted, series-parallel multigraphs.

Definition 3.3. *Let $r_c(n)$ be the number of 3-regular connected weighted series-parallel multigraphs on n vertices and*

$$R_c(z) = \sum_{n\geq 0} r_c(n) \frac{x^n}{n!}$$

its exponentially generating function (EGF).

As we want to use the symbolic method, we will count instead the following graphs.

Definition 3.4. *Let $\mathcal{C}$ be the class of all 3-regular rooted connected weighted series-parallel multigraphs. Such a graph is a connected graph $G = (V, E)$ as above with one special directed edge $(s, t) \in E$. This edge is called the root of the graph. Furthermore, the graph has weight $w(G) = 2^{-f_1 - f_2} 6^{-f_3}$, where f_1 is the number of loops, f_2 is the number of double edges and f_3 is the number of triple edges. Let $C(z) = \sum_{n\geq 0} c(n) \frac{x^n}{n!}$ be their generating function.*

As each edge can be the root and it can be directed in two directions, we have $c(n) = 2\frac{3n}{2} r_c(n)$. From this, we have $C(z) = 3x \frac{dR_c}{dz}(z)$. For using symbolic method, we will distinguish what sort of edge will be the root. Similar to [18] we will get four different cases:

Definition 3.5. *Let $e = (s, t)$ be the root of $G = (V, E)$. Then we call the graph a*

(i) b-graph, if $s = t$,

(ii) d-graph, if $G - e$ is not connected,

(iii) s-graph, if $G - e$ is connected, but there exists another edge $f \in E$ such that $G - \{e, f\}$ is not connected,

(iv) p-graph, if there is no cut edge in $G - e$, but (s, t) has another edge connecting them or $G - \{s, t\}$ is not connected.

e is then called a $b-$, $d-$, $s-$ or $p-$edge respectively. Also let $B(x)$, $D(x)$, $S(x)$ and $P(x)$ be their generating functions and $\mathcal{B}$, $\mathcal{D}$, $\mathcal{S}$ and $\mathcal{P}$ the corresponding graph classes respectively.

The last case of $h-$roots discussed in [18] cannot occur in series-parallel graphs.

Lemma 3.7. *Let $e = (s,t)$ be the root of $G = (V,E)$. Then e is in one of the categories above.*

Proof. Suppose e is an edge in G not contained in one of the categories above. Then $G-e$ is connected, there is no other edge f such that $G-\{e,f\}$ is not connected, and $G-\{s,t\}$ is connected.

As there is no such f, one can conclude, that $G-e$ remains 2-edge-connected and as all vertices have degree at most 3, this implies that there are two vertex-disjoint paths W_1 and W_2 from s to t in $G-e$. Furthermore, as e is a single edge (otherwise e would be a $p1$-edge), both paths have length at least 2. As $G-\{s,t\}$ is connected, there exists a connection between the two paths. Let $v_1 \in W_1$ and $v_2 \in W_2$ be vertices such, that there is a path between them edge-disjoint from W_1 and W_2. Using this, we have edge-disjoint paths connecting s, t, v_1 and v_2. We found a K_4-minor. This is a contradiction to G being series-parallel. This proves that such an edge cannot exist.

□

As we stay in the same connected component as at the beginning, this only counts connected series-parallel multigraphs.

From all this, we can get the following equations in a similar way to [18] and [27].

Theorem 3.8. *The following equations hold.*

$$C(z) = 3z\frac{dR_c}{dz}(z) = B(z) + D(z) + S(z) + P(z)$$
$$B(z) = (D(z) + S(z) + B(z) + P(z))\frac{z^2}{2} + \frac{z^2}{2}$$
$$D(z) = \frac{B(z)^2}{z^2}$$
$$S(z) = (B(z) + S(z) + P(z))(B(z) + P(z))$$
$$P(z) = z^2(B(z) + S(z) + P(z)) + \frac{z^2}{2}(B(z) + S(z) + P(z))^2 + \frac{z^2}{2}$$

Proof. As we have seen, the Class $\mathcal{C}$ is given as the disjoint union over the classes $\mathcal{B}$, $\mathcal{D}$, $\mathcal{S}$ and $\mathcal{P}$, implying

$$\mathcal{C} = \mathcal{B} + \mathcal{D} + \mathcal{S} + \mathcal{P}. \tag{23}$$

Also, as there are $\frac{3}{2}(2n)$ edges and each of them can be the root. We know that the number of unrooted graphs is $3n$ times the number of rooted graphs. As multiplication by n translates to taking the derivative and multiplying z, we get the first equation.

Suppose the graph is in class $\mathcal{B}$. Then the root e has exactly one neighbouring point v_0 ant this point has one other neighbour v_1 which again has two other exiting edges. There are the following cases. First these two edges might be the same, meaning they are a loop at v_1. Then this is the whole graph and it builds the class $Set_{=2}(\mathcal{Z})$. Otherwise, we can replace these edges by one new root edge between the other two neighbours of v_1. From this we get a labelled product of a set of two vertices and one arbitrary graph in $\mathcal{C}$. From this, we have:

$$\mathcal{B} = \mathrm{Set}_{=2}(\mathcal{Z}) + \mathrm{Set}_{=2}(\mathcal{Z}) * \mathcal{C}.$$

This yields the second equation in the theorem by using equation (23).

Let us suppose we are in the class $\mathcal{D}$. Then we can separate the root by including two new vertices, connecting them with the two endpoints of e and putting a loop as $b-$root on these vertices. Hence we have here:

$$\mathrm{Seq}_{=2}(\mathcal{Z}) * \mathcal{D} = \mathcal{B}.$$

Here we need a sequence as the root has a direction and therefore we can distinguish the two new vertices. From this we get the third formula.

As a next step suppose e is an $s-$root. Then there exists some other edge e' such that $G - \{e, e'\}$ is not connected. Let v, v' be the endpoints of e and e' in one component respectively and w, w' the endpoints in the other component. Take then two new edges (v, v') and (w, w') as roots for the two parts with this direction. As to make this decomposition unique, take the first possible of these edges in direction of e. In this case one can see that (v, v') cannot be a new $s-$root. From this we can deduce a partition (labelled product) of $\mathcal{S}$ as one part being either in $\mathcal{B}$ or in $\mathcal{P}$ and the other being either in $\mathcal{B}$, in $\mathcal{P}$ or in $\mathcal{P}$. This yields the following:

$$\mathcal{S} = (\mathcal{B} + \mathcal{P}) * (\mathcal{B} + \mathcal{P} + \mathcal{S}).$$

Finally, suppose e is a $p-$root. Then we have the following cases. Suppose e is in a triple-edge. Then these two vertices is the whole graph and we have the class $\mathrm{Set}_{=2}(\mathcal{Z})$. If e is a double edge (u,v), then u and v have another neighbour u' and v' respectively. These two points can also be the same. As a new root take a new edge from u' to v' and delete u and v. This edge can be a $b-$, $s-$ or $p-$root. Using the direction of the root, we have the class $\mathrm{Seq}_{=2}(\mathcal{Z}) * (\mathcal{B}+\mathcal{P}+\mathcal{S})$. Lastly, if $e=(u,v)$ is a single edge, we have that $G-\{u,v\}$ has two connected components. The corresponding neighbours of u are u' and u'' and of v are v' and v''. Take a new root edge in each of the two components as (u',v') and (u'',v'') respectively. With this, we can delete the vertices u and v. As both edges can be $b-$, $s-$ or $p-$roots, we get another part of $\mathrm{Set}_{=2}(\mathcal{B}+\mathcal{P}+\mathcal{S})$. Combining these three as a disjoint union yields:

$$\mathcal{P}=\mathrm{Set}_{=2}(\mathcal{Z})+\mathrm{Seq}_{=2}(\mathcal{Z}) * (\mathcal{B}+\mathcal{P}+\mathcal{S})+\mathrm{Set}_{=2}(\mathcal{B}+\mathcal{P}+\mathcal{S})$$

and from this we get the last equation.

□

Let $H(z)=\sum_{n\geq 0}\frac{h_n}{n!}z^n=B(z)+S(z)+P(z)$. From this and the above system of equations one gets:

$$\begin{aligned}0 &= 15H(z)^4z^4+6H(z)^5z^4+H(z)^6z^4+H(z)^3z^2\left(8+20z^2\right)\\ &\quad -H(z)\left(8-24z^2-6z^4\right)-H(z)^2\left(4-24z^2-15z^4\right)+z^2\left(8+z^2\right). \quad (24)\end{aligned}$$

Using singularity analysis on this equation, we will get:

$$\frac{h_n}{n!}=0.434n^{-\frac{3}{2}}2.392^n\left(1+o\left(1\right)\right).$$

.

Theorem 3.9. *The asymptotics of the coefficient $r_c(n)$ of*

$$R_c(z)=\sum_{n\geq 0}r_c(n)\frac{z^n}{n!}$$

is given by

$$\frac{r_c(n)}{n!}=\alpha n^{-\frac{5}{2}}\beta^n\left(1+o\left(1\right)\right),$$

where $\beta\approx 2.392$.

Proof. To show this, we will start with $H(z)$ and equation (24). The discriminant of this Polynomial is equal to $65536(-256z^{12}+7776z^{16}+19683z^{20})$. The roots of this polynomial are 0 with multiplicity twelve and $\{0.418, -0.418, 0.418i, -0.418i, 0.571+0.571i, 0.571-0.571i, -0.571+0.571i, -0.571-0.571i\}$ each with multiplicity one. The smallest positive value of these is $z = 0.418$, resulting in a double root at $H(0.418) = 0.612$. By setting $H := H + 0.612$ and $z := z + 0.418$, we get the following polynomial.

$$
\begin{aligned}
p(z, H) = {} & 5.856H^2 + 3.958H^3 + 1.191H^4 + 0.296H^5 + 0.031H^6 - 33.141z \\
& - 71.225Hz - 61.949H^2z - 31.175H^3z - 11.393H^4z - 2.827H^5z \\
& - 0.292H^6z + 51.896z^2 + 130.823Hz^2 + 144.881H^2z^2 + 95.846H^3z^2 \\
& + 40.875H^4z^2 + 10.144H^5z^2 + 1.049H^6z^2 - 29.329z^3 - 109.175Hz^3 \\
& - 169.331H^2z^3 - 140.071H^3z^3 - 65.176H^4z^3 - 16.174H^5z^3 \\
& - 1.672H^6z^3 + 17.537z^4 + 65.280Hz^4 + 101.250H^2z^4 + 83.754H^3z^4 \\
& + 38.971H^4z^4 + 9.671H^5z^4 + H^6z^4
\end{aligned}
$$

By replacing H by az^α, one can easily see that the main contribution is set by the terms H^2 and z and as a consequence we get $\alpha = \frac{1}{2}$ as branching type. Furthermore for cancellation, we need that the two coefficients sum up to zero, so $5.856a^2 - 33.141 = 0$ or $a = \pm 2.379$.

From this, using the binomial theorem and Theorem 2.21, we get:

$$
\begin{aligned}
[z^n]H(z) &= [z^n] \pm 2.379\sqrt{\rho - z} \\
&= [z^n] \pm 2.379 \sum_{i=0}^{\infty} \binom{\frac{1}{2}}{i} (-z)^i (\rho)^{\frac{1}{2}-i} \\
&= \pm 2.379 \binom{\frac{1}{2}}{n} (-1)^n \rho^{\frac{1}{2}-n} \\
&= \pm 2.379 (-1)^{n+1} \frac{1}{4^n (2n-1)} \binom{2n}{n} (-1)^n \rho^{\frac{1}{2}-n} \\
&= - \pm 2.379 \rho^{\frac{1}{2}} \frac{1}{4^n (2n-1) \rho^n} \binom{2n}{n}.
\end{aligned}
$$

As this coefficient has to be positive, we see that the sign of the main term in the Puiseux-expansion has to be negative. Also, using equation (5) for this binomial coefficient, we get:

$$
\begin{aligned}
\frac{h_n}{n!} &= 1.538 \frac{1}{4^n (2n-1) \rho^n} \frac{4^n}{\sqrt{\pi n}} (1 + o(1)) \\
&= 0.868 \frac{1}{2n^{\frac{3}{2}}} \rho^{-n} (1 + o(1)) \\
&= 0.434 n^{-\frac{3}{2}} 2.379^n (1 + o(1)) .
\end{aligned}
$$

Using this, we can get similar formulas for s_n, p_n, b_n and d_n, only differing in the constant factor. We can conclude, that:

$$
\begin{aligned}
[z^n] C_3^{(c)}(z) &= [z^n] \frac{1}{3z} \int H(z) + D(z)\, dz \\
&= \frac{2}{3(n+1)} \alpha' n^{-\frac{3}{2}} 2.392^n n! \left(1 + O\left(n^{-1}\right)\right) \\
&= \alpha n^{-\frac{5}{2}} 2.392^n n! \left(1 + O\left(n^{-1}\right)\right) .
\end{aligned}
$$

□

From this we can conclude that the number of weighted 3-regular connected series-parallel multigraphs is $r_c(n) = \alpha n^{-\frac{5}{2}} 2.392^n n! (1 + O(n^{-1}))$ for even n.

To get the number of all 3-regular series-parallel multigraphs, we see that a general multigraph is a set of its connected components. We have then:

$$
R = Set(R_c) \;\Rightarrow\; R(z) = \exp(R_c(z)) .
$$

From this we get the asymptotic formula:

$$
r(n) = \alpha_u n^{-\frac{5}{2}} 2.392^n n! \left(1 + O\left(n^{-1}\right)\right) , \tag{25}
$$

with $\alpha_u = \alpha \exp(R_c(0.418)) = \alpha \exp(0.612)$.

Let for the rest of this thesis be $g = \alpha \exp(0.612)$ and $\gamma = 2.392$ such that $r(n) = g\gamma^n n^{-\frac{5}{2}} n!(1 + O(n^{-1}))$.

3.3 Series-parallel graphs with minimum degree 3

In the last section, we looked at 3-regular graphs. As a next step we have a look at graphs with more edges than that. For this, we will look at the number of labelled weighted series-parallel multigraphs with n vertices, an excess of d and minimum degree three.

Definition 3.6. *Let $Q(n,d)$ be the set of all labelled weighted series-parallel multigraphs with minimum degree at least 3, with n vertices and deficiency d. Also let $q(n,d) = |Q(n,d)|$ be the number of such graphs.*

As $d = 2M - 3n$, such graphs have exactly $\frac{3n+d}{2}$ edges. For this to be possible, we have to have that $3n + d$ is even. Let us assume this for the rest of the chapter. Also $Q(n,0)$ is the set of all clean graphs and $q(n,0) = gn^{-\frac{5}{2}}\gamma^n n!$ as seen in Section 3.2.

We want to put $q(n,0)$ and $q(n,d)$ into some relation, as this is exactly what we need. To do this, we will use the same method as in [18] to show the following theorem.

Theorem 3.10. *Let $1 \leq d < n$. Then:*

$$q(n+d,0)\,36^{-d} \leq d!q(n,d) \leq q(n+d,0)\,9^d.$$

Also we get:

$$q(n,d) = \frac{\alpha^d}{d!} g(n+d)^{-\frac{5}{2}}\gamma^(n+d)(n+d)!,$$

where $\frac{1}{36} \leq \alpha \leq 9$ is some function depending on n and d.

Proof. To show this, consider the following.

Let G be a graph in $Q(n+d,0)$. We choose for each of the vertices with labels $\{n+1,\ldots,n+d\}$ one of its neighbours. Now, starting from vertex $n+d$ working down, merge the vertex with the chosen neighbour, if possible. This means, replacing the two vertices by one new vertex connected to all neighbours of the original vertices, labelled with the smaller of the two numbers. This procedure either stops at a graph in $Q(n,d)$ or fails before that, if one vertex nominated a neighbour with a higher label. Clearly, all graphs in $Q(n,d)$ can be reached in this way.

To see this we will expand a graph in $Q(n,d)$ in a way that choosing neighbours like above is possible. We can do this by replacing a vertex with degree $3+l$ by a tree of $l+1$ vertices. Then there is at least one possibility connecting all original neighbours with this tree such that each vertex has degree 3 so that the graph remains series-parallel. This can easily be seen inductively. At a vertex of degree 4 one has the following cases. If there is one bridge connected to v, this bridge can be paired with an arbitrary other edge to be neighbour of the new vertex. Otherwise, take two of the paths joining again before joining with one of the other paths. These two paths can be split to the new vertex. This also works recursively for vertices with degree ≥ 4.

From this, we have $q\left(n,d\right)$ is smaller or equal than the number $q(n+d,0)$ multiplied by the number of possible ways to do shrinking like above. First, we have to choose neighbours for the vertices $n+1$ to $n+d$. As each of them has at most three neighbours, we have at most 3^d possibilities to do that. Furthermore, if a graph H in $Q\left(n,d\right)$ is counted once for a graph G in $Q\left(n+d,0\right)$, then it is also counted for each other graph differing from G only in the labelling of the vertices $n+1$ to $n+d$ in the following way. Remove all labels $n+1$ to $n+d$ from vertices in G to obtain G'. Let w be the label of a vertex with degree more than 3 in H. This vertex has $1 \leq i \leq 3$ unlabelled vertices as neighbours in G'. As a consequence, there are at least $\frac{d\cdot\ldots\cdot(d-i+1)}{2^i}$ possible ways of doing this. Iterating this until all vertices are labelled, yields at least $d!2^{-d}$ graphs for which H is counted. Finally the rise in the weight by this operation is maximal, if one contracts a triple edge and replaces it with one vertex with two loops, gaining $\frac{6}{4}$ in weight. In all other cases the weight is smaller in G than in H. Therefore we get an upper bound of

$$\begin{aligned} q\left(n,d\right) &\leq q\left(n+d,0\right)\cdot 3^d \cdot \left(\frac{3}{2}\right)^d \frac{1}{d!2^{-d}} \\ &= \frac{q\left(n+d,0\right)9^d}{d!}. \end{aligned}$$

Conversely, for the lower bound, we estimate the number $q\left(n,d\right)$ by only those graphs in this class with maximum degree 4. This number is clearly smaller. Such a graph has exactly d vertices of degree 4 and $n-d$ vertices of degree 3. Each vertex of degree 4 can be split in two vertices of degree 3 leaving one of the new vertices unlabelled and labelling the other the same as the original vertex. This can be done in at most six ways for each vertex. We can identify each of the new vertices uniquely by the vertex it emerged from. Correspondingly there are $d!$ possibilities of labelling these vertices. Also the weight increases by at most a factor of 6, by splitting up a quadruple edge into two double edges. From this, we have a lower bound of

$$q\left(n,d\right) \geq q\left(n+d,0\right)\cdot \frac{1}{d!6^d}\frac{1}{6^d}$$

as claimed.

From this, we have:

$$q(n,d) = \frac{\alpha^d}{d!}g(n+d)^{-\frac{5}{2}}\gamma^{(n+d)}\left(n+d\right)!$$

or, as we will need it:

$$q\left(2l-d,d\right) = \frac{\alpha^d}{d!}g\left(2l\right)^{-\frac{5}{2}}\gamma^{(2l)}\left(2l\right)!\left(1+o\left(1\right)\right), \tag{26}$$

where $\alpha = \alpha(l, d)$ is some function with values in $[\frac{1}{36}, 9]$. $\square$

3.4 Complex graphs

From the last section we have $q(2l-d, d) = g\frac{\alpha^d}{d!}p_d^{2l}(2l)^{-\frac{5}{2}}(2l)!(1+o(1))$. Our first goal will be to find the number of all complex labelled series-parallel graphs with n vertices, M edges and deficiency d. To do this, let us define the following.

Definition 3.7. *Let $C_d(n, M)$ be the number of labelled series-parallel graphs with n vertices, M edges and deficiency d where all connected components are complex.*

For counting this number we will use combinatorial arguments. From these, we get:

Theorem 3.11. *For the number $C_d(k, k+l)$, where k is the number of vertices and l is the excess with $l \ll k$, we have:*

$$C_d(k, k+l) = \sum_i \binom{k}{i}\binom{i}{v} q(v, d)(i-v)!\binom{i-v-ml+e-1}{e-1} ik^{k-i-1}, \tag{27}$$

where v is the number of vertices of the kernel, e is the number of edges in the kernel and m is a constant with $0 \le m \le 6$. Furthermore, in such a graph $v = 2l - d$ and $e = 3l - d$.

Proof. For counting these graphs, we will first count all kernels, then we will expand these to cores by inserting vertices on all multiple edges and all loops. Finally, we will add rooted forests to the core for getting all graphs. For doing this, we have to select vertices $(c_1, \ldots, c_i)$ for the core and $(k_1, \ldots, k_v)$ from them for the kernel. The possibilities for this are $\binom{k}{i}\binom{i}{v}$. We have $q(v, d)$ possibilities for choosing a kernel. Following this we have to put the remaining core vertices on the edges in the kernel in such a way, that on each multiple edge at most one edge remains without a vertex and on each loop there are at least two vertices. For this, we have to choose one direction in each loop and an order for all multiple edges. As we weighted each loop with $\frac{1}{2}$ and each multiple edge with $\frac{1}{u!}$, where u is the number of edges. These weights guarantee that we don't count twice. This fixes $l \cdot m = 2f_1 + \sum_{j\ge 2}(j-1)f_j$ vertices on these edges, where f_1 is the number of loops and f_j is the number of multiple edges with cardinality j. For this we have to sort the vertices, giving $(i-v)!$ possibilities and then placing

$i - v - lm$ not yet fixed vertices on e edges. As we can put multiple vertices on the same edge, we get $\binom{(i-v-lm)+e-1}{e-1}$ possibilities. Finally, we have to put a rooted forest on this graph with exactly the core vertices as roots. By Cayley's formula [7] this is possible in exactly ik^{k-i-1} ways. After multiplying all this together and summing over all possible core-sizes, we get equation (27). As we will see from now on, the numbers v and e are already fixed by fixing k, l, and d. Only the size of the core is not fixed and as a consequence we have to sum over all possible values of i Finally it remains to show that $0 \leq m \leq 6$. As $m = \frac{1}{l}\left(2f_1 + \sum (j-1) f_j\right)$, m is clearly positive. For the upper bound, we have the following: $d = 2M - 3n \geq 0 \;\Rightarrow\; n \leq \frac{2}{3}M$. From this we get $l = M - n \geq \frac{1}{3}M$. Let f_0 be the number of single edges. Then $M = f_0 + \sum_{j\geq 1} j f_j$, as this is counting all edges with their respective multiplicity. Accordingly we have:

$$\begin{aligned} m &= \frac{1}{l}\left(2f_1 + \sum (j-1) f_j\right) \\ &\leq \frac{3}{M}\left(M - f_0 + f_1 - \sum_{j\geq 2} f_j\right) \\ &\leq \frac{3}{M}(M - 0 + M - 0) = 6. \end{aligned}$$

It remains to show the formulas for v and e. By definition of the deficiency, we have $d = 2e - 3v$. Furthermore, as by clipping away vertices of degree one and replacing vertices of degree two by edges, the number of vertices and edges both decrease by one, the excess of the graph is equal to the excess of the kernel. Hence $l = e - v$. Solving these two equations for v and e yields the desired result.

□

In the next part of this section we will simplify this formula to get:

Theorem 3.12. *Let k, l and d be as above. Then:*

$$C_d(k, k+l) = 2^{-3} g \gamma^{2l} l^{-\frac{5}{2}} \alpha^d \binom{2l}{d} k^{k+\frac{3l-d-1}{2}} e^{\frac{3l-d}{2}} (3l-d)^{-\frac{3l-d-1}{2}} \tag{28}$$

$$\times \exp\left(\left(-(m+1)\, l + \frac{2}{3}d\right)\sqrt{\frac{3l-d}{k}} + O\left(\frac{l^2}{k} + \frac{1}{l}\right)\right). \tag{29}$$

Furthermore, the main contribution of the sum over all i occurs at $i = i_0 + O(\sqrt{i_0})$ with $i_0 = \sqrt{k(3l-d)}$.

Proof. For this we will use $q(2l-d,d) = \alpha^d \left(1+o\left(\frac{1}{l}\right)\right) g(2l)^{-\frac{5}{2}} \gamma^{2l} (2l)!$ from the last chapter, where $\frac{1}{36} \leq \alpha \leq 9$. Replacing $v = 2l-d$ and $e = 3l-d$ from the previous theorem in formula (27) yields

$$\begin{aligned}
C_d(k,k+l) &= \sum_i \binom{k}{i}\binom{i}{2l-d}\frac{\alpha^d}{d!}\left(1+o\left(\frac{1}{l}\right)\right) g(2l)^{-\frac{5}{2}}\gamma^{2l}(2l)! \\
&\times (i-2l+d)!\binom{i+l(1-m)-1}{3l-d-1} i k^{k-i-1} \\
&= \frac{\alpha^d}{d!}\left(1+o\left(\frac{1}{l}\right)\right) g(2l)^{-\frac{5}{2}}\gamma^{2l}(2l)! \sum_i \frac{i!}{(2l-d)!(i-2l+d)!} \\
&\times \frac{(k)_i}{i!}(i-2l+d)!\binom{i+l(1-m)-1}{3l-d-1} i k^{k-i-1} \\
&= \frac{\alpha^d}{d!}\left(1+o\left(\frac{1}{l}\right)\right) g(2l)^{-\frac{5}{2}}\gamma^{2l}(2l)! \\
&\times \sum_i \frac{(k)_i}{(2l-d)!}\binom{i+l(1-m)-1}{3l-d-1} i k^{k-i-1}. \qquad (30)
\end{aligned}$$

Using equations (2) and (1) on $\binom{i+l(1-m)-1}{3l-d-1}$, we get:

$$\begin{aligned}
\binom{i+l(1-m)-1}{3l-d-1} &= \frac{(i+l(1-m)-1)_{3l-d-1}}{(3l-d-1)!}\frac{3l-d}{3l-d} \\
&= \frac{(3l-d)\, i^{3l-d-1}\prod_j \left(1+\frac{-(m-1)l-j}{i}\right)}{\left(1+O\left(\frac{1}{l}\right)\right)\sqrt{2\pi(3l-d)}\left(\frac{3l-d}{e}\right)^{3l-d}} \\
&= \frac{e^{3l-d} i^{3l-d-1}}{(3l-d)^{3l-d-\frac{1}{2}}\sqrt{2\pi}} \\
&\times \prod_j \left(\exp\left(\frac{-(m-1)l-j}{i} - \frac{1}{2}\left(\frac{-(m-1)l-j}{i}\right)^2 \right.\right. \\
&\left.\left. +O\left(\left(\frac{-(m-1)l-j}{i}\right)^3\right)\right)\right) \\
&= \frac{e^{3l-d} i^{3l-d-1}\exp\left(-\frac{(3l-d)^2}{2i} + \frac{3l-d+(m-1)l}{i} + \frac{l}{i} + \frac{1}{l}\right)}{(3l-d)^{3l-d-\frac{1}{2}}\sqrt{2\pi}}
\end{aligned}$$

after summing in the exponent.

Using this and equation (3) on $(k)_i$ in equation (30), we get:

$$C_d(k,k+l) = \alpha^d \left(1 + o\left(\frac{1}{l}\right)\right) g(2l)^{-\frac{5}{2}} \gamma^{2l}$$
$$\times \frac{e^{3l-d}(2l)!}{(2l-d)!d!(3l-d)^{3l-d-\frac{1}{2}}\sqrt{2\pi}} \sum_i \exp a(i), \qquad (31)$$

with

$$a(i) = (3l-d)\log(i) - \frac{i^2}{2k} - \frac{(3l-d)^2}{2i} + \frac{(3l-d)(1-m)l}{i}$$
$$+ \frac{1}{l} + \frac{l}{i} + O\left(\frac{i}{k} + \frac{i^3}{k^2}\right). \qquad (32)$$

In the next step, we want to calculate this sum. For this we will first look where the main contribution is. Then the rest will be fairly easy to compute. We get the main contribution for $\sum_i \exp(a(i))$ at the point:

$$i_0 = \sqrt{k(3l-d)} + O\left(\sqrt{k}\right). \qquad (33)$$

To proof this, we will use the following. The main contribution to this sum occurs in a neighbourhood of the maximum of $a(i)$ in equation (32), as exp is a monotonous function. One gets this maximum by making the derivative 0.

This derivative is:

$$a'(i) = \frac{3l-d}{i} - \frac{i}{k} - \frac{i^2}{2k^2} + \frac{(3l-d)^2}{2i^2} - \frac{(1-m)l(3l-d)}{i^2} - \frac{l}{i^2}. \qquad (34)$$

The main contribution to this is obtained from the first two terms. Setting them 0 yields $i_0 = \sqrt{k(3l-d)}$. For this choice, we get:

$$a(i_0) = (3l-d)\log(i_0) - \frac{k(3l-d)}{2k} - \frac{(3l-d)^2}{2\sqrt{k(3l-d)}} + \frac{(3l-d)(1-m)l}{\sqrt{3l-d}}$$
$$+ \frac{1}{l} + \frac{l}{\sqrt{k(3l-d)}} + O\left(\frac{\sqrt{kl}}{k} + \frac{(kl)^2}{k^3}\right)$$
$$= (3l-d)\log(i_0) - \frac{3l-d}{2} - \alpha\sqrt{\frac{l^3}{k}} + \frac{1}{l} + O\left(\sqrt{\frac{l}{k}} + \frac{l^2}{k}\right).$$

for some α. As can be seen here, the first two terms give the main contribution, as $\sqrt{\frac{l^3}{k}} \ll l$ for $l \ll k$.

Suppose $i = i_0 + O(k^\alpha)$ for some α. We will show that by choosing $\alpha = \frac{1}{2}$, the error made in the sum is smaller than some of the other error terms already existing. This results in the following sum:

$$\sum_i \exp(a(i)) = \exp(a(i_0)) \sum_{i=i_0+O(k^\alpha)} \exp(a(i) - a(i_0)),$$

with

$$\begin{aligned}\exp(a(i_0)) = (k(3l-d))^{(3l-d)/2} \exp\Bigg(& -\frac{3l-d}{2} - \frac{3l-d}{6}\sqrt{\frac{3l-d}{k}} \\ & -\frac{3l-d}{2}\sqrt{\frac{3l-d}{k}} + (1-m)\sqrt{\frac{3l-d}{k}} + O\left(\frac{l^2}{k} + \frac{1}{l}\right)\Bigg) \\ = (k(3l-d))^{(3l-d)/2} \exp\Bigg(& -\frac{3l-d}{2} \\ & +\sqrt{\frac{3l-d}{k}}\left((m+1)l + \frac{2}{3}d\right) + O\left(\frac{l^2}{k} + \frac{1}{l}\right)\Bigg).\end{aligned}$$

Furthermore, we can use the following to calculate $\exp(a(i) - a(i_0))$. By performing Taylor-expansion the term can be rewritten as $\log(i_0 + \delta i) = \log(i_0) + \frac{\Delta i}{i_0} - \frac{\Delta i^2}{2i_0^2} + O\left(\frac{\Delta i^3}{i_0^3}\right)$. Also, we have $\frac{a}{i_0+\Delta i} - \frac{a}{i_0} = -\frac{a\Delta i}{i_0^2}\left(1 + O\left(\frac{a\Delta_i}{i_0}\right)\right)$. Using this, we get:

$$\begin{aligned}\sum_{i=i_0+O(\sqrt{k})} e^{(a(i)-a(i_0))} &= \sum_{\Delta i = O\left(k^{\frac{1}{2}}\right)} \exp\left(-\frac{(\Delta i)^2 - 2i_0\Delta i}{k} + O\left(\frac{\Delta i^2 l^2}{i_0^3}\right)\right) \\ &= \int_{-\infty}^{\infty} \exp\left(-\frac{\left(xk^{\frac{1}{2}}\right)^2}{k}\right) d\left(xk^{\frac{1}{2}}\right) \exp\left(O\left(\frac{l}{k}\right)\right) \\ &= \sqrt{k\pi}\exp\left(O\left(\frac{l}{k}\right)\right).\end{aligned}$$

If $\Delta i = O\left(\sqrt{i_0}\right)$, then the additional order term is $\exp\left(O\left(\frac{l}{k}\right)\right)$ which is small enough to not give any additional order terms. Concluding this, counting over $\Delta i = O\left(\sqrt{i_0}\right)$ yields the main contribution. We can use this and equation (1) in equation (31) to get

$$C_d\left(k,k+l\right) = 2^{-3} g\gamma^{2l} l^{-\frac{5}{2}} \alpha^d \binom{2l}{d} k^{k+\frac{3l-d-1}{2}} e^{\frac{3l-d}{2}} \left(3l-d\right)^{-\frac{3l-d-1}{2}}$$
$$\times \exp\left(\left(-\left(m+1\right)l + \frac{2}{3}d\right)\sqrt{\frac{3l-d}{k}} + O\left(\frac{l^2}{k} + \frac{1}{l}\right)\right)$$

as claimed. □

As a next step, we will sum over all possible deficiencies to get a formula on the number $C(n, M)$ of all complex graphs. For this value, we have:

Theorem 3.13. *Let k, l be as above. Then:*

$$C(k,k+l) = 2^{-3} g\gamma^{2l} l^{-\frac{5}{2}} k^{k+\frac{3l-1}{2}} e^{\frac{3l}{2}} \left(3l\right)^{(3l-1)/2} \exp\left(O\left(\frac{1}{l}\right)\right)$$
$$\times \exp\left(\beta\sqrt{\frac{l^3}{k}} + O\left(\frac{l^2}{k}\right)\right). \tag{35}$$

Furthermore the main contribution to this value with respect to d is attained at $d_0 = \Theta\left(\sqrt{\frac{l^3}{k}}\right)$.

Proof. We are looking at:

$$\begin{aligned}
C\left(k,k+l\right) &= \sum_d C_d\left(k,k+l\right) \\
&= 2^{-3} g\gamma^{2l} l^{-\frac{5}{2}} k^{k+\frac{3l-1}{2}} e^{\frac{3l}{2}} \exp\left(O\left(\frac{l^2}{k} + \frac{1}{l}\right)\right) \\
&\times \sum_d \alpha^d \binom{2l}{d} k^{-\frac{d}{2}} e^{-\frac{d}{2}} \left(3l-d\right)^{-\frac{3l-d-1}{2}} && (36) \\
&\times \exp\left(\left(\left(1-m\right)l + \frac{2}{3}d\right)\sqrt{\frac{3l-d}{k}}\right). && (37)
\end{aligned}$$

In this sum, we can rewrite the following term with the help of equation (2):

$$\begin{aligned}
\left(3l-d\right)^{-\frac{3l-d-1}{2}} &= \left(3l\right)^{\frac{3l-d-1}{2}} \left(1-\frac{d}{3l}\right)^{-\frac{3l-d-1}{2}} \\
&= \left(3l\right)^{\frac{3l-d-1}{2}} \exp\left(-\frac{3l-d-1}{2}\left(-\frac{d}{3l} + O\left(\frac{d^2}{l^2}\right)\right)\right) \\
&= \left(3l\right)^{\frac{3l-d-1}{2}} \exp\left(\frac{d}{2} + O\left(\frac{d^2}{l}\right)\right).
\end{aligned}$$

With this, one can write the factor $(3l)^{\frac{3l-1}{2}}$ in front of the sum (37) and one has to look at:

$$\sum_d \alpha^d \binom{2l}{d} k^{-\frac{d}{2}} (3l)^{\frac{d}{2}} \exp\left(\left((1-m)\, l + \frac{2}{3}d\right) \sqrt{\frac{3l-d}{k}}\right). \tag{38}$$

We want to get the main contribution to this sum. From equation (1) we get the approximation $\left(\frac{2l}{d}\right)^d \leq \binom{2l}{d} \leq \left(\frac{2le}{d}\right)^d$. Using this one can rewrite the above equation as:

$$\sum_d \left(\frac{c}{d}\right)^d \exp\left(\left((1-m)\, l + \frac{2}{3}d\right) \sqrt{\frac{3l-d}{k}}\right),$$

with some constant $c = c\,(k,l) = 6lr\alpha\sqrt{\frac{l^3}{k}}$ and $1 \leq r \leq e$. One gets the main contribution to the sum as the main contribution of $\left(\frac{c}{d}\right)^d$ and this is at $d = c = \Theta\left(\sqrt{\frac{l^3}{k}}\right)$. With this the sum transforms into:

$$\begin{aligned}
&\sum_d \alpha^d \binom{2l}{d} k^{-\frac{d}{2}} (3l)^{\frac{d}{2}} \exp\left(\left((1-m)\, l + \frac{2}{3}d\right) \sqrt{\frac{3l-d}{k}}\right) \\
&\quad = \sum_d \alpha^d \binom{2l}{d} k^{-\frac{d}{2}} (3l)^{\frac{d}{2}} \exp\left(\left((1-m)\, l + \Theta\,(1) \sqrt{\frac{l^3}{k}}\right) \sqrt{\frac{3l}{k}}\right) \\
&\quad = \left(1 + \alpha\sqrt{\frac{3l}{k}}\right)^{2l} \exp\left((1-m)\, l\sqrt{\frac{3l}{k}} + O\left(\sqrt{\frac{l^3}{k}}\sqrt{\frac{3l}{k}}\right)\right) \\
&\quad = \exp\left(2l\alpha\sqrt{\frac{3l}{k}} + (1-m)\sqrt{\frac{3l^3}{k}} + O\left(\frac{l^2}{k}\right)\right) \\
&\quad = \exp\left((2\alpha - m + 1)\sqrt{\frac{3l^3}{k}} + O\left(\frac{l^2}{k}\right)\right).
\end{aligned}$$

As we know $\frac{1}{36} \leq \alpha \leq 9$ and $0 \leq m \leq 6$, we get:

$$\begin{aligned}
C\,(k, k+l) = {} & 2^{-3} g \gamma^{2l} l^{-\frac{5}{2}} k^{k+\frac{3l-1}{2}} e^{\frac{3l}{2}} (3l)^{(3l-1)/2} \exp\left(O\left(\frac{1}{l}\right)\right) \\
& \times \exp\left(\beta\sqrt{\frac{l^3}{k}} + O\left(\frac{l^2}{k}\right)\right),
\end{aligned}$$

with $\beta = (2\alpha - m + 1)\sqrt{3} \leq (2 * 9 - 0 + 1)\sqrt{3} = 19\sqrt{3} < 33$ and $\beta \geq \left(\frac{2}{36} - 6 + 1\right)\sqrt{3} > -9$. □

3.5 The asymptotic number of series-parallel graphs

In this section we will use the results of the previous sections to finally prove Theorem 3.1.

As we have seen in Chapter 3.1,

$$sp\,(n, M) = \sum_{k,l} \binom{n}{k} C\,(k, k+l)\,U\,(n-k, M-k-l)\,.$$

To start, we will rewrite the term for $U(n-k, M-k-l)$ from Theorem 3.3 to get the following:

Lemma 3.14. *Let n, l, n, M be as above. Then:*

$$\begin{aligned} U\,(n-k, M-k-l) = {} & \rho\,(n-k, M-k-l)\left(\frac{2\,(M-k)}{(n-k)^2}\right)^l \\ & \times \frac{(n-k)^{2(M-k)}\, e^{M-k+\frac{M-k-l}{n-k}+\frac{(M-k-l)^2}{(n-k)^2}}}{(2\,(M-k))^{M-k+\frac{1}{2}}\sqrt{2\pi}} \\ & \times \exp\left(O\left(\frac{M-k-l}{(n-k)^2} + \frac{1}{M-k-l} + \frac{l^2}{M} + \frac{l^3}{M^2}\right)\right). \end{aligned}$$

Proof. From Theorem 3.3, we have:

$$\begin{aligned} U\,(n-k, M-k-l) = {} & \frac{\rho\,(n-k, M-k-l)\,(n-k)^{2(M-k-l)}}{\sqrt{2\pi\,(M-k-l)}\,(2\,(M-k-l))^{M-k-l}} \\ & \times \exp\left(M-k-l - \frac{M-k-l}{n-k} - \frac{(M-k-l)^2}{(n-k)^2}\right) \\ & \exp\left(O\left(\frac{M-k-l}{(n-k)^2} + \frac{1}{M-k-l}\right)\right). \end{aligned}$$

We can rewrite the factor $(2\,(M-k-l))^{M-k-l+\frac{1}{2}}$ to get:

$$
\begin{aligned}
&(2(M-k-l))^{M-k-l+\frac{1}{2}} \\
&\qquad = (2(M-k))^{-l}(2(M-k))^{M-k+\frac{1}{2}}\left(1-\frac{l}{M-k}\right)^{M-k-l+\frac{1}{2}} \\
&\qquad = (2(M-k))^{-l}(2(M-k))^{M-k+\frac{1}{2}} \\
&\qquad \times \exp\left(\left(M-k-l+\frac{1}{2}\right)\left(-\frac{l}{M-k}+O\left(\frac{l^2}{(M-k)^2}\right)\right)\right) \\
&\qquad = (2(M-k))^{-l}(2(M-k))^{M-k+\frac{1}{2}} \\
&\qquad \times \exp\left(-l+O\left(\frac{l}{M}+\frac{l^2}{M}+\frac{l^3}{M^2}\right)\right).
\end{aligned}
$$

After cancelling out e^l and sorting all factors, one gets the desired result:

$$
\begin{aligned}
U(n-k, M-k-l) = \rho(n-k, M-k-l)&\left(\frac{2(M-k)}{(n-k)^2}\right)^l \\
&\times \frac{(n-k)^{2(M-k)}\, e^{M-k+\frac{M-k-l}{n-k}+\frac{(M-k-l)^2}{(n-k)^2}}}{(2(M-k))^{M-k+\frac{1}{2}}\sqrt{2\pi}}
\end{aligned}
$$

times all order terms above.

□

We can use this and Lemma 2.6 for approximating $\binom{n}{k}$ to calculate $sp(n, M)$. From that we get:

$$
\begin{aligned}
sp(n, M) &= \sum_{k,l}\binom{n}{k} C_{sp}(k, k+l)\, S(n-k, M-k-l) \\
&= \sum_{k,l}\frac{(1+O(k^{-1}))\, n^{n+\frac{1}{2}}}{\sqrt{2\pi}\,(n-k)^{n-k+\frac{1}{2}}\, k^{k+\frac{1}{2}}} \\
&\times 2^{-3} g\gamma^{2l} l^{-\frac{5}{2}} k^{k+(3l-1)/2} e^{3l/2}(3l)^{(3l-1)/2}\rho(n-k, M-k-l) \\
&\times \left(\frac{2(M-k)}{(n-k)^2}\right)^l \frac{(n-k)^{2(M-k)}\, e^{M-k+\frac{M-k-l}{n-k}+\frac{(M-k-l)^2}{(n-k)^2}}}{(2(M-k))^{M-k+\frac{1}{2}}\sqrt{2\pi}} \\
&\times \left(1+O\left(n^{-1}\right)\right)\exp\left(\beta\sqrt{\frac{l^3}{k}}+O\left(\frac{l}{k}+\frac{l^2}{k}+\frac{l^3}{k^2}+\frac{1}{l}\right)\right).
\end{aligned}
$$

Collecting all terms and approximating $M-k-l$ by $M-k$ in the exponent (the error to do this is small enough to vanish in the other order terms), one gets:

$$\begin{aligned}
sp(n,M) &= \left(1+O\left(n^{-1}\right)\right) n^{n+\frac{1}{2}} 2^{-\frac{7}{2}} 3^{-\frac{1}{2}} \pi^{-1} g e^{M} \\
&\times \sum_k \left(1+O\left(k^{-1}+kn^{-1}\right)\right) \frac{k^{k-\frac{1}{2}} (n-k)^{2(M-k)} e^{-k+\frac{M-k}{n-k}+\frac{(M-k)^2}{(n-k)^2}}}{(n-k)^{n-k+\frac{1}{2}} k^{k+\frac{1}{2}} (2(M-k))^{M-k+\frac{1}{2}}} \\
&\times \sum_l \gamma^{2l} l^{-2} k^{3l/2} e^{3l/2} (3l)^{3l/2} \rho(n-k, M-k-l) \left(\frac{2(M-k)}{(n-k)^2}\right)^l \\
&\times \exp\left(\beta\sqrt{\frac{l^3}{k}} + O\left(\frac{l}{k}+\frac{l^2}{k}+\frac{l^3}{k^2}+\frac{1}{l}\right)\right) \\
&= \left(1+O\left(n^{-1}\right)\right) n^{n+\frac{1}{2}} 2^{-\frac{7}{2}} (3\pi)^{-\frac{1}{2}} g e^{M} \\
&\times \sum_k \left(1+O\left(k^{-1}+kn^{-1}\right)\right) \frac{(n-k)^{2(M-k)} e^{-k+\frac{M-k}{n-k}+\frac{(M-k)^2}{(n-k)^2}}}{(n-k)^{n-k+\frac{1}{2}} k (2(M-k))^{M-k+\frac{1}{2}}} \\
&\times \sum_l \rho(n-k, M-k-l) \left(\frac{\gamma^2 e^{3/2} k^{3/2} 2(M-k)}{3^{3/2}(n-k)^2}\right)^l \\
&\times \exp\left(\beta\sqrt{\frac{l^3}{k}} + O\left(\frac{l}{k}+\frac{l^2}{k}+\frac{l^3}{k^2}+\frac{1}{l}\right)\right). \qquad (39)
\end{aligned}$$

As a next step, we will distinguish the different regions for M. As we have seen in Chapter 3.1, $M = \frac{n}{2} + s$, $s = o(n)$ is a point of change in the behaviour of random graphs. So substituting this, we get:

$$e^{\frac{M-k}{n-k}} = e^{\frac{1}{2}+o(1)}$$

and

$$e^{\left(\frac{M-k}{n-k}\right)^2} = e^{\frac{1}{4}+o(1)}$$

and from this

$$
\begin{aligned}
sp\left(n, \frac{n}{2}+s\right) = &\left(1+O\left(\frac{1}{n}\right)\right) 2^{-\frac{7}{2}} g n^{n+\frac{1}{2}} e^{\frac{n}{2}+s-\frac{3}{4}} \frac{\sqrt{3}}{\pi} \\
&\times \sum_k \frac{(n-k)^{2s-k-\frac{1}{2}}}{k\,(n+2s-2k)^{\frac{n}{2}+s-k+\frac{1}{2}}} e^{-k} \\
&\times \sum_l \left(\frac{\gamma^2 k^{\frac{3}{2}} e^{\frac{3}{2}}}{(3l)^{\frac{3}{2}}}\right)^l l^{-2} \left(\frac{n+2s-2k}{(n-k)^2}\right)^l \\
&\times \rho\left(n-k, \frac{n}{2}+s-k-l\right) \exp\left(\beta l \sqrt{\frac{l}{k}} + O\left(\frac{l^3}{k^2}+\frac{l^2}{k}\right)\right).
\end{aligned}
\tag{40}
$$

The main contribution to these sums depend on the value of s. We will distinguish here the cases $s \ll -n^{2/3}$, $s = cn^{2/3}$ for some constant c and $n^{2/3} \ll s \ll n$.

Firstly suppose $s \ll -n^{\frac{2}{3}}$. In this case, $G(n, M)$ is series-parallel a.a.s. Then we are in case one of Theorem 3.1 and have

$$
sp(n, M) = (1+o(1))\, G(n, M) = (1+o(1)) \frac{n^{n+2s} e^{\frac{n-1}{2}+s}}{\sqrt{\pi}(n+2s)^{\frac{n+1}{2}+s}}.
$$

We use the following fact proven by Łuczak [21, 22]and Łuczak, Pittel and Wierman [24] and also stated by Janson, Łuczak, and Ruciński [17].

Theorem 3.15. *If* $\frac{s^3}{n^2} \to -\infty$, *then, a.a.s.* $G\left(n, \frac{n}{2}+s\right)$ *is a collection of trees and unicyclic components.*

As such graphs are series-parallel, this part of the theorem follows.

Now suppose $s = cn^{\frac{2}{3}}$ for some constant c. Then Noy et al. [27] proved, that in this case the probability that a random graph is series-parallel tends to a limit depending only on c. If $c = 0$ this limit is approximately 0.98003. Using this, we have:

$$
sp(n, M) = \Theta(1) G(n, M) = \Theta(1) \frac{n^{n+2s} e^{\frac{n-1}{2}+s}}{\sqrt{\pi}(n+2s)^{\frac{n+1}{2}+s}}.
$$

Secondly let $n^{\frac{2}{3}} \ll s \ll n$. We will look at equation (40). First, we have to find the main contribution to the innermost sum over l. To do this, we will first state that for k and l around the main contribution, $\rho\left(n-k, \frac{n}{2}+s-k-l\right) = \rho\left(n-k, \frac{n}{2}+s-k\right)$.

Proposition 3.16. *Suppose $k = 2s + r$ for some r and $l \ll r$. Then:*

$$\rho\left(n-k, \frac{n}{2}+s-k-l\right) = \rho\left(n-k, \frac{n}{2}+s-k\right).$$

Proof. As we can distinguish $\rho\left(N, \frac{N}{2}+S\right)$ with respect to $\frac{S^3}{N^2}$, we take $N = n-k$ and

$$\begin{aligned} S &= \frac{n}{2}+s-k-l-\frac{N}{2} \\ &= \frac{n}{2}+s-2s-r-l-\frac{n-2s-r}{2} \\ &= -\frac{r+2l}{2}. \end{aligned}$$

If $l \ll r$, we have:

$$S^3 = -\left(\frac{r+2l}{2}\right)^3 = -\frac{(1+o(1))}{8}r^3,$$

and this value does not depend on l. As the value of ρ does only depend on the ratio $\frac{S^3}{N^2}$ as N tends to infinity, this proves the claim. □

From now on, we will replace $\rho\left(n-k, \frac{n}{2}+s-k-l\right)$ with $\rho\left(n-k, \frac{n}{2}+s-k\right)$. So for these calculations to be valid, we have to assert that in the end the conditions of proposition 3.16 are valid. For the main contribution with respect to l, we get the following estimations.

Lemma 3.17. *Suppose*

$$\phi = \phi(n,s,k) = \frac{\gamma^2 k^{\frac{3}{2}} e^{\frac{3}{2}} (n+2s-2k)}{3^{\frac{3}{2}} (n-k)^2}. \tag{41}$$

Then the main contribution to the sum in (40) *with respect to l is at $l = l_0 + O\left(\sqrt{l_0}\right)$ where $l_0 = \phi^{2/3}e^{-1}$.*

Proof. The sum concerning l in (40) can be written as $\sum_l \frac{\phi}{l^{3/2}}^l l^{-2}$.

We are looking for the main contribution in this sum. For this, we have to maximize the function

$$\frac{\phi}{l^{\frac{3}{2}}}^l l^{-2} = l^{-2}\exp\left(l\log\phi - \frac{3l}{2}\log l\right).$$

By making the derivative of this equal to 0, we get:

$$0 = \left(2l^{-3} - l^{-2}\left(\log\phi - \frac{3}{2}\left(\log\left(l\right) + 1\right)\right)\right)\exp\left(l\log\phi - \frac{3l}{2}\log l\right).$$

As $2l^{-3} = o\left(l^{-2}\left(\log\phi - \frac{3}{2}\left(\log\left(l\right)+1\right)\right)\right)$, it is sufficient to find a value for l such that the second term vanishes, as then the first term will automatically converge to 0 as $l \to \infty$. Hence we have to solve

$$0 = \left(\log\phi - \frac{3}{2}\left(\log\left(l\right) + 1\right)\right) \Leftrightarrow l = \frac{\phi^{\frac{2}{3}}}{e}. \tag{42}$$

From this we can conclude that $l = l_0$ will give the main contribution. Set $l = l_0 + \Delta l$ and $b\left(l\right) = \left(\frac{\phi}{l^{3/2}}\right)^l l^{-2}$. Using also $\log\left(a+b\right) = \log\left(a\right) + \frac{b}{a} - \frac{b^2}{2a^2} + O\left(\frac{b^3}{a^3}\right)$, one gets:

$$\begin{aligned}
\sum_l b\left(l\right) &= b\left(l_0\right)\sum_{\Delta l = l - l_0}\exp\left(\log\left(b\left(l\right)\right) - \log\left(b\left(l_0\right)\right)\right)\\
&= b\left(l_0\right)\sum_{\Delta l}\exp\left(\left(l - l_0\right)\log\left(\phi\right) - \frac{3}{2}\left(l\log\left(l\right) - l_0\log\left(l_0\right)\right)\right.\\
&\left.-2\left(\log\left(l\right) - \log\left(l_0\right)\right)\right)\\
&= b\left(l_0\right)\sum_{\Delta l}\exp\left(\Delta l\log\left(\phi\right) - \frac{3}{2}\left(\Delta l + \frac{\Delta l^2}{2l_0} + \Delta l\log\left(l_0\right)\right) - 2\frac{\Delta l}{l_0}\right)\\
&= b\left(l_0\right)\sum_{\Delta l}\exp\left(-\frac{3\Delta l^2}{4l_0} - \frac{2\Delta l}{l_0} + O\left(\frac{\Delta l^2}{l_0^2}\right)\right).
\end{aligned}$$

If $\Delta l = O\left(\sqrt{l_0}\right)$, the error term is a $\exp\left(O\left(\frac{1}{l_0}\right)\right)$ which does already exist. In this case, the error made in the summation is small enough to not yield an additional error term. The sum itself can be estimated by an integral. Taking $\Delta l = c\sqrt{l_0}$ with integrating over $c\sqrt{l_0}$ from $-\infty$ to ∞, we get:

$$\sum_{\Delta l} = \int_{-\infty}^{\infty}\exp\left(-\frac{3\left(c\sqrt{l_0} - l_0\right)^2}{4l_0}\right)dc\sqrt{l_0} = \sqrt{\frac{4\pi l_0}{3}},$$

and the sum over l is then:

$$\sum_{l}\left(\frac{\phi}{l^{\frac{3}{2}}}\right)^{l} l^{-2} = (1+o(1))\, l_0^{-2}\left(\frac{\phi}{l_0^{\frac{3}{2}}}\right)^{l_0} \sum_{\Delta l}\frac{b(l)}{b(l_0)}$$
$$= (1+o(1))\,\frac{6\sqrt{\pi}\,(n-k)^2}{k^{\frac{3}{2}}\gamma^2\,(n+2s-2k)}\exp\left(\frac{\gamma^{\frac{4}{3}}k\,(n+2s-2k)^{\frac{2}{3}}}{2\,(n-k)^{\frac{4}{3}}}\right).$$

From this we have:

$$l_0 = O\left(\phi^{\frac{2}{3}}\right) = O\left(\frac{kn^{\frac{2}{3}}}{n^{\frac{4}{3}}}\right) = O\left(kn^{-\frac{2}{3}}\right).$$

□

We will need this again later for checking the conditions of proposition 3.17. From the sum, we get in (40):

$$sp\left(n,\frac{n}{2}+s\right) = \left(1+O\left(\frac{1}{n}\right)\right)2^{-\frac{7}{2}}gn^{n+\frac{1}{2}}e^{\frac{n}{2}+s-\frac{3}{4}}\frac{\sqrt{3}}{\pi}$$
$$\times\sum_{k}\rho\left(n-k,\frac{n}{2}+s-k\right)\frac{(n-k)^{2s-k-\frac{1}{2}}}{k\,(n+2s-2k)^{\frac{n}{2}+s-k+\frac{1}{2}}}e^{-k}$$
$$\times\frac{6\sqrt{\pi}\,(n-k)^2}{k^{\frac{3}{2}}\gamma^2\,(n+2s-2k)}\exp\left(\frac{\gamma^{\frac{4}{3}}k\,(n+2s-2k)^{\frac{2}{3}}}{2\,(n-k)^{\frac{4}{3}}}\right)$$
$$= \left(1+O\left(\frac{1}{n}\right)\right)2^{-\frac{5}{2}}\gamma^{-2}gn^{n+\frac{1}{2}}e^{\frac{n}{2}+s-\frac{3}{4}}3^{\frac{3}{2}}\pi^{-\frac{1}{2}}$$
$$\times\sum_{k}\rho\left(n-k,\frac{n}{2}+s-k\right)\frac{(n-k)^{2s-k+\frac{3}{2}}}{k^{\frac{5}{2}}\,(n+2s-2k)^{\frac{n}{2}+s-k+\frac{3}{2}}}e^{-k}$$
$$\times\exp\left(\frac{\gamma^{\frac{4}{3}}k\,(n+2s-2k)^{\frac{2}{3}}}{2\,(n-k)^{\frac{4}{3}}}\right).$$

In the next step, we have to evaluate the sum over k. To do this, we will show the following result.

Lemma 3.18. *Let k, n and s be as above. Then the main contribution for k in the sum above is at*

$$k = 2s + O\left(n^{2/3}\right). \tag{43}$$

To show this, we will use the substitution $k = 2s + r$. Substituting this, we have to look at the sum $\sum_r \rho\left(n - 2s - r, \frac{n}{2} - s - r\right)\psi(r)$ where

$$\begin{aligned}
\psi(r) &= \frac{(n-k)^{2s-k+\frac{3}{2}}}{k^{\frac{5}{2}}(n+2s-2k)^{\frac{n}{2}+s-k+\frac{3}{2}}} e^{-k} \exp\left(\frac{\gamma^{\frac{4}{3}} k (n+2s-2k)^{\frac{2}{3}}}{2(n-k)^{\frac{4}{3}}}\right) \\
&= \frac{(n-k)^{2s-k+\frac{3}{2}}}{(n+2s-2k)^{\frac{n}{2}+s-k+\frac{3}{2}}} (2s+r)^{-\frac{5}{2}} \\
&\times \exp\left(-2s - r + \frac{\gamma^{\frac{4}{3}}(2s+r)(n-r)^{\frac{2}{3}}}{2(n-2s+r)^{\frac{4}{3}}}\right).
\end{aligned}$$

Here we have factors e^{-2s} and $\exp\left(\gamma^{4/3} s n^{-2/3}\right)$ not dependent on r which we can move in front of the sum. To find the desired asymptotics of this formula, in a next step we will simplify the factors $\frac{(n-k)^{2s-k}}{(n+2s-2k)^{\frac{n}{2}+s-k}}$ and $\left(\frac{(n-k)}{(n+2s-2k)}\right)^{\frac{3}{2}}$.

For these, we get:

$$\begin{aligned}
\frac{(n-k)^{2s-k}}{(n+2s-2k)^{\frac{n}{2}+s-k}} &= \frac{(n-2s-r)^{-r}}{(n-2s-2r)^{\frac{n}{2}-s-r}} \\
&= (n-2s)^{-\frac{n}{2}+s}\left(\frac{n-2s-r}{n-2s}\right)^{-r}\left(\frac{n-2s-2r}{n-2s}\right)^{r-\frac{n}{2}-s} \\
&= (n-2s)^{-\frac{n}{2}+s}\left(1 - \frac{r}{n-2s}\right)^{-r}\left(1 - \frac{2r}{n-2s}\right)^{r-\frac{n}{2}-s} \\
&= (n-2s)^{-\frac{n}{2}+s}\exp\left(r\left(\frac{r}{n-2s} + \dots\right)\right. \\
&\left. - \left(r - \frac{n}{2} - s\right)\left(\frac{2r}{n-2s} + \dots\right)\right) \\
&= (n-2s)^{-\frac{n}{2}+s}\exp\left(r - \frac{r^3}{6(n-2s)^2}\right) \\
&\times \exp\left(O\left(\frac{r^4}{(n-2s)^3}\right)\right),
\end{aligned}$$

giving a factor $(n-2s)^{-\frac{n}{2}+s}$ not dependent on r and

$$\left(\frac{(n-k)}{(n+2s-2k)}\right)^{\frac{3}{2}} = \left(1 - \frac{2s-k}{n+2s-2k}\right)^{\frac{3}{2}} = 1 + O\left(n^{-\frac{3}{2}}\right).$$

We have to find the main contribution of the sum

$$\sum_r \rho\left(n-2s-r, \frac{n}{2}-s-r\right)\left(s+\frac{r}{2}\right)^{-\frac{5}{2}}$$
$$\times \exp\left(\frac{\gamma^{\frac{4}{3}}r}{2\left(n-2s\right)^{\frac{2}{3}}}+\frac{r^3}{6\left(n-2s\right)^2}+O\left(\frac{r}{n}+\frac{r^4}{\left(n-2s\right)^3}\right)\right).$$

For this, we will need some term, where ρ is not small. To get this we have $N = n-2s-r$ and $S = \frac{n}{2}-s-r-\frac{n-2s-r}{2} = \frac{r}{2}$. Hence we get from Theorem 3.4 that $\rho\left(n-2s-r, \frac{n}{2}-s-r\right) = \nu\left(\frac{x}{2}\right)$, if $\frac{r}{(n-2s-r)^{\frac{2}{3}}} \to x$ with $x \in \mathbb{R}$, if $r = O\left(n^{\frac{2}{3}}\right)$. As this is also the region in which the main contribution of the exponential part is maximal, $r = O\left(n^{\frac{2}{3}}\right)$ is the main contribution of this sum. Also in this region we have $\left(s+\frac{r}{2}\right)^{-\frac{5}{2}} = s^{-\frac{5}{2}}\left(1+o\left(1\right)\right)$. Using this and rewriting the sum again as integral with $r = n^{2/3}x$, we have:

$$\sum_k = \left(1+o\left(1\right)\right)n^{2/3}s^{-\frac{5}{2}}\int_{-\infty}^{\infty}\nu\left(-\frac{x}{2}\right)\exp\left(\frac{-x^3}{6}+\frac{\gamma^{\frac{4}{3}}x}{2}\right)dx.$$

This integral does not depend on n or s and so is a constant I. With this, we have:

$$sp = \left(1+o\left(1\right)\right)2^{-\frac{5}{2}}\gamma^{-2}g\pi^{-\frac{1}{2}}In^{n+\frac{7}{6}}e^{\frac{n}{2}-s-\frac{3}{4}}3^{\frac{3}{2}}$$
$$\times \exp\left(\gamma^{\frac{4}{3}}sn^{-\frac{2}{3}}\right)\left(n-2s\right)^{-\frac{n}{2}+s}s^{-\frac{5}{2}}.$$

Also, as $r = O\left(n^{\frac{2}{3}}\right) = o\left(s\right)$, we have $k = 2s+r = 2s\left(1+o\left(1\right)\right)$ and the conditions in Lemma 3.17 are fulfilled, as $l = O\left(kn^{-\frac{1}{3}}\right) = O\left(n^{\frac{1}{3}}\right)$.

For the next case, suppose $s = \alpha n$ with $0 < \alpha < \frac{1}{2}$. For this calculation, nearly all steps taken for $M = \frac{n}{2}+s$, $n^{2/3} << s << n$ do also work. The only point where we needed that $s = o\left(n\right)$ was right in the beginning giving new estimates for the following statements:

$$e^{\frac{M-k}{n-k}} = e^{\frac{1}{2}+\alpha+o(1)}$$

and

$$e^{\left(\frac{M-k}{n-k}\right)^2} = e^{\left(\frac{1}{2}+\alpha\right)^2+o(1)}.$$

This implies that the constant factor is changing, but the rest of the calculations work the same way. So we have:

$$\begin{aligned} sp\left(n,\alpha n\right) &= \theta\left(1\right) n^{n+\frac{7}{6}} e^{\frac{n}{2}-\alpha(n)-\frac{3}{4}} \exp\left(\gamma^{\frac{4}{3}}\alpha n^{\frac{1}{3}}\right)\left(n\left(1-2\alpha\right)\right)^{-\frac{n}{2}+\alpha n}\left(\alpha n\right)^{-\frac{5}{2}} I \\ &= \theta\left(1\right) n^{n+\frac{7}{6}} e^{\left(\frac{1}{2}-\alpha\right)n} \exp\left(\gamma^{\frac{4}{3}}\alpha n^{\frac{1}{3}}\right)\left(n\left(1-2\alpha\right)\right)^{\left(-\frac{1}{2}+\alpha\right)n} n^{-\frac{5}{2}} \\ &= \theta\left(1\right) n^{n\left(\frac{1}{2}+\alpha\right)-\frac{4}{3}} \left(\frac{e}{n\left(1-2\alpha\right)}\right)^{\left(\frac{1}{2}-\alpha\right)n} \exp\left(\gamma^{\frac{4}{3}}\alpha n^{\frac{1}{3}}\right). \end{aligned}$$

Also, all main contributions will occur at the same values for l and r. This method does not work anymore for $s = \frac{n}{2}+o\left(n\right)$, as in this case, $n-2s = o\left(1\right)$ in the denominator of some fractions and so the error terms occurring will get uncontrollably high. Taking this into consideration we will look at this case in a way a bit different from this one.

Let us look at $M = n+t$ with $t = o(n)$. As was shown by Kang and Łuczak in [18], planar graphs have a second critical phase in this region. It is also known that the general random graph $G\left(n,M\right)$ does not have this second critical phase. We will show here that series-parallel graphs do have the same second critical phase as planar graphs with only some minor differences. For this, we will again look at the number of series-parallel graphs in formula (19). Taking $M = n+t$ with $t = o\left(n\right)$, we get:

$$
\begin{aligned}
sp(n, n+t) &= \sum_{k,l} \binom{n}{k} C(n-k, n-k+l)\, U(k, k+t-l) \\
&= \sum_{k,l} \frac{\left(1+O\left(\frac{1}{k}\right)\right) n^{n-\frac{1}{2}}}{\sqrt{2\pi}\,(n-k)^{n-k-\frac{1}{2}}\, k^{k+\frac{1}{2}}} 2^{-3} g \gamma^{2l} l^{-\frac{5}{2}} (n-k)^{n-k+\frac{3l-1}{2}} \\
&\times e^{\frac{3l}{2}} (3l)^{-\frac{3l+1}{2}} \exp\left(\beta \sqrt{\frac{l^3}{n-k}} + O\left(\frac{l^3}{k^2} + \frac{l}{k}\right)\right) \\
&\times \rho(k, k+t-l) \frac{k^{2(k+t-l)}}{\sqrt{\pi}\,(2(k+t-l))^{k+t-l+\frac{1}{2}}} \\
&\times \exp\left(k+t-l - \frac{k+t-l}{k} + \frac{(k+t-l)^2}{k^2}\right) \\
&\times \exp\left(O\left(\frac{1}{k} + \frac{k+t-l}{k^2}\right)\right) \\
&= (1+o(1)) \frac{g\sqrt{3}}{\pi 2^4} n^{n-\frac{1}{2}} \sum_l \left(\frac{\gamma^2 e^{\frac{3}{2}}}{3^{\frac{3}{2}}}\right)^l l^{-\frac{3l}{2}-2} \sum_k \delta(k), \qquad (44)
\end{aligned}
$$

with

$$
\begin{aligned}
\delta(k) &= \rho(k, k+t-l)\,(k(k+t-l))^{-\frac{1}{2}} (n-k)^{\frac{3l}{2}-1} \\
&\times k^{k+2t-2l} \left(\frac{e}{2(k+t-l)}\right)^{k+t-l} \exp\left(-\frac{k+t-l}{k} - \frac{(k+l-t)^2}{k^2}\right) \\
&\times \exp\left(\beta\sqrt{\frac{l^3}{n-k}} + O\left(\frac{1}{k} + \frac{l^3}{k^2} + \frac{l}{k}\right)\right). \qquad (45)
\end{aligned}
$$

As a next step we want to get the main contribution for the sum $\sum_k \delta(k)$.

Lemma 3.19. *The main contribution for the sum $\sum_k \delta(k)$ is attained at*

$$
k_0 = 2(l-t) + O\left((2(l-t))^{\frac{2}{3}}\right).
$$

Proof. To show this, we will transform this sum by replacing $k = 2(l-t) + x$ and summing over x. Then it remains to show that the main contribution is made by $x = O\left((2(l-t))^{\frac{2}{3}}\right)$. Thus, the sum gets:

$$\sum_k \delta(k) = \sum_r \rho(2(l-t)+x, 2(l-t)+x+t-l)$$
$$\times ((2(l-t)+x)(2(l-t)+x+t-l))^{-\frac{1}{2}}(n-2(l-t)+x)^{\frac{3l}{2}-1}$$
$$\times (2(l-t)+x)^{2(l-t)+x+2t-2l}\left(\frac{e}{2(2(l-t)+x+t-l)}\right)^{2(l-t)+x+t-l}$$
$$\times \exp\left(-\frac{2(l-t)+x+t-l}{k} - \frac{(2(l-t)+x+l-t)^2}{(2(l-t)+x)^2}\right)$$
$$\times \exp\left(\beta\sqrt{\frac{l^3}{n-2(l-t)+x}} + O\left(\frac{1+l}{2(l-t)} + \frac{l^3}{(2(l-t))^2}\right)\right).$$

Using several times the exp-log-transformation (2) yield the following terms dependent on x:

$$\sum_x \tau(x) = \sum_x \rho(2(l-t)+x, l-t+x)\exp\left(\frac{x^3}{6(2(l-t))^2}\right)$$
$$\times \exp\left(-\frac{3xl}{2(n-2(l-t))} + O\left(\frac{x}{l-t} + \frac{x}{n-2(l-t)}\right)\right).$$

For ρ we know from Theorem 3.4, that its value depends on the value of $\lim_{n\to\infty}\frac{s^3}{n^2}$ where $s = M - \frac{n}{2}$. For these values we have $n = 2(l-t)+x$, $M = l-t+x$ and so $s = \frac{x}{2}$. We have then the following cases:

- $\lim_{n\to\infty}\frac{s^3}{n^2} = -\infty$. Here we have $x \ll -(2(l-t)+x)^{\frac{2}{3}}$. Also we have $\rho(2(l-t)+x, l-t+x) = \left(1 + O\left(\frac{(2(l-t)+x)^2}{|(x/2)|^3}\right)\right)$ where the order term is exactly the negative inverse from the condition, so it tends to zero as n tends to infinity. So $\rho(2(l-t)+x, l-t+x) = 1 + o(1)$.
- $\lim_{n\to\infty}\frac{s^3}{n^2} = \infty$. Then we have:

  $$\rho(2(l-t)+x, l-t+x) \le \exp\left(-\frac{s^3}{n^2}\right).$$

  In this case the summands tend to 0 exponentially fast and as such do not have an additional input to the sum.

Using this, the main part of the sum attained for is $x = O\left((2(l-t))^{\frac{2}{3}}\right)$. □

Using this and approximating $\sum_{x=O\left((2(l-t))^{\frac{2}{3}}\right)}$ by $\int_{-\infty}^{\infty} dc$ with $x = c \cdot (2(l-t))^{\frac{2}{3}}$ we get:

$$\begin{aligned}\sum_x \tau(x) &= \int_{-\infty}^{\infty} \rho(2(l-t)+x, l-t+x) \\ &\times \exp\left(\frac{x^3}{6(2(l-t))^2} - \frac{3xl}{2(n-2(l-t))}\right) dc \\ &= \int_{-\infty}^{\infty} \rho\left(2(l-t)+c(2(l-t))^{\frac{2}{3}}, l-t+c(2(l-t))^{\frac{2}{3}}\right) \\ &\times \exp\left(\frac{\left(c(2(l-t))^{\frac{2}{3}}\right)^3}{6(2(l-t))^2} - \frac{3c(2(l-t))^{\frac{2}{3}} l}{2(n-2(l-t))}\right) dc.\end{aligned}$$

Then we can use that by Theorem 3.4 $\rho(2(l-t)+x, l-t+x) = \nu\left(\frac{c}{2}\right)$ in this range, we get:

$$\sum_x \tau(x) = \int_{-\infty}^{\infty} \nu\left(\frac{c}{2}\right) \exp\left(\frac{c^3}{6} - \frac{3c(2(l-t))^{\frac{2}{3}} l}{2(n-2(l-t))}\right) dc. \tag{46}$$

For the last part in the integral, we look for the part where the main contribution with respect to l is. This will be calculated in the next lemma.

Lemma 3.20. *Let $l = l_0 + O(l_0^{\alpha})$ be the main contribution to the sum (44). Then, for t fixed, l_0 satisfies the equation:*

$$l_0 = \frac{\gamma^{\frac{4}{3}}(n-2(l_0-t))}{3(2(l_0-t))^{\frac{2}{3}}}.$$

Proof. We have seen that:

$$\begin{aligned}\sum_k \delta(k) &= (1+o(1)) e^{-3/4-t} 2^{1/2} \frac{e^l (n-2(l-t))^{3l/2}}{(2(l-t))^{l-t+1}} \\ &\times \int_{-\infty}^{\infty} \nu\left(\frac{c}{2}\right) \exp\left(\frac{c^3}{6} - \frac{3c(2(l-t))^{\frac{2}{3}} l}{2(n-2(l-t))}\right) dc.\end{aligned}$$

As the integral depends only marginally on l, we can treat it as a constant for calculating the main contribution of l. From this we have as additional factor the fraction in this term.

Let $a\left(l\right) := \left(\frac{\gamma^2 e^{\frac{5}{2}}(n-2(l-t))^{3/2}}{3^{\frac{3}{2}}(2(l-t))}\right)$. Then we have to find the main contribution of the sum :

$$\sum_{l} a(l)^l l^{-\frac{3}{2}l-2} \left(l-t\right)^{t-1}. \tag{47}$$

By utilizing monotony of the logarithm, we can also maximize the logarithm of this sum. For this, we will take the derivative:

$$\begin{aligned}&(\log(a(l)^l l^{-\frac{3}{2}l-2} \left(l-t\right)^{t-1}))' \\ &\qquad = \log\left(a(l)\right) + \frac{la'\left(l\right)}{a\left(l\right)} - \frac{3}{2}\log\left(l\right) - \frac{3}{2} - \frac{2}{l} + \frac{t-1}{l-t} \overset{!}{=} 0.\end{aligned}$$

As l tends to infinity with n, $-\frac{2}{l} + \frac{t-1}{l-t}$ tends to zero. Also $\frac{la'(l)}{a(l)} = l\left(\frac{1}{(-l+t)} - \frac{3}{(-2l+n+2t)}\right)$ tends to $\frac{1}{2}$ as $l \to \infty$. More exactly it is $\frac{1}{2} + O\left(\frac{1}{l}\right)$. For getting the main contribution, we then have:

$$\log\left(a(l)\right) - \frac{3}{2}\log\left(l\right) = 0 \Leftrightarrow l = a(l)^{\frac{2}{3}}.$$

Concluding this, we have $l_0 = a\left(l_0\right)^{\frac{2}{3}}$ as claimed. □

Using this in the integral, we get:

$$\sum_{x} \tau\left(x\right) = \int_{-\infty}^{\infty} \nu\left(\frac{c}{2}\right) \exp\left(\frac{c^3}{6} - \frac{\gamma^{\frac{4}{3}} c}{2}\right) dc.$$

As γ is constant, this integral is also a constant with value I.

Now using all of this, we can rewrite the number of series-parallel graphs $sp\left(n, n+t\right)$ as:

$$
\begin{aligned}
sp(n,n+t) &= (1+o(1))\frac{g\sqrt{3}}{\pi 2^4} n^{n-\frac{1}{2}} \sum_l \left(\frac{\gamma^2 e^{\frac{3}{2}}}{3^{\frac{3}{2}}}\right)^l l^{-\frac{3l}{2}-2} \sum_k \delta(k) \\
&= (1+o(1))\frac{g\sqrt{3}}{\pi 2^4} I n^{n-\frac{1}{2}} \sum_l \left(\frac{\gamma^2 e^{\frac{3}{2}}}{3^{\frac{3}{2}}}\right)^l l^{-\frac{3l}{2}-2} \\
&\times (1+o(1)) e^{-3/4-t} 2^{1/2} \frac{e^l (n-2(l-t))^{3l/2}}{(2(l-t))^{l-t+1}} \\
&\times \exp\left(\beta\sqrt{\frac{l_0^3}{n-2(l_0-t)}} + O\left(\frac{l_0^2}{n-2(l_0-t)}\right)\right) \\
&= (1+o(1))\frac{g\sqrt{3}e^{-3/4-t}}{\pi 2^{7/2}} I n^{n-\frac{1}{2}} D(n,t) \\
&\times \exp\left(\beta\sqrt{\frac{l_0^3}{n-2(l_0-t)}} + O\left(\frac{l_0^2}{n-2(l_0-t)}\right)\right), \qquad (48)
\end{aligned}
$$

with

$$
D(n,t) = \sum_l \left(\frac{\gamma^2 e^{\frac{3}{2}}}{3^{\frac{3}{2}}}\right)^l l^{-\frac{3l}{2}-2} \frac{e^l (n-2(l-t))^{3l/2}}{(2(l-t))^{l-t+1}}.
$$

As a next step we want to find the main contribution of $D(n,t)$. We know already from Lemma 3.20, that the main contribution to the sum is achieved for l_0 satisfying:

$$
l_0 = \frac{\gamma^{\frac{4}{3}}(n-2(l_0-t))}{3(2(l_0-t))^{\frac{2}{3}}}.
$$

At first suppose t to be constant. In this case, we have:

$$
3l_0(2(l_0-t))^{\frac{2}{3}} = \gamma^{\frac{4}{3}}(n-2(l_0-t)),
$$

or $l_0^{\frac{5}{3}} = \Theta(n+l_0) = \Theta(n)$. As to derive the asymptotics of $(l-t)$ in the formula above, we will distinguish whether $t \ll -n^{\frac{3}{5}}$, $t = cn^{\frac{3}{5}}$ or $t \gg n^{\frac{3}{5}}$.

Secondly suppose $t \ll -n^{\frac{3}{5}}$. Accordingly, we also have $-t \gg l_0$. In this case, we get:

$$
l_0 - t = -t\left(1+\frac{l_0}{-t}\right) = -t(1+o(1)).
$$

From this we have the main contribution at

$$l_0 = \frac{\gamma^{\frac{4}{3}} \left(n - 2\left(l_0 - t\right)\right)}{3\left(2\left(l_0 - t\right)\right)^{\frac{2}{3}}} = \frac{\gamma^{\frac{4}{3}} \left(n + 2t\right)}{3\left(-2t\right)^{\frac{2}{3}}}. \tag{49}$$

With this, we can calculate $D(n,t)$. This can be done by rewriting the sum as:

$$\begin{aligned} D(n,t) &= \sum_{l=l_0+O\left(\sqrt{l_0}\right)} \exp\left(\log\left(d\left(n,t,l\right)\right)\right) \\ &= d\left(n,t,l_0\right) \sum_{\Delta l \in O\sqrt{l_0}} \exp\left(\log\left(d\left(n,t,l\right)\right) - \log\left(d\left(n,t,l_0\right)\right)\right). \end{aligned}$$

Using this we get:

$$\begin{aligned} d(n,t,l_0) &= \left(\frac{\gamma^2 e^{3/2}}{3^{3/2}}\right)^{l_0} \frac{e^{l_0-t}\left(n - \left(2\left(l_0-t\right)\right)\right)^{3l_0/2}}{l_0^{3l_0/2+2}\left(2\left(l_0-t\right)\right)^{l_0-t+1/3}} \\ &= e^{5l_0/2-t}\left(2\left(l_0-t\right)\right)^{-l_0} l_0^{-2}\left(2\left(l_0-t\right)\right)^{t-1/3}\left(\frac{\gamma^2\left(n-2\left(l_0-t\right)\right)^{3/2}}{\left(3l_0\right)^{3/2}}\right)^{l_0} \\ &= e^{5l_0/2-t}\left(2\left(l_0-t\right)\right)^{-l_0} l_0^{-2}\left(2\left(l_0-t\right)\right)^{t-1/3}\left(2t\left(1-\frac{2\gamma^{4/3}}{3t^{2/3}}\right)^{3/2}\right)^{l_0} \\ &= e^{5l_0/2-t}\left(l_0-t\right)^{-l_0} l_0^{-2}\left(2\left(l_0-t\right)\right)^{t-1/3} t^{l_0} \exp\left(\frac{3}{2}l_0\left(\frac{2l_0}{3\left(n+2t\right)}\right)\right). \end{aligned}$$

By using the fact that $\log(a+b) = \log(a) + \frac{b}{a} + O\left(\frac{b^2}{a^2}\right)$, we can rewrite the rest of the sum as follows:

$$\log\left(d\left(n,t,l\right)\right) - \log\left(d\left(n,t,l_0\right)\right) = -\Delta l^2\left(\frac{3}{n+2t-2l_0} + \frac{3}{2l_0} + \frac{1}{l_0-t}\right),$$

plus terms of lower order. Using that for $\Delta l \in O\left(\sqrt{l_0}\right)$

$$\frac{3\Delta l^2}{n+2t-2l_0} = O\left(\frac{l_0}{n}\right) = O\left(\frac{n}{tn}\right) = o(1)$$

we can calculate the rest sum as:

$$\sum_{\Delta l \in O\sqrt{l_0}} \exp(\log(d(n,t,l)) - \log(d(n,t,l_0)))$$
$$= \sum_{\Delta l \in O\sqrt{l_0}} \exp\left(-\Delta l^2 \left(\frac{3}{2l_0} + \frac{1}{l_0 - t}\right)\right).$$

Substituting $\Delta l = x\sqrt{l_0}$ and integrating yields to:

$$\sum_{\Delta l \in O\sqrt{l_0}} \exp(\log(d(n,t,l)) - \log(d(n,t,l_0))) = (1 + o(1)) \frac{\sqrt{2l_0\pi(l-t)}}{5l_0 - 3t}.$$

Combining these results, one gets:

$$D(n,t) = (1 + o(1)) \sqrt{2\pi} \frac{(2(l_0 - t))^{t-1/3+1/2}}{\sqrt{5l_0 - 3t}}$$
$$\times\, l_0^{-3/2} t^{l_0} (l_0 - t)^{-l_0} \exp\left(\frac{5l_0}{2} - t - \frac{3l_0^2}{n + 2t}\right).$$

Replacing this in the sum (48), we get the result stated in Theorem 3.1. In this case, we have the main contribution with respect to k at $k_0 = 2(l_0 - t)$ and with respect to l at $l_0 = \frac{\gamma^{\frac{4}{3}}(n+2t)}{3(-2t)^{\frac{2}{3}}}$.

Thirdly suppose $t = cn^{\frac{3}{5}}$ for some constant $c \in \mathbb{R}$. Let us also suppose:

$$l_0 = bn^{\frac{3}{5}}. \tag{50}$$

Then:

$$l_0 = \frac{\gamma^{\frac{4}{3}}(n - 2(l_0 - t))}{3(2(l_0 - t))^{\frac{2}{3}}}$$
$$\Leftrightarrow\ bn^{\frac{3}{5}} = \frac{\gamma^{\frac{4}{3}}\left(n - 2n^{\frac{3}{5}}(b - c)\right)}{3\left(2n^{\frac{3}{5}}(b - c)\right)^{\frac{2}{3}}}$$
$$\Leftrightarrow\ bn^{\frac{3}{5}} 3\left(2n^{\frac{3}{5}}(b - c)\right)^{\frac{2}{3}} = \gamma^{\frac{4}{3}}\left(n - 2n^{\frac{3}{5}}(b - c)\right)$$
$$\Leftrightarrow\ 3 \cdot 2^{\frac{2}{3}} nb(b - c)^{\frac{2}{3}} = \gamma^{\frac{4}{3}} n\left(1 - 2n^{\frac{-2}{5}}(b - c)\right)$$
$$\Leftrightarrow\ 3 \cdot 2^{\frac{2}{3}} nb(b - c)^{\frac{2}{3}} = \gamma^{\frac{4}{3}} n(1 + o(1)).$$

To fulfil this, b has to be a solution of the equation $3{\cdot}2^{\frac{2}{3}}b\left(b-c\right)^{\frac{2}{3}}=\gamma^{\frac{4}{3}}$. As $b\left(b-c\right)^{\frac{2}{3}}$ is a strictly increasing function in the interval $[c,\infty)$ with values from 0 to ∞ and negative values in the interval $(0,c)$, there is a unique positive solution b_0 for this equation. We can again calculate $D\left(n,t\right)$. For this, using $l_0=\frac{b}{c}t=bn^{3/5}$, we have:

$$\begin{aligned}
d\left(n,t,l_0\right) &= \left(\frac{\gamma^2 e^{3/2}}{3^{3/2}}\right)^{l_0}\frac{e^{l_0-t}\left(n-\left(2\left(l_0-t\right)\right)\right)^{3l_0/2}}{l_0^{3l_0/2+2}\left(2\left(l_0-t\right)\right)^{l_0-t+1/3}}\\
&= e^{\frac{5l_0}{2}-t}\left(\frac{\gamma^2\left(n-2\left(b-c\right)n^{3/5}\right)^{3/2}}{3^{\frac{3}{2}}l_0^{3/2}2\left(l_0-t\right)}\right)l_0^{-2}\left(2\left(l_0-t\right)\right)^{t-1/3}\\
&= e^{\frac{5b}{2c}t-t}\left(\frac{\left(n^{2/5}-2\left(b-c\right)\right)^{3/2}}{n^{3/5}}\right)^{l_0}b^{-2}c^2t^{-2}\left(2\left(\frac{b}{c}t-t\right)\right)^{t-1/3}\\
&= e^{\frac{5b}{2c}t-t}\left(1-2\left(b-c\right)n^{-2/5}\right)^{3l_0/2}b^{-2}c^2t^{-2}\left(2\left(\frac{b}{c}t-t\right)\right)^{t-1/3}\\
&= \exp\left(\left(\frac{5b}{2c}-1\right)t-\frac{3l_0}{2}2\left(b-c\right)n^{-2/5}\right)\\
&\times b^{-2}c^2t^{-2-1/3}t^t\left(2\left(\frac{b}{c}-1\right)\right)^{t-1/3}\\
&= \exp\left(\left(\frac{5b}{2c}-1\right)t-3\frac{t^2}{n}\frac{b}{c}\left(\frac{b}{c}-1\right)\right)\\
&\times b^{-2}c^2t^{-2-1/3}t^t\left(2\left(\frac{b}{c}-1\right)\right)^{t-1/3}.
\end{aligned}$$

The rest of the sum yields exactly the same term as in the subcritical phase:

$$\begin{aligned}
\sum_{\Delta l\in O\sqrt{l_0}}\exp\left(\log\left(d\left(n,t,l\right)\right)-\log\left(d\left(n,t,l_0\right)\right)\right) &= \left(1+o\left(1\right)\right)\frac{\sqrt{2l_0\pi\left(l-t\right)}}{5l_0-3t}\\
&= \left(1+o\left(1\right)\right)\frac{\sqrt{2\frac{b}{c}t\pi\left(\frac{b}{c}-1\right)t}}{\left(5b-3c\right)\frac{t}{c}}.
\end{aligned}$$

Collecting these terms yields to:

$$D(n,t) = (1+o(1))\sqrt{2\pi}\frac{c^2\left(\frac{b}{c}-1\right)^{1/6}t^{-11/6}}{b^{3/2}\sqrt{(5b-3c)}}\left(2\left(\frac{b}{c}-1\right)t\right)^t$$
$$\times \exp\left(\left(\frac{5b}{2c}-1\right)t-\frac{3t^2}{n}\frac{b}{c}\left(\frac{b}{c}-1\right)\right).$$

Using this and (48), we get the result for this region of Theorem 3.1

Finally, assume $t \gg n^{\frac{3}{5}}$. Suppose $l_0 = t+x$. Then we get:

$$l_0 = \frac{\gamma^{\frac{4}{3}}(n-2(l_0-t))}{3(2(l_0-t))^{\frac{2}{3}}}$$
$$\Leftrightarrow\ t+x = \frac{\gamma^{\frac{4}{3}}(n-2x)}{3(2x)^{\frac{2}{3}}}$$
$$\Leftrightarrow\ x^{\frac{2}{3}}t = \frac{\gamma^{\frac{4}{3}}n}{2^{\frac{2}{3}}3}\left(1-\frac{2x}{n}-\frac{2^{\frac{2}{3}}3x^{\frac{5}{3}}}{\gamma^{\frac{4}{3}}n}\right).$$

If here $x = o\left(n^{\frac{3}{5}}\right)$, then the last equation would be $x^{\frac{2}{3}} = \frac{\gamma^{\frac{4}{3}}n}{2^{\frac{2}{3}}3t}(1+o(1))$. In this case, we would have $x = \frac{\gamma^2 n^{\frac{3}{2}}}{2\cdot 3^{\frac{3}{2}}t^{\frac{3}{2}}}(1+o(1))$. As $t \gg n^{\frac{3}{5}}$, we have $x = o\left(\frac{n^{\frac{3}{2}}}{n^{\frac{3\cdot 3}{5\cdot 2}}}\right) = o\left(n^{\frac{3}{5}}\right)$. Therefore the condition is fulfilled and this x is indeed the solution. In this case, we get

$$d(n,t,l_0) = \left(\frac{\gamma^2 e^{3/2}}{3^{3/2}}\right)^{l_0}\frac{e^{l_0-t}(n-(2(l_0-t)))^{3l_0/2}}{l_0^{3l_0/2+2}(2(l_0-t))^{l_0-t+1/3}}$$
$$= e^{\frac{5l_0}{2}-t}\left(\frac{\gamma^2(n-2z)^{3/2}}{3^{\frac{3}{2}}2z}\right)^{l_0} l_0^{-2-\frac{3l_0}{2}}(2(l_0-t))^{t-1/3}$$
$$= e^{\frac{5z}{2}+\frac{3}{2}t}\left(\frac{2zt^{3/2}(n-2z)^{3/2}}{n^{\frac{3}{2}}2z}\right)^{l_0}(t+z)^{-2-\frac{3(t+z)}{2}}(2z)^{t-1/3}$$
$$= e^{\frac{5z}{2}+\frac{3}{2}t}\left(1-\frac{2z}{n}\right)^{l_0} t^{3l_0/2}l_0^{-2-\frac{3l_0}{2}}(2z)^{t-1/3}$$
$$= e^{\frac{5z}{2}+\frac{3}{2}t}\exp\left(\frac{3l_0 z}{n}\right)t^{3(t+z)/2}(t+z)^{-2-\frac{3(t+z)}{2}}(2z)^{t-1/3},$$

and the rest of the sum is again:

$$\sum_{\Delta l \in O\sqrt{l_0}} \exp\left(\log\left(d\left(n,t,l\right)\right) - \log\left(d\left(n,t,l_0\right)\right)\right) = \left(1+o\left(1\right)\right) \frac{\sqrt{2l_0\pi\left(l-t\right)}}{5l_0-3t}$$
$$= \left(1+o\left(1\right)\right) \frac{\sqrt{2\left(z+t\right)\pi z}}{5\left(z+t\right)-3t}.$$

Combining these results, we have:

$$D\left(n,t\right) = \left(1+o\left(1\right)\right)\sqrt{2\pi}\frac{\left(2z\right)^{t+1/6}}{\sqrt{5z+2t}}\left(z+t\right)^{-3/2} t^{3(z+t)/2}$$
$$\times \left(z+t\right)^{-3(z+t)/2} \exp\left(\frac{5z}{2}+\frac{3t}{2}-\frac{3\left(z+t\right)z}{n}\right),$$

and substituting this again in equation (48), we also get the final result of Theorem 3.1.

4 Properties of random series-parallel graphs

In this section we will study prperties of random series-parallel graph. These results will again be split into different ranges as seen in theorem 3.1.

4.1 Deficiency and excess

First, we will look at the deficiency. As was defined in Chapter 2.2, the deficiency d of a graph G is equal to $2l-3k$ where l is the number of edges in the kernel of G and k is the number of vertices in the same kernel. In Chapter 3.4, we defined $C_d(n,M)$ as the number of complex graphs with n vertices, M edges and deficiency d. As trees and unicyclic graphs do not have a kernel, they do not contribute to the value of d. Concluding this, we will look at the main value of d contributing to $sp(n,M)$ for the different regions of M, getting the following theorem.

Theorem 4.1. *Suppose G is a random series-parallel graph with n vertices and M edges. Then:*

(i) if $M = \frac{n}{2} - s$ with $s \gg n^{\frac{2}{3}}$, G has a.a.s. deficiency $d=0$,

(ii) if $M = \frac{n}{2} + s$ with $n \gg s \gg n^{\frac{2}{3}}$, the expected deficiency is $d = \Theta\left(\frac{s}{n}\right)$,

(iii) if $M = \alpha n$ with $\frac{1}{2} < \alpha < 1$, the expected deficiency is $d = \Theta(1)$,

(iv) if $M = n - s$ *with* $s \gg n^{\frac{3}{5}}$*, but* $s \ll n$*, then* $d = \Theta\left(\frac{n}{s}\right)$*,*

(v) if $M = n + s$ *with* $s = cn^{\frac{3}{5}}$*, then* $d = \Theta(n^{2/5})$*,*

(vi) if $M = n + s$ *with* $n \gg s \gg n^{\frac{3}{5}}$*, then* $d = \Theta\left(\frac{s^{3/2}}{n^{1/2}}\right)$*.*

Proof. At first, note that trees and unicyclic graphs do not have a kernel. From this, we can immediately conclude that for $M = \frac{n}{2} - s$ with $s \gg n^{\frac{2}{3}}$, the deficiency is a.a.s. zero, as such graphs have a.a.s. only these components. As a consequence the number of these graphs does not give a contribution to the deficiency. Furthermore, we have seen in formula (19), that

$$sp\,(n, M) = \sum_{k,l} \binom{n}{k} C\,(k, k + l)\, S\,(n - k, M - k - l)\,.$$

In order to get the expected deficiency, we have to look at the main terms k_0 for k and l_0 for l in this sum. With these we can find the expected deficiency for $C(k_0, k_0 + l_0)$, which, as seen in Theorem 3.13, is equal to $\Theta\left(\sqrt{\frac{l_0^3}{k_0}}\right)$.

If M is as in 4.1, we saw in Lemma 3.17 that $l_0 = \Theta(kn^{-2/3})$ and in Lemma 3.18 that $k_0 = \Theta(s)$. From all these conditions we have $d = \Theta(sn^{-1})$. As the above mentioned lemmas also hold for $s = (\alpha - \frac{1}{2})n$, we can conclude that in this case, we have a deficiency of $\Theta(1)$.

However, in the case of $M = n + s$, we have

$$sp\,(n, M) = \sum_{k,l} \binom{n}{k} C\,(n - k, n - k + l)\, S\,(k, k + s - l)\,.$$

So, in order to find the deficiency as above, we have to calculate $\sqrt{\frac{l_0^3}{n - k_0}}$. Throughout this part of the proof, we have $k_0 = \Theta(l - s)$, as seen in Lemma 3.19. Additionally, we know from the derivations of Lemma 3.20, that for $s \ll -n^{3/5}$, $l_0 = \Theta(ns^{-2/3})$ and thus we can conclude that $d = \Theta(n\,|s|^{-1})$ for this region. If $l = cs = cn^{3/5}$, we get by using the estimates from Lemma 3.20 a deficiency of $d = n^{2/5}$. Finally, for $s \gg n^{3/5}$, we have $l = \Theta\,(s)$ resulting in a deficiency of $d = \sqrt{\frac{s^3}{n}}$. □

So the behaviour of the deficiency is as follows. For $M = (\frac{1}{2} + o(1))n$, the deficiency is very small tending to zero as n goes to infinity. Accordingly, in this case G has a clean kernel a.a.s. If $M = \alpha n$ with $\frac{1}{2} < \alpha < 1$, the deficiency is $\Theta(1)$, and hence not dependent on the size n of the graph. And finally, if $M = (1 + o(1))n$, the deficiency tends to infinity as n tends to infinity resulting in kernels with a greater number of edges.

The expected excess can be obtained in a similar fashion. As the excess is given by the difference of the number of vertices and edges, this number is given in advance. Another, more interesting number would be ex_c, the excess of the complex part of the graph. For this we have the following statement.

Theorem 4.2. *Suppose G is a random series-parallel graph with n vertices and M edges. Then:*

(i) if $M = \frac{n}{2} - s$ with $s \gg n^{\frac{2}{3}}$, G has a.a.s. no complex components, and consequently $\mathrm{ex}_c = 0$,

(ii) if $M = \frac{n}{2} + s$ with $n^{\frac{2}{3}} \ll s$, the excess is $\mathrm{ex}_c = \left(\frac{2\gamma^{4/3}}{3} + o(1)\right) sn^{-2/3}$,

(iii) if $M = n + s$ with $-s \gg n^{\frac{3}{5}}$, but s so that $n + s \gg \frac{n}{2} + n^{2/3}$, then $\mathrm{ex}_c = \frac{\gamma^{4/3}(n+2s)}{3(2s)^{2/3}}$,

(iv) if $M = n + s$ with $s = cn^{\frac{3}{5}}$, then $\mathrm{ex} = bn^{3/5}(1 + o(1))$,

(v) if $M = n + s$ with $n^{2/3} \gg s \gg n^{\frac{3}{5}}$, then $\mathrm{ex} = s\left(1 + O\left(n^{3/2}s^{-5/2}\right)\right)$.

Proof. At first, note that trees and unicyclic graphs do not have complex components. From this, we can immediately conclude that for $M = \frac{n}{2} - s$ with $s \gg n^{\frac{2}{3}}$, $\mathrm{ex}_c = 0$ a.a.s., as such graphs have a.a.s. only these components. Therefore the number of these graphs does not give a contribution to ex_c. Furthermore, we have seen in formula (19) that:

$$sp(n, M) = \sum_{k,l} \binom{n}{k} C(k, k+l)\, S(n-k, M-k-l).$$

In order to get the expected deficiency, we have to look at the main terms k_0 for k and l_0 for l in this sum. With these we can find the expected deficiency for $C(k_0, k_0 + l_0)$, which is by definition equal to $(k_0 + l_0) - k_0 = l_0$.

If $M = \frac{n}{2} + s$, we saw in Lemma 3.17 that $l_0 = \frac{\gamma^{4/3}}{3} kn^{-2/3}(1 + o(1))$ and in Lemma 3.18 that $k_0 = 2s(1 + o(1))$. Using these two conditions, we get $l_0 = \frac{2\gamma^{4/3}}{3}(sn^{-2/3})$ up until the point where $s = (\alpha - \frac{1}{2})n$, resulting in an excess of $\mathrm{ex}_c = \Theta(n^{1/3})$.

Similarly, in the case of $M = n + s$, we have:

$$sp(n, M) = \sum_{k,l} \binom{n}{k} C(n-k, n-k+l)\, S(k, k+s-l).$$

From this we have again l as the value for ex_c. We know from Lemma 3.20 that

$$l_0 = \frac{\gamma^{\frac{4}{3}} \left(n - 2\left(l_0 - t\right)\right)}{3\left(2\left(l_0 - t\right)\right)^{\frac{2}{3}}}.$$

Redoing the calculations after this lemma, we get exactly the results stated in the theorem.

□

4.2 Kernel and core

In this section we will calculate the size of the average kernel and the average core. To calculate the expected number of vertices v and edges e of the kernel, note that by definition the deficiency $d = 2e - 3v$ and from Lemma 2.12 we have $\mathrm{ex}_c = e - v$. From these two equations, we can compute $v = 2\,\mathrm{ex}_c - d$ and $e = 3\,\mathrm{ex} - d$. Using the results of the previous section, we get the following theorem:

Theorem 4.3. *Suppose G is a random series-parallel graph with n vertices and M edges. Then $\mathrm{ex}_c \gg d$ and the kernel has on average $2\,\mathrm{ex}_c(1 + o(1))$ vertices and $3\,\mathrm{ex}_c(1 + o(1))$ edges.*

Proof. By comparing the results of theorems 4.1 and 4.2 we see that in all regions for M, d is asymptotically smaller than ex_c. Now, as stated at the beginning of this section, the kernel has $2\,\mathrm{ex}_c - d = 2\,\mathrm{ex}_c(1 + o(1))$ vertices and $3\,\mathrm{ex}_c(1 + (1))$ edges. □

Conversely, to calculate the number of vertices in the core, we use the way we counted all complex planar graphs and in particular Theorem 3.11 to get the following result.

Theorem 4.4. *Let S be a random series-parallel graph with n vertices and M edges. Then the following holds:*

(i) if $M = \frac{n}{2} + s$ with $n \gg s \gg n^{\frac{2}{3}}$, the core has $O\left(sn^{-1/3}\right)$ vertices,

(ii) if $M = n + s$ with $-s \gg n^{\frac{3}{5}}$, if $s = -\alpha n + s_1$ such that $0 \leq \alpha < \frac{1}{2}$ and $s_1 \ll n$, the core has $O\left(ns^{-1/3}\right)$ vertices. Especially, if $\alpha > 0$, the core has $\theta\left(n^{2/3}\right)$ vertices,

(iii) if $M = n + s$ with $s = cn^{\frac{3}{5}}$, the core has $O\left(n^{4/5}\right)$ vertices,

(iv) if $M = n + s$ with $n \gg s \gg n^{\frac{3}{5}}$, the core has $O\left(\sqrt{ns}\right)$ vertices.

Proof. In the proof of Theorem 3.12, more exactly in (33), we have seen that the expected number of vertices in the core is $\sqrt{k(3l-d)}$. As in the previous section, $d \ll l$ in the whole region. Therefore we have to look at the number $\sqrt{3kl}$.

Firstly, we know from the previous section that $k = 2s + O\left(n^{2/3}\right)$ and $l = 2\gamma^{4/3} s n^{-2/3} (1 + o(1))$. From this, we get:

$$\begin{aligned}\sqrt{3kl} &= \sqrt{4\gamma^{4/3} s^2 n^{-2/3} (1 + o(1))} \\ &= 2\gamma^{2/3} s n^{-1/3} (1 + o(1)).\end{aligned}$$

Secondly, as in the previous section, we will again distinguish three cases. First, let $n^{3/5} \ll -s \ll n$. Then $l = \frac{\gamma^{4/3} n}{3(2s)^{2/3}} (1 + o(1))$ and $n - k = n(1 + o(1))$. From this we get

$$\begin{aligned}\sqrt{3l(n-k)} &= \sqrt{3n \frac{\gamma^{4/3} n}{3(2s)^{2/3}}} (1 + o(1)) \\ &= \frac{\gamma^{2/3}}{2^{1/3}} n s^{-1/3} (1 + o(1)).\end{aligned}$$

If $s = -\alpha n + s_1$ with $0 < \alpha < \frac{1}{2}$ and $s1 \ll n$, we have:

$$l = \frac{\gamma^{4/3} n (1 - 2\alpha)}{3(2s)^{2/3}} (1 + o(1))$$

and as a consequence of that:

$$\sqrt{3l(n-k)} = \frac{\gamma^{2/3} (1 - 2\alpha)^{1/2}}{2^{1/3}} n s^{-1/3} (1 + o(1)).$$

Also, if $s = -\frac{n}{2} + s_1$, we get the same asymptotics as in the case for (ii). Suppose $s = cn^{3/5}$. Then $l = bn^{3/5}(1 + o(1))$ and we get:

$$\begin{aligned}\sqrt{3l(n-k)} &= \sqrt{3bn^{3/5} n} (1 + o(1)) \\ &= \sqrt{3b} n^{4/5} (1 + o(1)).\end{aligned}$$

Finally, for $n \gg s \gg n^{\frac{3}{5}}$ we have $l = s(1 + o(1))$ and hence we get:

$$\sqrt{3l(n-k)} = \sqrt{3sn} (1 + o(1)).$$

□

In this case as well, we see two critical regions. In the first at $M = \frac{n}{2} + s$, the size of the core changes from $sn^{-1/3}$ to $ns^{-1/3}$ and in the second at $M = n$, the size changes from $n|s|^{-1/3}$ to $\sqrt{n|s|}$.

4.3 The largest component

In this section we will compute the size of the largest component $L_1(G)$ of a random series-parallel graph, showing that there are indeed two critical phases as claimed in the introduction. We will first give a result on the size of the largest component in random 3-regular multigraphs. From this we will conclude a statement of the size of $L_1(G)$ where G is a random complex series-parallel graph satisfying certain conditions. This will be enough to prove the first phase transition. Finally, to see the second phase transition, we will have to count the number of vertices not in the largest component.

Starting with 3-regular multigraphs, we have the following lemma.

Lemma 4.5. *Let $G(n)$ be a random complex 3-regular series-parallel multigraph as in Section 3.2. Further suppose $j \leq \frac{n}{2}$. Then the probability that the largest component $L_1(G)$ has exactly $n-j$ vertices is:*

$$P(L_1(G) = n-j) = \left(1 + O\left(\frac{1}{n}\right)\right) g_c \left(j - \frac{j^2}{n}\right)^{-5/2}.$$

Proof. As $n - j \geq \frac{n}{2}$, we have $g_c(n-j)$ different choices for the largest component and $g(j)$ different choices for the rest, as in this rest there cannot be a bigger component. Furthermore, one can choose the $n-j$ vertices in $\binom{n}{j}$ different ways. With this we have:

$$P(|L_1| = n-j) = \binom{n}{j} \frac{g_c(n-j)\, g(j)}{g(n)}.$$

Using Theorem 3.6, we get:

$$\begin{aligned} P(|L_1| = n-j) &= \left(1 + O\left(n^{-1}\right)\right) \frac{n! g_c (n-j)^{-5/2} \gamma^{n-j} (n-j)! g j^{-5/2} \gamma^j j!}{j!\,(n-j)! g n^{-5/2} \gamma^n n!} \\ &= g_c \left(\frac{j\,(n-j)}{n}\right)^{-5/2}. \end{aligned}$$

□

Furthermore, using this we also get the following:

Lemma 4.6. *There exist c, n_0 constants such that for all $n \geq n_0$ and all j*

$$P(|L_1| \leq n-j) \leq C j^{-\frac{3}{2}}. \tag{51}$$

Proof. As we have seen in the previous lemma, we have for any fixed $j \leq i \leq \frac{n}{2}$

$$P(|L_1| = n - i) = g_c \left(\frac{i(n-i)}{n} \right)^{-5/2} \leq C' i^{-5/2}$$

for some constant C', as $0 < \left(\frac{(n-i)}{n} \right)^{-5/2} \leq 1$. From this we get:

$$\sum_{i=j}^{\frac{n}{2}} P(|L_1| = n - i) = P\left(\frac{n}{2} \leq |L_1| \leq n - j \right) \leq C' \sum_{i=j}^{\frac{n}{2}} i^{-5/2} \leq C'' j^{-\frac{3}{2}},$$

as one can approximate the sum by the integral.

Thus one has to bind the term $P\left(|L_1| \leq \frac{n}{2}\right)$. To do this, we can observe that in this case, there exists a partitioning of the vertex set in two sets V_1, V_2 such that both have size in $[\frac{n}{3}, \frac{2n}{3}]$, $|V_1| + |V_2| = n$ and there is no edge between the sets. Using the number of possibilities for these sets gives an upper bound on the number of graphs with largest component smaller than $\frac{n}{2}$. So

$$\begin{aligned} P\left(|L_1| \leq \frac{n}{2}\right) &\leq \frac{\sum_{i=\frac{n}{3}}^{\frac{n}{2}} \binom{n}{i} g(i)\, g(n-i)}{g(n)} \\ &= \left(1 + O\left(n^{-1}\right)\right) g \sum_{i=\frac{n}{3}}^{\frac{n}{2}} \left(\frac{i(n-i)}{n} \right)^{-5/2} \\ &\leq \bar{c} n^{-3/2} \end{aligned}$$

for n big enough. Using these two estimates, we get the result. □

With these two lemmas, we can move on to complex series-parallel graphs.

Lemma 4.7. *Let $C(k, k+l)$ be as in Section 3.1. If $l = O\left(k^{1/3}\right)$, then the size of the largest component L_1 is $k - O\left(\frac{k}{l}\right)$.*

Proof. From Lemma 4.6 for cubic graphs we have $P(|L_1| \leq n - j) \leq C j^{-\frac{3}{2}}$ for all j. From this, we have that this probability tends to 0, if j tends to infinity. The largest component has then a size of $n - O(1)$. Furthermore, if $l = O\left(k^{1/3}\right)$, the exponential factor in formula (35) for $C(k, k+l)$ is bound by some constant. In this case we have $C(k, k+l) = c\gamma^{2l} l^{-\frac{5}{2}} k^{k + \frac{3l-1}{2}} e^{\frac{3l}{2}}$. We can do the same thing as in Lemma 4.6 to get the size of the kernel of the biggest component to be $2l - O(1)$, as the total kernel size is $2l - d$ with $d = O\sqrt{l^3 k^{-1}} = O(1)$. The rest of the vertices are then added to the kernel as trees. The expected size of such a tree on each vertex of the kernel is $\frac{k-2l+O(1)}{2l-O1}$ and hence the size of the biggest component is expected to be $(2l - O(1)) \frac{k-2l+O(1)}{2l-O1} = k - O(k/l)$. □

Suppose we are before or in the first critical phase. Then the structure of the biggest components is the same as in general random graphs, as a general random graph is series-parallel with some fixed probability bounded away from 0. In the subcritical phase the $j-$th largest component has $L_j = \left(\frac{1}{2} + o(1)\right) \frac{n^2}{s^2} \log\left(\frac{|s^3|}{n^2}\right)$ vertices for any fixed j and is a tree . In the critical range, the $j-$th largest component has $L_j = \theta\left(n^{2/3}\right)$ vertices and all complex components (if any) have this size. For these statements see Łuczak [21, 22]and Łuczak, Pittel and Wierman [24]. For finding the size of the biggest component in the first supercritical range, we will use Lemma 4.7 which states that the size of the biggest component is asymptotically $k - O(k/l)$, if $l = O\left(k^{1/3}\right)$.

Theorem 4.8. *Let $S(n, M)$ be a random series-parallel graph with n vertices and $M = \frac{n}{2} + s$ edges, where $n^{2/3} \ll s \ll n$. Then its biggest component has $2s + O\left(n^{2/3}\right)$ vertices. If $M = \alpha n$ for $\frac{1}{2} < \alpha < 1$, then the largest component has $(2\alpha - 1 + o(1))\, n$ vertices. Also, a.a.s. this component is complex.*

Proof. In this region, we have $k = 2s(1 + (1))$ and $l = O\left(k\frac{(n+2s-2k)^{2/3}}{(n-k)^{4/3}}\right) = O\left(kn^{-2/3}\right) = o\left(ss^{-2/3}\right) = o\left(s^{1/3}\right) = o\left(k^{1/3}\right)$. We can use Lemma 4.7 and get in this case the following statement:

$$\begin{aligned} k - O\left(\frac{k}{l}\right) &= 2s + O\left(n^{2/3}\right) - O\left(\frac{s}{sn^{-2/3}}\right) \\ &= 2s + O\left(n^{2/3}\right) \end{aligned}$$

vertices.

In the middle range for $M = \alpha n$, $\frac{1}{2} < \alpha < 1$, we have $l = O\left(n^{1/3}\right)$ and $k = n(2\alpha - 1 + o(1))$. Here we can again use Theorem 4.7 to get for the size of the biggest component:

$$
\begin{aligned}
L_1 &= k - O\left(\frac{k}{l}\right) \\
&= (2\alpha - 1 + o(1))\, n - O\left(n^{\frac{2}{3}}\right) \\
&= (2\alpha - 1 + o(1))\, n.
\end{aligned}
$$

□

Concluding this, we have that at the point $M = \frac{n}{2} + cn^{2/3}$ the size of the largest component jumps from $L_1 = \left(\frac{1}{2} + o(1)\right) \frac{n^2}{s^2} \log\left(\frac{|s^3|}{n^2}\right)$ for $s \ll -n^{2/3}$ as one of many of this size to a single largest component of size $2s + O\left(n^{2/3}\right)$ for $s \gg n^{2/3}$, the giant component. Nonetheless, the excess of the largest component remains at $O(n^{1/3})$ throughout the range $M = \alpha n$, $\frac{1}{2} < \alpha < 1$, as seen in Theorem 4.2. This shows that, although complex, the largest component does remain relatively sparse throughout this region. Suppose $M = \alpha n$ for some $\alpha = 2 - \epsilon$. Then the graph is a.a.s. connected, as a series-parallel graph has less than $2n$ edges. In this case, the excess of this component is of order n. Therefore at some point this behaviour has to change. We will show that this change does occur at $M = n + cn^{3/5}$.

As we have seen in Theorem 4.1, for $M = n(1 + o(1))$ the deficiency is asymptotically bigger than one and so Lemma 4.7 cannot be used. Kang and Łuczak proposed in [18] that one can expand the region as to where this lemma holds, but were not able to prove it. One has to find another way of calculating the largest component. Looking at the $M = \alpha n$, $\frac{1}{2} < \alpha < 1$, one can conclude, that the size of the largest component is asymptotically the same as the number of vertices in the complex component, as both have a size of $(2\alpha - 1)n + O(n^{2/3})$, as seen in theorems 3.18 and 4.8. From this one can conclude that also for $M = (1 + o(1))n$, the main terms of these will be the same, as by adding more edges the size of the largest component will not decrease. Using this, we get the following result.

Theorem 4.9. *Let $M = n + s$ with $-n \ll s \ll n^{2/3}$. Then at most $O(l - s)$ vertices are not in the complex component where l is the expected excess of the complex part.*

Proof. Like before, asymptotically almost all vertices of the complex part of the graph are in the giant component. From this we have that the asymptotic number of vertices not in the giant component is determined by the number of vertices in all trees and unicyclic components. As we have seen in Lemma

3.19 the number of vertices in these components is $2(l-s)+O((l-s)^{2/3})$ where l is as described as above. $\square$

We can use this to get the results on the structure of L_1.

Theorem 4.10. *Let $M = n+s$ and L_1 be the size of the giant component. Then the following holds.*

(i) If $-n \ll s \ll -n^{3/5}$, then the giant component has $n-(2+o(1))\,|s|$ vertices and an excess of $\left(\frac{\gamma^{4/3}}{3^{3/2}2}+o(1)\right)ns^{-2/3}$.

(ii) If $s = cn^{3/5}$ and b such that $b^{3/2}(b-c) = \frac{\gamma^{4/3}}{2\cdot 3^{3/2}}$, then the giant component has $n-(2(b-c)+o(1))n^{3/5}$ vertices and an excess of $(b+o(1))n^{3/5}$.

(iii) If $n^{3/5} \ll s \ll n^{2/3}$, then the expected number of vertices in the giant component is $n-\left(\frac{\gamma^2}{3^{3/2}}+o(1)\right)n^{3/2}s^{-3/2}$ and it has an excess of $s\left(1+O\left(n^{3/2}s^{-5/2}\right)\right)$.

Proof. In all three cases, the excess is an immediate consequence of Theorem 4.2 and the fact that the giant component is approximately the entire complex part of the graph. As to obtain the size of the giant component, we use the values for ex_c from the same theorem and the fact from the previous theorem that L_1 is approximately $n-(2(\mathrm{ex}_c-s)+O((\mathrm{ex}_c-s)^{2/3}))$. $\square$

The reason for limiting s with $s \ll n^{2/3}$ is due to the fact that for s bigger than this value the error term for the size of the giant component is bigger than $O(\sqrt{n})$ which is too big for exact structural properties. For the rest of the region, we get a sudden increase in the density at this values of M from $\mathrm{ex}_c = \Theta(n^{1/3})$ at $s \to -n$ up to $\Theta(n^{2/3})$ for $s = \Theta(n^{2/3})$. Also this is the right parameterisation of the region as at $s = cn^{3/5}$ the asymptotics of the excess changes from $\frac{n}{s^{2/3}}$ to s.

5 Discussion

5.1 Outerplanar graphs

Outerplanar graphs is a class of graphs, which contains forests, but is contained in the class of series-parallel graphs. They are defined as follows.

Definition 5.1. *A graph G is called outerplanar if there exists an embedding in the plane such that there exists a face, which is adjacent to all vertices in the graph.*

It has been shown that there is also a characterization for outerplanar graphs in terms of forbidden minors (e.g. in [8]). This characterization states clearly why in most cases series-parallel graphs and outerplanar graphs are looked at simultaneously.

Theorem 5.1. *A graph G is outerplanar iff G does not contain K_4 and $K_{2,3}$ as minor.*

Taking this into consideration, one would think that the method stated in this thesis would also yield similar results for the class of outerplanar graphs, but there are some unexpected difficulties that arise if one tries the same way to prove similar statements.

The first part, as seen in Section 3.2, works exactly the same with the following changes.

Theorem 5.2. *Let $G(x)$, $B(x)$, $D(x)$, $S(x)$ as in the case of series-parallel graphs and let*

- *$P_1(x)$ be the EGF of all graphs with root e such that e is a multiple edge with f and $G - \{e, f\}$ is connected.*
- *$P_2(x)$ be the EGF of all graphs with root e such that e is a single edge and $G - \{s, t\}$ is not connected.*

Then the system of equations for these generating functions is

$$3x\frac{dR}{dz}(z) = B(z) + D(z) + S(z) + P_1(z) + P_2(z) \tag{52}$$

$$B(z) = (D(z) + S(z) + B(z) + P_1(z))\frac{z^2}{2} + \frac{z^2}{2} \tag{53}$$

$$D(z) = \frac{B(z)^2}{z^2} \tag{54}$$

$$S(z) = (B(z) + S(z) + P_1(z))(B(z) + P_1(z)) \tag{55}$$

$$P_1(z) = z^2(B(z) + S(z) + P_1(z)) + \frac{z^2}{2} \tag{56}$$

$$P_2(z) = \frac{z^2}{2}(B(z) + S(z) + P_1(z))^2. \tag{57}$$

Proof. The difference between outerplanar graphs and series-parallel graphs is that $K_{2,3}$ is not allowed as minor. One has to restrict the system for series-parallel graphs in a way that a $K_{2,3}-$minor cannot occur.

As P_1 and P_2 cover all graphs covered in P in definition 3.5, if one replaces P_1 by $P_1 + P_2$ on all right sides above (except the first), one gets the system for series-parallel graphs. Therefore one has to show that by generating a P_2-root, one would have a $K_{2,3}-$minor and by all other constructions one cannot get a $K_{2,3}-$minor.

By replacing a $b-$, $s-$, p_1- or p_2-root, one replaces a path of at least length 2 by a new root. In the case where this root is a p_2-root, there are at least two additional paths of length 2 from s to t. As a consequence we would have had a $K_{2,3}-$minor before the substitution.

If the root is a p_2-root, the path deleted was only a path of length one, as it can only be the first edge used as root and does not occur later in the process. Hence it does not delete any $K_{2,3}$.

Also, the substitution of $b-$, $d-$, $s-$ and p_1-roots cannot delete possible $K_{2,3}-$minors from G, as these replacements do affect at most one of the three paths needed for a $K_{2,3}$. Also, if the corresponding path shrinks to a path of length one, this edge is the root in the next step and again only a $b-$, $d-$, $s-$ or p_1-root. As such it does again only use one possible path.

It remains to show that these substitutions only affect one path. $d-$roots do not affect $K_{2,3}-$minors at all, as their substitution does not shorten any paths possibly occurring in such a minor. Itself cannot be part of one also as it is a bridge and as such cannot occur in a 2-connected component.

Similarly, $b-$roots affect only the length of one path from 2 to 1. The third possible path from the deleted vertex is the one leading only to the $b-$root. Therefore the deleted vertex cannot be one of the two vertices connected by the three different paths. Therefore only one path was affected. A replacement of a p_1-root deletes two vertices. But these were connected by one non-root edge. This edge did not emerge by replacing more than one edge. So also these two vertices cannot be one of the vertices connected by three different paths in $K_{2,3}$.

Finally, the replacement of an $s-$root partitions the graph in two parts. The two corresponding vertices have to be in the same part, as there were only two connections between the parts. Consequently the cut off part was only on one path. A $K_{2,3}-$minor occurring at the start would be also present in the end, but the only graphs at the end are a graph with two vertices with three edges between them and a graph with two vertices, one loop at each end and one connection between them. Both of these do not contain a $K_{2,3}-$minor. □

Again, using the same method as in the series-parallel case, let $H(z) = B(z) + S(z) + P_1(z)$. One then gets the equation

$$\begin{aligned} 0 &= 4H^4z^4 + H^3z^2\left(4+12z^2\right) + H^2\left(-4+20z^2+13z^4\right) \\ &\quad + H\left(-8+24z^2+6z^4\right) + z^2\left(8+z^2\right). \end{aligned} \tag{58}$$

The discriminant of this polynomial is $1024(80z^4 - 640z^6 + 1076z^8 + 504z^{10} - 61z^{12} + 2z^{14})$ and rises the main singularity at $z = 0.440$ with the corresponding value $H = 0.787$. Again the main terms cancelling in $p(H + 0.787, z+0.440)$ are H^2 and z. We get again $H(z) = \pm 3.380(z-0.440)^{1/2}(1+o(1))$. Analogously we get for R_{opl}, as in the series-parallel case, $r_n = 0.422n^{-5/2}\gamma_o^n(1 + o(1))$, where $\gamma_o = \frac{1}{0.440} = 2.273$. Comparing this with the growth rate of series-parallel graphs, $\gamma = 2.392$, this is a believable difference, as the class of series-parallel graphs is clearly bigger.

We will try to prove a statement similar to Theorem 3.11. This will not work exactly the same, as one cannot put vertices on all the edges in the kernel, because of the possibility of getting $K_{2,3}$-minors. To do this, we would have to replace the number of edges e in the kernel by the number of edges e' in the kernel on which one is allowed to put additional vertices. For this number holds the following inequality.

Proposition 5.3. *In the setting of Theorem 3.11 for outerplanar graphs one has to replace* e *by* e' *with* $\frac{e}{3} \leq e' \leq e$.

Proof. Obviously $e' \leq e$ as the number of allowed edges is a subset of all the edges. We want to find a lower bound on the number of allowed edges. Suppose the kernel is partitioned in its 3-connected components and the edges between them. Clearly all edges between such components are allowed, as $K_{2,3}$ is 2-connected. Now each 2-connected outerplanar multigraph has a unique Hamiltonian cycle (up to multiple edges). This is true because of the following observations. Each 2-connected outerplanar graph has a Hamiltonian cycle, because all vertices have to lie at the outer face and the border of this face is a Hamiltonian cycle. However, if there would be two Hamiltonian cycles differing by a permutation of the vertices, then the edges of these cycles would form a K_4-minor. All the edges (one per multiple edge) from this Hamiltonian cycle are allowed, and all the others are not. This holds, because, a vertex on a diagonal would imply a $K_{2,3}$-minor, as would vertices on two edges in a multiple edge. Furthermore, after inserting all vertices, the core is a graph without multiple edges or loops. Hence all diagonals are single edges and all multiple edges on the Hamiltonian cycle have at most two edges and therefore we have n allowed edges and at most $n + (n - 3)$ forbidden edges per 2-connected component with n vertices. As a consequence at least $\frac{1}{3}$ of all edges are allowed. □

The problem is that this factor of $\frac{1}{3} \leq \delta \leq 1$ does appear in the leading terms of all subsequent calculations not only as a factor but actively changing the asymptotics. Nonetheless it is strongly expected that nearly the same estimates as for series-parallel graphs do also hold in the case of outerplanar graphs.

5.2 Comparisons to planar graphs

If one compares the results of this thesis with the results on planar graphs given by Kang and Łuczak in [18], one will find that the results do not differ by very much. The most notable difference is in at the beginning of the counting for 3-regular graphs, where series-parallel graphs have an asymptotic of $g(n) = (1 + O(n^{-1}))gn^{-5/2}\gamma^n n!$ an planar graphs have an asymptotic of $g(n) = (1+O(n^{-1}))gn^{-7/2}\gamma^n n!$. The differences in the number of such graphs is as follows.

Let $\delta(n, M) := \frac{sp(n,M)}{pl(n,M)}$ where $pl(n, M)$ is the number of planar graphs with n vertices and M edges. Then:

(i) $\delta(n, \frac{n}{2} + s) = 1 + o(1)$, if $s \ll -n^{2/3}$,

(ii) $\delta(n, \frac{n}{2} + s) = \Theta(1)$, if $s = cn^{2/3}$,

(iii) $\delta(n, \frac{n}{2} + s) = \Theta(1)n^{-2/3}s\exp(-1.88sn^{-2/3})$, if $s \gg n^{2/3}$,

(iv) $\delta(n, \alpha n) = \Theta(1)n^{1/3}\exp(-Cn^{1/3})$ for some positive value C dependent only on α for $\frac{1}{2} < \alpha < 1$.

Also in this region, the excess of the complex part and the size of the core differ by a factor of $\Theta(1)$. Furthermore if $M = n(1 + o(1))$ the excess of the complex component and the core differ again by a factor of $\Theta(1)$. As in this case the values of $pl(n, M)$ and $sp(n, M)$ depend strongly on the value of the excess, these values do strongly differ in this region, although they are structurally nearly the same. Finally, the number of vertices in the largest component is a.a.s. the same in both graph models. These statements can easily be seen by comparing the results from this thesis with [18].

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