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Approximation of Schrödinger operators with δ-interactions supported on hypersurfaces

MASTERARBEIT

zur Erlangung des akademischen Grades

Diplom-Ingenieur

Masterstudium Technomathematik

eingereicht an der

Technischen Universität Graz

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Institut für Numerische Mathematik
Graz, Oktober 2014

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Acknowledgement

At this place, I would like to thank the people who supported me during my studies and in writing this thesis. First of all, I thank Prof. Dr. Jussi Behrndt for giving me the interesting topic of this work and for the excellent supervision of this thesis. Furthermore, I would like to thank him for giving me the possibility to take part at the 3rd Najman Conference on spectral problems for operators and matrices in Biograd in September 2013 and at the 85th Annual Meeting of the International Association of Applied Mathematics and Mechanics in Erlangen in March 2014. Here, I also thank the Austrian Science Fund (FWF) for financial support for the travel and accommodation costs.

Furthermore, I gratefully thank my second advisor Dr.tech. Vladimir Lotoreichik for always taking time for debates on the thesis. Moreover, I would like to thank the other members of the group of Prof. Dr. Jussi Behrndt, Dipl.-Math. Christian Kühn and Dipl.-Math. Dr.rer.nat. Jonathan Rohleder, who helped me in some questions with useful hints. I also thank Prof. Dr. Pavel Exner, who discussed the main ideas of this thesis with me during his stay in Graz in December 2013, and Dr. Alessandro Michelangeli and the Center for Advanced Studies in Munich for inviting me to a workshop in Munich in February 2014.

Finally, I thank Thomas Holzmann and Christine Dollinger for proof-reading my thesis and my parents Roswitha and Christian Holzmann for supporting me during the whole studies.

1 Introduction

In this thesis, we show that a Schrödinger operator $A_{\delta,\alpha}$ with a δ-interaction of strength α supported on a hypersurface Σ can be approximated by a family of Hamiltonians with local scaled short-range potentials. Here, the differential operator $A_{\delta,\alpha}$ is viewed as a self-adjoint realization of the formal differential expression $-\Delta - \alpha\langle\delta_\Sigma, \cdot\rangle\delta_\Sigma$ and $\Sigma \subset \mathbb{R}^d$ is a compact and closed C^2-hypersurface.

Schrödinger operators with singular δ-interactions provide an important concept in the field of mathematical physics and have gained a lot of attention in the last decades. Such operators are formally given as

$$\mathcal{L}_{\alpha,\Sigma} := -\Delta - \alpha\langle\delta_\Sigma, \cdot\rangle\delta_\Sigma,$$

where Σ is generally a subset of $\mathbb{R}^d$ of Lebesgue measure zero, δ_Σ is the δ-distribution supported on Σ and $\alpha : \Sigma \to \mathbb{R}$ is called the strength of the interaction. Differential operators associated to $\mathcal{L}_{\alpha,\Sigma}$ are used as idealized models to solve approximately the spectral problem for classical Hamiltonians $H = -\Delta - V$, where the potentials V are real-valued, supported in a small neighborhood of Σ and have relatively large values. Such operators appear in quantum mechanics in the description of many body systems or in models for so-called leaky quantum graphs that describe the motion of a particle confined to a graph Σ in a way such that quantum tunnelling effects between different parts of it are allowed. Moreover, such operators arise in the theory of sound and electromagnetic wave propagation, where δ-potentials are used to model high contrast objects in dielectric media.

The first time, when a differential operator with δ-interactions was treated in the literature, was in 1931. In [40] de Kronig and Penney constructed a simple model for the motion of a nonrelativistic electron in a one-dimensional crystal using the differential expression $\mathcal{L}_{\alpha,\Sigma}$ with $\Sigma = \mathbb{Z}$ and $\alpha = \text{const}$. In the subsequent decades Hamiltonians with point interactions (i.e. Σ is a set of points) were treated in various dimensions in a heuristic way. Then in 1961, Berezin and Faddeev published the first rigorous mathematical work on a differential operator associated to the formal differential expression $\mathcal{L}_{\alpha,\Sigma}$. In [13] these authors considered the case $d = 3$ and $\Sigma = \{0\}$ and they constructed a differential operator in the Hilbert space $L^2(\mathbb{R}^3)$ associated to $\mathcal{L}_{\alpha,\Sigma}$, which will be denoted from now on by $A_{\delta,\alpha}$, as a self-adjoint extension of $-\Delta$ restricted to $C_c^\infty(\mathbb{R}^3 \setminus \{0\})$. In the following decades, differential operators with δ-interactions supported on a finite or an infinite set of points in space dimension $d = 1, 2, 3$ were investigated extensively in the literature, see for instance the textbook [3] or [2, 4, 6, 19, 33, 36, 37, 39].

In numerous applications it is required to consider operators associated to $\mathcal{L}_{\alpha,\Sigma}$ in the case that Σ is a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$, a surface in $\mathbb{R}^3$ or more generally a submanifold in $\mathbb{R}^d$ with codimension $1, 2$ or 3. The analysis of such operators is more complicated as for operators with point interactions, as the spectral properties of these Hamiltonians are connected to the geometrical properties of Σ. Hence, the available results are less complete as in the point interaction case, see the review paper [20] and for instance [12, 15, 21, 24, 38].

From the mathematical point of view, it requires a justification that the spectral properties of the Hamiltonian $A_{\delta,\alpha}$ corresponding to the formal differential expression $\mathcal{L}_{\alpha,\Sigma}$ are close to those of a classical Schrödinger operator of the form $H = -\Delta - V$, where V is a real-valued potential supported in a neighborhood of Σ with relatively large values, and thus that $A_{\delta,\alpha}$ can be used as an idealized model for H. One way to justify this is to show that $A_{\delta,\alpha}$ can be approximated in the norm resolvent sense by a family of Hamiltonians of the form

$$H_{\varepsilon,\Sigma} := -\Delta - V_\varepsilon \tag{1.1}$$

for an appropriately constructed family of potentials V_ε, as the spectral properties of $H_{\varepsilon,\Sigma}$ are then close to those of $A_{\delta,\alpha}$.

The approximation of Hamiltonians with singular interactions supported on a set of points was intensively studied in the past. Already in 1935 Thomas published an influential paper [47] that contained an approximation procedure of a Schrödinger operator with a δ-point interaction in $\mathbb{R}^3$. Starting in the 1970s, the approximation of differential operators with δ-interactions supported on a finite or an infinite set of points in $\mathbb{R}^d$, $d \in \{1, 2, 3\}$, was treated systematically in the literature, see the textbook [3] or [2, 4, 5, 10, 16, 27, 34, 35]; the available results can be seen as quite complete. There are also several recent results available for the approximation of models with more singular point interactions in $\mathbb{R}$, for instance of δ'-type, that yield more realistic models in some applications, cf. [7, 17, 23, 29, 30, 31, 32, 48].

Concerning Hamiltonians with δ-interactions supported on manifolds in $\mathbb{R}^d$, $d \geq 2$, the literature on approximation results is not that complete. Whereas to the best of our knowledge there are no results on the approximation of Schrödinger operators with δ-potentials supported on manifolds with codimension larger than one available, there exist a few results in the case that the interaction is supported on a hypersurface (i.e. a manifold with codimension 1) in $\mathbb{R}^2$ and $\mathbb{R}^3$. The first work in this context was published in 1992 by Shimada [46]. He proved the convergence of operators $H_{\varepsilon,\Sigma}$ of the form (1.1) to $A_{\delta,\alpha}$ in the norm resolvent sense, if Σ is a sphere in $\mathbb{R}^3$ and α is continuous. Popov extended this result for hypersurfaces $\Sigma \subset \mathbb{R}^d$, $d \in \{2, 3\}$, that can be parametrized by polar coordinates [44]. Figotin and Kuchment investigated the convergence of operators $H_{\varepsilon,\Sigma}$ in the case that Σ is an unbounded and periodic hypersurface in $\mathbb{R}^d$. Their motivation came from the theory of electromagnetic waves and they used a different notion of convergence which employs the periodicity of Σ, cf. [25, 26]. Finally, Exner, Ichinose and Kondej considered the case of unbounded hypersurfaces in $\mathbb{R}^2$ and $\mathbb{R}^3$ with a global parametrization and constant strength $\alpha \in \mathbb{R}$ in [21, 22].

Nevertheless, the available literature on the approximation of $A_{\delta,\alpha}$ is not complete, since the known results were obtained under restrictions on the space dimension d, the hypersurface Σ or the strength α. It is our main objective in this thesis to extend the established results. More precisely, we prove that $A_{\delta,\alpha}$ can be approximated by a family of Hamiltonians with local scaled short-range potentials in the norm resolvent sense for any space dimension $d \geq 2$, for any compact and closed C^2-smooth hypersurface $\Sigma \subset \mathbb{R}^d$ and for any strength $\alpha \in L^\infty(\Sigma)$.

In what follows, we give a short description of our approximation procedure and we state our main result. Let $d \geq 2$ and let $\Sigma \subset \mathbb{R}^d$ be a closed, connected, compact and C^2-smooth hypersurface that separates $\mathbb{R}^d$ into a bounded part $\Omega_{\rm i}$ and an unbounded part $\Omega_{\rm e}$ (i.e. Σ is the boundary of the bounded C^2-domain $\Omega_{\rm i}$). It is known from [12] that for a strength $\alpha \in L^\infty(\Sigma)$ the operator $A_{\delta,\alpha}$ has the explicit form

$$\begin{aligned} A_{\delta,\alpha} f &= -\Delta f, \\ \operatorname{dom} A_{\delta,\alpha} &= \big\{ f \in H^{3/2}_\Delta(\mathbb{R}^d \setminus \Sigma) : f_{\rm i}|_\Sigma = f_{\rm e}|_\Sigma,\ \alpha f_{\rm i}|_\Sigma = \partial_{\nu_{\rm e}} f_{\rm e}|_\Sigma + \partial_{\nu_{\rm i}} f_{\rm i}|_\Sigma \big\}, \end{aligned} \tag{1.2}$$

where $f_{\rm i}$ and $f_{\rm e}$ stand for the restrictions of a function f on $\mathbb{R}^d$ onto $\Omega_{\rm i}$ and $\Omega_{\rm e}$, respectively, and $\partial_{\nu_{\rm i}} f_{\rm i}|_\Sigma$ and $\partial_{\nu_{\rm e}} f_{\rm e}|_\Sigma$ denote the derivatives in direction of the unit outward normal $\nu_{\rm i}$ and $\nu_{\rm e}$ of $\Omega_{\rm i}$ and $\Omega_{\rm e}$, respectively.[1] Observe that self-adjoint Hamiltonians with δ-interactions supported on non-closed surfaces are naturally contained in the above scheme as α may be zero on subsets of Σ. This is one of the main reasons to allow general strength coefficients $\alpha \in L^\infty(\Sigma)$. The following approximation procedure of $A_{\delta,\alpha}$ is inspired by [3, 21, 22]. For a small $\beta > 0$ we set

$$\Omega_\beta := \{ x_\Sigma + t\nu_{\rm i}(x_\Sigma) : x_\Sigma \in \Sigma, t \in (-\beta, \beta) \} \tag{1.3}$$

and we choose a fixed real-valued potential $V \in L^\infty(\mathbb{R}^d)$ supported in Ω_β. Moreover, we define for $\varepsilon \in (0, \beta]$ the scaled potential $V_\varepsilon \in L^\infty(\mathbb{R}^d)$ as

$$V_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon} V\left(x_\Sigma + \frac{\beta}{\varepsilon} t \nu_{\rm i}(x_\Sigma) \right), & \text{if } x = x_\Sigma + t\nu_{\rm i}(x_\Sigma) \text{ with } x_\Sigma \in \Sigma,\ t \in (-\varepsilon, \varepsilon), \\ 0, & \text{else.} \end{cases} \tag{1.4}$$

Note, that V_ε is well-defined; this follows from a theorem in Section 3.3. Then our main result, which is proved in Chapter 7, reads as follows:

Theorem I. *Let $d \geq 2$, let $V \in L^\infty(\mathbb{R}^d)$ be real-valued such that the support of V is contained in Ω_β, let V_ε be given as in (1.4), let $H_{\varepsilon,\Sigma}$ be given as in (1.1) and let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Moreover, define the coupling $\alpha \in L^\infty(\Sigma)$ as*

$$\alpha(x_\Sigma) := \int_{-1}^{1} V(x_\Sigma + \beta s \nu_{\rm i}(x_\Sigma)) \mathrm{d}s$$

for almost all $x_\Sigma \in \Sigma$ and $A_{\delta,\alpha}$ as in (1.2). Then there exists a constant $c > 0$ such that

$$\left\| (H_{\varepsilon,\Sigma} - \lambda)^{-1} - (A_{\delta,\alpha} - \lambda)^{-1} \right\| \leq c \varepsilon^{\frac{1}{2d}}$$

holds for all sufficiently small $\varepsilon > 0$. In particular, $H_{\varepsilon,\Sigma}$ converges to $A_{\delta,\alpha}$ in the norm resolvent sense as $\varepsilon \to 0+$.

[1] The space $H^{3/2}_\Delta(\mathbb{R}^d \setminus \Sigma)$ in (1.2) consists of functions $f = (f_{\rm i}, f_{\rm e})^\top$, where the components $f_{\rm i}$ and $f_{\rm e}$ belong to the fractional order Sobolev spaces $H^{3/2}(\Omega_{\rm i})$ and $H^{3/2}(\Omega_{\rm e})$, respectively, and satisfy $\Delta f_{\rm i} \in L^2(\Omega_{\rm i})$ and $\Delta f_{\rm e} \in L^2(\Omega_{\rm e})$.

Let us provide an overview of the content of this thesis. In Chapter 2, the basic notions and notations of the spectral theory of linear operators are introduced. In particular, we discuss convergence in the norm resolvent sense, a notion which is appropriate to investigate the convergence of a sequence of unbounded self-adjoint operators.

In Chapter 3, we discuss closed connected hypersurfaces in a form that is convenient for our purposes. In Section 3.1, we define hypersurfaces and various geometrical properties of them. Then in Section 3.2, we introduce integration on hypersurfaces with respect to the Hausdorff measure. Finally, in Section 3.3, we investigate tubes Ω_β of the form (1.3) around closed compact C^2-hypersurfaces.

Then, in Chapter 4, we discuss the function spaces which are required for the definition and the analysis of our differential operators. We consider classical function spaces of continuously differentiable and Lebesgue measurable functions and introduce Sobolev spaces of weakly differentiable functions defined in open domains and on hypersurfaces. Furthermore, we introduce the Dirichlet and the Neumann trace of a weakly differentiable function and we state a generalized version of Green's identity.

In Chapter 5, we state several results about auxiliary operators which will play an important part in our considerations. In Section 5.1, we introduce multiplication operators in $L^2(X, \mu)$. Then, in Section 5.2, we discuss integral operators and we give upper bounds for the operator norms of such operators assuming that the integral kernels of these operators fulfill suitable conditions. In particular, we prove the Schur-Holmgren bound for the norm of an integral operator - this is one of the main tools for our proof of the convergence of $H_{\varepsilon,\Sigma}$ to $A_{\delta,\alpha}$ in the norm resolvent sense. In Section 5.3, we state some well-known results about the spectral properties of classical Schrödinger operators. Finally, in Section 5.4, we introduce the single layer potential associated to the differential expression $-\Delta + 1$ and the hypersurface Σ.

In Chapter 6, we introduce Hamiltonians $A_{\delta,\alpha}$ with a δ-interaction supported on a hypersurface Σ of strength $\alpha \in L^\infty(\Sigma)$ in a mathematically rigorous way. Here, we follow an extension theoretic approach from [12]. In particular, we derive a suitable resolvent formula for $A_{\delta,\alpha}$.

Finally, in Chapter 7, we prove the main result of this thesis, namely Theorem I. For this purpose, we derive a suitable resolvent formula for the Hamiltonians $H_{\varepsilon,\Sigma}$ given by (1.1) and we show the convergence of this family of resolvents. Since the proof of the convergence is very technical and long, we outsource some parts of it into Appendix A.

2 Basic concepts of operator theory

The main aim in this thesis is to analyze the convergence of a certain family of self-adjoint Hamiltonians to a Schrödinger operator with a δ-interaction supported on a hypersurface in $\mathbb{R}^d$. In this chapter, we introduce the basic concepts of the operator theory which are needed for this purpose. In particular, we have to define an appropriate notion of convergence of a sequence of (unbounded) self-adjoint operators. But initially, we constitute some notations, introduce linear operators in Hilbert spaces and state some important basic results of the spectral theory of linear operators in Hilbert spaces.

2.1 Linear operators in Hilbert spaces

Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces over $\mathbb{C}$. The inner product is denoted by $(\cdot,\cdot)_{\mathcal{H}}$ and $(\cdot,\cdot)_{\mathcal{K}}$. If it is clear, which Hilbert space is meant, we drop the subindex and write just $(\cdot,\cdot)$. The Cartesian product of these Hilbert spaces is denoted by $\mathcal{H} \oplus \mathcal{K}$ and the inner product in $\mathcal{H} \oplus \mathcal{K}$ is given by $(\cdot,\cdot)_{\mathcal{H}} + (\cdot,\cdot)_{\mathcal{K}}$.

A mapping $S : \operatorname{dom} S \to \mathcal{K}$, which is defined on a linear subspace $\operatorname{dom} S$ of $\mathcal{H}$ and which satisfies $S(\alpha x + \beta y) = \alpha S x + \beta S y$ for all $x, y \in \operatorname{dom} S$ and $\alpha, \beta \in \mathbb{C}$, is called a linear operator from $\mathcal{H}$ to $\mathcal{K}$. If $\mathcal{K} = \mathcal{H}$, we will also say that S is a linear operator in $\mathcal{H}$. We will often omit the term linear, as we are only considering linear operators. An operator S from $\mathcal{H}$ to $\mathcal{K}$ is said to be everywhere defined, if $\operatorname{dom} S = \mathcal{H}$, and it is called densely defined, if $\overline{\operatorname{dom} S} = \mathcal{H}$. The graph of a linear operator S is given by $\left\{ \left(\begin{smallmatrix} x \\ Sx \end{smallmatrix} \right) : x \in \operatorname{dom} S \right\}$ and is a linear subset of $\mathcal{H} \oplus \mathcal{K}$. In the following we will identify the graph of a linear operator with the operator itself, so we will also write S for the graph. In this context, for two linear operators S and T from $\mathcal{H}$ to $\mathcal{K}$ the notation $S \subset T$ is understood as $\operatorname{dom} S \subset \operatorname{dom} T$ and $Sx = Tx$ for all $x \in \operatorname{dom} S$.

The range and the kernel of a linear operator S from $\mathcal{H}$ to $\mathcal{K}$ are defined as

$$
\begin{aligned}
\operatorname{ran} S &:= \{ y \in \mathcal{K} : \exists x \in \operatorname{dom} S : Sx = y \}, \\
\ker S &:= \{ x \in \operatorname{dom} S : Sx = 0 \}.
\end{aligned}
$$

A linear operator S is said to be closed, if the graph of S is a closed subspace of $\mathcal{H} \oplus \mathcal{K}$. Moreover, an operator S is called bounded, if

$$
\|S\| := \sup_{0 \neq x \in \operatorname{dom} S} \frac{\|Sx\|_{\mathcal{K}}}{\|x\|_{\mathcal{H}}} < \infty. \tag{2.1}
$$

Recall, that the space of all bounded and everywhere defined linear operators equipped with the operator norm (2.1) is a Banach space [50, Satz 2.12] and that a linear operator is continuous, if and only if it is bounded [50, Satz 2.1]. An important connection of closed and bounded operators is stated in the following theorem, which is known as the closed graph theorem [50, Satz 4.4]:

Theorem 2.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let S be a closed operator from $\mathcal{H}$ to $\mathcal{K}$. If* $\operatorname{dom} S \subset \mathcal{H}$ *is closed, then S is bounded.*

Next we define, what is understood by the spectrum and the resolvent set of a linear operator.

Definition 2.2. *Let $\mathcal{H}$ be a Hilbert space and let S be a closed linear operator in $\mathcal{H}$.*

(i) $\lambda \in \mathbb{C}$ *belongs to the resolvent set* $\rho(S)$ *of* S *if and only if* $S - \lambda$ *is injective and* $(S-\lambda)^{-1}$ *is bounded and everywhere defined. In this case, the operator* $(S-\lambda)^{-1}$ *is called "resolvent".*

(ii) *The spectrum of* S *is defined as* $\sigma(S) := \mathbb{C} \setminus \rho(S)$.

(iii) *A number* $\lambda \in \sigma(S)$ *is called an eigenvalue of* S, *if* $S - \lambda$ *is not injective, so if there exists an eigenvector* $0 \neq x \in \operatorname{dom} S$ *such that* $Sx = \lambda x$ *holds. The set of all eigenvalues of* S *will be denoted by* $\sigma_{\mathrm{p}}(S)$.

In the following step we define the adjoint of a linear operator S.

Definition 2.3. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let S be a densely defined linear operator from $\mathcal{H}$ to $\mathcal{K}$. We define the adjoint operator S^* of S, which is a linear operator from $\mathcal{K}$ to $\mathcal{H}$, via*

$$\operatorname{dom} S^* := \{y \in \mathcal{K} : \exists y' \in \mathcal{H} : \forall x \in \operatorname{dom} S : (Sx, y)_{\mathcal{K}} = (x, y')_{\mathcal{H}}\}$$

and $S^* y := y'$.

We remark that S^* is a well-defined linear operator from $\mathcal{K}$ to $\mathcal{H}$, as S is densely defined. If S is a bounded and everywhere defined operator, one can show that also S^* is bounded and everywhere defined, see for example [50, Satz 2.36]. Some properties of the adjoint operator are summarized in the next corollary.

Corollary 2.4. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let S and T be densely defined linear operators from $\mathcal{H}$ to $\mathcal{K}$. Then the following holds:*

(i) S^* *is a closed linear operator;*

(ii) $S \subset T$ *implies* $T^* \subset S^*$;

(iii) *if* $S + T$ *is densely defined, then* $S^* + T^* \subset (S+T)^*$. *Further, if* T *is bounded and everywhere defined, then* $S^* + T^* = (S+T)^*$.

The proof of assertion (i) and (ii) can be found in [50, Satz 2.48] and item (iii) is shown in [50, Satz 2.45]. Very important classes of operators are symmetric and self-adjoint operators:

Definition 2.5. *Let $\mathcal{H}$ be a Hilbert space and let S be a densely defined linear operator in $\mathcal{H}$. Then S is called*

(i) *symmetric, if* $S \subset S^*$;

(ii) *self-adjoint, if* $S = S^*$.

In what follows we state numerous properties of symmetric and self-adjoint operators. The first proposition treats an equivalent condition to the symmetry of a linear operator. The simple proof of this statement is left to the reader.

Proposition 2.6. *Let* $\mathcal{H}$ *be a Hilbert space and let* S *be a densely defined linear operator in* $\mathcal{H}$. *Then the following are equivalent:*

(i) S *is symmetric;*

(ii) $(Sx, y) = (x, Sy)$ *holds for all* $x, y \in \operatorname{dom} S$.

Next, we mention some equivalent conditions to the self-adjointness of a symmetric operator. The proof of this statement can be found in [50, Satz 5.14 a)].

Theorem 2.7. *Let* $\mathcal{H}$ *be a Hilbert space and let* S *be a symmetric operator in* $\mathcal{H}$. *Then the following are equivalent:*

(i) S *is self-adjoint;*

(ii) $\operatorname{ran}(S - \lambda) = \mathcal{H} = \operatorname{ran}(S - \overline{\lambda})$ *holds for one, and hence for all,* $\lambda \in \mathbb{C} \setminus \mathbb{R}$*;*

(iii) $\mathbb{C} \setminus \mathbb{R} \subset \rho(S)$.

Now, we want to investigate the spectrum of a self-adjoint operator in a more detailed way:

Definition 2.8. *Let* A *be a self-adjoint operator in a Hilbert space* $\mathcal{H}$.

(i) *The essential spectrum* $\sigma_{\text{ess}}(A)$ *of* A *is defined as the set of all eigenvalues* λ *of* A *with* $\dim \ker(A - \lambda) = \infty$ *and all accumulation points of* $\sigma(A)$.

(ii) *A number* $\lambda \in \sigma(A)$ *is contained in the discrete spectrum* $\sigma_{\text{disc}}(A)$ *of* A*, if and only if* λ *is an eigenvalue of* A*, which satisfies* $\dim \ker(A - \lambda) < \infty$ *and additionally is isolated in* $\sigma(A)$.

Note, that it holds $\sigma(A) = \sigma_{\text{ess}}(A) \mathbin{\dot\cup} \sigma_{\text{disc}}(A)$ for any self-adjoint operator A. Next, we introduce another important class of linear operators, the unitary operators:

Definition 2.9. *Let* $\mathcal{H}$ *be a Hilbert space and let* U *be a linear operator in* $\mathcal{H}$. *Then* U *is called unitary, if* $\operatorname{dom} U = \operatorname{ran} U = \mathcal{H}$ *and* $\|Ux\| = \|x\|$ *holds for all* $x \in \mathcal{H}$.

Observe that each unitary operator is bounded and bijective. One property of unitary operators, which will result to be important for our considerations, is stated in the following proposition. The simple proof of this statement is left to the reader.

Proposition 2.10. *Let $\mathcal{H}$ be a Hilbert space and let T be a closed operator in $\mathcal{H}$. Further, let U be a unitary operator in $\mathcal{H}$. Then it holds*

$$\sigma(T) = \sigma(UTU^{-1}).$$

Finally, the following proposition describes a way, in which the domain of definition of an extension T of a linear operator A can be decomposed. Here, we use the notation $\dot{+}$ for the direct sum of two sets.

Proposition 2.11. *Let $\mathcal{H}$ be a Hilbert space and let T be a linear operator in $\mathcal{H}$. Moreover, let A be a closed linear operator in $\mathcal{H}$ such that $A \subset T$ and $\rho(A) \neq \emptyset$ are fulfilled. Then it holds*

$$\operatorname{dom} T = \ker(T - \lambda) \dot{+} \operatorname{dom} A$$

for all $\lambda \in \rho(A)$.

Proof. Let $\lambda \in \rho(A)$ be fixed. The inclusion $\operatorname{dom} T \supset \ker(T - \lambda) + \operatorname{dom} A$ is trivial. In order to show $\operatorname{dom} T \subset \ker(T - \lambda) + \operatorname{dom} A$, let $x \in \operatorname{dom} T$ be arbitrary. Since $\lambda \in \rho(A)$, there exists $y \in \operatorname{dom} A$ such that $(T - \lambda)x = (A - \lambda)y$ holds. If we define now $z := x - y$, we see

$$(T - \lambda)z = (T - \lambda)(x - y) = (T - \lambda)x - (A - \lambda)y = 0,$$

where we used $A \subset T$ and the definition of y. Hence, we find $z \in \ker(T - \lambda)$, which shows that $x = z + y \in \ker(T - \lambda) + \operatorname{dom} A$ is true. Thus, we also proved $\operatorname{dom} T \subset \ker(T - \lambda) + \operatorname{dom} A$, which implies now

$$\operatorname{dom} T = \ker(T - \lambda) + \operatorname{dom} A.$$

It remains to verify, that the sum is direct. For this purpose, let $x \in \ker(T - \lambda) \cap \operatorname{dom} A$. Then it holds

$$0 = (T - \lambda)x = (A - \lambda)x,$$

as $A \subset T$ and $x \in \operatorname{dom} A$. Since we have $\lambda \in \rho(A)$ by assumption, the operator $A - \lambda$ is bijective and thus $x = 0$ must be true. Hence, we find $\ker(T - \lambda) \cap \operatorname{dom} A = \{0\}$, which finishes the proof of this proposition. □

2.2 Compact operators

In this section, we discuss compact operators. The set of all compact operators is a subclass of the set of the bounded operators. Compact operators are important, as they will help us in the following chapters in the spectral analysis of Schrödinger operators.

In order to define compact operators, recall that a set M is said to be relatively compact, if its closure $\overline{M}$ is compact.

Definition 2.12. *Let $\mathcal{H}$ be a Hilbert space. A linear operator $K : \mathcal{H} \to \mathcal{H}$ is called compact, if it maps bounded sets onto relatively compact sets.*

Note, that any compact operator is bounded, as it maps the unit ball $B(0,1) \subset \mathcal{H}$ onto a relatively compact set, which is also bounded. Moreover, we mention that the set of all compact operators is obviously linear.

The next theorem is the famous Fredholm's alternative - a result about $(1-K)^{-1}$ for a compact operator K. This result follows for instance from [51, Satz VI.2.4].

Theorem 2.13. *Let $\mathcal{H}$ be a Hilbert space and let K be a compact operator in $\mathcal{H}$. Then either the operator $(1-K)^{-1}$ exists and is bounded and everywhere defined or* $\ker(1-K)$ *is a finite dimensional subspace of $\mathcal{H}$.*

In the next proposition, we collect some useful facts about compact operators. The proofs of these statements can be found in [50, Satz 3.2 and Satz 3.3].

Proposition 2.14. *Let $\mathcal{H}$ be a Hilbert space. Then the following assertions are true:*

(i) *Let K be compact and let T be a bounded and everywhere defined operator in $\mathcal{H}$. Then the operators KT and TK are compact.*

(ii) *Let (K_n) be a sequence of compact operators and let K be bounded and everywhere defined in $\mathcal{H}$. If $\|K_n - K\| \to 0$ in the operator norm* (2.1)*, then K is also compact.*

Note, that Proposition 2.14 brings us to the following result: the set of all compact operators is a closed ideal in the set of all bounded and everywhere defined operators.

The next result is an important tool in the perturbation theory of linear operators. It allows us to compare the essential spectra of two self-adjoint operators, if they are close to each other in a certain sense [50, Satz 9.15]:

Proposition 2.15. *Let $\mathcal{H}$ be a Hilbert space and let A and B be self-adjoint operators in $\mathcal{H}$. If there exists $\lambda \in \rho(A) \cap \rho(B)$ such that $(A-\lambda)^{-1} - (B-\lambda)^{-1}$ is compact, then it holds $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B)$.*

2.3 Convergence of unbounded operators

In this section, we introduce a notion for the convergence of a sequence of unbounded operators, which is needed to analyze the approximation of a differential operator with a δ-interaction supported on a hypersurface by a family of self-adjoint Schrödinger operators with regular potentials. Note, that for the set of all bounded and everywhere defined operators we have a topology which is induced by the operator norm

$$\|T\| := \sup_{0 \neq x \in \mathcal{H}} \frac{\|Tx\|_{\mathcal{H}}}{\|x\|_{\mathcal{H}}}.$$

So, a sequence of bounded and everywhere defined operators (T_n) converges to T, if and only if $\lim_{n\to\infty} \|T_n - T\| = 0$.

On the other hand, for a sequence of unbounded operators, as it is considered in this thesis, a convergence as above does not make sense. An appropriate notion of convergence for a sequence of self-adjoint operators (T_n) is the convergence in the norm resolvent sense, which will be discussed now in detail in this section following [50, Section 9.3].

Throughout this section, let $\mathcal{H}$ always be a Hilbert space and let (A_n) be a sequence of self-adjoint, in general unbounded operators in $\mathcal{H}$. Moreover, let A be another self-adjoint operator in $\mathcal{H}$. We define the set $\mathcal{M} \subset \mathbb{C}$ as

$$\mathcal{M} := \left(\bigcap_{n\in\mathbb{N}} \rho(A_n) \right) \cap \rho(A) \tag{2.2}$$

and observe $\mathbb{C} \setminus \mathbb{R} \subset \mathcal{M}$. Now, we define the notion of convergence in the norm resolvent sense.

Definition 2.16. *Let $\mathcal{H}$ be a Hilbert space and let A be a self-adjoint operator in $\mathcal{H}$. Then a sequence of self-adjoint operators (A_n) in $\mathcal{H}$ converges to A in the norm resolvent sense, if there exists a complex number $\lambda \in \mathcal{M}$ with $\mathcal{M}$ as above, such that*

$$\left\| (A_n - \lambda)^{-1} - (A - \lambda)^{-1} \right\| \to 0$$

as $n \to \infty$.

At first glance, Definition 2.16 seems to depend on a special choice of $\lambda \in \mathcal{M}$. In fact, the following proposition shows that this is not true.

Proposition 2.17. *Let $\mathcal{H}$ be a Hilbert space, let A be a self-adjoint operator in $\mathcal{H}$, let (A_n) be a sequence of self-adjoint operators in $\mathcal{H}$ and let $\mathcal{M}$ be given as in* (2.2)*. If there exists a complex number $\lambda_0 \in \mathcal{M}$ such that*

$$\left\| (A_n - \lambda_0)^{-1} - (A - \lambda_0)^{-1} \right\| \to 0,$$

then it holds for all $\lambda \in \mathcal{M}$

$$\left\| (A_n - \lambda)^{-1} - (A - \lambda)^{-1} \right\| \to 0$$

as $n \to \infty$.

The proof of this proposition is given in [50, Satz 9.20]. The following result indicates, that a sequence (T_n) of bounded and everywhere defined, but not necessarily self-adjoint operators, which converges in the operator norm, also converges in the norm resolvent sense. Therefore, the concept of convergence in the norm resolvent sense is a generalization of the usual convergence notion for bounded operators.

Proposition 2.18. *Let $\mathcal{H}$ be a Hilbert space and let T_n and T be bounded and everywhere defined operators in $\mathcal{H}$ such that $\|T_n - T\| \to 0$. Then it holds for all sufficiently large n and all $\lambda \in \left(\bigcap_{n\in\mathbb{N}} \rho(T_n)\right) \cap \rho(T)$*

$$\left\| (T_n - \lambda)^{-1} - (T - \lambda)^{-1} \right\| \leq \frac{\|(T - \lambda)^{-1}\|^2}{1 - \|T - T_n\| \cdot \|(T - \lambda)^{-1}\|} \|T - T_n\|.$$

In particular, $\left\| (T_n - \lambda)^{-1} - (T - \lambda)^{-1} \right\| \to 0$ as $n \to \infty$.

Proof. Let $\lambda \in \left(\bigcap_{n\in\mathbb{N}} \rho(T_n)\right) \cap \rho(T)$ and set $R(\lambda) := (T-\lambda)^{-1}$ and $R_n(\lambda) := (T_n-\lambda)^{-1}$. Using the resolvent identity [50, Satz 5.4]

$$R_n(\lambda) - R(\lambda) = R_n(\lambda)(T-T_n)R(\lambda), \tag{2.3}$$

we find

$$\|R_n(\lambda) - R(\lambda)\| = \|R_n(\lambda)(T-T_n)R(\lambda)\| \leq \|R_n(\lambda)\| \cdot \|T-T_n\| \cdot \|R(\lambda)\|.$$

So it remains to show, that $\|R_n(\lambda)\|$ is bounded. Using the resolvent identity (2.3) again, we see

$$\|R_n(\lambda)\| \leq \|R(\lambda)\| + \|R_n(\lambda)\| \cdot \|T-T_n\| \cdot \|R(\lambda)\|.$$

Choosing n sufficiently large, such that $\|T-T_n\| \cdot \|R(\lambda)\| < 1$ is satisfied, we find $1 - \|T - T_n\| \cdot \|R(\lambda)\| > 0$ and hence, we conclude from the above calculation

$$\|R_n(\lambda)\| \leq \frac{\|R(\lambda)\|}{1 - \|T-T_n\| \cdot \|R(\lambda)\|}.$$

This yields finally

$$\|R_n(\lambda) - R(\lambda)\| \leq \|R_n(\lambda)\| \cdot \|T-T_n\| \cdot \|R(\lambda)\| \leq \frac{\|R(\lambda)\|^2}{1 - \|T-T_n\| \cdot \|R(\lambda)\|} \|T-T_n\|,$$

which is the claimed result. $\square$

Finally, we state a result on the connection of the spectra of A_n and A, if A_n converges to A in the norm resolvent sense [50, Satz 9.24].

Proposition 2.19. *Let $\mathcal{H}$ be a Hilbert space and let A be a self-adjoint operator in $\mathcal{H}$. If (A_n) is a sequence of self-adjoint operators in $\mathcal{H}$ converging to A in the norm resolvent sense, it follows that $\sigma(A_n)$ converges to $\sigma(A)$. This means that $\lambda \in \sigma(A)$ if and only if there exists a sequence (λ_n) with $\lambda_n \in \sigma(A_n)$ such that λ_n converges to λ.*

3 Compact hypersurfaces and their basic properties

In this chapter, we introduce hypersurfaces Σ in a form which is suitable for our application, namely as the support of a δ-interaction of a Schrödinger operator $A_{\delta,\alpha}$ associated to the formal differential expression $-\Delta - \alpha\langle\delta_\Sigma, \cdot\rangle\delta_\Sigma$. Moreover, we introduce various notations and discuss some properties of hypersurfaces, which are needed for our main purpose, the approximation of $A_{\delta,\alpha}$ by a family of Schrödinger operators $-\Delta - V_\varepsilon$ with regular potentials V_ε that have support in a suitable neighborhood of Σ. In this chapter, we follow the presentation of [41, Kapitel 3], but we generalize the notations and some proofs from the three- to the d-dimensional case and adapt numerous notions to our needs.

3.1 Hypersurfaces and their basic properties

In the following section we investigate hypersurfaces in the Euclidean space $\mathbb{R}^d$ equipped with the inner product $\langle\cdot,\cdot\rangle$. First, we give a definition of a closed hypersurface, which is suitable for our needs. For this purpose, we need the notion of a C^k-hypograph:

Definition 3.1. *Let $d \geq 2$ and let $k \in \mathbb{N}$. A set $\Omega \subset \mathbb{R}^d$ is called a C^k-hypograph, if there exists $U \subset \mathbb{R}^{d-1}$ and a C^k-smooth function $\varphi_d : U \to \mathbb{R}$ such that*

$$\Omega = \left\{(x_1, \dots, x_d) \in \mathbb{R}^d : (x_1, \dots, x_{d-1}) \in U, x_d < \varphi_d(x_1, \dots, x_{d-1})\right\}.$$

Now, we introduce hypersurfaces in the form that is suitable for our purposes.

Definition 3.2. *Let $d \geq 2$ and let $k \in \mathbb{N}$. We call a set $\Sigma \subset \mathbb{R}^d$ a closed connected C^k-hypersurface or a closed connected hypersurface of C^k-smoothness, if the following conditions are satisfied:*

(i) *there exists a bounded, open and connected set $\Omega \subset \mathbb{R}^d$ such that Σ is the boundary of Ω, i.e. $\Sigma = \partial\Omega$;*

(ii) *there exists a finite index set I and $\{\varphi_i, U_i, V_i\}_{i\in I}$ such that*

   (a) *$U_i \subset \mathbb{R}^{d-1}$ and $V_i \subset \mathbb{R}^d$ are open sets and $\varphi_i : U_i \to V_i$ is a C^k-smooth function for any $i \in I$;*

   (b) rank $D\varphi_i(u) = d-1$ *holds for all $u \in U_i$;*

   (c) *$\varphi_i(U_i) = V_i \cap \Sigma$ and $\varphi_i : U_i \to V_i \cap \Sigma$ is a homeomorphism;*

   (d) *$\Sigma \subset \bigcup_{i\in I} V_i$;*

   (e) *For any $i \in I$ there exists a set $\Omega_i \subset \mathbb{R}^d$, which can be transformed to a C^k-hypograph by a rotation and a translation, such that $\Omega \cap V_i = \Omega_i \cap V_i$ holds.*

$\{\varphi_i, U_i, V_i\}_{i\in I}$ is called a parametrization of Σ.

Remark 3.3. Let $\Sigma \subset \mathbb{R}^d$ be a hypersurface in the sense of Definition 3.2 and let $\{\varphi_i, U_i, V_i\}_{i\in I}$ be a parametrization of Σ.

(i) Σ is compact, as Σ is obviously bounded and closed.

(ii) Condition (e) in Definition 3.2 requires that $\varphi_i(U_i)$ has, up to a rotation and a translation, the form

$$\Big\{(x_1,\dots,x_d)\in\mathbb{R}^d : \tilde{x} := (x_1,\dots,x_{d-1})\in\tilde{V}, x_d=\psi\big(\tilde{x}\big)\Big\} \tag{3.1}$$

for a set $\tilde{V}\subset\mathbb{R}^{d-1}$ and a C^k-smooth function $\psi:\tilde{V}\to\mathbb{R}$. We would like to point out that this is no restriction on φ_i.

In fact, if an arbitrary parametrization $\{\varphi_i,U_i,V_i\}_{i\in I}$ of Σ is given, the graph of φ_i has locally the form (3.1) up to a rotation and a translation. In order to see this, let $u\in U_i$. Since rank $D\varphi_i(u)=d-1$ by requirement, there exists a coordinate j_u such that the mapping $\tilde{\varphi}_i:U_i\to\mathbb{R}^{d-1}$ consisting of all coordinates except coordinate j_u of φ_i is differentiable and has a full rank Jacobian in u and by continuity also in a neighborhood U_u of u. We set $V_{u,1}:=\tilde{\varphi}_i\big(U_u\big)$. Now, it follows from the inverse function theorem, see for instance [49, Satz 4.6], that $\tilde{\varphi}_i$ has locally a k-times differentiable inverse $\tilde{\varphi}_i^{-1}:V_{u,2}\to U_i$ defined in a neighborhood $V_{u,2}$ of $\varphi_i(u)$. We set $V_u:=V_{u,1}\cap V_{u,2}$ and note that $\tilde{\varphi}_i^{-1}$ is bijective on V_u. Setting $B_u:=\varphi_i\big(\tilde{\varphi}_i^{-1}(V_u)\big)$ and $\psi_u:=\varphi_{i,j_u}\circ\tilde{\varphi}_i^{-1}|_{V_u}$, we find

$$\operatorname{graph}\varphi_i\cap B_u=\big\{x\in\mathbb{R}^d : \tilde{x}:=(x_1,\dots,x_{j_u-1},x_{j_u+1},\dots,x_d)\in V_u, x_{j_u}=\psi_u\big(\tilde{x}\big)\big\}$$

and thus, graph φ_i is locally the boundary of a C^k-hypograph up to a rotation and a translation.

From the above considerations it follows that $\{\varphi_i,U_i,V_i\}_{i\in I}$ can be replaced by another parametrization, where the graphs of the corresponding functions φ_i have the form (3.1) up to a rotation and a translation. In fact, since Σ is compact, there exist finitely many points $u_{i,j}\in U_i$ such that $\Sigma\subset\bigcup B_{u_{i,j}}$, where $B_{u_{i,j}}$ is given as above. Replacing now the original parametrization $\{\varphi_i,U_i,V_i\}_{i\in I}$ by

$$\Big\{\varphi_i|_{\varphi_i^{-1}(B_{u_{i,j}})},\varphi_i^{-1}(B_{u_{i,j}}),B_{u_{i,j}}\Big\},$$

we get a new parametrization where the graphs of $\varphi_i|_{\varphi_i^{-1}(B_{u_{i,j}})}$ have the form (3.1) up to a rotation and a translation.

(iii) In view of the previous considerations in this remark, condition (e) in Definition 3.2 means that Ω lies locally on one side of Σ.

Suppose that we have two parametrizations $\{\varphi_i,U_i,V_i\}_{i\in I}$ and $\{\tilde{\varphi}_j,\tilde{U}_j,\tilde{V}_j\}_{j\in J}$ of a closed hypersurface Σ as in the definition above. An interesting relation between the mappings φ_i and $\tilde{\varphi}_j$ is contained in the following proposition, which follows the ideas of a theorem in [49, § 8.3].

Proposition 3.4. *Let $k \in \mathbb{N}$, let $U_1, U_2 \subset \mathbb{R}^{d-1}$ and let $\varphi_1 : U_1 \to \mathbb{R}^d$ and $\varphi_2 : U_2 \to \mathbb{R}^d$ be C^k-smooth homeomorphisms satisfying $\varphi_1(U_1) = \varphi_2(U_2)$ and $\operatorname{rank} D\varphi_1 = \operatorname{rank} D\varphi_2 = d-1$ everywhere in U_1 and U_2, respectively. Then there exists a diffeomorphism $\psi : U_1 \to U_2$ (i.e. ψ is bijective and ψ and ψ^{-1} are continuously differentiable), such that $\varphi_1 = \varphi_2 \circ \psi$ holds.*

Proof. We define $\psi := \varphi_2^{-1} \circ \varphi_1$. Then ψ is by definition a homeomorphism, so it remains to show that ψ and ψ^{-1} are differentiable. Here, it is sufficient to verify the statement for ψ, the statement for ψ^{-1} follows then by symmetry.

Since differentiability is a local property, it is sufficient to show that ψ is differentiable for any u. Let $u \in U_1$, $x = \varphi_1(u)$ and $v = \varphi_2^{-1}(x)$. Since $\operatorname{rank} D\varphi_2(v) = d - 1$, there exists a coordinate, say w.l.o.g. coordinate d, such that the mapping $\tilde{\varphi}_2 : U_2 \to \mathbb{R}^{d-1}$ consisting of the first $d - 1$ coordinates of φ_2 is differentiable and has a full rank Jacobian in v. Hence, it follows from the inverse function theorem, see for instance [49, Satz 4.6], that $\tilde{\varphi}_2$ has locally a differentiable inverse. Defining $\tilde{\varphi}_1$ also as the function consisting of the first $d - 1$ coordinates of φ_1, it follows $\psi = \tilde{\varphi}_2^{-1} \circ \tilde{\varphi}_1$ in a neighborhood of u, implying that ψ is differentiable in u. □

Remark 3.5. Let $\{\varphi_i, U_i, V_i\}_{i \in I}$ and $\{\tilde{\varphi}_j, \tilde{U}_j, \tilde{V}_j\}_{j \in J}$ be two parametrizations of a closed hypersurface Σ in the sense of Definition 3.2. Then Proposition 3.4 has two important consequences:

(i) If $\varphi_i(U_i) \cap \varphi_j(U_j) \neq \emptyset$, then there exists a diffeomorphism $\psi_{ij} : \varphi_i^{-1}(V_i \cap V_j) \to \varphi_j^{-1}(V_i \cap V_j)$ such that $\varphi_i|_{\varphi_i^{-1}(V_i \cap V_j)} = \varphi_j|_{\varphi_j^{-1}(V_i \cap V_j)} \circ \psi_{ij}$ holds.

(ii) For any $i \in I$ and $j \in J$ there exists a diffeomorphism $\psi_{ij} : \varphi_i^{-1}(V_i \cap \tilde{V}_j) \to \tilde{\varphi}_j^{-1}(V_i \cap \tilde{V}_j)$ satisfying $\varphi_i|_{\varphi_i^{-1}(V_i \cap V_j)} = \tilde{\varphi}|_{\tilde{\varphi}_j^{-1}(V_i \cap V_j)} \circ \psi_{ij}$.

Next, we define the tangential space and the normal vector field associated to a hypersurface Σ. Here, we use the notation $\partial_j f := \frac{\partial f}{\partial u_j}$ for $j \in \{1, \dots, d-1\}$.

Definition 3.6. *Let $k \in \mathbb{N}$, let $\Sigma \subset \mathbb{R}^d$ be a closed connected C^k-hypersurface in the sense of Definition 3.2 and let $\{\varphi_i, U_i, V_i\}_{i \in I}$ be a parametrization of Σ.*

(i) *For $x \in \Sigma$ with $x = \varphi_i(u)$, $u \in U_i$, the tangential space T_x of Σ in x is defined as*

$$T_x = \operatorname{span}\{\partial_1 \varphi_i(u), \dots, \partial_{d-1}\varphi_i(u)\}.$$

Note that T_x is a vector space of dimension $d - 1$, as the Jacobian $D\varphi_i(u)$ has full rank by definition.

(ii) *For $x \in \Sigma$, the normal vector field of Σ at x is defined as the one-dimensional orthogonal complement of T_x. Moreover, the unit normal vector of Σ at x, which points outwards of the bounded part $\Omega \subset \mathbb{R}^d$ with $\Sigma = \partial\Omega$, will be denoted by $\nu(x)$.*

Note that Proposition 3.4 and the chain rule imply that the tangential space T_x at $x \in \Sigma$ is well-defined and independent from the parametrization of Σ. Next, we show that the normal vector ν is differentiable, if the hypersurface Σ has at least C^2-smoothness:

Proposition 3.7. *Let $\Sigma \subset \mathbb{R}^d$ be a closed connected hypersurface that is at least C^2-smooth and let $\{\varphi_i, U_i, V_i\}_{i\in I}$ be a parametrization of Σ. Then the following assertions are true:*

(i) *The mapping $U_i \ni u \mapsto \nu(\varphi_i(u))$ is continuously differentiable for any $i \in I$.*

(ii) *It holds $\partial_j \nu(\varphi_i(u)) \in T_{\varphi_i(u)}$ for any $u \in U_i$, $j \in \{1, \ldots, d-1\}$ and $i \in I$.*

Proof. (i) Let $i \in I$ and $u \in U_i$ be fixed. Note that there exists a canonical basis vector $e_k \in \mathbb{R}^d$ such that $(e_k, t) \neq 0$ is true for any $t \in T_{\varphi_i(u)}$ and by continuity also for any $t \in T_{\varphi_i(v)}$ for all v in a small neighborhood U_u of u. Then a normal vector at $\varphi_i(v)$ with $v \in U_u$ is given by

$$\tilde{\nu}(\varphi_i(v)) = e_k - \sum_{j=1}^{d-1} \frac{\langle e_k, \partial_j \varphi_i(v)\rangle}{\langle \partial_j \varphi_i(v), \partial_j \varphi_i(v)\rangle} \partial_j \varphi_i(v)$$

and the normal unit vector $\nu(\varphi_i(v))$ is then

$$\nu(\varphi_i(v)) = \pm \frac{\tilde{\nu}(\varphi_i(v))}{\|\tilde{\nu}(\varphi_i(v))\|},$$

where the sign has to be chosen in such a way that $\nu(\varphi_i(v))$ points outwards of the bounded domain Ω with $\Sigma = \partial\Omega$ and it is the same for all $v \in U_u$, as Ω is on one side of Σ by definition. Hence, due to our assumptions on the smoothness of Σ and of φ_i the mapping $U_i \ni u \mapsto \nu(\varphi_i(u))$ is evidently continuously differentiable.

(ii) Since

$$1 = \|\nu(\varphi_i(u))\|^2 \tag{3.2}$$

holds by the definition of $\nu(\varphi_i(u))$ for any $i \in I$ and $u \in U_i$, a differentiation of (3.2) implies immediately $0 = \langle \partial_j \nu(\varphi_i(u)), \nu(\varphi_i(u))\rangle$ and hence $\partial_j \nu(\varphi_i(u)) \in T_{\varphi_i(u)}$. □

Next, we introduce the first fundamental form associated to a hypersurface Σ. This mapping will be essential for the definition of an integral over Σ in Section 3.2.

Definition 3.8. *Let $\Sigma \subset \mathbb{R}^d$ be a closed connected hypersurface. Then the first fundamental form I_x associated to Σ is a mapping defined on Σ that acts for any $x \in \Sigma$ as a bilinear form operating on the tangential space T_x and it is pointwise defined as*

$$I_x(a, b) = \langle a, b\rangle$$

for $a, b \in T_x$. If $\{\varphi_i, U_i, V_i\}_{i\in I}$ is a parametrization of Σ, $x = \varphi_i(u)$ with $u \in U_i$ and a, b are represented via the basis $\{\partial_j \varphi_i(u)\}$ of T_x, then the representing matrix $G_i(u)$ of I_x is given by

$$G_i(u) = \big(\langle \partial_k \varphi_i(u), \partial_l \varphi_i(u)\rangle\big)_{k,l=1}^{d-1}.$$

The next proposition, together with Proposition 3.4, describes the connection of the representing matrices of the first fundamental form associated to different parametrizations:

Proposition 3.9. *Let $U, V \subset \mathbb{R}^{d-1}$, let $\tilde{\varphi} : V \to \mathbb{R}^d$ be continuously differentiable and let $\psi : U \to V$ be a diffeomorphism. Let $\varphi := \tilde{\varphi} \circ \psi$ and set for $u \in U$ and $v \in V$*

$$G(u) := \big(\langle \partial_k \varphi(u), \partial_l \varphi(u)\rangle\big)_{k,l=1}^{d-1} \quad \text{and} \quad \tilde{G}(v) := \big(\langle \partial_k \tilde{\varphi}(v), \partial_l \tilde{\varphi}(v)\rangle\big)_{k,l=1}^{d-1}.$$

Then it holds $G(u) = (D\psi(u))^\top \tilde{G}(\psi(u)) D\psi(u)$ for all $u \in U$.

Proof. An easy calculation verifies $G(u) = (D\varphi(u))^\top D\varphi(u)$ and $\tilde{G}(v) = (D\tilde{\varphi}(v))^\top D\tilde{\varphi}(v)$ for any $u \in U$ and $v \in V$. Hence, according to the chain rule it follows

$$\begin{aligned}
G(u) &= (D\varphi_i(u))^\top D\varphi(u) = (D(\tilde{\varphi} \circ \psi)(u))^\top D(\tilde{\varphi} \circ \psi)(u) \\
&= (D\tilde{\varphi}(\psi(u)) \cdot D\psi(u))^\top (D\tilde{\varphi}(\psi(u)) \cdot D\psi(u)) \\
&= (D\psi(u))^\top (D\tilde{\varphi}(\psi(u)))^\top D\tilde{\varphi}(\psi(u)) D\psi(u) = (D\psi(u))^\top \tilde{G}(\psi(u)) D\psi(u)
\end{aligned}$$

for any $u \in U$, which shows the claimed result. □

Finally, we introduce the notion of the Weingarten map, which is also known as shape operator. This map is essential to describe a suitable integral transform onto a neighborhood of a hypersurface that is necessary later in our approximation procedure. In order to analyze the properties of the Weingarten map, we also need the notion of the second fundamental form associated to a hypersurface Σ.

Definition 3.10. *Let $\Sigma \subset \mathbb{R}^d$ be a closed connected hypersurface that is at least C^2-smooth and let $\{\varphi_i, U_i, V_i\}_{i \in I}$ be a parametrization of Σ.*

(i) *The Weingarten map W defines for any $x \in \Sigma$ a linear operator $W(x) : T_x \simeq \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$, which acts for $\varphi_i(u) = x$, $u \in U_i$, on basis vectors $\partial_j \varphi_i(u)$ of T_x as $W(x)\partial_j \varphi_i(u) = -\partial_j \nu(\varphi_i(u))$. The matrix associated to the linear mapping $W(x)$ corresponding to the basis $\{\partial_j \varphi_i(u)\}$ of T_x will be denoted by $L_i(u)$.*

(ii) *The second fundamental form II_x associated to Σ is for any $x \in \Sigma$ a bilinear form acting on the tangential space T_x and it is defined as*

$$II_x(a, b) = \langle W(x)a, b\rangle$$

for $a, b \in T_x$. If $x = \varphi_i(u)$ with $u \in U_i$ and a, b are represented via the basis $\{\partial_j \varphi_i(u)\}$ of T_x, then the representing matrix $H_i(u)$ of II_x is given by

$$H_i(u) = -\big(\langle \partial_k \nu(\varphi_i(u)), \partial_l \varphi_i(u)\rangle\big)_{k,l=1}^{d-1}.$$

From the definition it is not obvious that the Weingarten map $W(x)$ is independent from the parametrization and how the representing matrices L_i and $\tilde{L}_j$ corresponding to different parametrizations are connected. These points, and the fact that the eigenvalues of the matrix of the Weingarten map are bounded, are discussed in the following proposition:

Proposition 3.11. *Let $\Sigma \subset \mathbb{R}^d$ be a closed connected hypersurface that is at least C^2-smooth. Then the following holds:*

(i) *The Weingarten map W is well-defined and independent from the parametrization.*

(ii) *Let $U_1, U_2 \subset \mathbb{R}^{d-1}$ be such that there exists a diffeomorphism $\psi : U_2 \to U_1$, let $\varphi_1 : U_1 \to \Sigma$ and set $\varphi_2 = \varphi_1 \circ \psi$. Further, let L_1 and L_2 be the matrices of the Weingarten map associated to the two "parametrizations" φ_1 and φ_2, respectively, of a subset of Σ. Then it holds*

$$L_2 = (D\psi)^{-1} L_1 D\psi.$$

(iii) *Let $\mu_1(x), \dots, \mu_{d-1}(x)$ be the eigenvalues of L_i, which are independent from the parametrization of Σ by* (ii). *Then the mapping $x \mapsto \mu_j(x)$ is continuous for any $j \in \{1, \dots, d-1\}$. In particular, $\mu_1(x), \dots, \mu_{d-1}(x)$ are uniformly bounded in $x \in \Sigma$.*

Proof. (i) Let $x \in \Sigma$ and let $U_1, U_2 \subset \mathbb{R}^{d-1}$, $\varphi_1 : U_1 \to \Sigma$, $\varphi_2 : U_2 \to \Sigma$ and $u_1 \in U_1, u_2 \in U_2$ be such that $\varphi_1(u_1) = \varphi_2(u_2) = x$. Then according to Proposition 3.4 there exist subsets $\tilde{U}_1 \subset U_1$ and $\tilde{U}_2 \subset U_2$ with $u_1 \in \tilde{U}_1$ and $u_2 \in \tilde{U}_2$ and a diffeomorphism $\psi : \tilde{U}_1 \to \tilde{U}_2$ such that $\varphi_1|_{\tilde{U}_1} = \varphi_2|_{\tilde{U}_2} \circ \psi$. W.l.o.g. we assume $\tilde{U}_1 = U_1$ and $\tilde{U}_2 = U_2$. Let W_1 and W_2 be the Weingarten maps corresponding to φ_1 and φ_2, respectively, for the part of Σ parametrized by φ_1 and φ_2. Because of the chain rule, any basis vector $\partial_j \varphi_1(u_1)$ of T_x with $x = \varphi_1(u_1) = \varphi_2(u_2)$ can be represented as

$$\partial_j \varphi_1(u_1) = D\varphi_2(u_2) \cdot \partial_j \psi(u_1).$$

Hence, we find

$$\begin{aligned} W_2 \partial_j \varphi_1(u_1) &= W_2 D\varphi_2(u_2) \cdot \partial_j \psi(u_1) = -D(\nu \circ \varphi_2)(u_2) \cdot \partial_j \psi(u_1) \\ &= -\partial_j (\nu \circ \varphi_1)(u_1) = W_1 \partial_j \varphi_1(u_1), \end{aligned}$$

where we used again the chain rule. Thus, $W_1 = W_2$.

(ii) This result follows from basic linear algebra facts, because the matrices L_1 and L_2 represent the same linear mapping in the tangential space T_x corresponding to the basis $\{\partial_j \varphi_1\}$ and $\{\partial_j \varphi_2\}$, respectively, with the transformation matrix $D\psi$.

(iii) Let $\{\varphi_i, U_i, V_i\}_{i \in I}$ be a parametrization of Σ, let $x \in \Sigma$ and let $u \in U_i$ such that $\varphi_i(u) = x$. We prove that the entries $l_{jk}(u)$ of the matrix $L_i(u)$ depend locally continuous on x, which implies then the claimed result, as the eigenvalues of a matrix depend continuously on its entries. For this purpose, it is sufficient to prove that $l_{jk}(u)$ is continuous in u, as φ_i is a homeomorphism by definition.

In order to prove that the entries of $L_i(u)$ depend continuously on u, we write $g_{jk} = \langle \partial_j \varphi_i(u), \partial_k \varphi_i(u) \rangle$ for the entries of the matrix of the first fundamental form and $h_{jk} = -\langle \partial_j \nu(\varphi_i(u)), \partial_k \varphi_i(u) \rangle$ for the entries of the matrix of the second fundamental form. Note that $G_i(u)$ is positive definite and hence invertible and that the entries of $G_i(u)^{-1}$ depend continuously on g_{jk} and hence on u. Now, by the definition of the second fundamental form, it holds

$$\begin{aligned} h_{jk} &= -\langle \partial_j \nu(\varphi_i(u)), \partial_k \varphi_i(u)\rangle = \langle W(\varphi_i(u))\partial_j \varphi_i(u), \partial_k \varphi_i(u)\rangle \\ &= \sum_{m=1}^{d-1} l_{mj}(u)\langle \partial_m \varphi_i(u), \partial_k \varphi_i(u)\rangle = \sum_{m=1}^{d-1} l_{mj}(u) g_{mk} = \sum_{m=1}^{d-1} g_{km} l_{mj}(u), \end{aligned}$$

where we used the symmetry of $G_i(u)$ in the last step, and hence $H_i(u) = G_i(u)L_i(u)$. This implies $L_i(u) = G_i(u)^{-1}H_i(u)$, which shows that the entries l_{jk} are continuous in u, because the entries of $G_i(u)^{-1}$ and $H_i(u)$ are continuous, as Σ is sufficiently smooth by requirement. □

Finally, we make a short remark about the geometrical interpretation of the Weingarten map, cf. [41, Definition 3.12]:

Remark 3.12. Let a C^2-smooth hypersurface Σ be given and let $x \in \Sigma$. Moreover, let c be a curve that is contained in Σ such that $x \in \operatorname{ran} c$ and let $t := c'(x)$ be the tangential vector of c at x, where the derivative is taken with respect to the arc-length. Then the curvature of c at x is given by $\kappa = |c''(x)|$. Now, one can show that the orthogonal projection $\langle c''(x), \nu(x)\rangle \nu(x)$ of $c''(x)$ onto the normal space is independent from the curve c and it depends just on the tangential vector t. Moreover, it holds $\langle c''(x), \nu(x)\rangle = II(t,t)$ and this value is called normal curvature.

Now, the $d-1$ eigenvalues of the matrix of the Weingarten map, that are independent from the parametrization of Σ by the last proposition, are the so-called principal curvatures and describe the geometrical properties of Σ around x. In particular, the biggest and the largest eigenvalue are the maximal and the minimal value of the normal curvature for any possible curve c with $x \in \operatorname{ran} c$ and the corresponding tangential vectors are the corresponding eigenvectors of W.

3.2 Integration on hypersurfaces

Our goal in this section is to discuss a suitable notion of an integral for functions defined on a closed hypersurface $\Sigma \subset \mathbb{R}^d$. The main idea behind this is as follows: given a parametrization $\{\varphi_i, U_i, V_i\}_{i\in I}$ of Σ, one defines the integral locally in $\Sigma \cap V_i$ via a suitably weighted $(d-1)$-dimensional integral over U_i. In order to do this, we need an appropriate decomposition of a function f defined on Σ into parts f_i with support contained in $\Sigma \cap V_i$, which is done via a so-called partition of unity. This concept is introduced in the following lemma, which can be found in a more general form for instance in [8, Section 2.20]:

Lemma 3.13. *Let $k \in \mathbb{N}$, let $\Sigma \subset \mathbb{R}^d$ be a closed connected C^k-hypersurface in the sense of Definition 3.2 and let $\{V_i\}_{i\in I}$ be a finite cover of open sets $V_i \subset \mathbb{R}^d$ of Σ. Then there exists a neighborhood V of Σ and a family of functions $\{\chi_i\}_{i\in I} \subset C^\infty(V)$ with the following properties:*

(i) $\chi_i \geq 0$ *holds for any* $i \in I$*;*

(ii) $\operatorname{supp} \chi_i \cap \Sigma \subset V_i \cap \Sigma$ *and* $\operatorname{supp} \chi_i \cap \Sigma$ *is compact in* Σ*;*

(iii) $\sum_{i \in I} \chi_i(x) = 1$ *holds for any* $x \in \Sigma$.

The family $\{\chi_i\}_{i \in I}$ *is called a partition of unity for* $\{V_i\}_{i \in I}$.

Proof. W.l.o.g. we assume that the index set I has the form $I = \{1, \ldots, n\}$ and that all sets V_i are bounded (otherwise, choose $R > 0$ such that $\Sigma \subset B(0, R)$ and replace V_i by $V_i \cap B(0, R)$).

Step 1: First, we construct via induction a family of open sets $\{W_i\}_{i \in I}$ such that

$$W_i \subset \overline{W_i} \subset V_i \text{ holds for any } i \in I \text{ and } \Sigma \subset \bigcup_{i \in I} W_i.$$

For $m \geq 1$, we assume that it holds

$$\Sigma \subset \bigcup_{i<m} W_i \cup \bigcup_{i \geq m} V_i,$$

which is true for $m = 1$, as $\{V_i\}$ is an open cover of Σ by assumption. This implies

$$\partial V_i \cap \Sigma \subset \bigcup_{i<m} W_i \cup \bigcup_{i>m} V_i$$

and hence, there exists $\delta_m > 0$ sufficiently small such that

$$\underbrace{\{x \in \mathbb{R}^d : \operatorname{dist}(x, \partial V_m) \leq \delta_m\}}_{=:\overline{B(\partial V_m, \delta_m)}} \cap \Sigma \subset \bigcup_{i<m} W_i \cup \bigcup_{i>m} V_i.$$

Defining then $W_m := V_m \setminus \overline{B(\partial V_i, \delta_m)}$, we find $W_m \neq \emptyset$ for δ_m sufficiently small and

$$\Sigma \subset \bigcup_{i<m} W_i \cup \underbrace{W_m \cup \overline{B(\partial V_i, \delta_m)}}_{=V_m} \cup \bigcup_{i>m} V_i = \bigcup_{i \leq m} W_i \cup \bigcup_{i>m} V_i,$$

which finishes the induction.

Step 2: According to [8, Section 2.19] there exists for any $i \in I$ a function $\eta_i \in C_c^\infty(\mathbb{R}^d)$ such that

$$0 \leq \eta_i \leq 1, \quad \operatorname{supp} \eta_i \subset V_i \quad \text{and} \quad \eta_i|_{W_i} \equiv 1$$

hold. Since $\Sigma \subset \bigcup_{i \in I} W_i$, it holds $\sum_{i \in I} \eta_i > 0$ on Σ and by continuity also on a neighborhood V of Σ. Setting

$$\chi_i := \frac{\eta_i}{\sum_{i \in I} \eta_i},$$

we have constructed the desired partition of unity. □

Now, we are prepared to define a suitable notion of integration on a hypersurface Σ:

Definition 3.14. *Let $k \in \mathbb{N}$, let $\Sigma \subset \mathbb{R}^d$ be a closed connected C^k-hypersurface with parametrization $\{\varphi_i, U_i, V_i\}_{i\in I}$ and let $\{\chi_i\}_{i\in I}$ be a partition of unity for $\{V_i\}_{i\in I}$.*

(i) *A function $f : \Sigma \to \mathbb{C}$ is said to be measurable (or integrable), if*

$$U_i \ni u \mapsto \chi_i(\varphi_i(u))f(\varphi_i(u))$$

is measurable (or integrable, respectively) for any $i \in I$.

(ii) *If $f : \Sigma \to \mathbb{C}$ is integrable, we define the integral of f over Σ as*

$$\int_\Sigma f(x)\mathrm{d}\sigma(x) := \sum_{i\in I} \int_{U_i} \chi_i(\varphi_i(u))f(\varphi_i(u))\sqrt{\det G_i(u)}\mathrm{d}u,$$

where $\mathrm{d}u := \mathrm{d}\Lambda_{d-1}(u)$ *denotes the $(d-1)$-dimensional Lebesgue measure and $G_i(u)$ is the matrix of the first fundamental form associated to Σ and its parametrization $\{\varphi_i, U_i, V_i\}_{i\in I}$. The measure σ is called "Hausdorff measure" on Σ.*

It is not obvious that $\int_\Sigma f(x)\mathrm{d}\sigma(x)$ given as in Definition 3.14 is independent from the parametrization. This will be shown in the following proposition:

Proposition 3.15. *Let $k \in \mathbb{N}$ and let $\Sigma \subset \mathbb{R}^d$ be a closed connected C^k-hypersurface. Then*

$$\int_\Sigma f(x)\mathrm{d}\sigma(x)$$

is independent from the parametrization of Σ and the partition of unity.

Proof. Let $\{\varphi_i, U_i, V_i\}_{i\in I}$ and $\{\tilde\varphi_j, \tilde U_j, \tilde V_j\}_{j\in J}$ be two parametrizations of Σ with corresponding matrices $G_i(u)$ and $\tilde G_j(u)$, respectively, of the first fundamental form. Moreover, let $\{\chi_i\}_{i\in I}$ and $\{\tilde\chi_j\}_{j\in J}$ be two partitions of unity for $\{V_i\}_{i\in I}$ and $\{\tilde V_j\}_{j\in J}$, respectively. According to Proposition 3.4, there exists for any $i \in I$ and $j \in J$ a diffeomorphism $\psi_{ij} : \varphi_i^{-1}(V_i \cap \tilde V_j) \to \tilde\varphi_j^{-1}(V_i \cap \tilde V_j)$ such that $\varphi_i|_{\varphi_i^{-1}(V_i\cap\tilde V_j)} = \tilde\varphi_j|_{\tilde\varphi_j^{-1}(V_i\cap\tilde V_j)} \circ \psi_{ij}$ holds. Note that the matrix of the first fundamental form transforms as

$$G_i(u) = (D\psi_{ij}(u))^\top \tilde G_j(\psi_{ij}(u))D\psi_{ij}(u)$$

for any $u \in \varphi_i^{-1}(V_i \cap \tilde{V}_j)$, cf. Proposition 3.9. Hence, we find

$$\begin{aligned}
&\sum_{i\in I}\int_{U_i}\chi_i(\varphi_i(u))f(\varphi_i(u))\sqrt{\det G_i(u)}\mathrm{d}u \\
&\qquad = \sum_{i\in I}\int_{U_i}\sum_{j\in J}\tilde{\chi}_j(\varphi_i(u))\chi_i(\varphi_i(u))f(\varphi_i(u))\sqrt{\det G_i(u)}\mathrm{d}u \\
&\qquad = \sum_{i\in I}\sum_{j\in J}\int_{\varphi_i^{-1}(V_i\cap\tilde{V}_j)}\tilde{\chi}_j(\tilde{\varphi}_j(\psi_{ij}(u)))\chi_i(\tilde{\varphi}_j(\psi_{ij}(u)))f(\tilde{\varphi}_j(\psi_{ij}(u))) \\
&\qquad\qquad\qquad\qquad \cdot\sqrt{\det\left((D\psi_{ij}(u))^\top\tilde{G}_j(\psi_{ij}(u))D\psi_{ij}(u)\right)}\mathrm{d}u \\
&\qquad = \sum_{i\in I}\sum_{j\in J}\int_{\varphi_i^{-1}(V_i\cap\tilde{V}_j)}\tilde{\chi}_j(\tilde{\varphi}_j(\psi_{ij}(u)))\chi_i(\tilde{\varphi}_j(\psi_{ij}(u)))f(\tilde{\varphi}_j(\psi_{ij}(u))) \\
&\qquad\qquad\qquad\qquad \cdot|\det D\psi_{ij}(u)|\sqrt{\det\tilde{G}_j(\psi_{ij}(u))}\mathrm{d}u \\
&\qquad = \sum_{i\in I}\sum_{j\in J}\int_{\tilde{\varphi}_j^{-1}(V_i\cap\tilde{V}_j)}\tilde{\chi}_j(\tilde{\varphi}_j(v))\chi_i(\tilde{\varphi}_j(v))f(\tilde{\varphi}_j(v))\sqrt{\det\tilde{G}_j(v)}\mathrm{d}v \\
&\qquad = \sum_{j\in J}\int_{\tilde{U}_j}\tilde{\chi}_j(\tilde{\varphi}_j(v))f(\tilde{\varphi}_j(v))\sqrt{\det\tilde{G}_j(v)}\mathrm{d}v,
\end{aligned}$$

which proves the statement of this proposition. □

Remark 3.16. According to our assumptions on the hypersurface Σ, the Hausdorff measure σ, which is well-defined by the previous proposition, is obviously a finite measure.

3.3 Tubes around hypersurfaces

Let Σ be a closed connected hypersurface which is at least C^2-smooth and let us write throughout this section elements of Σ with a subscript Σ. Our goal in this section is to discuss neighborhoods Ω_β of Σ of the form

$$\Omega_\beta := \{x_\Sigma + t\nu(x_\Sigma) : x_\Sigma \in \Sigma, t \in (-\beta,\beta)\}\,, \tag{3.3}$$

where $\beta > 0$. In particular, we are going to prove that for $\beta > 0$ sufficiently small the mapping

$$\iota_{\Sigma,\beta} : \Sigma \times (-\beta,\beta) \to \Omega_\beta, \quad (x_\Sigma, t) \mapsto x_\Sigma + t\nu(x_\Sigma) \tag{3.4}$$

is bijective. This will allow us to identify functions supported on Ω_β with functions defined on $\Sigma \times (-\beta,\beta)$ via

$$f(x) = f(x_\Sigma + t\nu(x_\Sigma)) = \tilde{f}(x_\Sigma, t)$$

for $x = x_\Sigma + t\nu(x_\Sigma)$. Finally, we will derive a transformation formula for integrals of f defined in Ω_β with respect to the d-dimensional Lebesgue measure Λ_d and of functions $\tilde{f}$ defined in $\Sigma \times (-\beta,\beta)$ with respect to the measure $\sigma \times \Lambda_1$ with f and $\tilde{f}$ as above. In the following proposition, we collect the basic properties of $\iota_{\Sigma,\beta}$:

Proposition 3.17. *Let $\Sigma \subset \mathbb{R}^d$, $d \geq 2$, be a closed connected hypersurface in the sense of Definition 3.2 with parametrization $\{\varphi_i, U_i, V_i\}_{i\in I}$, which is at least C^2-smooth. Further, let $\beta > 0$ and define the mapping $\iota_{i,\beta}$ as*

$$\iota_{i,\beta} : U_i \times (-\beta, \beta) \to \mathbb{R}^d, \quad (u,t) \mapsto \varphi_i(u) + t\nu(\varphi_i(u)).$$

Then the following hold:

(i) *For any $i \in I$ the mapping $\iota_{i,\beta}$ is differentiable and its Jacobian is given by*

$$D\iota_{i,\beta}(u,t) = \begin{pmatrix} D_u\varphi_i(u)(1 - tL_i(u)) & \nu(\varphi_i(u)) \end{pmatrix},$$

where L_i denotes the matrix of the Weingarten map W.

(ii) *It holds*

$$|\det D\iota_{i,\beta}(u,t)| = \left|\det(1 - tL_i(u))\sqrt{\det G_i(u)}\right|,$$

where G_i denotes the matrix of the first fundamental form associated to Σ and the given parametrization.

Proof. (i) The fact that $\iota_{i,\beta}$ is differentiable follows from the smoothness of Σ and from Proposition 3.7. In order to compute the derivatives of $\iota_{i,\beta}$ a simple calculation shows

$$\frac{\partial \iota_{i,\beta}}{\partial t} = \nu(\varphi_i(u))$$

and

$$\frac{\partial \iota_{i,\beta}}{\partial u_j} = \frac{\partial \varphi_i(u)}{\partial u_j} + t\frac{\partial \nu(\varphi_i(u))}{\partial u_j} = (1 - tW)\frac{\partial \varphi_i(u)}{\partial u_j}$$

for $j \in \{1, \dots, d-1\}$, where we used just the definition of the Weingarten map W, cf. Definition 3.10. Denoting the entries of the matrix of the Weingarten map $L_i(u)$ by $l_{kj}(u)$ and its j-th column by $l_{\cdot j}(u)$, we find

$$W\frac{\partial \varphi_i(u)}{\partial u_j} = \sum_{k=1}^{d-1} l_{kj}(u)\frac{\partial \varphi_i(u)}{\partial u_k} = D_u\varphi_i(u) l_{\cdot j}(u),$$

which implies finally $\frac{\partial \iota_{i,\beta}}{\partial u} = D_u\varphi_i(u)(1 - tL_i(u))$.

(ii) In order to compute the determinant of $D\iota_{i,\beta}(u,t)$, we mention first

$$|\det D\iota_{i,\beta}(u,t)| = \left(\det(D\iota_{i,\beta}(u,t))^2\right)^{1/2} = \Big(\det\big(\underbrace{(D\iota_{i,\beta}(u,t))^\top D\iota_{i,\beta}(u,t)}_{=:S}\big)\Big)^{1/2}.$$

To compute the entries s_{kl} of the symmetric matrix S, we find first

$$s_{dd} = \big\langle \nu(\varphi_i(u)), \nu(\varphi_i(u)) \big\rangle = 1$$

and, using the result from assertion (i),

$$s_{kd} = s_{dk} = \left\langle (1 - tW(\varphi_i(u)))\frac{\partial \varphi_i(u)}{\partial u_j}, \nu(\varphi_i(u)) \right\rangle = 0$$

for $k \in \{1, \dots, d-1\}$, as the Weingarten map maps any vector from the tangential space $T_{\varphi_i(u)}$ into $T_{\varphi_i(u)}$. Since it holds

$$\begin{aligned}
&\big(D_u\varphi_i(u)(1 - tL_i(u))\big)^\top D_u\varphi_i(u)(1 - tL_i(u)) \\
&\qquad\qquad = \big(1 - tL_i(u)^\top\big)\big(D_u\varphi_i(u)\big)^\top D_u\varphi_i(u)(1 - tL_i(u)) \\
&\qquad\qquad = \big(1 - tL_i(u)^\top\big)G_i(u)(1 - tL_i(u)),
\end{aligned}$$

we find

$$S = \begin{pmatrix} \big(1 - tL_i(u)^\top\big)G_i(u)(1 - tL_i(u)) & 0 \\ 0 & 1 \end{pmatrix}.$$

This implies finally

$$\begin{aligned}
|\det D\iota_{i,\beta}(u,t)| &= \sqrt{\det S} = \sqrt{\det\Big(\big(1 - tL_i(u)^\top\big)G_i(u)(1 - tL_i(u))\Big)} \\
&= \left|\det(1 - tL(\varphi_i(u)))\sqrt{\det G_i(u)}\right|.
\end{aligned}$$

□

Remark 3.18. According to Proposition 3.11, the mapping $\det(1-tL_i(u))$ is independent from the parametrization of Σ, as the eigenvalues of $L_i(u)$ have this property. Hence, we will denote for $x_\Sigma = \varphi_i(u)$ from now on $\det(1 - tL_i(u))$ by $\det(1 - tW(x_\Sigma))$ and we regard it as a function in $x_\Sigma \in \Sigma$.

Using the results from Proposition 3.17, we are now able to prove that $\iota_{\Sigma,\beta}$ is bijective for $\beta > 0$ sufficiently small:

Theorem 3.19. *Let $\Sigma \subset \mathbb{R}^d$, $d \geq 2$, be a closed connected hypersurface in the sense of Definition 3.2, which is at least C^2-smooth. Further, let for $\beta > 0$ the set Ω_β be defined as in (3.3) and let $\iota_{\Sigma,\beta}$ be given by (3.4). Then there exists $\beta_0 > 0$ such that $\iota_{\Sigma,\beta}$ is injective for all $\beta \in (0, \beta_0)$.*

Proof. We prove the claimed result by a reduction to a contradiction. So we assume that for any $n \in \mathbb{N}$ there exist $x_n, y_n \in \Sigma$ and $a_n, b_n \in \big(-\frac{1}{n}, \frac{1}{n}\big)$ with $(x_n, a_n) \neq (y_n, b_n)$ such that

$$x_n + a_n\nu(x_n) = \iota_{\Sigma,\beta}(x_n, a_n) = \iota_{\Sigma,\beta}(y_n, b_n) = y_n + b_n\nu(y_n) \tag{3.5}$$

holds. Since Σ is compact, there exist convergent subsequences of (x_n) and (y_n). W.l.o.g. we assume that these subsequences are (x_n) and (y_n) and we set $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$. Note that (3.5) implies immediately $x = y$.

Let $\{\varphi_i, U_i, V_i\}_{i\in I}$ be a parametrization of Σ and let the mapping $\iota_{i,\beta}$ be defined as in Proposition 3.17. Since $x_n \to x$ and $y_n \to y = x$, it follows that there exists $N \in \mathbb{N}$ and $i \in I$ such that $x_n \in V_i$ and $y_n \in V_i$ hold for all $n \geq N$. Therefore, the sequences $u_n := \varphi_i^{-1}(x_n)$ and $\tilde{u}_n := \varphi_i^{-1}(y_n)$, $n \geq N$, are well-defined and they fulfill

$$\begin{aligned}\iota_{i,\beta}(u_n, a_n) &= \varphi_i(u_n) + a_n\nu(\varphi_i(u_n)) = x_n + a_n\nu(x_n) = y_n + b_n\nu(y_n)\\ &= \varphi_i(\tilde{u}_n) + b_n\nu(\varphi_i(\tilde{u}_n)) = \iota_{i,\beta}(\tilde{u}_n, b_n).\end{aligned}$$

Hence, $\iota_{i,\beta}$ is not injective for any $\beta > 0$. On the other hand, it holds by Proposition 3.17, statement (ii), for a sufficiently small $\beta_0 > 0$

$$\big|\det D\iota_{i,\beta}(u,t)\big| = \det(1 - tL_i(u))\sqrt{\det G_i(u)} > 0$$

for all $t \in (-\beta_0, \beta_0)$ and all $\beta < \beta_0$, as the entries of $L_i(u)$ are bounded and thus $\det(1 - tL_i(u)) \approx 1$ is true in this case. This is now a contradiction. Hence, $\iota_{\Sigma,\beta}$ is injective for all $\beta < \beta_0$. $\square$

In the following proposition, we show that the inverse of $\iota_{i,\beta}$, which exists for sufficiently small β by Theorem 3.19, is locally Lipschitz continuous.

Proposition 3.20. *Let $\Sigma \subset \mathbb{R}^d$, $d \geq 2$, be a closed connected hypersurface in the sense of Definition 3.2 with parametrization $\{\varphi_i, U_i, V_i\}_{i\in I}$, which is at least C^2-smooth. Further, let $\iota_{i,\beta}$ be given as in Proposition 3.17 and let $\beta > 0$ be sufficiently small, such that* $\det(1 - tL_i(u))$ *is uniformly positive for all $t \in (-\beta, \beta)$. Finally, let $K \subset V_i \cap \Sigma$ be compact in Σ. Then there exists a positive constant C depending on i and K, such that*

$$|\iota_{i,\beta}(u,t) - \iota_{i,\beta}(v,s)| \geq C\left(|u - v|^2 + |t - s|^2\right)^{1/2}$$

holds for all $u, v \in \varphi_i^{-1}(K)$ and all $s, t \in \left(-\frac{\beta}{3}, \frac{\beta}{3}\right)$.

Proof. Since $K \subset V_i \cap \Sigma$ is compact, there exists a constant $c > 0$ such that

$$\{x_\Sigma \in \Sigma : \exists y_\Sigma \in K : |x_\Sigma - y_\Sigma| < 2c\} \cap \{x_\Sigma \in \Sigma : x_\Sigma \notin V_i\} = \emptyset.$$

We set

$$\tilde{\Omega} := \left\{x_\Sigma + r\nu(x_\Sigma) : x_\Sigma \in \Sigma,\ \exists y_\Sigma \in K : |x_\Sigma - y_\Sigma| \leq c \text{ and } r \in \left[-\frac{2\beta}{3}, \frac{2\beta}{3}\right]\right\} \subset \Omega_\beta,$$

where Ω_β is defined as in (3.3), and we choose $\beta_1 > 0$ such that $B_x := \overline{B(x, \beta_1)}$ is contained in $\tilde{\Omega}$ for any $x = x_\Sigma + t\nu(x_\Sigma)$ with $x_\Sigma \in K$ and $t \in \left(-\frac{\beta}{3}, \frac{\beta}{3}\right)$.

Let $(u,t) \in \varphi_i^{-1}(K) \times \left(-\frac{\beta}{3}, \frac{\beta}{3}\right)$ be fixed, let $(v,s) \in \varphi_i^{-1}(K) \times \left(-\frac{\beta}{3}, \frac{\beta}{3}\right)$ and set $x = \iota_{i,\beta}(u,t)$ and $y = \iota_{i,\beta}(v,s)$. We distinguish two cases: $|x - y| \leq \beta_1$ and $|x - y| > \beta_1$.

Note that in the first case y is contained in the convex set B_x. Since $\tilde{\Omega}$ is a compact subset of $\operatorname{ran} \iota_{i,\beta}$, it follows from our assumptions that

$$|\det D\iota_{i,\beta}(u,t)| = \det(1 - tL_i(u))\sqrt{\det G_i(u)}$$

is uniformly bounded from below on $\iota_{i,\beta}^{-1}(\tilde{\Omega})$ and hence, there exists a constant $C_1 > 0$ such that

$$\left\| D\iota_{i,\beta}^{-1}(z) \right\| = \left\| \left((D\iota_{i,\beta})\iota_{i,\beta}^{-1}(z) \right)^{-1} \right\| \leq C_1$$

is true for all $z \in \tilde{\Omega}$. Since B_x is convex, it holds $x + \xi(y-x) \in B_x$ for all $\xi \in [0,1]$ and thus, we find

$$\begin{aligned}(u,t) - (v,s) = \iota_{i,\beta}^{-1}(x) - \iota_{i,\beta}^{-1}(y) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\xi}\iota_{i,\beta}^{-1}(x + \xi(y-x))\mathrm{d}\xi \\ &= \int_0^1 D\iota_{i,\beta}^{-1}(x + \xi(y-x)) \cdot (y-x)\mathrm{d}\xi,\end{aligned}$$

where we used the chain rule in the last step. This implies

$$\begin{aligned}\left(|u-v|^2 + |t-s|^2\right)^{1/2} = |\iota_{i,\beta}^{-1}(x) - \iota_{i,\beta}^{-1}(y)| &\leq \int_0^1 \left| D\iota_{i,\beta}^{-1}(x + \xi(y-x)) \cdot (y-x) \right| \mathrm{d}\xi \\ &\leq \max\left\{ \left\| D\iota_{i,\beta}^{-1}(z) \right\| : z \in B_x \right\} \cdot |x-y| \leq C_1 |\iota_{i,\beta}(u,t) - \iota_{i,\beta}(v,s)|,\end{aligned}$$

so the claimed assertion holds in this case.

In the second case, we use the fact that $\iota_{i,\beta}^{-1}$ is continuously differentiable by the inverse function theorem [49, Satz 4.6] and hence $\iota_{i,\beta}^{-1}$ is uniformly bounded on the compact set K. Setting $C_2 := \max\left\{ |\iota_{i,\beta}^{-1}(z)| : z \in K \right\}$, we find

$$\left(|u-v|^2 + |t-s|^2\right)^{1/2} = |\iota_{i,\beta}^{-1}(x) - \iota_{i,\beta}^{-1}(y)| \leq \frac{2C_2}{|x-y|}|x-y| \leq \frac{2C_2}{\beta_1}|\iota_{i,\beta}(u,t) - \iota_{i,\beta}(v,s)|,$$

which is the claimed result in the second case.

Setting finally

$$C := \min\left\{ \frac{1}{C_1}, \frac{\beta_1}{2C_2} \right\},$$

the result of this proposition follows. □

From Proposition 3.17, we conclude also the following corollary about integration in a tube Ω_β:

Corollary 3.21. *Let $\Sigma \subset \mathbb{R}^d$, $\iota_{\Sigma,\beta}$ and Ω_β be as in Theorem 3.19 and let $\beta > 0$ be sufficiently small. Then a function $f : \Omega_\beta \to \mathbb{C}$ is integrable with respect to the d-dimensional Lebesgue measure Λ_d if and only if $\tilde{f} : \Sigma \times (-\beta, \beta) \to \mathbb{C}$ defined as $\tilde{f}(x_\Sigma, t) = f(\iota_{\Sigma,\beta}(x_\Sigma, t))$ is integrable with respect to the measure $\sigma \times \Lambda_1$. In this case, it holds*

$$\int_{\Omega_\beta} f(x)\mathrm{d}x = \int_\Sigma \int_{-\beta}^{\beta} \tilde{f}(x_\Sigma, t) \det(1 - tW(x_\Sigma))\mathrm{d}t\mathrm{d}\sigma,$$

where W is the Weingarten map associated to Σ, and there exist positive constants c_1 and c_2 such that

$$c_1 \int_{\Omega_\beta} |f(x)|\mathrm{d}x \leq \int_\Sigma \int_{-\beta}^{\beta} \left| \tilde{f}(x_\Sigma, t) \right| \mathrm{d}t\mathrm{d}\sigma \leq c_2 \int_{\Omega_\beta} |f(x)|\mathrm{d}x$$

is true.

Proof. Let $\{\varphi_i, U_i, V_i\}_{i\in I}$ be a parametrization of Σ and let $\{\chi_i\}_{i\in I}$ be a partition of unity for $\{V_i\}_{i\in I}$. Note that it is sufficient to consider functions $f : \Omega_\beta \to [0,\infty)$ to prove the statement of this corollary, as an arbitrary complex-valued function g can be decomposed as

$$g = (\mathrm{Re}g)_+ - (\mathrm{Re}g)_- + i(\mathrm{Im}g)_+ - i(\mathrm{Im}g)_-$$

with nonnegative functions $(\mathrm{Re}g)_+, (\mathrm{Re}g)_-, (\mathrm{Im}g)_+$ and $(\mathrm{Im}g)_-$. Using the transformation $\iota_{i,\beta}$ defined as in Proposition 3.17 and setting $\tilde{\chi}_i(x_\Sigma + t\nu(x_\Sigma)) = \chi_i(x_\Sigma)$ for $t \in (-\beta, \beta)$, we find with $x = x_\Sigma + t\nu(x_\Sigma) = \varphi_i(u) + t\nu(\varphi_i(u))$

$$\begin{aligned}
\int_{\Omega_\beta} f(x)\mathrm{d}x &= \int_{\Omega_\beta} \sum_{i\in I} \tilde{\chi}_i(\iota_{\Sigma,\beta}(x_\Sigma, t)) f(\iota_{\Sigma,\beta}(x_\Sigma, t))\mathrm{d}x \\
&= \int_{\Omega_\beta} \sum_{i\in I} \tilde{\chi}_i(\iota_{i,\beta}(u, t)) f(\iota_{i,\beta}(u, t))\mathrm{d}x \\
&= \sum_{i\in I} \int_{-\beta}^{\beta} \int_{U_i} \chi_i(\varphi_i(u)) \tilde{f}(\varphi_i(u), t) \det(1 - tL_i(u)) \sqrt{\det G_i(u)}\mathrm{d}u\mathrm{d}t \\
&= \int_{-\beta}^{\beta} \int_{\Sigma} \tilde{f}(x_\Sigma, t) \det(1 - tW(x_\Sigma))\mathrm{d}\sigma\mathrm{d}t
\end{aligned}$$

due to the definition of the Hausdorff measure σ, cf. Definition 3.14. Note that it holds $\det(1 - tW(x_\Sigma)) \approx 1$ for sufficiently small t, as the eigenvalues of $L_i(u)$ are bounded by Proposition 3.11, in particular $\det(1 - tW(x_\Sigma))$ is uniformly bounded from above and from below. Hence, all claimed results from this corollary follow from the above calculation. $\square$

4 Function spaces

In this chapter, we introduce some notations and the function spaces, in which our operators act and which are essential for the definition of differential operators with δ-interactions in a suitable Hilbert space setting. In particular, in Section 4.1 we provide our notations for differentiable and Lebesgue measurable functions. Then in Section 4.2 we introduce the concept of weak differentiability and Sobolev spaces defined on open subsets of $\mathbb{R}^d$. Finally, in Section 4.3 we define various function spaces on hypersurfaces and we state a generalized version of Green's first integral formula.

4.1 Classical function spaces

In the following section, we introduce classical function spaces of continuously differentiable and Lebesgue measurable functions and we provide numerous notations connected to these function spaces. Furthermore, we collect some well-known results which will be needed for our further investigations.

4.1.1 Continuous and continuously differentiable functions

Let $d \in \mathbb{N}$ and let Ω be an open subset of $\mathbb{R}^d$. The set of all continuous functions $u : \Omega \to \mathbb{C}$ is denoted by $C(\Omega)$, the set of all k times continuously differentiable functions is denoted by $C^k(\Omega)$. For a continuous function u its support is defined by $\operatorname{supp} u := \overline{\{x \in \Omega \mid u(x) \neq 0\}}$. Furthermore, we set

$$C_c(\Omega) := \{u \in C(\Omega) : \operatorname{supp} u \text{ is compact}\}$$

and for $k \in \mathbb{N}$ we write $C_c^k(\Omega) := C_c(\Omega) \cap C^k(\Omega)$. Finally, we define the space $C_c^\infty(\Omega)$ of test functions in Ω by

$$C_c^\infty(\Omega) := \bigcap_{k \in \mathbb{N}} C_c^k(\Omega)$$

and we set

$$C_c^\infty\left(\overline{\Omega}\right) := \left\{u|_\Omega : u \in C_c^\infty(\mathbb{R}^d)\right\}.$$

4.1.2 The function spaces $L^p(X, \mu)$

Let (X, μ) be a σ-finite measure space, i.e. there exists an at most countable family of sets $\{X_n\}_{n \in \mathbb{N}}$ such that

$$X = \bigcup_{n \in \mathbb{N}} X_n \quad \text{and} \quad \mu(X_n) < \infty$$

hold. In what follows, we introduce the Banach spaces $L^p(X, \mu)$ for $p \in [1, \infty]$. For this, we set

$$\mathcal{N} := \left\{f : X \to \mathbb{C} : f \text{ is measurable and } \int_X |f| \mathrm{d}\mu = 0\right\}.$$

Then the spaces $L^p(X, \mu)$ are defined as follows:

Definition 4.1. *Let (X, μ) be a σ-finite measure space and let $\mathcal{N}$ be given as above.*

(i) *For $p \in [1, \infty)$ the function space $L^p(X, \mu)$ is defined as*

$$L^p(X, \mu) := \left\{ f : X \to \mathbb{C} : f \text{ is measurable, } \int_X |f|^p \mathrm{d}\mu < \infty \right\} \Big/ \mathcal{N}.$$

(ii) *The function space $L^\infty(X, \mu)$ is given by*

$$L^\infty(X, \mu) := \big\{ f : X \to \mathbb{C} : f \text{ is measurable, } \operatorname{ess\,sup} f < \infty \big\} / \mathcal{N}$$

with

$$\operatorname{ess\,sup} f := \inf \big\{ \alpha \geq 0 : \mu\left(\{x \in X : |f(x)| > \alpha\}\right) = 0 \big\}.$$

Let us mention that the elements of $L^p(X, \mu)$ are actually equivalence classes of functions, but we will identify functions with their equivalence classes and vice versa and we will calculate with the elements of $L^p(X, \mu)$ as with usual functions. Moreover, we will call the elements of $L^p(X, \mu)$ functions. In this sense, we say that a function $f \in L^p(X, \mu)$ is supported on a set $Y \subset X$, if $f = 0$ holds almost everywhere in $X \setminus Y$. In particular, we say that f is compactly supported, if this is true for a compact set Y.

The next theorem contains the well-known facts that $L^p(X, \mu)$ is a Banach space and that in particular $L^2(X, \mu)$ is a Hilbert space [50, Satz 1.38 and Satz 1.41].

Theorem 4.2. *Let (X, μ) be a σ-finite measure space. Moreover, define the norms*

$$\|f\|_{L^p} := \left(\int_X |f|^p \mathrm{d}\mu \right)^{1/p}$$

for $p \in [1, \infty)$ and $\|f\|_{L^\infty} := \operatorname{ess\,sup} f$. Then the following assertions are true:

(i) $\big(L^p(X, \mu), \|\cdot\|_{L^p}\big)_{L^p}$ *is a Banach space for any $p \in [1, \infty]$.*

(ii) $L^2(X, \mu)$, *equipped with the inner product*

$$(f, g)_{L^2} := \int_X f \overline{g} \mathrm{d}\mu \quad \text{for } f, g \in L^2(X, \mu),$$

is a Hilbert space.

Let us have a look on the special case that $X = \Omega$ is a subset of $\mathbb{R}^d$ and μ is the d-dimensional Lebesgue measure, which will be denoted by Λ_d in this thesis. Since most of the integrals appearing in this thesis are integrals with respect to the Lebesgue measure, we simply write $\mathrm{d}x$ instead of $\mathrm{d}\Lambda_d(x)$. Furthermore, we just write $L^p(\Omega)$ instead of $L^p(\Omega, \Lambda_d)$. The following theorem contains the important result that the set of test functions $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$ for $p \in [1, \infty)$, see for instance [43, Theorem 3.4].

Theorem 4.3. *Let $d \in \mathbb{N}$, let Ω be an open subset of $\mathbb{R}^d$ and let $p \in [1, \infty)$. Then $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$.*

Finally, we introduce the Fourier transform in $L^2(\mathbb{R}^d)$. For this purpose, we define first the Fourier transform for integrable functions and using the fact that $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, we extend this definition to the whole space $L^2(\mathbb{R}^d)$. For the definition of the Fourier transform for integrable functions, recall that the Euclidean scalar product in $\mathbb{R}^d$ is denoted by $\langle \cdot, \cdot \rangle$.

Definition 4.4. *For $f \in L^1(\mathbb{R}^d)$ the Fourier transform $\mathcal{F}_1 f : \mathbb{R}^d \to \mathbb{C}$ is defined as*

$$(\mathcal{F}_1 f)(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-i\langle x, y \rangle} f(y) \mathrm{d}y.$$

In the next theorem, which is known as the theorem of Plancherel, we extend the definition of the Fourier $\mathcal{F}_1$ to the whole space $L^2(\mathbb{R}^d)$, cf. [50, Satz 11.9].

Theorem 4.5. *The Fourier transform $\mathcal{F}_1$ given as in Definition 4.4 can be extended uniquely from $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ to a unitary operator $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$. Moreover, it holds $(\mathcal{F}^{-1} f)(x) = (\mathcal{F} f)(-x)$ for almost all $x \in \mathbb{R}^d$ and all $f \in L^2(\mathbb{R}^d)$.*

4.2 Sobolev spaces

In the following section, we introduce the Sobolev spaces $H^s(\Omega)$. This concept contains a generalization of differentiation which will allow us to define differential operators in the Hilbert space $L^2(\Omega)$ for $\Omega \subset \mathbb{R}^d$. We introduce Sobolev spaces in two different ways: on the one hand via weak derivatives, and on the other hand via the Fourier transform. Finally, we provide the function spaces $H^s_\Delta(\Omega)$. For $f \in H^s_\Delta(\Omega)$ there exists $\Delta f \in L^2(\Omega)$ in a weak sense, and for such functions we will generalize in Section 4.3 Green's first identity in a setting that is appropriate for our needs. In this section, we follow the presentation from [43].

4.2.1 Sobolev spaces - definition via weak derivatives

Throughout this section, let Ω be a nonempty open subset of $\mathbb{R}^d$. Let us arrange some notations for multi-indices: for $(\alpha_1, \dots, \alpha_d)^\top = \alpha \in \mathbb{N}_0^d$ and $x \in \mathbb{R}^d$ we write $|\alpha| := \sum_{k=1}^d \alpha_k$, $x^\alpha := \prod_{k=1}^d x_k^{\alpha_k}$ and $D_\alpha \varphi := \frac{\partial^{|\alpha|} \varphi}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$ for $\varphi \in C_c^\infty(\Omega)$. In particular, we denote $D_0 \varphi := \varphi$.

Definition 4.6. *Let $\Omega \subset \mathbb{R}^d$ be open, let $\alpha \in \mathbb{N}_0^d$ and let $f \in L^2(\Omega)$. Assume that there exists a function $g \in L^2(\Omega)$ such that*

$$\int_\Omega f(x) D_\alpha \varphi(x) \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega g(x) \varphi(x) \mathrm{d}x$$

holds for all $\varphi \in C_c^\infty(\Omega)$. Then f is said to be weakly differentiable of order α and $D_\alpha f := g$ is called the weak α-th derivative of f.

Let us give a few remarks on the definition of weak differentiation:

Remark 4.7.

(i) Let f be weakly differentiable with respect to the multi-index $\alpha \in \mathbb{N}_0^d$. Then it follows from Theorem 4.3 that the weak derivative $D_\alpha f$ is unique as an element of $L^2(\Omega)$.

(ii) If f is k times continuously differentiable and $D_\alpha f \in L^2(\Omega)$ holds for $|\alpha| \leq k$, then $D_\alpha f$ also exists in the weak sense and the weak and the classical derivative coincide. Hence, differentiation in the weak sense is a generalization of classical differentiation.

Now, we are already prepared to give a first definition of Sobolev spaces via weak derivatives:

Definition 4.8. *Let $\Omega \subset \mathbb{R}^d$ be open and let $k \in \mathbb{N}$. Then the Sobolev space $W^k(\Omega)$ of order k is defined as*

$$W^k(\Omega) := \left\{ f \in L^2(\Omega) : D_\alpha f \in L^2(\Omega) \ \forall |\alpha| \leq k \right\}.$$

Here it is contained that for $f \in W^k(\Omega)$ the weak derivatives $D_\alpha f$ exist for $|\alpha| \leq k$. Furthermore, we define for $f, g \in W^k(\Omega)$

$$(f, g)_{W^k} := \sum_{|\alpha| \leq k} (D_\alpha f, D_\alpha g)_{L^2}$$

and

$$\|f\|_{W^k} := \sqrt{(f, f)_{W^k}}.$$

Remark 4.9.

(i) By definition it holds $L^2(\Omega) \supset W^1(\Omega) \supset W^2(\Omega) \supset \dots$ and for $l \leq k$ the embedding from $W^k(\Omega)$ to $W^l(\Omega)$ is continuous.

(ii) Set $W^0(\Omega) = L^2(\Omega)$. Then the Sobolev spaces $W^k(\Omega)$ can be introduced recursively in the following way:

$$W^k(\Omega) = \left\{ f \in W^1(\Omega) : D_{e_j} f \in W^{k-1}(\Omega) \ \forall j \in \{1, \dots, d\} \right\},$$

where e_j is a canonical basis vector in $\mathbb{R}^d$.

The next theorem contains the well-known result that $W^k(\Omega)$ is a Hilbert space, see [14, Theorem 2.15.1].

Theorem 4.10. *Let $\Omega \subset \mathbb{R}^d$ be open and let $k \in \mathbb{N}$. Then $W^k(\Omega)$, equipped with $(\cdot, \cdot)_{W^k}$, is a Hilbert space.*

Finally, we state a result on the Fourier transform of the weak derivatives of a function belonging to $W^k(\mathbb{R}^d)$, cf. [9, Satz 6.45]:

Proposition 4.11. *Let $k \in \mathbb{N}$ and let the Fourier transform $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be defined as in Theorem 4.5. Then it holds*

$$(\mathcal{F} D_\alpha f)(x) = (ix)^\alpha (\mathcal{F} f)(x)$$

for all $f \in W^k(\mathbb{R}^d)$, all multi-indices $|\alpha| \leq k$ and almost all $x \in \mathbb{R}^d$.

4.2.2 Sobolev spaces - definition via the Fourier transform

In this subsection, we give a second definition of Sobolev spaces $H^s(\Omega)$ via the Fourier transform, which is motivated by Proposition 4.11 and which can be extended in a natural way to any real $s \geq 0$. Furthermore, we will show that under some weak assumptions on Ω the spaces $H^s(\Omega)$ and $W^s(\Omega)$ coincide for any $s \in \mathbb{N}$. First, let us state the definition of $H^s(\mathbb{R}^d)$:

Definition 4.12. *Let $d \in \mathbb{N}$, let $s \in [0, \infty)$ and let the Fourier transform $\mathcal{F}$ be defined as in Theorem 4.5. Then the Sobolev space $H^s(\mathbb{R}^d)$ of order s is defined as*

$$H^s(\mathbb{R}^d) := \left\{ f \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} (1 + |x|^2)^s |\mathcal{F}f(x)|^2 \mathrm{d}x < \infty \right\}.$$

Moreover, we define on $H^s(\mathbb{R}^d)$ the inner product $(\cdot, \cdot)_{H^s(\mathbb{R}^d)}$ via

$$(f, g)_{H^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} (1 + |x|^2)^s \mathcal{F}f(x) \overline{\mathcal{F}g(x)} \mathrm{d}x, \quad f, g \in H^s(\mathbb{R}^d).$$

Note that it holds by definition $H^{s_1}(\mathbb{R}^d) \subset H^{s_2}(\mathbb{R}^d)$, if $0 \leq s_2 \leq s_1$, as it holds in this case $(1 + |x|^2)^{s_2} \leq (1 + |x|^2)^{s_1}$ for any $x \in \mathbb{R}^d$. Moreover, we see immediately $H^0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$. It is not difficult to see that $H^s(\mathbb{R}^d)$ is a Hilbert space for any $s \geq 0$, cf. the discussion in [50, Section 11.4]:

Corollary 4.13. *Let $d \in \mathbb{N}$ and let $s \in [0, \infty)$. Then $\big(H^s(\mathbb{R}^d), (\cdot, \cdot)_{H^s(\mathbb{R}^d)}\big)$ is a Hilbert space.*

Next, we extend the definition of $H^s(\Omega)$ from $\Omega = \mathbb{R}^d$ to any open set $\Omega \subset \mathbb{R}^d$.

Definition 4.14. *Let $d \in \mathbb{N}$, let $s \in [0, \infty)$, let $H^s(\mathbb{R}^d)$ be constituted as in Definition 4.12 and let $\Omega \subset \mathbb{R}^d$ be open. Then $H^s(\Omega)$ is defined as*

$$H^s(\Omega) := \left\{ f \in L^2(\Omega) : f = F|_\Omega \textit{ for some } F \in H^s(\mathbb{R}^d) \right\}.$$

In order to get a Hilbert space structure on $H^s(\Omega)$, we need numerous preparations. Let $\Omega \subset \mathbb{R}^d$ be open, set $M := \mathbb{R}^d \setminus \Omega$ and let $s \geq 0$. We set

$$H^s_M := \left\{ f \in H^s(\mathbb{R}^d) : \operatorname{supp} f \subset M \right\}.$$

It is not difficult to see that H^s_M is a closed subspace of $H^s(\mathbb{R}^d)$. Hence, there exists an orthogonal projection $P_{\Omega,s}$ acting in $H^s(\mathbb{R}^d)$ onto H^s_M. Clearly, $P_{\Omega,s}$ satisfies

$$P_{\Omega,s} f|_\Omega = 0 \quad \text{and} \quad (1 - P_{\Omega,s}) f|_\Omega = f|_\Omega$$

for any $f \in H^s(\mathbb{R}^d)$. Now, we set for $f, g \in H^s(\Omega)$

$$(f, g)_{H^s(\Omega)} := \big((1 - P_{\Omega,s})F, (1 - P_{\Omega,s})G \big)_{H^s(\mathbb{R}^d)}, \tag{4.1}$$

where $F, G \in H^s(\mathbb{R}^d)$ are given in such a way that $f = F|_\Omega$ and $g = G|_\Omega$ hold. One can show that $(\cdot, \cdot)_{H^s(\Omega)}$ is a well-defined inner product on $H^s(\Omega)$, cf. [43, page 77]. If it is clear, which set Ω is meant, we just write $(\cdot, \cdot)_{H^s}$ instead of $(\cdot, \cdot)_{H^s(\Omega)}$. Now, the following result holds [43]:

Theorem 4.15. *Let $d \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^d$ be open and let $s \geq 0$. Then $\big(H^s(\Omega), (\cdot,\cdot)_{H^s(\Omega)}\big)$ is a Hilbert space. Moreover, it holds*

$$\|f\|_{H^s(\Omega)} := \sqrt{(f,f)_{H^s(\Omega)}} = \inf\left\{\|F\|_{H^s(\mathbb{R}^d)} : F \in H^s(\mathbb{R}^d), f = F|_\Omega\right\}$$

for any $f \in H^s(\Omega)$.

Let us continue with a density result that can be found in [43, page 77]:

Proposition 4.16. *Let $\Omega \subset \mathbb{R}^d$ be open. Then $C_c^\infty\big(\overline{\Omega}\big)$ is dense in $H^s(\Omega)$ for any $s \geq 0$.*

Finally, we give a condition on Ω, which will always be fulfilled in our applications, under which $H^k(\Omega)$ and $W^k(\Omega)$ given as in Section 4.2.1 coincide for any $k \in \mathbb{N}$. This result follows from [43, Theorem 3.18 and Theorem A.4].

Theorem 4.17. *Let $2 \leq d \in \mathbb{N}$, let $k \in \mathbb{N}$ and let $\Omega \subset \mathbb{R}^d$ be open. Moreover, let $W^k(\Omega)$ be constituted as in Definition 4.8 and let $H^k(\Omega)$ be given as in Definition 4.12. If $\partial\Omega$ is a C^1-smooth hypersurface in the sense of Definition 3.2, then $W^k(\Omega) = H^k(\Omega)$ holds and the norms $\|\cdot\|_{W^k}$ and $\|\cdot\|_{H^k}$ are equivalent.*

4.2.3 The space $H^s_\Delta(\Omega)$

In this section, we investigate specific subspaces of $H^s(\Omega)$. A functions f belonging to $H^s_\Delta(\Omega)$ has the property that Δf exists in a weak sense and belongs to $L^2(\Omega)$. For such functions we will state in Section 4.3 a version of Green's integral identity - a generalization of classical integration by parts. Let us start with the definition of $H^s_\Delta(\Omega)$:

Definition 4.18. *Let $\Omega \subset \mathbb{R}^d$ be open and let $s \geq 0$. Then the function space $H^s_\Delta(\Omega)$ is defined as*

$$H^s_\Delta(\Omega) := \left\{ f \in H^s(\Omega) : \exists \Delta f := g \in L^2(\Omega) : \int_\Omega f \Delta\varphi \mathrm{d}x = \int_\Omega g\varphi \mathrm{d}x \ \forall \varphi \in C_c^\infty(\Omega) \right\}.$$

Moreover, we equip $H^s_\Delta(\Omega)$ with the inner product

$$(f,g)_{H^s_\Delta(\Omega)} := (f,g)_{H^s(\Omega)} + (\Delta f, \Delta g)_{L^2(\Omega)}.$$

Note that for $s \geq 2$ the spaces $H^s_\Delta(\Omega)$ and $H^s(\Omega)$ coincide. Next, we state the fact that $H^s_\Delta(\Omega)$ is a Hilbert space. This can be shown in a similar way as Theorem 4.10, see also [14, Theorem 2.15.1]. The details of this simple proof are left to the reader.

Proposition 4.19. *Let $\Omega \subset \mathbb{R}^d$ be open and let $s \geq 0$. Then $\big(H^s_\Delta(\Omega), (\cdot,\cdot)_{H^s_\Delta(\Omega)}\big)$ is a Hilbert space.*

Finally, we state that $C_c^\infty\big(\overline{\Omega}\big)$ is dense in $H^s_\Delta(\Omega)$. This fact will be needed later to prove an appropriate version of Green's formula.

Proposition 4.20. *Let $\Omega \subset \mathbb{R}^d$ be open such that $\partial\Omega$ is a C^1-smooth hypersurface in the sense of Definition 3.2, and let $s \geq 0$. Then $C_c^\infty(\overline{\Omega})$ is dense in $\left(H^s_\Delta(\Omega), (\cdot,\cdot)_{H^s_\Delta(\Omega)}\right)$.*

Proof. For $s \in [0,2)$ the statement is mentioned in the proof of Lemma 2.4 in [28]; see also [18], where the statement is shown for space dimension $d = 2, 3$, and [42, Section 6.4], where a proof is given for a C^∞-smooth boundary $\partial\Omega$ and $s = 0$. For $s \geq 2$ there is nothing to show, as it holds $H^s_\Delta(\Omega) = H^s(\Omega)$ in this case and the associated norms are equivalent. $\square$

4.3 Sobolev spaces on hypersurfaces

In this section, we introduce Sobolev spaces $H^s(\Sigma)$ consisting of functions defined on a hypersurface Σ. Throughout this section, we assume that Ω is an open subset of $\mathbb{R}^d$ and that its boundary $\Sigma := \partial\Omega$ is a hypersurface in the sense of Definition 3.2. The importance of functions in $H^s(\Sigma)$ is the fact that they are the boundary values of functions in $H^t(\Omega)$.

4.3.1 The space $H^s(\Sigma)$

Let $d \geq 2$ be a natural number and let $\Sigma \subset \mathbb{R}^d$ be a hypersurface as in Definition 3.2 with parametrization $\{U_i, V_i, \varphi_i\}_{i\in I}$. Further, recall the idea of a partition of unity from Lemma 3.13 and the definition of the Hausdorff measure σ and the induced integral notion on Σ from Section 3.2. In what follows, we construct the Sobolev spaces $H^s(\Sigma)$, $s \geq 0$, as subspaces of $L^2(\Sigma) := L^2(\Sigma, \sigma)$:

Definition 4.21. *Let $k \in \mathbb{N}$ and let $\Sigma \subset \mathbb{R}^d$ be a hypersurface that is C^k-smooth. Moreover, let $\{U_i, V_i, \varphi_i\}_{i\in I}$ be a parametrization of Σ and let $\{\chi_i\}_{i\in I}$ be a partition of unity for $\{V_i\}_{i\in I}$. For $\xi \in L^2(\Sigma)$ and $i \in I$ we define the functions $\xi_i \in L^2(U_i)$ as*

$$\xi_i : U_i \to \mathbb{C}, \quad \xi_i(u) = \chi_i(\varphi_i(u)) \cdot \xi(\varphi_i(u)).$$

Then for $0 \leq s \leq k$ the Sobolev space $H^s(\Sigma)$ is defined as

$$H^s(\Sigma) := \left\{\xi \in L^2(\Sigma) : \xi_i \in H^s(U_i) \ \forall i \in I\right\}.$$

Moreover, we define on $H^s(\Sigma)$ the inner product

$$(\xi, \zeta)_{H^s(\Sigma)} := \sum_{i\in I} (\xi_i, \zeta_i)_{H^s(U_i)}$$

for $\xi, \zeta \in H^s(\Sigma)$.

We would like to point out, that $H^s(\Sigma)$ can only be defined for $s \leq k$, if Σ is C^k-smooth. The definition of the inner product in $H^s(\Sigma)$ depends on the choice of the parametrization of Σ and on the choice of the partition of unity. Nevertheless, the definition of $H^s(\Sigma)$ is independent from these quantities and it is a Hilbert space [14, Remark 8.13.1 and Theorem 8.13.4]:

Proposition 4.22. *Let $k \in \mathbb{N}$, let $\Sigma \subset \mathbb{R}^d$ be a C^k-smooth hypersurface, let $s \in [0, k]$ and let $H^s(\Sigma)$ be given as in Definition 4.21. Then the following assertions are true:*

(i) *The definition of $H^s(\Sigma)$ does not depend on the choice of the parametrization of Σ and the partition of unity. Moreover, given two parametrizations of Σ with corresponding partitions of unity, the induced inner products in Definition 4.21 yield equivalent norms.*

(ii) $\big(H^s(\Sigma), (\cdot,\cdot)_{H^s(\Sigma)}\big)$ *is a Hilbert space.*

In the next proposition, we state that $H^s(\Sigma)$ is compactly embedded in $L^2(\Sigma)$ for $s > 0$. This statement can be found in [28, Appendix A].

Proposition 4.23. *Let $\Sigma \subset \mathbb{R}^d$ be a hypersurface in the sense of Definition 3.2 which is at least C^1-smooth. Then the embedding $H^s(\Sigma) \to L^2(\Sigma)$ is compact for any $s \in (0, 1]$.*

4.3.2 Trace operators

In this subsection, we introduce the trace operators, that generalize the boundary values of continuously differentiable functions to weakly differentiable functions. Further, we discuss the connection of the Sobolev spaces $H^s(\Sigma)$ and $H^t(\Omega)$. Using these trace operators, we state a generalized version of the classical integration by parts formula. Let us start with the definition and the properties of the Dirichlet trace operator [43, Theorem 3.37]:

Theorem 4.24. *Let $k \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^d$ be an open set such that $\Sigma := \partial\Omega$ is a C^k-smooth hypersurface and let $s \in \big(\frac{1}{2}, k + \frac{1}{2}\big]$. Then there exists a unique bounded and surjective linear operator $\gamma_{D,s} : H^s(\Omega) \to H^{s-1/2}(\Sigma)$ such that*

$$\gamma_{D,s}\varphi = \varphi|_\Sigma$$

holds for all $\varphi \in C_c^\infty\big(\overline{\Omega}\big)$.

We would like to point out that the statement of Theorem 4.24 is only true for $s > \frac{1}{2}$, for $s = \frac{1}{2}$ the result does not hold, i.e. there is no continuous trace operator

$$\gamma_{D,1/2} : H^{1/2}(\Omega) \to L^2(\Sigma).$$

Later, we will use the notation $f|_\Sigma$ for the trace of $f \in H^s(\Omega)$, so $f|_\Sigma = \gamma_{D,s} f$.

Next, we are going to introduce a generalized Neumann trace, i.e. an operator γ_N that satisfies

$$\gamma_N \varphi = \sum_{j=1}^{d} \nu_j \frac{\partial \varphi}{\partial x_j}\bigg|_\Sigma \tag{4.2}$$

for any $\varphi \in C_c^\infty\big(\overline{\Omega}\big)$, where $\nu = (\nu_1, \dots, \nu_d)$ denotes the unit normal vector pointing outwards of Ω. We are going to do this in two different ways, namely once as an operator $\tilde{\gamma}_N : H^2(\Omega) \to L^2(\Sigma)$, where we use the Dirichlet trace for the definition, and once as $\gamma_N : H^{3/2}_\Delta(\Omega) \to L^2(\Sigma)$, which is more important for our applications and which requires some more involved ideas for the construction. Let us start with the definition of $\tilde{\gamma}_N$:

Proposition 4.25. *Let $\Omega \subset \mathbb{R}^d$ be an open set such that $\Sigma := \partial\Omega$ is a C^2-smooth hypersurface, let $\nu = (\nu_1, \dots, \nu_d)$ be the unit normal vector pointing outwards of Ω and let $\gamma_{D,1}$ be given as in Theorem 4.24. Moreover, define the linear operator $\tilde{\gamma}_N : H^2(\Omega) \to \Sigma$ as*

$$\tilde{\gamma}_N f := \sum_{j=1}^{d} \nu_j \gamma_{D,1} D_{e_j} f, \quad f \in H^2(\Omega),$$

where $e_1, \dots, e_d$ denote the canonical basis vectors of $\mathbb{R}^d$. Then $\tilde{\gamma}_N$ is bounded.

Proof. First, we mention that $\tilde{\gamma}_N$ is well-defined, as $D_{e_j} f$ belongs to $H^1(\Omega)$ for any $f \in H^2(\Omega)$ because of our assumptions on Ω and Theorem 4.17.

In order to prove that $\tilde{\gamma}_N$ is bounded, we observe first that

$$D_{e_j} : H^2(\Omega_j) \to H^1(\Omega_j), \quad f \mapsto D_{e_j} f$$

is continuous for any canonical basis vector e_j of $\mathbb{R}^d$, which can be seen by the definition of the norm $\| \cdot \|_{W^k}$, that is equivalent to $\| \cdot \|_{H^k}$ by Theorem 4.17, in Definition 4.8. Moreover, by Theorem 4.24 and Proposition 4.23 the mapping $\gamma_{D,1}$ is continuous from $H^1(\Omega_j)$ to $L^2(\Sigma)$. Hence, we find

$$\begin{aligned}\|\tilde{\gamma}_N f\|_{L^2(\Sigma)} &= \left\| \sum_{j=1}^{d} \nu_j \gamma_{D,1} D_{e_j} f \right\|_{L^2(\Sigma)} \leq \sum_{j=1}^{d} \left\| \nu_j \gamma_{D,1} D_{e_j} f \right\|_{L^2(\Sigma)} \\ &\leq \|\nu\| \left(\sum_{j=1}^{d} \left\| \gamma_{D,1} D_{e_j} f \right\|^2_{L^2(\Sigma)} \right)^{1/2} \leq c_1 \left(\sum_{j=1}^{d} \left\| D_{e_j} f \right\|^2_{H^1(\Omega)} \right)^{1/2} \leq c_2 \|f\|_{H^2(\Omega_j)}\end{aligned}$$

for any $f \in H^2(\Omega)$, where we used the Cauchy-Schwarz inequality for sums and the fact that ν is normed by definition. This shows that $\tilde{\gamma}_N$ is bounded. $\square$

In what follows, we introduce the Neumann trace operator $\gamma_N : H^{3/2}_\Delta(\Omega) \to L^2(\partial\Omega)$. This operator will allow us to derive a version of Green's identity for functions belonging to $H^{3/2}_\Delta(\Omega)$ that is needed in our applications. This non-standard construction of γ_N can be found for instance in [28, Lemma 2.4].

Theorem 4.26. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set such that $\Sigma := \partial\Omega$ is a C^1-smooth hypersurface. Then there exists a unique bounded and surjective linear operator $\gamma_N : H^{3/2}_\Delta(\Omega) \to L^2(\Sigma)$ such that (4.2) holds for all $\varphi \in C^\infty_c(\overline{\Omega})$.*

Theorem 4.26 gives only a definition of γ_N, when Ω is a *bounded* domain with a smooth boundary $\Sigma = \partial\Omega$. But in our applications, we also need the Neumann trace of functions $f \in H^{3/2}_\Delta(\Omega)$ in the case that $\mathbb{R}^d \setminus \overline{\Omega}$ is bounded. We are going to extend the definition of γ_N to this case:

Let $\Omega \subset \mathbb{R}^d$ be open such that $\mathbb{R}^d \setminus \overline{\Omega}$ fulfills the assumptions of Theorem 4.26. Hence, $\Sigma := \partial\Omega$ is compact and there exists $R > 0$ such that $\Sigma \subset B(0, R)$. If we define the set $\Omega_R := \Omega \cap B(0,R)$, we see that Ω_R satisfies the assumptions of Theorem 4.26 and $\partial\Omega_R = \Sigma \cup \partial B(0,R)$ and thus, there exists a Neumann trace $\gamma_{N,R} : H^{3/2}_\Delta(\Omega_R) \to L^2(\partial\Omega_R)$. In order to define the Neumann trace γ_N on $H^{3/2}_\Delta(\Omega)$, we introduce the projections

$$P_R : H^{3/2}_\Delta(\Omega) \to H^{3/2}_\Delta(\Omega_R), \quad P_R f = f|_{\Omega_R}$$

and

$$P_\Sigma : L^2(\partial\Omega_R) \to L^2(\Sigma), \quad P_\Sigma \xi = \xi|_\Sigma.$$

Now, we define $\gamma_N := P_\Sigma \gamma_{N,R} P_R$ and see that $\gamma_N : H^{3/2}_\Delta(\Omega) \to L^2(\Sigma)$ is bounded and that (4.2) is fulfilled for any $\varphi \in C_c^\infty\big(\overline{\Omega}\big)$. Hence, γ_N is the desired Neumann trace-operator for Ω. Later, we will use the notation $\partial_\nu f|_\Sigma = \gamma_N f$ for the Neumann trace.

Using the operators $\gamma_{D,s}$ from Theorem 4.24 and γ_N from Theorem 4.26, we are going to state a generalized version of Green's first integral formula:

Theorem 4.27. *Let $\Omega \subset \mathbb{R}^d$ be an open set such that $\Sigma := \partial\Omega$ is a C^1-smooth hypersurface, let $\gamma_D := \gamma_{D,1}$ be defined as in Theorem 4.24 and let γ_N be given as in Theorem 4.26. Moreover, let e_j be the canonical basis vectors of $\mathbb{R}^d$. Then it holds*

$$\int_\Omega \sum_{j=1}^d D_{e_j} f D_{e_j} g \mathrm{d}x + \int_\Omega (\Delta f) g \mathrm{d}x = \int_\Sigma \gamma_N f \gamma_D g \mathrm{d}\sigma$$

for all $f \in H^{3/2}_\Delta(\Omega)$ and $g \in H^1(\Omega)$.

Proof. Let $f \in H^{3/2}_\Delta(\Omega)$ and $g \in H^1(\Omega)$ be fixed. According to Proposition 4.16, there exists a sequence $(g_n) \subset C_c^\infty\big(\overline{\Omega}\big)$ such that g_n converges to g in $H^1(\Omega)$. Hence, it holds $g_n \to g$ and $D_{e_j} g_n \to D_{e_j} g$ in $L^2(\Omega)$. Furthermore, by Proposition 4.20, there exists a sequence $(f_n) \subset C_c^\infty\big(\overline{\Omega}\big)$ such that $f_n \to f$ in $H^{3/2}_\Delta(\Omega)$. This implies $\Delta f_n \to \Delta f$ and $D_{e_j} f_n \to D_{e_j} f$ in $L^2(\Omega)$. Note furthermore that it holds $\gamma_D g_n \to \gamma_D g$ and $\gamma_N f_n \to \gamma_N f$ in $L^2(\Sigma)$, as γ_D and γ_N are continuous by Theorem 4.24 and Theorem 4.26. Using the classical integral formula of Green which is valid for $C_c^\infty\big(\overline{\Omega}\big)$-functions, we find

$$\int_\Omega \sum_{j=1}^d D_{e_j} f_n D_{e_j} g_n \mathrm{d}x + \int_\Omega (\Delta f_n) g_n \mathrm{d}x = \int_\Sigma \gamma_N f_n \gamma_D g_n \mathrm{d}\sigma.$$

Thus, we get finally

$$\begin{aligned}\int_\Sigma \gamma_N f \gamma_D g \mathrm{d}\sigma &= \lim_{n\to\infty} \int_\Sigma \gamma_N f_n \gamma_D g_n \mathrm{d}\sigma = \lim_{n\to\infty} \left(\int_\Omega \sum_{j=1}^d D_{e_j} f_n D_{e_j} g_n \mathrm{d}x + \int_\Omega (\Delta f_n) g_n \mathrm{d}x \right) \\ &= \int_\Omega \sum_{j=1}^d D_{e_j} f D_{e_j} g \mathrm{d}x + \int_\Omega (\Delta f) g \mathrm{d}x,\end{aligned}$$

which is the claimed result. □

5 Some classes of auxiliary operators

In this chapter, we study several special types of operators that act between Hilbert spaces of the form $L^2(X,\mu)$ introduced in Section 4.1 and that will appear frequently in our applications. First, we discuss multiplication operators which act as $f \mapsto uf$ for a fixed function u, then we introduce integral operators of the form

$$f \mapsto \int_X k(\cdot, y) f(y) \mathrm{d}\mu(y)$$

for a fixed integral kernel k and we consider classical Schrödinger operators $-\Delta + V$ for a potential $V \in L^\infty(\mathbb{R}^d)$. Finally, we investigate the single layer potential associated to the differential expression $-\Delta + 1$, which is essential in the investigation of the δ-operator $A_{\delta,\alpha}$.

5.1 Multiplication operators in $L^2(X,\mu)$

In the following section, we discuss operators that act formally as $f \mapsto uf$, where u is a fixed function. To be more precise, let (X,μ) be a σ-finite measure space and let $u : X \to \mathbb{C}$ be a measurable function. Then the associated, maximal multiplication operator M_u in $L^2(X,\mu)$ is defined as

$$M_u f = uf, \quad \operatorname{dom} M_u := \big\{ f \in L^2(X,\mu) : uf \in L^2(X,\mu) \big\}. \tag{5.1}$$

The main properties of this operator are summarized in the following theorem, cf. [50, Satz 6.1 and Satz 6.2]:

Theorem 5.1. *Let (X,μ) be a σ-finite measure space, let $u : X \to \mathbb{C}$ be a measurable function and let M_u be the associated multiplication operator given by* (5.1). *Then the following assertions are true:*

(i) *M_u is a densely defined and closed operator and its adjoint is given by $M_u^* = M_{\overline{u}}$.*

(ii) *M_u is bounded, if and only if $u \in L^\infty(X,\mu)$. In this case, it holds $\|M_u\| = \|u\|_{L^\infty}$.*

(iii) *$\lambda \in \mathbb{C}$ belongs to the spectrum of M_u, if and only if*

$$\mu\big(\{x \in X : |u(x) - \lambda| < \varepsilon\}\big) > 0$$

holds for all $\varepsilon > 0$. In particular, if u is continuous and μ is the Lebesgue measure, it holds $\sigma(M_u) = \overline{\operatorname{ran} u}$.

5.2 Integral operators

Let (X_1, μ_1) and (X_2, μ_2) be two σ-finite measure spaces. We want to study operators K defined in $L^2(X_1, \mu_1)$ and mapping into $L^2(X_2, \mu_2)$, which act formally as

$$(Kf)(x_2) = \int_{X_1} k(x_1, x_2) f(x_1) \mathrm{d}\mu_1(x_1)$$

for almost all $x_2 \in X_2$. Here, the function $k : X_1 \times X_2 \to \mathbb{C}$, which is assumed to be $\mu_1 \times \mu_2$-measurable, is called the integral kernel of K. In particular, we are going to give some criteria on k, such that the operator K is well-defined on the whole space $L^2(X_1, \mu_1)$ and bounded or even compact and we are going to give various upper bounds for the norm of K.

A first result is stated in the following theorem, which will be the main tool to prove the convergence of a family of Schrödinger operators with regular potentials to a Hamiltonian with a δ-interaction supported on a hypersurface. Since this result is very important for our purposes, we give a proof of it which follows [50, Satz 6.9].

Theorem 5.2. *Let (X_1, μ_1) and (X_2, μ_2) be two σ-finite measure spaces and let $k : X_1 \times X_2 \to \mathbb{C}$ be $\mu_1 \times \mu_2$ measurable. Assume that there exist constants $C_1, C_2 > 0$ such that*

$$\int_{X_1} |k(x_1, x_2)| \mathrm{d}\mu_1(x_1) \leq C_1$$

is true for μ_2-almost all $x_2 \in X_2$ and

$$\int_{X_2} |k(x_1, x_2)| \mathrm{d}\mu_2(x_2) \leq C_2$$

holds for μ_1-almost all $x_1 \in X_1$. Then the operator $K : L^2(X_1, \mu_1) \to L^2(X_2, \mu_2)$, which acts as

$$(Kf)(x_2) = \int_{X_1} k(x_1, x_2) f(x_1) \mathrm{d}\mu_1(x_1)$$

for almost all $x_2 \in X_2$ and all $f \in L^2(X_1, \mu_1)$, is well-defined and bounded. Moreover, it holds $\|K\|^2 \leq C_1 C_2$.

Remark 5.3. Let K, C_1 and C_2 be as in Theorem 5.2. Then $\sqrt{C_1 C_2}$ is called "Schur-Holmgren bound" for $\|K\|$.

Proof of Theorem 5.2. Step 1: Let $f \in L^2(X_1, \mu_1)$ be fixed. We prove that $Kf(x_2)$ is finite for almost all $x_2 \in X_2$ and that Kf is a μ_2-measurable function in X_2. Since X_2 is σ-finite by assumption, there exist countably many sets $\tilde{X}_n$, such that $X_2 = \bigcup_n \tilde{X}_n$ and $\mu_2(\tilde{X}_n) < \infty$ are satisfied. Hence, it is sufficient to prove for any $n \in \mathbb{N}$ that $Kf(x_2)$ is finite for almost any $x_2 \in \tilde{X}_n$ and that $Kf|_{\tilde{X}_n}$ is measurable. According to the theorem of Fubini-Tonelli [50, Satz A.21] this is true, if

$$\int_{\tilde{X}_n \times X_1} |k(x_1, x_2) f(x_1)| \mathrm{d}(\mu_2 \times \mu_1)(x_2, x_1) < \infty.$$

In order to prove this, we use that

$$a \leq 2a \leq 1 + a^2$$

holds for any $a \geq 0$, which yields

$$\begin{aligned}
\int_{\tilde{X}_n \times X_1} |k(x_1,x_2)f(x_1)| \mathrm{d}(\mu_2 \times \mu_1)(x_2,x_1) &= \int_{\tilde{X}_n} \int_{X_1} |k(x_1,x_2)f(x_1)| \mathrm{d}\mu_1(x_1) \mathrm{d}\mu_2(x_2) \\
&\leq \int_{\tilde{X}_n} \left(1 + \left(\int_{X_1} |k(x_1,x_2)f(x_1)| \mathrm{d}\mu_1(x_1) \right)^2 \right) \mathrm{d}\mu_2(x_2) \\
&= \mu_2\bigl(\tilde{X}_n\bigr) + \int_{\tilde{X}_n} \left(\int_{X_1} |k(x_1,x_2)f(x_1)| \mathrm{d}\mu_1(x_1) \right)^2 \mathrm{d}\mu_2(x_2).
\end{aligned}$$

Now, we need to look for an estimate for the second integral. For this purpose, we use the Cauchy-Schwarz inequality, our assumptions on k and the theorem of Fubini-Tonelli and find

$$\begin{aligned}
&\int_{\tilde{X}_n} \left(\int_{X_1} |k(x_1,x_2)f(x_1)| \mathrm{d}\mu_1(x_1) \right)^2 \mathrm{d}\mu_2(x_2) \\
&\qquad \leq \int_{X_2} \left(\int_{X_1} |k(x_1,x_2)|^{1/2} |k(x_1,x_2)|^{1/2} |f(x_1)| \mathrm{d}\mu_1(x_1) \right)^2 \mathrm{d}\mu_2(x_2) \\
&\qquad \leq \int_{X_2} \underbrace{\int_{X_1} |k(x_1,x_2)| \mathrm{d}\mu_1(x_1)}_{\leq C_1} \int_{X_1} |k(x_1,x_2)| |f(x_1)|^2 \mathrm{d}\mu_1(x_1) \mathrm{d}\mu_2(x_2) \\
&\qquad \leq C_1 \int_{X_1} \underbrace{\int_{X_2} |k(x_1,x_2)| \mathrm{d}\mu_2(x_2)}_{\leq C_2} |f(x_1)|^2 \mathrm{d}\mu_1(x_1) \leq C_1 C_2 \|f\|_{L^2}^2.
\end{aligned}$$

Hence, we find

$$\begin{aligned}
&\int_{\tilde{X}_n \times X_1} |k(x_1,x_2)f(x_1)| \mathrm{d}(\mu_2 \times \mu_1)(x_2,x_1) \\
&\qquad \leq \mu_2\bigl(\tilde{X}_n\bigr) + \int_{\tilde{X}_n} \left(\int_{X_1} |k(x_1,x_2)f(x_1)| \mathrm{d}\mu_1(x_1) \right)^2 \mathrm{d}\mu_2(x_2) \leq \mu_2\bigl(\tilde{X}_n\bigr) + C_1 C_2 \|f\|_{L^2}^2,
\end{aligned}$$

which is finite, as $\mu_2\bigl(\tilde{X}_n\bigr) < \infty$.

Step 2: Let $f \in L^2(X_1, \mu_1)$ be fixed. It remains to prove that Kf is an element of $L^2(X_2, \mu_2)$ and that K is bounded with $\|K\| \leq \sqrt{C_1 C_2}$. Using the Cauchy-Schwarz inequality, our assumptions on k and the theorem of Fubini-Tonelli, we find

$$\begin{aligned}\int_{X_2} |Kf(x_2)|^2 \,\mathrm{d}\mu_2(x_2) &= \int_{X_2} \left| \int_{X_1} k(x_1, x_2) f(x_1) \mathrm{d}\mu_1(x_1) \right|^2 \mathrm{d}\mu_2(x_2) \\ &\leq \int_{X_2} \left(\int_{X_1} |k(x_1, x_2)|^{1/2} |k(x_1, x_2)|^{1/2} |f(x_1)| \mathrm{d}\mu_1(x_1) \right)^2 \mathrm{d}\mu_2(x_2) \\ &\leq \int_{X_2} \underbrace{\int_{X_1} |k(x_1, x_2)| \mathrm{d}\mu_1(x_1)}_{\leq C_1} \int_{X_1} |k(x_1, x_2)| |f(x_1)|^2 \mathrm{d}\mu_1(x_1) \mathrm{d}\mu_2(x_2) \\ &\leq C_1 \int_{X_1} \underbrace{\int_{X_2} |k(x_1, x_2)| \mathrm{d}\mu_2(x_2)}_{\leq C_2} |f(x_1)|^2 \mathrm{d}\mu_1(x_1) \leq C_1 C_2 \|f\|_{L^2}^2.\end{aligned}$$

Hence, it follows $Kf \in L^2(X_2, \mu_2)$, so K is well-defined, and $\|K\| \leq \sqrt{C_1 C_2}$. Thus, all claimed statements of this theorem are true. □

Finally, we give a criterion on the integral kernel k under that the corresponding integral operator K is compact [50, Satz 3.19]:

Proposition 5.4. *Let (X_1, μ_1) and (X_2, μ_2) be two σ-finite measure spaces and let $k \in L^2(X_1 \times X_2, \mu_1 \times \mu_2)$. Then the operator $K : L^2(X_1, \mu_1) \to L^2(X_2, \mu_2)$, which acts as*

$$(Kf)(x_2) = \int_{X_1} k(x_1, x_2) f(x_1) \mathrm{d}\mu_1(x_1)$$

for almost all $x_2 \in X_2$ and all $f \in L^2(X_1, \mu_1)$, is well-defined and compact. Moreover, it holds $\|K\| \leq \|k\|_{L^2}$.

5.3 Schrödinger operators $-\Delta + V$ in $\mathbb{R}^d$

In this section, we introduce Schrödinger operators of the form $-\Delta + V$ for a compactly supported and real-valued potential $V \in L^\infty(\mathbb{R}^d)$ and discuss their basic properties. First, we are going to investigate the free Laplace operator $-\Delta$ in $\mathbb{R}^d$, which is defined on the Sobolev space $H^2(\mathbb{R}^d)$.

Remark 5.5. If the symbol $-\Delta$ is used for an operator in this thesis, then it is always meant with the domain of definition $H^2(\mathbb{R}^d)$.

In the following proposition we summarize various basic facts about modified Bessel functions of the second kind K_ν that are needed for the analysis of $-\Delta$ and that can be found in [1, Chapter 9.6 and 9.7]. For the definition and a detailed discussion of the properties of K_ν we refer to [1].

Proposition 5.6. *Let $\nu \geq 0$ and let K_ν be a modified Bessel function of the second kind. Then the following assertions are true:*

(i) *The mapping* $\mathbb{C} \setminus (-\infty, 0] \ni z \mapsto K_\nu(z)$ *is holomorphic.*

(ii) $K_\nu'(z) = K_{\nu+1}(z) + \frac{\nu}{z} K_\nu(z)$.

(iii) *For* $z \to 0$ *it holds*

$$K_\nu(z) \sim \begin{cases} -\ln z, & \nu = 0, \\ \frac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}, & \nu > 0. \end{cases}$$

(iv) *For* $|z| \to \infty$ *with* $|\arg z| < \frac{3}{2}\pi$ *it holds*

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right).$$

For our considerations on $-\Delta$, we need the following notation for the square root of a complex number:

Definition 5.7. *Let* $\lambda \in \mathbb{C} \setminus [0, \infty)$*. Then* $\sqrt{\lambda}$ *is defined as the complex number satisfying* $\left(\sqrt{\lambda}\right)^2 = \lambda$ *and* $\operatorname{Im}\sqrt{\lambda} > 0$*.*

In the following lemma we introduce the function G_λ. It will turn out that this function is the integral kernel of $(-\Delta - \lambda)^{-1}$.

Lemma 5.8. *Let* $d \geq 2$ *and let* $\lambda \in \mathbb{C} \setminus [0, \infty)$*. Moreover, define the function* G_λ *as*

$$G_\lambda(x) := \frac{1}{(2\pi)^{d/2}} \left(\frac{|x|}{-i\sqrt{\lambda}}\right)^{1-d/2} K_{d/2-1}\left(-i\sqrt{\lambda}|x|\right), \quad x \in \mathbb{R}^d \setminus \{0\}.$$

Then it holds $G_\lambda \in L^1(\mathbb{R}^d)$*.*

Proof. In order to prove the statement of this lemma, we note first that

$$\int_{\mathbb{R}^d} |G_\lambda(x)| \mathrm{d}x = \int_{B(0,1/2)} |G_\lambda(x)| \mathrm{d}x + \int_{\mathbb{R}^d \setminus B(0,1/2)} |G_\lambda(x)| \mathrm{d}x$$

holds and we show that both integrals on the right hand side are bounded.

According to Proposition 5.6 (iv), there exists a constant $c_1 > 0$ such that

$$|G_\lambda(x)| \leq c_1 e^{-\operatorname{Im}\sqrt{\lambda}|x|}$$

holds for any $x \notin B(0, 1/2)$ and hence we find $\int_{\mathbb{R}^d \setminus B(0,1/2)} |G_\lambda(x)| \mathrm{d}x < \infty$.

In order to prove $\int_{B(0,1/2)} |G_\lambda(x)| \mathrm{d}x < \infty$, we consider first the case $d = 2$. Here, it holds by Proposition 5.6, item (iii),

$$|G_\lambda(x)| \leq c_2 |\ln |x||$$

for all $x \in B(0,1/2)$ with a constant $c_2 > 0$. Hence, we find

$$\int_{B(0,1/2)} |G_\lambda(x)|\mathrm{d}x \leq c_2 \int_{B(0,1/2)} |\ln|x||\mathrm{d}x = -2\pi c_2 \int_0^{1/2} \ln r \cdot r\mathrm{d}r < \infty,$$

where we used a substitution to polar coordinates.

In the case $d \geq 3$ it holds

$$|G_\lambda(x)| \leq c_3|x|^{2-d}$$

for all $x \in B(0,1/2)$ with a constant $c_3 > 0$. Thus, we get

$$\int_{B(0,1/2)} |G_\lambda(x)|\mathrm{d}x \leq c_3 \int_{B(0,1/2)} |x|^{2-d}\mathrm{d}x = c_4 \int_0^{1/2} r^{2-d} \cdot r^{d-1}\mathrm{d}r < \infty,$$

where we used again a substitution to spherical coordinates. So, we get in all cases

$$\int_{\mathbb{R}^d} |G_\lambda(x)|\mathrm{d}x = \int_{B(0,1/2)} |G_\lambda(x)|\mathrm{d}x + \int_{\mathbb{R}^d\setminus B(0,1/2)} |G_\lambda(x)|\mathrm{d}x < \infty$$

and thus, $G_\lambda \in L^1(\mathbb{R}^d)$. □

Basic properties of the free Laplacian are stated in the following theorem:

Theorem 5.9. *Consider for $d \geq 2$ the Laplacian $-\Delta$ with $\operatorname{dom}(-\Delta) = H^2(\mathbb{R}^d)$. Then the following assertions are true:*

(i) *$-\Delta$ is an unbounded, self-adjoint linear operator in $L^2(\mathbb{R}^d)$ and its spectrum is given by*

$$\sigma(-\Delta) = \sigma_{\mathrm{ess}}(-\Delta) = [0,\infty).$$

(ii) *Let $\lambda \in \rho(-\Delta) = \mathbb{C} \setminus [0,\infty)$ and let G_λ be defined as in Lemma 5.8. Then it holds*

$$(-\Delta - \lambda)^{-1}f = \int_{\mathbb{R}^d} G_\lambda(\cdot - y)f(y)\mathrm{d}y$$

for all $f \in L^2(\mathbb{R}^d)$.

Proof. (i) Let $q(x) = |x|^2$ and define in $L^2(\mathbb{R}^d)$ the multiplication operator M_q as

$$M_q f = qf, \quad \operatorname{dom} M_q = \left\{f \in L^2(\mathbb{R}^d) : qf \in L^2(\mathbb{R}^d)\right\}.$$

Then by Theorem 5.1 the operator M_q is unbounded, self-adjoint and it holds $\sigma(M_q) = [0,\infty)$.

Let $\mathcal{F}$ be the Fourier transform given as in Theorem 4.5. Then, using the definition of $H^2(\mathbb{R}^d)$ and Proposition 4.11 it is easy to see that $-\Delta = \mathcal{F}^{-1}M_q\mathcal{F}$. Hence, $-\Delta$ is self-adjoint and it follows immediately from Proposition 2.10 that $\sigma(-\Delta) = [0,\infty)$, as $\mathcal{F}$

is unitary in the space $L^2(\mathbb{R}^d)$. Finally, recalling the definition of the essential spectrum from Section 2.1 we find $\sigma(-\Delta) = \sigma_{\text{ess}}(-\Delta) = [0, \infty)$.

(ii) Let $\lambda \in \rho(-\Delta) = \mathbb{C} \setminus [0, \infty)$. In order to prove a resolvent formula for $(-\Delta - \lambda)^{-1}$, we use [45, Theorem IX.29]. This result says that

$$(-\Delta - \lambda)^{-1} f = \int_{\mathbb{R}^d} K_\lambda(\cdot - y) f(y) \mathrm{d}y$$

holds for any $f \in L^2(\mathbb{R}^d)$, if the function K_λ satisfies $K_\lambda \in L^1(\mathbb{R}^d)$ and $\mathcal{F}_1 K_\lambda = \big((\cdot)^2 - \lambda\big)^{-1}$, where $\mathcal{F}_1$ is the Fourier transform constituted as in Definition 4.4, see also Example 2 in [45, Section IX.7].

Let us consider the function G_λ defined as in Lemma 5.8. Then, it holds $G_\lambda \in L^1(\mathbb{R}^d)$. Moreover, using formula 9.6.4 from [1], we find

$$G_\lambda(x) = \frac{i\sqrt{\lambda}^{d-2}}{2(2\pi)^{(d-1)/2}} \sqrt{\frac{\pi}{2}} \frac{1}{|x|^{d/2-1}} H^{(1)}_{d/2-1}\left(\sqrt{\lambda}|x|\right),$$

where $H^{(1)}_{d/2-1}$ is a Hankel function of the first kind and order $\frac{d}{2} - 1$. Using now the last representation of G_λ and [43, Theorem 9.4 and (6.12)], we get

$$\mathcal{F}_1 G_\lambda = \big((\cdot)^2 - \lambda\big)^{-1}.$$

Thus, we find $K_\lambda = G_\lambda$ and hence, the claimed result is true. $\square$

In the rest of this section, we discuss the operator

$$H_V f = -\Delta f + V f, \quad \operatorname{dom} H_V = H^2(\mathbb{R}^d), \tag{5.2}$$

where the potential $V \in L^\infty(\mathbb{R}^d)$ is compactly supported and real-valued. Note that it follows immediately from Corollary 2.4 that H_V is self-adjoint, as $-\Delta$ is self-adjoint by Theorem 5.9 and the multiplication operator associated with V is bounded, everywhere defined and self-adjoint, see Theorem 5.1. In order to find more properties of H_V, we need the following lemma:

Lemma 5.10. *Let $u \in L^\infty(\mathbb{R}^d)$ be compactly supported, let M_u be the associated multiplication operator and let $\lambda \in \mathbb{C} \setminus [0, \infty)$. Then the operator $M_u(-\Delta - \lambda)^{-1}$ is compact.*

Proof. Let χ_ε be the characteristic function for $\mathbb{R}^d \setminus B(0, \varepsilon)$, let G_λ be as in Lemma 5.8 and let $G^\varepsilon_\lambda := \chi_\varepsilon G_\lambda$. Due to the asymptotic properties of

$$G_\lambda = \frac{1}{(2\pi)^{d/2}} \left(\frac{|\cdot|}{-i\sqrt{\lambda}}\right)^{1-d/2} K_{d/2-1}\left(-i\sqrt{\lambda}|\cdot|\right),$$

cf. Proposition 5.6, items (iii) and (iv), it holds $G^\varepsilon_\lambda \in L^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Hence, the operator K_ε, which acts as

$$K_\varepsilon f = u \int_{\mathbb{R}^d} G^\varepsilon_\lambda(\cdot - y) f(y) \mathrm{d}y,$$

is compact in $L^2(\mathbb{R}^d)$ for any $\varepsilon > 0$, see Proposition 5.4. Since

$$\int_{\mathbb{R}^d} \left| u(x) \left(G_\lambda^\varepsilon(x-y) - G_\lambda(x-y) \right) \right| \mathrm{d}y \leq \|u\|_{L^\infty} \int_{B(0,\varepsilon)} |G_\lambda(y)| \mathrm{d}y$$

holds for any $x \in \mathbb{R}^d$ due to the translation invariance of the Lebesgue measure,

$$\int_{\mathbb{R}^d} \left| u(x) \left(G_\lambda^\varepsilon(x-y) - G_\lambda(x-y) \right) \right| \mathrm{d}x \leq \|u\|_{L^\infty} \int_{B(0,\varepsilon)} |G_\lambda(x)| \mathrm{d}x$$

is fulfilled for any $y \in \mathbb{R}^d$ and

$$\int_{B(0,\varepsilon)} |G_\lambda(x)| \mathrm{d}x \to 0, \quad \varepsilon \to 0+,$$

is true by the dominated convergence theorem, as $G_\lambda \in L^1(\mathbb{R}^d)$ by Lemma 5.8, we find, using a Schur-Holmgren bound, see Theorem 5.2,

$$\begin{aligned} \left\| K_\varepsilon - M_u(-\Delta - \lambda)^{-1} \right\|^2 &\leq \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left| u(x) \left(G_\lambda^\varepsilon(x-y) - G_\lambda(x-y) \right) \right| \mathrm{d}y \\ &\quad \cdot \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left| u(x) \left(G_\lambda^\varepsilon(x-y) - G_\lambda(x-y) \right) \right| \mathrm{d}x \\ &\leq \|u\|_{L^\infty}^2 \left(\int_{B(0,\varepsilon)} |G_\lambda(z)| \mathrm{d}z \right)^2 \to 0. \end{aligned}$$

Therefore, $M_u(-\Delta - \lambda)^{-1}$ is the limit of a sequence of compact operators and hence, it is also compact by Proposition 2.14 (ii). □

Using the result from Lemma 5.10, we can prove a detailed result about the spectrum of H_V.

Theorem 5.11. *Let $V \in L^\infty(\mathbb{R}^d)$ be real-valued and compactly supported and let H_V be defined as in* (5.2)*. Moreover, set*

$$u := |V|^{1/2} \quad \text{and} \quad v := \operatorname{sign} V \cdot |V|^{1/2}$$

and let M_u and M_v be the associated multiplication operators. Then the following assertions hold:

(i) $\sigma_{\mathrm{ess}}(H_V) = [0, \infty)$.

(ii) *$\lambda \in (-\infty, 0)$ is a discrete eigenvalue of H_V, if and only if -1 is an eigenvalue of $M_u(-\Delta - \lambda)^{-1} M_v$. Otherwise, the operator $\left(1 + M_u(-\Delta - \lambda)^{-1} M_v\right)^{-1}$ is bounded and everywhere defined.*

Proof. (i) Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and let M_V be the multiplication operator associated to V. Using a resolvent identity from [50, Satz 5.4], which can be shown by a simple calculation, we find

$$\begin{aligned}(-\Delta - \lambda)^{-1} - (H_V - \lambda)^{-1} &= (H_V - \lambda)^{-1}(H_V - (-\Delta))(-\Delta - \lambda)^{-1} \\ &= (H_V - \lambda)^{-1} M_V (-\Delta - \lambda)^{-1}.\end{aligned}$$

Since $M_V(-\Delta - \lambda)^{-1}$ is compact by Lemma 5.10, it follows that $(-\Delta - \lambda)^{-1} - (H_V - \lambda)^{-1}$ is compact, cf. Proposition 2.14. Using this and the results from Proposition 2.15 and Theorem 5.9, we find $\sigma_{\mathrm{ess}}(H_V) = \sigma_{\mathrm{ess}}(-\Delta) = [0, \infty)$.

(ii) Let $\lambda \in (-\infty, 0)$ be an eigenvalue of H_V, i.e. there exists $0 \neq f \in \operatorname{dom} H_V$ such that $H_V f = \lambda f$. Note that $Vf \neq 0$, as otherwise λ would be an eigenvalue of $-\Delta$, which is not possible. Hence, it follows from $H_V f = \lambda f$

$$\begin{aligned}(-\Delta - \lambda) f = -M_v M_u f &\Rightarrow f = -(-\Delta - \lambda)^{-1} M_v M_u f \\ &\Rightarrow -M_u f = M_u(-\Delta - \lambda)^{-1} M_v M_u f,\end{aligned}$$

i.e. $0 \neq M_u f \in L^2(\mathbb{R}^d)$ is an eigenfunction of $M_u(-\Delta - \lambda)^{-1} M_v$ corresponding to the eigenvalue -1.

Conversely, if $0 \neq f \in L^2(\mathbb{R}^d)$ is an eigenfunction corresponding to the eigenvalue -1 of $M_u(-\Delta - \lambda)^{-1} M_v$ for $\lambda \in (-\infty, 0)$, then $0 \neq (-\Delta - \lambda)^{-1} M_v f \in H^2(\mathbb{R}^d) = \operatorname{dom} H_V$ satisfies

$$(H_V - \lambda)(-\Delta - \lambda)^{-1} M_v f = M_v f + \underbrace{M_V(-\Delta - \lambda)^{-1} M_v f}_{= -M_v f} = 0.$$

Hence, $(-\Delta - \lambda)^{-1} M_v f$ is an eigenfunction of H_V corresponding to the eigenvalue λ.

Finally, if -1 is not an eigenvalue of $M_u(-\Delta - \lambda)^{-1} M_v$, it follows from Fredholm's alternative, cf. Theorem 2.13, that $\big(1 + M_u(-\Delta - \lambda)^{-1} M_v\big)^{-1}$ is bounded and everywhere defined, as $M_u(-\Delta - \lambda)^{-1} M_v$ is compact by Lemma 5.10. □

5.4 The single layer potential associated to $-\Delta + 1$

In this section, we introduce the single layer potential associated to the differential expression $-\Delta + 1$ and a hypersurface Σ. This operator has some properties that are essential in the investigation of the δ-operator $A_{\delta,\alpha}$. Here, we follow the presentation of [43].

Let us fix some general assumptions for the whole section. Let $d \geq 2$ and let $\Sigma \subset \mathbb{R}^d$ be a hypersurface in the sense of Definition 3.2 that is at least C^2-smooth and that splits $\mathbb{R}^d$ into a bounded interior part Ω_{i} and an unbounded exterior part Ω_{e}. Moreover, we fix a cutoff function $\eta \in C_c^\infty(\mathbb{R}^d)$ which satisfies $\eta \equiv 1$ in a neighborhood of Ω_{i}; note that such a function η exists, see for instance [8, Section 2.19].

In order to define the single layer potential, we recall the definition of the Dirichlet trace $\gamma_{D,j} := \gamma_{D,2} : H^2(\Omega_j) \to H^{3/2}(\Sigma)$ for $j \in \{\mathrm{i}, \mathrm{e}\}$. According to Theorem 4.24, $\gamma_{D,j}$ is bounded and everywhere defined and thus its adjoint $\gamma_{D,j}^* : H^{3/2}(\Sigma) \to H^2(\Omega_j)$ is also bounded and everywhere defined. Furthermore, we set $H^s(\mathbb{R}^d \setminus \Sigma) := H^s(\Omega_{\mathrm{i}}) \oplus H^s(\Omega_{\mathrm{e}})$ and $H^s_\Delta(\mathbb{R}^d \setminus \Sigma) := H^s_\Delta(\Omega_{\mathrm{i}}) \oplus H^s_\Delta(\Omega_{\mathrm{e}})$ for $s \geq 0$ and we define

$$\gamma^* : H^{3/2}(\Sigma) \to H^2(\mathbb{R}^d \setminus \Sigma), \quad \gamma^*\xi = \gamma^*_{D,\mathrm{i}}\xi \oplus \gamma^*_{D,\mathrm{e}}\xi.$$

Finally, recall that $-1 \in \rho(-\Delta)$ is true. Now, we are prepared to introduce the single layer potential for sufficiently smooth functions:

Definition 5.12. *Let the adjoint trace γ^* be defined as above. Then the single layer potential $\tilde{\mathrm{SL}} : H^{3/2}(\Sigma) \to H^{3/2}(\mathbb{R}^d)$ associated to the differential expression $-\Delta + 1$ and the hypersurface Σ is defined as*

$$\tilde{\mathrm{SL}}\xi := (-\Delta + 1)^{-1}\gamma^*\xi, \quad \xi \in H^{3/2}(\Sigma).$$

In the next step, we extend the definition of the single layer potential from $H^{3/2}(\Sigma)$ to $L^2(\Sigma)$, cf. [43, Corollary 6.14]:

Theorem 5.13. *Let $\tilde{\mathrm{SL}}$ be constituted as in Definition 5.12 and let $\chi \in C_c^\infty(\mathbb{R}^d)$. Then the operator $\chi\tilde{\mathrm{SL}}$ can be extended to a bounded linear operator*

$$\chi \mathrm{SL} : L^2(\Sigma) \to H^{3/2}(\mathbb{R}^d \setminus \Sigma).$$

In what follows, we investigate the jump relations of $\eta\mathrm{SL}\xi$ for $\xi \in L^2(\Sigma)$, i.e. we consider

$$[\eta\mathrm{SL}\xi]_\Sigma := (\eta\mathrm{SL}\xi|_{\Omega_{\mathrm{i}}})\,|_\Sigma - (\eta\mathrm{SL}\xi|_{\Omega_{\mathrm{e}}})\,|_\Sigma$$

and

$$[\partial_\nu\eta\mathrm{SL}\xi]_\Sigma := \partial_{\nu_{\mathrm{i}}}(\eta\mathrm{SL}\xi|_{\Omega_{\mathrm{i}}})\,|_\Sigma + \partial_{\nu_{\mathrm{e}}}(\eta\mathrm{SL}\xi|_{\Omega_{\mathrm{e}}})\,|_\Sigma.$$

Here, ν_j denotes the unit normal vector of Σ pointing outwards of Ω_j for $j \in \{\mathrm{i}, \mathrm{e}\}$. Note that it holds $\nu_{\mathrm{i}} = -\nu_{\mathrm{e}}$. In order to apply the Neumann trace defined as in Theorem 4.26 to $\eta\mathrm{SL}\xi \in H^{3/2}(\mathbb{R}^d \setminus \Sigma)$, we have to show $\eta\mathrm{SL}\xi \in H^{3/2}_\Delta(\mathbb{R}^d \setminus \Sigma)$, i.e. $\Delta(\eta\mathrm{SL}\xi)|_{\Omega_j} \in L^2(\Omega_j)$.

Proposition 5.14. *Let the single layer potential SL be defined as in Theorem 5.13. Then it holds $\eta\mathrm{SL}\xi \in H^{3/2}_\Delta(\mathbb{R}^d \setminus \Sigma)$ for any $\xi \in L^2(\Sigma)$.*

Proof. Let $\xi \in L^2(\Sigma)$ be fixed. According to Theorem 5.13, it holds $\eta\mathrm{SL}\xi \in H^{3/2}(\mathbb{R}^d \setminus \Sigma)$. Hence, it remains to show that there exists for $j \in \{\mathrm{i}, \mathrm{e}\}$ a function $f_j \in L^2(\Omega_j)$ such that

$$\int_{\Omega_j} \eta\mathrm{SL}\xi\Delta\varphi \mathrm{d}x = \int_{\Omega_j} f_j\varphi \mathrm{d}x$$

is true for any $\varphi \in C_c^\infty(\Omega_j)$. Let $\varphi \in C_c^\infty(\Omega_j)$ be fixed. We consider the term

$$\int_{\Omega_j} (\eta\mathrm{SL}\xi)(-\Delta + 1)\varphi \mathrm{d}x = \int_{\Omega_j} \mathrm{SL}\xi\big(\eta(-\Delta + 1)\varphi\big)\mathrm{d}x.$$

Using the product rule, it is easy to verify that $-\eta\Delta\varphi = -\Delta(\eta\varphi) + 2\langle\nabla\eta, \nabla\varphi\rangle + \varphi\Delta\eta$ is true. Hence, we find

$$\int_{\Omega_j} (\eta \mathrm{SL}\xi)(-\Delta+1)\varphi \mathrm{d}x = \int_{\Omega_j} \mathrm{SL}\xi\big((-\Delta+1)(\eta\varphi) + 2\langle\nabla\eta, \nabla\varphi\rangle + \varphi\Delta\eta\big)\mathrm{d}x.$$

First, we mention that it holds $\Delta\eta\mathrm{SL}\xi \in H^{3/2}(\Omega_j) \subset L^2(\Omega_j)$ by Theorem 5.13, as $\Delta\eta \in C_c^\infty(\mathbb{R}^d)$. Next, we consider the term

$$\int_{\Omega_j} \mathrm{SL}\xi \cdot 2\langle\nabla\eta, \nabla\varphi\rangle \mathrm{d}x.$$

Note that it holds $\frac{\partial\eta}{\partial x_k}\mathrm{SL}\xi \in H^{3/2}(\Omega_j)$ by Theorem 5.13, in particular $\frac{\partial\eta}{\partial x_k}\mathrm{SL}\xi$ is weakly differentiable and hence it holds

$$\int_{\Omega_j} \mathrm{SL}\xi \cdot 2\langle\nabla\eta, \nabla\varphi\rangle \mathrm{d}x = -2\int_{\Omega_j} \operatorname{div}\left(\nabla\eta\mathrm{SL}\xi\right)\varphi \mathrm{d}x$$

with $\operatorname{div}(\nabla\eta\mathrm{SL}\xi) \in L^2(\Omega_j)$, where the divergence is understood in the weak sense. It remains to analyze

$$\int_{\Omega_j} \mathrm{SL}\xi(-\Delta+1)(\eta\varphi)\mathrm{d}x = \int_{\Omega_j} \tilde\eta\mathrm{SL}\xi(-\Delta+1)(\eta\varphi)\mathrm{d}x,$$

where $\tilde\eta \in C_c^\infty(\mathbb{R}^d)$ is chosen in such a way that $\tilde\eta|_{\operatorname{supp}\eta} \equiv 1$ is satisfied. By Theorem 5.13 the operator $\tilde\eta\mathrm{SL} : L^2(\Sigma) \to H^{3/2}(\Omega_j)$ is bounded and everywhere defined. Therefore, also $\tilde\eta\mathrm{SL} : L^2(\Sigma) \to L^2(\Omega_j)$ is bounded and everywhere defined and hence, it has a bounded and everywhere defined adjoint $(\tilde\eta\mathrm{SL})^* : L^2(\Omega_j) \to L^2(\Sigma)$ satisfying

$$\int_{\Omega_j} \tilde\eta\mathrm{SL}\xi(-\Delta+1)(\eta\varphi)\mathrm{d}x = \int_\Sigma \xi\overline{(\tilde\eta\mathrm{SL})^*\,\overline{(-\Delta+1)(\eta\varphi)}}\mathrm{d}\sigma.$$

Recall that the single layer potential acts as $\mathrm{SL}\zeta = (-\Delta+1)^{-1}\gamma^*\zeta$, if ζ is sufficiently smooth. Thus, as $\overline{(-\Delta+1)(\eta\varphi)} \in C_c^\infty(\mathbb{R}^d)$, the adjoint acts as

$$(\tilde\eta\mathrm{SL})^*\,\overline{(-\Delta+1)(\eta\varphi)} = \gamma_{D,j}(-\Delta+1)^{-1}\tilde\eta\overline{(-\Delta+1)(\eta\varphi)}.$$

Using $\tilde\eta|_{\operatorname{supp}\eta} \equiv 1$, we find

$$(\tilde\eta\mathrm{SL})^*\,\overline{(-\Delta+1)(\eta\varphi)} = \gamma_{D,j}(-\Delta+1)^{-1}\overline{(-\Delta+1)(\eta\varphi)} = \gamma_{D,j}\overline{\eta\varphi} = 0,$$

as $\varphi \in C_c^\infty(\Omega_j)$ is compactly supported. Therefore, we get

$$\int_{\Omega_j} \mathrm{SL}\xi(-\Delta+1)(\eta\varphi)\mathrm{d}x = 0.$$

Putting together these observations, we find

$$\begin{aligned}\int_{\Omega_j}(\eta\mathrm{SL}\xi)(-\Delta+1)\varphi\mathrm{d}x &= \int_{\Omega_j}\mathrm{SL}\xi\big((-\Delta+1)(\eta\varphi)+2\langle\nabla\eta,\nabla\varphi\rangle+\varphi\Delta\eta\big)\mathrm{d}x\\ &= \int_{\Omega_j}\left(\Delta\eta\mathrm{SL}\xi-2\,\mathrm{div}\,(\nabla\eta\mathrm{SL}\xi)\right)\varphi\mathrm{d}x.\end{aligned}$$

Hence, $\Delta(\eta\mathrm{SL}\xi)=(\eta\mathrm{SL}\xi)-\Delta\eta\mathrm{SL}\xi+2\,\mathrm{div}\,(\nabla\eta\mathrm{SL}\xi)$ exists in a weak sense and belongs to $L^2(\Omega_j)$, which implies $\eta\mathrm{SL}\xi\in H^{3/2}_{\Delta}(\Omega_j)$. □

Since the Dirichlet and the Neumann trace of $\eta\mathrm{SL}\xi$ are well-defined for $\xi\in L^2(\Sigma)$ by the previous proposition, we can investigate the jump properties of $\eta\mathrm{SL}\xi$ now. This result is stated in [43, Theorem 6.11].

Theorem 5.15. *Let the single layer potential* SL *be defined as in Theorem 5.13. Then it holds for any* $\xi\in L^2(\Sigma)$

$$[\eta\mathrm{SL}\xi]_\Sigma=(\eta\mathrm{SL}\xi|_{\Omega_\mathrm{i}})\,|_\Sigma-(\eta\mathrm{SL}\xi|_{\Omega_\mathrm{e}})\,|_\Sigma=0$$

and

$$[\partial_\nu\eta\mathrm{SL}\xi]_\Sigma=\partial_{\nu_\mathrm{i}}\,(\eta\mathrm{SL}\xi|_{\Omega_\mathrm{i}})\,|_\Sigma+\partial_{\nu_\mathrm{e}}\,(\eta\mathrm{SL}\xi|_{\Omega_\mathrm{e}})\,|_\Sigma=\xi.$$

6 Differential operators $A_{\delta,\alpha}$ with δ-interactions supported on compact hypersurfaces

In this chapter, we introduce Schrödinger operators $A_{\delta,\alpha}$ in $L^2(\mathbb{R}^d)$ with δ-interactions supported on a closed compact hypersurface Σ that are formally given as

$$-\Delta - \alpha\langle\delta_\Sigma, \cdot\rangle\delta_\Sigma$$

with a real-valued function $\alpha \in L^\infty(\Sigma)$, which is often called the strength of the interaction, in a mathematical rigorous way. Here, we follow an extension-theoretic approach from [12], where the abstract tool of quasi boundary triples is used to construct $A_{\delta,\alpha}$. Since it is sufficient for our purposes to prove the required results directly in a similar way as in [12], we will not discuss the general theory of quasi boundary triples here. We mention that $A_{\delta,\alpha}$ can also be defined via quadratic forms [15].

Let $d \geq 2$ and let $\Sigma \subset \mathbb{R}^d$ be a closed and compact hypersurface in the sense of Definition 3.2, which is C^2-smooth and which separates $\mathbb{R}^d$ into a bounded interior domain Ω_i and an unbounded exterior domain Ω_e, implying $\mathbb{R}^d = \Omega_\mathrm{i}\dot\cup\Sigma\dot\cup\Omega_\mathrm{e}$. Hence, it holds $L^2(\mathbb{R}^d) = L^2(\Omega_\mathrm{i}) \oplus L^2(\Omega_\mathrm{e})$ and according to this decomposition we write for $f \in L^2(\mathbb{R}^d)$ also $f = f_\mathrm{i} \oplus f_\mathrm{e}$ with $f_\mathrm{i} := f|_{\Omega_\mathrm{i}} \in L^2(\Omega_\mathrm{i})$ and $f_\mathrm{e} := f|_{\Omega_\mathrm{e}} \in L^2(\Omega_\mathrm{e})$. As in Section 4.3.2 we use for $f_j \in H^{3/2}_\Delta(\Omega_j)$, $j \in \{\mathrm{i}, \mathrm{e}\}$, the notation $f_j|_\Sigma$ for the Dirichlet trace and $\partial_{\nu_j} f_j|_\Sigma$ for the Neumann trace with the unit normal vector ν_j pointing outwards of Ω_j. Note that $\nu_\mathrm{i} = -\nu_\mathrm{e}$. Finally, we define

$$H^{3/2}_\Delta(\mathbb{R}^d \setminus \Sigma) := H^{3/2}_\Delta(\Omega_\mathrm{i}) \oplus H^{3/2}_\Delta(\Omega_\mathrm{e}).$$

With these notations in hands, we introduce the operator T in $L^2(\mathbb{R}^d)$ via

$$Tf := (-\Delta f_\mathrm{i}) \oplus (-\Delta f_\mathrm{e}), \quad \mathrm{dom}\, T := \left\{ f = f_\mathrm{i} \oplus f_\mathrm{e} \in H^{3/2}_\Delta(\mathbb{R}^d \setminus \Sigma) : f_\mathrm{i}|_\Sigma = f_\mathrm{e}|_\Sigma \right\}. \tag{6.1}$$

This operator T is essential in the construction of $A_{\delta,\alpha}$. Moreover, we define the operators $\Gamma_0, \Gamma_1 : \mathrm{dom}\, T \to L^2(\Sigma)$ as

$$\Gamma_0 f := \partial_{\nu_\mathrm{i}} f_\mathrm{i}|_\Sigma + \partial_{\nu_\mathrm{e}} f_\mathrm{e}|_\Sigma \quad \text{and} \quad \Gamma_1 f := f|_\Sigma, \quad f = f_\mathrm{i} \oplus f_\mathrm{e} \in \mathrm{dom}\, T. \tag{6.2}$$

Some basic properties of T, Γ_0 and Γ_1 that are needed for the construction of $A_{\delta,\alpha}$ are stated in the following proposition:

Proposition 6.1. *Let T, Γ_0 and Γ_1 be defined as above. Then the following assertions are true:*

(i) $\mathrm{ran}\, \Gamma_0 = L^2(\Sigma)$.

(ii) *The identity*

$$(Tf, g)_{L^2(\mathbb{R}^d)} - (f, Tg)_{L^2(\mathbb{R}^d)} = (\Gamma_1 f, \Gamma_0 g)_{L^2(\Sigma)} - (\Gamma_0 f, \Gamma_1 g)_{L^2(\Sigma)}$$

holds for all $f, g \in \mathrm{dom}\, T$.

Proof. (i) First, we mention that it holds $\partial_{\nu_j} f_j|_\Sigma \in L^2(\Sigma)$ for any $f_j \in H^{3/2}_\Delta(\Omega_j)$, $j \in \{\mathrm{i}, \mathrm{e}\}$, by Theorem 4.26 and thus $\operatorname{ran}\Gamma_0 \subset L^2(\Sigma)$. So, it remains to prove that Γ_0 is surjective. For this, we consider for an arbitrary, but fixed $\xi \in L^2(\Sigma)$ the function $f := \eta \mathrm{SL}\xi$, where $\eta \in C_c^\infty(\mathbb{R}^d)$ is chosen such that $\eta \equiv 1$ holds in a neighborhood of $\overline{\Omega_\mathrm{i}}$ and SL is the single layer potential associated to $-\Delta + 1$ constituted as in Theorem 5.13. Then, according to Proposition 5.14 the function f belongs to $H^{3/2}_\Delta(\mathbb{R}^d \setminus \Sigma)$ and by Theorem 5.15 it fulfills $f_\mathrm{i}|_\Sigma = f_\mathrm{e}|_\Sigma$ and hence, we have $f \in \operatorname{dom} T$. Moreover, f satisfies

$$\Gamma_0 f = \partial_{\nu_\mathrm{i}} f_\mathrm{i}|_\Sigma + \partial_{\nu_\mathrm{e}} f_\mathrm{e}|_\Sigma = \xi$$

again by Theorem 5.15, which shows that Γ_0 is surjective.

(ii) Let $f, g \in \operatorname{dom} T$ and let $j \in \{\mathrm{i}, \mathrm{e}\}$. We write $f_j := f|_{\Omega_j}$ and $g_j := g|_{\Omega_j}$. Using Theorem 4.27 and the facts that $f_j, g_j \in H^{3/2}_\Delta(\Omega_j)$, $(Tf)|_{\Omega_j} = -\Delta f_j$ and $(Tg)|_{\Omega_j} = -\Delta g_j$ hold, we find

$$\big((Tf)|_{\Omega_j}, g_j\big)_{L^2(\Omega_j)} = (-\Delta f_j, g_j)_{L^2(\Omega_j)} = (\nabla f_j, \nabla g_j)_{L^2(\Omega_j)} - \big(\partial_{\nu_j} f_j|_\Sigma, g_j|_\Sigma\big)_{L^2(\Sigma)}$$

and

$$\big(f_j, (Tg)|_{\Omega_j}\big)_{L^2(\Omega_j)} = (f_j, -\Delta g_j)_{L^2(\Omega_j)} = (\nabla f_j, \nabla g_j)_{L^2(\Omega_j)} - \big(f_j|_\Sigma, \partial_{\nu_j} g_j|_\Sigma\big)_{L^2(\Sigma)}.$$

Putting together these results and using $f, g \in \operatorname{dom} T$, which implies $f_\mathrm{i}|_\Sigma = f_\mathrm{e}|_\Sigma =: f|_\Sigma$ and $g_\mathrm{i}|_\Sigma = g_\mathrm{e}|_\Sigma =: g|_\Sigma$, we get

$$\begin{aligned}(Tf, g)_{L^2(\mathbb{R}^d)} - (f, Tg)_{L^2(\mathbb{R}^d)} &= (f|_\Sigma, \partial_{\nu_\mathrm{i}} f_\mathrm{i}|_\Sigma + \partial_{\nu_\mathrm{e}} f_\mathrm{e}|_\Sigma)_{L^2(\Sigma)} - (\partial_{\nu_\mathrm{i}} f_\mathrm{i}|_\Sigma + \partial_{\nu_\mathrm{e}} f_\mathrm{e}|_\Sigma, g|_\Sigma)_{L^2(\Sigma)} \\ &= (\Gamma_1 f, \Gamma_0 g)_{L^2(\Sigma)} - (\Gamma_0 f, \Gamma_1 g)_{L^2(\Sigma)},\end{aligned}$$

which is the desired result. □

With the help of the boundary mappings Γ_0 and Γ_1 we are now able to introduce the δ-operator $A_{\delta,\alpha}$:

Definition 6.2. *Let $\alpha \in L^\infty(\Sigma)$ be real-valued, let M_α be the associated multiplication operator in $L^2(\Sigma)$, let T be defined as in (6.1) and let Γ_0 and Γ_1 be given by (6.2). Then the Schrödinger operator $A_{\delta,\alpha}$ with a δ-interaction supported on Σ of strength α is defined as*

$$A_{\delta,\alpha} := T|_{\ker(M_\alpha \Gamma_1 - \Gamma_0)}.$$

This can be written in a more explicit way as

$$\begin{aligned} A_{\delta,\alpha} f &= (-\Delta f_\mathrm{i}) \oplus (-\Delta f_\mathrm{e}), \\ \operatorname{dom} A_{\delta,\alpha} &= \Big\{ f = f_\mathrm{i} \oplus f_\mathrm{e} \in H^{3/2}_\Delta(\mathbb{R}^d \setminus \Sigma) : f_\mathrm{i}|_\Sigma = f_\mathrm{e}|_\Sigma,\ \partial_{\nu_\mathrm{i}} f_\mathrm{i}|_\Sigma + \partial_{\nu_\mathrm{e}} f_\mathrm{e}|_\Sigma = \alpha f|_\Sigma \Big\}. \end{aligned}$$

Proving self-adjointness of $A_{\delta,\alpha}$, which will be done in Theorem 6.7 at the end of this chapter, is non-trivial and requires numerous preparations. One of these is showing symmetry of $A_{\delta,\alpha}$, which is done in the following lemma:

Lemma 6.3. *Let $\alpha \in L^\infty(\Sigma)$ be real-valued and let $A_{\delta,\alpha}$ be constituted as in Definition 6.2. Then $A_{\delta,\alpha}$ is symmetric.*

Proof. Let $f, g \in \operatorname{dom} A_{\delta.\alpha}$. Using Proposition 6.1, assertion (ii), the definition of $A_{\delta,\alpha}$ and the fact that the multiplication operator M_α associated to $\alpha \in L^\infty(\Sigma)$ is bounded, everywhere defined and self-adjoint in $L^2(\Sigma)$, cf. Theorem 5.1, we find

$$\begin{aligned}(A_{\delta,\alpha} f, g)_{L^2(\mathbb{R}^d)} - (f, A_{\delta,\alpha} g)_{L^2(\mathbb{R}^d)} &= (\Gamma_1 f, \Gamma_0 g)_{L^2(\Sigma)} - (\Gamma_0 f, \Gamma_1 g)_{L^2(\Sigma)} \\ &= (\Gamma_1 f, M_\alpha \Gamma_1 g)_{L^2(\Sigma)} - (M_\alpha \Gamma_1 f, \Gamma_1 g)_{L^2(\Sigma)} = 0.\end{aligned}$$

Therefore, $A_{\delta,\alpha}$ is symmetric by Proposition 2.6. □

The operator $A_{\delta,0}$ that is a δ-operator with strength "0" will be of special importance for our considerations. In the following corollary, we prove that $A_{\delta,0}$ is just the free Laplacian in $\mathbb{R}^d$.

Corollary 6.4. *Let $A_{\delta,0}$ be given as in Definition 6.2. Then $A_{\delta,0} = -\Delta$, i.e.*

$$A_{\delta,0} f = -\Delta f, \quad \operatorname{dom} A_{\delta,0} = H^2(\mathbb{R}^d).$$

Proof. It is sufficient to verify $-\Delta \subset A_{\delta,0}$, as in this case the self-adjoint operator $-\Delta$ is contained in the symmetric operator $A_{\delta,0}$ and hence, they must coincide. Let T be defined as in (6.1) and let Γ_0 be defined as in (6.2). Note that it holds $H^2(\mathbb{R}^d) \subset \operatorname{dom} T$. We claim $\Gamma_0(H^2(\mathbb{R}^d)) = 0$. For this purpose, we define for $j \in \{\mathrm{i}, \mathrm{e}\}$ the mapping

$$\tilde{\Gamma}_{0,j} : H^2(\Omega_j) \to L^2(\Sigma), \quad \tilde{\Gamma}_{0,j} f = \partial_{\nu_j} f|_\Sigma = \sum_{k=1}^d \nu_{j,k} (D_{e_k} f)|_\Sigma,$$

where $\nu_j = (\nu_{j,1}, \ldots, \nu_{j,d})$ is the unit normal vector of Σ pointing outwards of Ω_j and $e_1, \ldots, e_d$ are the canonical basis vectors of $\mathbb{R}^d$. We observe that $\tilde{\Gamma}_{0,j}$ is bounded, cf. Proposition 4.25. Using this, we find

$$\Gamma_0 f = \tilde{\Gamma}_{0,\mathrm{i}} f_\mathrm{i} + \tilde{\Gamma}_{0,\mathrm{e}} f_\mathrm{e} = \partial_{\nu_\mathrm{i}} f_\mathrm{i}|_\Sigma + \partial_{\nu_\mathrm{e}} f_\mathrm{e}|_\Sigma = 0$$

for any $f \in C_c^\infty(\mathbb{R}^d)$. Since $C_c^\infty(\mathbb{R}^d)$ is dense in $H^2(\mathbb{R}^d)$ by Proposition 4.16, the last equation is also true for all $f \in H^2(\mathbb{R}^d)$, as $\tilde{\Gamma}_{0,\mathrm{i}}$ and $\tilde{\Gamma}_{0,\mathrm{e}}$ are continuous. Therefore, we find $\operatorname{dom}(-\Delta) = H^2(\mathbb{R}^d) \subset \ker \Gamma_0$ and hence $-\Delta \subset T|_{\ker \Gamma_0} = A_{\delta,0}$. □

In order to prove self-adjointness of $A_{\delta,\alpha}$ and to derive a suitable resolvent formula for this operator, we need the following technical lemma:

Lemma 6.5. *Let T be defined as in* (6.1) *and let Γ_0 and Γ_1 be defined as in* (6.2)*. Moreover, let $\lambda \in \mathbb{C} \setminus [0, \infty)$, denote the integral kernel of $(-\Delta - \lambda)^{-1}$ by G_λ, cf. Theorem 5.9, and write $\mathcal{N}_\lambda(T) := \ker(T - \lambda)$. Then the following assertions hold:*

(i) *The operator*

$$\gamma(\lambda) : L^2(\Sigma) \to L^2(\mathbb{R}^d), \quad \gamma(\lambda)\xi := \big(\Gamma_0|_{\mathcal{N}_\lambda(T)}\big)^{-1}\xi = \int_\Sigma G_\lambda(\cdot - y)\xi(y)\mathrm{d}\sigma(y)$$

is well-defined and bounded.

(ii) *The adjoint* $\gamma(\lambda)^* : L^2(\mathbb{R}^d) \to L^2(\Sigma)$ *of* $\gamma(\lambda)$ *is given by* $\gamma(\lambda)^* = \Gamma_1\big(-\Delta - \overline{\lambda}\big)^{-1}$ *and it has the explicit representation*

$$\gamma(\lambda)^* f = \int_{\mathbb{R}^d} G_{\overline{\lambda}}(\cdot - y)f(y)\mathrm{d}y.$$

(iii) *The operator*

$$M(\lambda) : L^2(\Sigma) \to L^2(\Sigma), \quad M(\lambda)\xi := \Gamma_1\left(\Gamma_0|_{\mathcal{N}_\lambda(T)}\right)^{-1}\xi = \int_\Sigma G_\lambda(\cdot - y)\xi(y)\mathrm{d}\sigma(y)$$

is well-defined and compact.

(iv) *For any* $f \in \mathcal{N}_\lambda(T)$ *it holds* $M(\lambda)\Gamma_0 f = \Gamma_1 f$.

(v) $M^*(\lambda) = M\big(\overline{\lambda}\big)$ *is true for any* $\lambda \in \mathbb{C} \setminus [0,\infty)$.

Proof. Let $\lambda \in \mathbb{C} \setminus [0,\infty)$ be fixed. First, we mention that the decomposition

$$\mathrm{dom}\, T = \ker(T - \lambda) \dot{+} \mathrm{dom}\,(-\Delta)$$

holds for any $\lambda \in \rho(-\Delta) = \mathbb{C} \setminus [0,\infty)$ by Proposition 2.11. Since $\ker\Gamma_0 = \mathrm{dom}\,(-\Delta)$, cf. Corollary 6.4, it follows that the mapping $\Gamma_0|_{\mathcal{N}_\lambda(T)}$ is bijective and that the operator $\gamma(\lambda) := \big(\Gamma_0|_{\mathcal{N}_\lambda(T)}\big)^{-1} : L^2(\Sigma) \to L^2(\mathbb{R}^d)$ is well defined.

Next, we prove $\gamma(\lambda)^* = \Gamma_1\big(-\Delta - \overline{\lambda}\big)^{-1}$. For this, let $\xi \in L^2(\Sigma)$ and $f \in L^2(\mathbb{R}^d)$. Note that a simple calculation shows

$$-\Delta\big(-\Delta - \overline{\lambda}\big)^{-1} f = f + \overline{\lambda}\big(-\Delta - \overline{\lambda}\big)^{-1} f.$$

Using this, $\mathrm{dom}\,(-\Delta) = \ker\Gamma_0$, $-\Delta \subset T$, $\gamma(\lambda)\xi \in \mathcal{N}_\lambda(T)$ and Proposition 6.1, assertion (ii), we find

$$\begin{aligned}
(\gamma(\lambda)\xi, f)_{L^2(\mathbb{R}^d)} &= \Big(\gamma(\lambda)\xi, \big(1 + \overline{\lambda}\big(-\Delta - \overline{\lambda}\big)^{-1}\big)f\Big)_{L^2(\mathbb{R}^d)} - \Big(\lambda\gamma(\lambda)\xi, \big(-\Delta - \overline{\lambda}\big)^{-1} f\Big)_{L^2(\mathbb{R}^d)} \\
&= \Big(\gamma(\lambda)\xi, T\big(-\Delta - \overline{\lambda}\big)^{-1} f\Big)_{L^2(\mathbb{R}^d)} - \Big(T\gamma(\lambda)\xi, \big(-\Delta - \overline{\lambda}\big)^{-1} f\Big)_{L^2(\mathbb{R}^d)} \\
&= \Big(\Gamma_0\gamma(\lambda)\xi, \Gamma_1\big(-\Delta - \overline{\lambda}\big)^{-1} f\Big)_{L^2(\Sigma)} - \Big(\Gamma_1\gamma(\lambda)\xi, \Gamma_0\big(-\Delta - \overline{\lambda}\big)^{-1} f\Big)_{L^2(\Sigma)} \\
&= \Big(\xi, \Gamma_1\big(-\Delta - \overline{\lambda}\big)^{-1} f\Big)_{L^2(\Sigma)}.
\end{aligned}$$

Since this is true for any $\xi \in L^2(\Sigma)$, it follows $\gamma(\lambda)^* = \Gamma_1\big(-\Delta - \overline{\lambda}\big)^{-1}$. Moreover, as f was allowed to be any function in $L^2(\mathbb{R}^d)$ in the above calculation, we find that $\gamma(\lambda)^*$ is everywhere defined in $L^2(\mathbb{R}^d)$, and since $\gamma(\lambda)^*$ is closed by Corollary 2.4, $\gamma(\lambda)^*$ is bounded by the closed graph theorem, see Theorem 2.1. Thus, it follows that also $\gamma(\lambda) = \gamma(\lambda)^{**}$ is bounded and everywhere defined and that the representations of $\gamma(\lambda)$ and $\gamma(\lambda)^*$ in (i) and (ii) hold.

It remains to verify the claimed assertions on $M(\lambda)$. From our previous considerations in this proof, it follows that $M(\lambda)$ is well-defined and that the representation in (iii) is true. Moreover, the formula $M(\lambda)\Gamma_0 f = \Gamma_1 f$ for $f \in \mathcal{N}_\lambda(T)$ follows directly from the definition of $M(\lambda)$.

In what follows, we prove that $M^*(\lambda) = M\big(\overline{\lambda}\big)$ is true for any $\lambda \in \mathbb{C} \setminus [0,\infty)$. For this purpose, let $\xi, \zeta \in L^2(\Sigma)$ and define the functions $f, g \in \operatorname{dom} T \subset L^2(\mathbb{R}^d)$ as $f := \gamma\big(\overline{\lambda}\big)\xi$ and $g := \gamma(\lambda)\zeta$. We observe that $f \in \mathcal{N}_{\overline{\lambda}}(T)$ and $g \in \mathcal{N}_\lambda(T)$ are satisfied. Based on this, the definition of $\gamma(\cdot)$, the result from assertion (iv) and Proposition 6.1, assertion (ii), we find

$$\begin{aligned}
\big(M\big(\overline{\lambda}\big)\xi, \zeta\big)_{L^2(\Sigma)} &= \big(M\big(\overline{\lambda}\big)\Gamma_0 f, \Gamma_0 g\big)_{L^2(\Sigma)} \\
&= \big(M\big(\overline{\lambda}\big)\Gamma_0 f, \Gamma_0 g\big)_{L^2(\Sigma)} - \big(\Gamma_0 f, M(\lambda)\Gamma_0 g\big)_{L^2(\Sigma)} + \big(\Gamma_0 f, M(\lambda)\Gamma_0 g\big)_{L^2(\Sigma)} \\
&= \big(\Gamma_1 f, \Gamma_0 g\big)_{L^2(\Sigma)} - \big(\Gamma_0 f, \Gamma_1 g\big)_{L^2(\Sigma)} + \big(\Gamma_0 f, M(\lambda)\Gamma_0 g\big)_{L^2(\Sigma)} \\
&= \big(Tf, g\big)_{L^2(\mathbb{R}^d)} - \big(f, Tg\big)_{L^2(\mathbb{R}^d)} + \big(\Gamma_0 f, M(\lambda)\Gamma_0 g\big)_{L^2(\Sigma)} \\
&= \big(\overline{\lambda} f, g\big)_{L^2(\mathbb{R}^d)} - \big(f, \lambda g\big)_{L^2(\mathbb{R}^d)} + \big(\xi, M(\lambda)\zeta\big)_{L^2(\Sigma)} = \big(\xi, M(\lambda)\zeta\big)_{L^2(\Sigma)}.
\end{aligned}$$

Since $\zeta \in L^2(\Sigma)$ was arbitrary, it follows $\xi \in \operatorname{dom} M(\lambda)^*$ and $M(\lambda)^*\xi = M\big(\overline{\lambda}\big)\xi$. Hence, $M(\lambda)^* = M\big(\overline{\lambda}\big)$ is true and also statement (v) is shown.

It remains to prove that $M(\lambda)$ is compact. For this, we note first that $M(\lambda)$ is everywhere defined and because of $M(\lambda) = M\big(\overline{\lambda}\big)^*$ and Corollary 2.4, assertion (i), it follows that $M(\lambda)$ is closed. Therefore, Theorem 2.1 implies that $M(\lambda)$ is a bounded operator in $L^2(\Sigma)$.

Next, we note that by the definition of $M(\lambda)$ and Theorem 4.24 it holds $\operatorname{ran} M(\lambda) \subset H^1(\Sigma)$. We also claim that $M(\lambda) : L^2(\Sigma) \to H^1(\Sigma)$ is bounded. Since $M(\lambda)$ is everywhere defined, the closed graph theorem implies that it is sufficient to show that $M(\lambda)$ is closed as a mapping from $L^2(\Sigma)$ to $H^1(\Sigma)$. In order to show this, we consider a sequence $(\xi_n) \subset \operatorname{dom} M(\lambda) = L^2(\Sigma)$ such that $\xi_n \to \xi$ holds in $L^2(\Sigma)$ and $M(\lambda)\xi_n \to \zeta$ in $H^1(\Sigma)$. Since $M(\lambda)$ is continuous as an operator in $L^2(\Sigma)$, we get $M(\lambda)\xi_n \to M(\lambda)\xi$ in $L^2(\Sigma)$ and as convergence in $H^1(\Sigma)$ implies convergence in $L^2(\Sigma)$, we get $M(\lambda)\xi = \zeta$. Hence, $M(\lambda)$ is a closed operator from $L^2(\Sigma)$ to $H^1(\Sigma)$ and thus, by the above considerations, it is also bounded.

Now, as the embedding $\iota : H^1(\Sigma) \to L^2(\Sigma)$ is compact by Proposition 4.23, we find that $M(\lambda)$ regarded as a linear operator in $L^2(\Sigma)$ is compact. This concludes the proof of this proposition. $\square$

In the next theorem, which is inspired by [11, Theorem 2.8], we state an appropriate resolvent formula for $A_{\delta,\alpha}$:

Theorem 6.6. *Let $\alpha \in L^\infty(\Sigma)$ be real-valued, let M_α be the associated multiplication operator in $L^2(\Sigma)$, let $A_{\delta,\alpha}$ be given as in Definition 6.2 and let $\lambda \in \mathbb{C} \setminus [0,\infty)$. Moreover, let $\gamma(\lambda)$, $\gamma(\overline{\lambda})^*$ and $M(\lambda)$ be defined as in Lemma 6.5. Then the following assertions hold:*

(i) $\lambda \in \sigma_{\mathrm{p}}(A_{\delta,\alpha}) \setminus [0,\infty)$ *if and only if* $\ker(1 - M_\alpha M(\lambda)) \neq \{0\}$.

(ii) *If $\lambda \notin (\sigma_{\mathrm{p}}(A_{\delta,\alpha}) \cup [0,\infty))$, then $\lambda \in \rho(A_{\delta,\alpha})$ and it holds*

$$(A_{\delta,\alpha} - \lambda)^{-1} = (-\Delta - \lambda)^{-1} + \gamma(\lambda)(1 - M_\alpha M(\lambda))^{-1} M_\alpha \gamma(\overline{\lambda})^*.$$

Proof. (i) Assume first that $\lambda \in \mathbb{C} \setminus [0,\infty)$ is an eigenvalue of $A_{\delta,\alpha}$, i.e. there exists $0 \neq f \in \operatorname{dom} A_{\delta,\alpha}$ such that $(A_{\delta,\alpha} - \lambda)f = 0$. Note that $\Gamma_0 f \neq 0$, as otherwise it would hold $f \in \operatorname{dom} A_{\delta,0} = \operatorname{dom}(-\Delta)$, which implies $f = 0$, as $\lambda \in \mathbb{C} \setminus [0,\infty) = \rho(-\Delta)$, cf. Theorem 5.9. Using $f \in \mathcal{N}_\lambda(T)$, Lemma 6.5, assertion (iv), and the fact $f \in \operatorname{dom} A_{\delta,\alpha}$, we find

$$(1 - M_\alpha M(\lambda))\Gamma_0 f = \Gamma_0 f - M_\alpha M(\lambda)\Gamma_0 f = \Gamma_0 f - M_\alpha \Gamma_1 f = 0.$$

Thus, $\ker\big(1 - M_\alpha M(\lambda)\big) \neq \{0\}$.

Conversely, if $\ker(1 - M_\alpha M(\lambda)) \neq \{0\}$, there exists $0 \neq \xi \in L^2(\Sigma)$ such that $(1 - M_\alpha M(\lambda))\xi = 0$. Defining $0 \neq f := \gamma(\lambda)\xi$, we find $\Gamma_0 f = \xi$ and $f \in \mathcal{N}_\lambda(T)$. Moreover, it holds

$$\Gamma_0 f - M_\alpha \Gamma_1 f = \xi - M_\alpha M(\lambda)\xi = (1 - M_\alpha M(\lambda))\xi = 0$$

and hence $f \in \operatorname{dom} A_{\delta,\alpha}$. Therefore, $f \in \operatorname{dom} A_{\delta,\alpha} \cap \mathcal{N}_\lambda(T)$ implies that λ is an eigenvalue of $A_{\delta,\alpha}$.

(ii) Let $\lambda \notin (\sigma_{\mathrm{p}}(A_{\delta,\alpha}) \cup [0,\infty))$. From (i), we know that $A_{\delta,\alpha} - \lambda$ is injective. We prove that this operator is also surjective and then that the inverse $(A_{\delta,\alpha} - \lambda)^{-1}$ is bounded.

Let $g \in L^2(\mathbb{R}^d)$ be arbitrary. In order to prove that $A_{\delta,\alpha} - \lambda$ is surjective, we mention first that the operator $M_\alpha M(\lambda)$ is compact in $L^2(\Sigma)$, as $M(\lambda)$ is compact by Lemma 6.5 and M_α is bounded, see Theorem 5.1. Hence, the operator $(1 - M_\alpha M(\lambda))^{-1}$ is bounded and everywhere defined in $L^2(\Sigma)$ by Fredholm's alternative (Theorem 2.13) and item (i) of this theorem. Thus, the function

$$f := (-\Delta - \lambda)^{-1} g + \gamma(\lambda)(1 - M_\alpha M(\lambda))^{-1} M_\alpha \gamma(\overline{\lambda})^* g$$

is well-defined in $L^2(\mathbb{R}^d)$. We show $f \in \operatorname{dom} A_{\delta,\alpha}$. For this, we compute

$$\Gamma_0 f = \underbrace{\Gamma_0(-\Delta - \lambda)^{-1} g}_{=0} + \Gamma_0 \gamma(\lambda)(1 - M_\alpha M(\lambda))^{-1} M_\alpha \gamma(\overline{\lambda})^* g = (1 - M_\alpha M(\lambda))^{-1} M_\alpha \gamma(\overline{\lambda})^* g,$$

which is true by the definition of the γ-field, and

$$\begin{aligned}\Gamma_1 f &= \Gamma_1(-\Delta - \lambda)^{-1} g + \Gamma_1 \gamma(\lambda)(1 - M_\alpha M(\lambda))^{-1} M_\alpha \gamma(\overline{\lambda})^* g \\ &= \gamma(\overline{\lambda})^* g + M(\lambda)(1 - M_\alpha M(\lambda))^{-1} M_\alpha \gamma(\overline{\lambda})^* g,\end{aligned}$$

where we used Lemma 6.5 (ii). Hence, we find

$$
\begin{aligned}
M_\alpha \Gamma_1 f - \Gamma_0 f &= M_\alpha \gamma\big(\overline{\lambda}\big)^* g + M_\alpha M(\lambda)(1 - M_\alpha M(\lambda))^{-1} M_\alpha \gamma\big(\overline{\lambda}\big)^* g \\
&\qquad\qquad\qquad\qquad - (1 - M_\alpha M(\lambda))^{-1} M_\alpha \gamma\big(\overline{\lambda}\big)^* g \\
&= M_\alpha \gamma\big(\overline{\lambda}\big)^* g - (1 - M_\alpha M(\lambda))(1 - M_\alpha M(\lambda))^{-1} M_\alpha \gamma\big(\overline{\lambda}\big)^* g = 0
\end{aligned}
$$

and thus $f \in \operatorname{dom} A_{\delta,\alpha}$. Now, using the facts $A_{\delta,\alpha} \subset T$, $-\Delta \subset T$ and $\gamma(\lambda)(1 - M_\alpha M(\lambda))^{-1} M_\alpha \gamma\big(\overline{\lambda}\big)^* g \in \mathcal{N}_\lambda(T)$, we find

$$
\begin{aligned}
(A_{\delta,\alpha} - \lambda) f &= (T - \lambda) f \\
&= (T - \lambda)(-\Delta - \lambda)^{-1} g + (T - \lambda)\gamma(\lambda)(1 - M_\alpha M(\lambda))^{-1} M_\alpha \gamma\big(\overline{\lambda}\big)^* g = g
\end{aligned}
$$

and hence $g \in \operatorname{ran} A_{\delta,\alpha}$. Therefore, $A_{\delta,\alpha}$ is surjective and also the formula

$$
(A_{\delta,\alpha} - \lambda)^{-1} = (-\Delta - \lambda)^{-1} + \gamma(\lambda)(1 - M_\alpha M(\lambda))^{-1} M_\alpha \gamma\big(\overline{\lambda}\big)^*
$$

for the inverse holds. Finally, since all operators on the right hand side of the above formula are bounded and everywhere defined, we conclude that also $(A_{\delta,\alpha} - \lambda)^{-1}$ is bounded and everywhere defined, which shows $\lambda \in \rho(A_{\delta,\alpha})$. $\square$

Using the result from Theorem 6.6, we finally can prove self-adjointness of $A_{\delta,\alpha}$:

Theorem 6.7. *Let $\alpha \in L^\infty(\Sigma)$ be real-valued and let $A_{\delta,\alpha}$ be defined as in Definition 6.2. Then $A_{\delta,\alpha}$ is self-adjoint.*

Proof. We mention first that $A_{\delta,\alpha}$ is symmetric by Lemma 6.3. In order to prove that $A_{\delta,\alpha}$ is self-adjoint, we use Theorem 2.7 and verify that $\operatorname{ran}(A_{\delta,\alpha} - \lambda) = L^2(\mathbb{R}^d)$ holds for any $\lambda \in \mathbb{C}\backslash\mathbb{R}$. For this purpose, we note that λ is not an eigenvalue of $A_{\delta,\alpha}$, as $A_{\delta,\alpha}$ is symmetric and thus all of its eigenvalues are real-valued. Hence, we conclude from Theorem 6.6 (ii), that $\lambda \in \rho(A_{\delta,\alpha})$ and thus $\operatorname{ran}(A_{\delta,\alpha} - \lambda) = L^2(\mathbb{R}^d)$. $\square$

7 Approximation of $A_{\delta,\alpha}$ by Hamiltonians with local scaled short-range potentials

In this chapter, we prove the main result of this thesis, namely that the Schrödinger operator $A_{\delta,\alpha}$ with a δ-interaction supported on a hypersurface Σ of strength α, which was introduced in Chapter 6, can be approximated in the norm resolvent sense by a family of Hamiltonians $H_{\varepsilon,\Sigma} = -\Delta - V_\varepsilon$, $\varepsilon > 0$, where $\{V_\varepsilon\}_{\varepsilon>0}$ is a set of suitably scaled potentials. Note that results of a similar type are known in the one-dimensional case, cf. [3] and the references therein, and in the two- and three-dimensional case, when Σ can be parametrized by polar coordinates [44, 46] and when Σ is unbounded and the strength α is constant [21, 22]. In our considerations, we follow the approach of [21, 22].

Let $d \geq 2$ and let $\Sigma \subset \mathbb{R}^d$ be a compact, closed and connected C^2-smooth hypersurface in the sense of Definition 3.2. Moreover, we denote by $\nu(x_\Sigma)$ the unit normal vector of Σ at $x_\Sigma \in \Sigma$ which points outwards of the bounded part of $\mathbb{R}^d$ with boundary Σ. Recall that according to Theorem 3.19 there exists $\beta > 0$ such that the mapping

$$\Sigma \times (-\beta, \beta) \ni (x_\Sigma, t) \mapsto x_\Sigma + t\nu(x_\Sigma)$$

is injective. For $\varepsilon \in (0, \beta]$ we set

$$\Omega_\varepsilon := \big\{x_\Sigma + t\nu(x_\Sigma) : x_\Sigma \in \Sigma, t \in (-\varepsilon, \varepsilon)\big\}. \tag{7.1}$$

Let $V \in L^\infty(\mathbb{R}^d)$ be fixed such that V is real-valued and the support of V is contained in Ω_β. Then we define the scaled potential $V_\varepsilon \in L^\infty(\mathbb{R}^d)$ as

$$V_\varepsilon(x) := \begin{cases} \frac{1}{\varepsilon} V\left(x_\Sigma + \frac{\beta}{\varepsilon} t\nu(x_\Sigma)\right), & \text{if } x = x_\Sigma + t\nu(x_\Sigma) \in \Omega_\varepsilon, \\ 0, & \text{else,} \end{cases}$$

and the associated Schrödinger operator $H_{\varepsilon,\Sigma}$ as

$$H_{\varepsilon,\Sigma} f = -\Delta f - V_\varepsilon f, \quad \operatorname{dom} H_{\varepsilon,\Sigma} = H^2(\mathbb{R}^d). \tag{7.2}$$

Note that $H_{\varepsilon,\Sigma}$ is a self-adjoint operator in $L^2(\mathbb{R}^d)$, as $-\Delta$ is self-adjoint by Theorem 5.9 and the multiplication operator corresponding to V_ε is bounded, everywhere defined and self-adjoint, cf. Theorem 5.1.

With these preparatory considerations we are already able to formulate the main result of this thesis:

Theorem 7.1. *Let $d \geq 2$, let $V \in L^\infty(\mathbb{R}^d)$ be real-valued such that the support of V is contained in Ω_β, let $H_{\varepsilon,\Sigma}$ be given as in* (7.2) *and let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Define the coupling $\alpha \in L^\infty(\Sigma)$ as*

$$\alpha(x_\Sigma) := \int_{-1}^{1} V(x_\Sigma + \beta s\nu(x_\Sigma))\mathrm{d}s$$

for almost all $x_\Sigma \in \Sigma$ and $A_{\delta,\alpha}$ as in Definition 6.2. Then there exists a constant $c > 0$ depending on λ, the space dimension d, the potential V and Σ such that

$$\left\|(H_{\varepsilon,\Sigma} - \lambda)^{-1} - (A_{\delta,\alpha} - \lambda)^{-1}\right\| \leq c\varepsilon^{\frac{1}{2d}}$$

holds for all sufficiently small $\varepsilon > 0$. In particular, $H_{\varepsilon,\Sigma}$ converges to $A_{\delta,\alpha}$ in the norm resolvent sense as $\varepsilon \to 0+$.

Remark 7.2. Let $V \in L^\infty(\mathbb{R}^d)$ be as in the above theorem. In the end of this chapter it will turn out that

$$\alpha := \int_{-1}^{1} V(\cdot + \beta s\nu(\cdot))\mathrm{d}s$$

is a well-defined element of $L^\infty(\Sigma)$, i.e. for different representatives of V we get functions that have the same values on Σ up to a set of Hausdorff measure 0. This follows from Lemma 7.8 by considering $\hat{V}w$ with $w(x_\Sigma, s) = |V(x_\Sigma + \beta s\nu(x_\Sigma))|^{1/2}$.

The aim of this chapter is to prove Theorem 7.1, for what we need some preparations. First, we need to derive a suitable resolvent formula for $H_{\varepsilon,\Sigma}$. For this purpose, we start deriving an auxiliary resolvent formula and then we transform it to another one which is convenient for the investigation of the convergence. To get the first resolvent formula, we introduce the functions

$$u_\varepsilon := |V_\varepsilon|^{1/2} \quad \text{and} \quad v_\varepsilon := \operatorname{sign} V_\varepsilon \cdot |V_\varepsilon|^{1/2} \tag{7.3}$$

and we denote the associated multiplication operators in $L^2(\mathbb{R}^d)$ by M_{u_ε} and M_{v_ε}. Note that M_{u_ε} and M_{v_ε} are everywhere defined and bounded, as $u_\varepsilon, v_\varepsilon \in L^\infty(\mathbb{R}^d)$, see Theorem 5.1. Moreover, we define the projections

$$P_\varepsilon : L^2(\mathbb{R}^d) \to L^2(\Omega_\varepsilon), \quad P_\varepsilon f = f|_{\Omega_\varepsilon} \tag{7.4}$$

and

$$P_\varepsilon^* : L^2(\Omega_\varepsilon) \to L^2(\mathbb{R}^d), \quad P_\varepsilon^* h = \begin{cases} h & \text{in } \Omega_\varepsilon \\ 0 & \text{in } \mathbb{R}^d \setminus \Omega_\varepsilon. \end{cases} \tag{7.5}$$

Note that it holds $M_{v_\varepsilon} = M_{v_\varepsilon} P_\varepsilon^* P_\varepsilon$ and $M_{u_\varepsilon} = P_\varepsilon^* P_\varepsilon M_{u_\varepsilon}$, as $\operatorname{supp} u_\varepsilon \subset \Omega_\varepsilon$ and $\operatorname{supp} v_\varepsilon \subset \Omega_\varepsilon$.

Proposition 7.3. *Let $H_{\varepsilon,\Sigma}$ be defined as in (7.2) and let $u_\varepsilon, v_\varepsilon, P_\varepsilon$ and P_ε^* be given as above. Then it holds $\sigma(H_{\varepsilon,\Sigma}) = [0,\infty) \cup \sigma_{\mathrm{disc}}(H_{\varepsilon,\Sigma})$ and $(1 - P_\varepsilon M_{u_\varepsilon}(-\Delta - \lambda)^{-1} M_{v_\varepsilon} P_\varepsilon^*)^{-1}$ is a bounded and everywhere defined operator in $L^2(\Omega_\varepsilon)$ for all $\lambda \in \rho(H_{\varepsilon,\Sigma})$. Moreover,*

$$\begin{aligned}(H_{\varepsilon,\Sigma} - \lambda)^{-1} &= (-\Delta - \lambda)^{-1} \\ &\quad + (-\Delta - \lambda)^{-1} M_{v_\varepsilon} P_\varepsilon^* \left(1 - P_\varepsilon M_{u_\varepsilon}(-\Delta - \lambda)^{-1} M_{v_\varepsilon} P_\varepsilon^*\right)^{-1} P_\varepsilon M_{v_\varepsilon}(-\Delta - \lambda)^{-1}\end{aligned}$$

is true for all $\lambda \in \rho(H_{\varepsilon,\Sigma})$.

Proof. Let $\lambda \in \rho(H_{\varepsilon,\Sigma})$. We note that it holds $\sigma_{\mathrm{ess}}(H_{\varepsilon,\Sigma}) = [0,\infty)$ by Theorem 5.11 and thus $\lambda \in \mathbb{C} \setminus [0,\infty)$. We set $R(\lambda) := (-\Delta - \lambda)^{-1}$.

Step 1: First, we prove the formula

$$(H_{\varepsilon,\Sigma} - \lambda)^{-1} = R(\lambda) + R(\lambda) M_{v_\varepsilon} \left(1 - M_{u_\varepsilon} R(\lambda) M_{v_\varepsilon}\right)^{-1} M_{v_\varepsilon} R(\lambda).$$

Note that according to Theorem 5.11 the operator $(1 - M_{u_\varepsilon} R(\lambda) M_{v_\varepsilon})^{-1}$ exists and is bounded and everywhere defined in $L^2(\mathbb{R}^d)$, as $\lambda \in \mathbb{C} \setminus [0,\infty)$ is not an eigenvalue of $H_{\varepsilon,\Sigma}$ by assumption. Moreover, it holds

$$\begin{aligned}
(H_{\varepsilon,\Sigma} - \lambda) &\left(R(\lambda) + R(\lambda) M_{v_\varepsilon} \left(1 - M_{u_\varepsilon} R(\lambda) M_{v_\varepsilon}\right)^{-1} M_{v_\varepsilon} R(\lambda)\right) f \\
&= (-\Delta - \lambda - M_{v_\varepsilon} M_{u_\varepsilon}) \left(R(\lambda) + R(\lambda) M_{v_\varepsilon} \left(1 - M_{u_\varepsilon} R(\lambda) M_{v_\varepsilon}\right)^{-1} M_{v_\varepsilon} R(\lambda)\right) f \\
&= f + M_{v_\varepsilon} \left(1 - M_{u_\varepsilon} R(\lambda) M_{v_\varepsilon}\right)^{-1} M_{v_\varepsilon} R(\lambda) f - M_{u_\varepsilon} M_{v_\varepsilon} R(\lambda) f \\
&\quad - M_{v_\varepsilon} (1 - 1 + M_{u_\varepsilon} R(\lambda) M_{v_\varepsilon}) \left(1 - M_{u_\varepsilon} R(\lambda) M_{v_\varepsilon}\right)^{-1} M_{v_\varepsilon} R(\lambda) f \\
&= f + M_{v_\varepsilon} \left(1 - M_{u_\varepsilon} R(\lambda) M_{v_\varepsilon}\right)^{-1} M_{v_\varepsilon} R(\lambda) f - M_{u_\varepsilon} M_{v_\varepsilon} R(\lambda) f \\
&\quad - M_{v_\varepsilon} \left(1 - M_{u_\varepsilon} R(\lambda) M_{v_\varepsilon}\right)^{-1} M_{v_\varepsilon} R(\lambda) f + M_{u_\varepsilon} M_{v_\varepsilon} R(\lambda) f = f
\end{aligned}$$

for any $f \in L^2(\mathbb{R}^d)$, which implies the claimed result.

Step 2: It remains to transform the resolvent formula which was shown in *Step 1* into the formula stated in the proposition. For this purpose, recall that it holds $M_{v_\varepsilon} = M_{v_\varepsilon} P_\varepsilon^* P_\varepsilon$ and $M_{u_\varepsilon} = P_\varepsilon^* P_\varepsilon M_{u_\varepsilon}$. Hence, considering the operator $(1 - M_{u_\varepsilon} R(\lambda) M_{v_\varepsilon})^{-1}$ with respect to the orthogonal decomposition $L^2(\mathbb{R}^d) = L^2(\Omega_\varepsilon) \oplus L^2(\mathbb{R}^d \setminus \Omega_\varepsilon)$, we find

$$\left(1 - M_{u_\varepsilon} R(\lambda) M_{v_\varepsilon}\right)^{-1} = \begin{pmatrix} \left(1 - P_\varepsilon M_{u_\varepsilon} R(\lambda) M_{v_\varepsilon} P_\varepsilon^*\right)^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, it holds

$$P_\varepsilon \left(1 - M_{u_\varepsilon} R(\lambda) M_{v_\varepsilon}\right)^{-1} P_\varepsilon^* = \left(1 - P_\varepsilon M_{u_\varepsilon} R(\lambda) M_{v_\varepsilon} P_\varepsilon^*\right)^{-1}$$

and we find that $(1 - P_\varepsilon M_{u_\varepsilon} R(\lambda) M_{v_\varepsilon} P_\varepsilon^*)^{-1}$ is a bounded and everywhere defined operator in $L^2(\Omega_\varepsilon)$. Putting all these observations together, we finally get

$$\begin{aligned}
(H_{\varepsilon,\Sigma} - \lambda)^{-1} &= R(\lambda) + R(\lambda) M_{v_\varepsilon} \left(1 - M_{u_\varepsilon} R(\lambda) M_{v_\varepsilon}\right)^{-1} M_{v_\varepsilon} R(\lambda) \\
&= R(\lambda) + R(\lambda) M_{v_\varepsilon} P_\varepsilon^* P_\varepsilon \left(1 - M_{u_\varepsilon} R(\lambda) M_{v_\varepsilon}\right)^{-1} P_\varepsilon^* P_\varepsilon M_{v_\varepsilon} R(\lambda) \\
&= R(\lambda) + R(\lambda) M_{v_\varepsilon} P_\varepsilon^* \left(1 - P_\varepsilon M_{u_\varepsilon} R(\lambda) M_{v_\varepsilon} P_\varepsilon^*\right)^{-1} P_\varepsilon M_{v_\varepsilon} R(\lambda).
\end{aligned}$$

□

In order to transform the resolvent formula from Proposition 7.3 into another one, which is more convenient to investigate the convergence of $(H_{\varepsilon,\Sigma} - \lambda)^{-1}$, we need numerous preparations. First, we introduce for $\varepsilon \in (0,\beta]$ the embedding operator

$$\mathcal{I}_{\varepsilon,\Sigma} : L^2(\Sigma \times (-\varepsilon,\varepsilon), \sigma \times \Lambda_1) \to L^2(\Omega_\varepsilon), \quad (\mathcal{I}_{\varepsilon,\Sigma}\Phi)(x_\Sigma + t\nu(x_\Sigma)) = \Phi(x_\Sigma, t), \tag{7.6}$$

where $\sigma \times \Lambda_1$ is the product measure of the Hausdorff measure σ and the one-dimensional Lebesgue measure Λ_1, and the scaling operator

$$S_\varepsilon : L^2(\Sigma \times (-1,1), \sigma \times \Lambda_1) \to L^2(\Sigma \times (-\varepsilon,\varepsilon), \sigma \times \Lambda_1), \quad (S_\varepsilon \Xi)(x_\Sigma, t) = \frac{1}{\sqrt{\varepsilon}} \Xi\left(x_\Sigma, \frac{t}{\varepsilon}\right). \tag{7.7}$$

The basic properties of $\mathcal{I}_{\varepsilon,\Sigma}$ and S_ε are stated in the following proposition:

Proposition 7.4. *Let $\mathcal{I}_{\varepsilon,\Sigma}$ and S_ε be defined as above and let $\varepsilon > 0$ be sufficiently small. Then the following assertions are true:*

(i) *$\mathcal{I}_{\varepsilon,\Sigma}$ is bounded and bijective. Furthermore, its inverse is bounded and given by*

$$\mathcal{I}_{\varepsilon,\Sigma}^{-1} : L^2(\Omega_\varepsilon) \to L^2(\Sigma \times (-\varepsilon,\varepsilon), \sigma \times \Lambda_1), \quad (\mathcal{I}_{\varepsilon,\Sigma}^{-1} h)(x_\Sigma, t) = h(x_\Sigma + t\nu(x_\Sigma))$$

for almost all $x_\Sigma \in \Sigma$ and $t \in (-\varepsilon,\varepsilon)$.

(ii) *S_ε is unitary and its inverse is given by*

$$\begin{aligned} &S_\varepsilon^{-1} : L^2(\Sigma \times (-\varepsilon,\varepsilon), \sigma \times \Lambda_1) \to L^2(\Sigma \times (-1,1), \sigma \times \Lambda_1), \\ &(S_\varepsilon^{-1}\Phi)(x_\Sigma, t) = \sqrt{\varepsilon}\Phi(x_\Sigma, \varepsilon t) \end{aligned}$$

for almost all $x_\Sigma \in \Sigma$ and $t \in (-1,1)$.

Proof. (i) Since the embedding

$$\Sigma \times (-\varepsilon,\varepsilon) \ni (x_\Sigma, t) \mapsto x_\Sigma + t\nu(x_\Sigma) \in \Omega_\varepsilon$$

is invertible for all $\varepsilon \leq \beta$ by our general assumptions, the operator $\mathcal{I}_{\varepsilon,\Sigma}$ is also invertible. Moreover, as the norms in $L^2(\Sigma \times (-\varepsilon,\varepsilon), \sigma \times \Lambda_1)$ and $L^2(\Omega_\varepsilon)$ are equivalent by Corollary 3.21 for sufficiently small $\varepsilon > 0$, it follows immediately that $\mathcal{I}_{\varepsilon,\Sigma}$ and $\mathcal{I}_{\varepsilon,\Sigma}^{-1}$ are bounded.

(ii) First, we show that S_ε is isometric. For this, we compute for $\Xi \in L^2(\Sigma \times (-1,1), \sigma \times \Lambda_1)$

$$\begin{aligned} \|S_\varepsilon \Xi\|^2_{L^2(\Sigma\times(-\varepsilon,\varepsilon))} &= \int_\Sigma \int_{-\varepsilon}^{\varepsilon} |S_\varepsilon \Xi(x_\Sigma, t)|^2 \,\mathrm{d}t\mathrm{d}\sigma = \int_\Sigma \int_{-\varepsilon}^{\varepsilon} \left| \frac{1}{\sqrt{\varepsilon}} \Xi\left(x_\Sigma, \frac{t}{\varepsilon}\right)\right|^2 \mathrm{d}t\mathrm{d}\sigma \\ &= \int_\Sigma \int_{-1}^{1} |\Xi(x_\Sigma, s)|^2 \,\mathrm{d}s\mathrm{d}\sigma = \|\Xi\|^2_{L^2(\Sigma\times(-1,1))}, \end{aligned}$$

where we used the substitution $s = \frac{t}{\varepsilon}$. This calculation also implies that S_ε is everywhere defined and bounded. A similar computation shows that for an arbitrary $\Phi \in L^2(\Sigma \times (-\varepsilon,\varepsilon), \sigma \times \Lambda_1)$ the function Θ, which is defined as

$$\Theta(x_\Sigma, t) := \sqrt{\varepsilon}\Phi(x_\Sigma, \varepsilon t)$$

for almost all $x_\Sigma \in \Sigma$ and $t \in (-1,1)$, belongs to $L^2(\Sigma \times (-1,1), \sigma \times \Lambda_1)$. Moreover, it holds

$$S_\varepsilon \Theta(x_\Sigma, t) = S_\varepsilon \sqrt{\varepsilon}\Phi(x_\Sigma, \varepsilon t) = \Phi(x_\Sigma, t)$$

in the sense of $L^2(\Sigma \times (-\varepsilon,\varepsilon), \sigma \times \Lambda_1)$, which proves that S_ε is surjective and that the claimed formula for S_ε^{-1} holds. Hence, S_ε is unitary. □

The next lemma contains the main factors for the transformation of the resolvent formula for $H_{\varepsilon,\Sigma}$ from Proposition 7.3 into another one that is more convenient for the investigation of its convergence. In order to define several needed operators, we introduce the functions $u, v \in L^\infty(\Sigma \times (-1,1), \sigma \times \Lambda_1)$ as

$$u(x_\Sigma, t) := |V(x_\Sigma + \beta t\nu(x_\Sigma))|^{1/2} \quad \text{and} \quad v(x_\Sigma, t) := \operatorname{sign}\big(V(x_\Sigma + \beta t\nu(x_\Sigma))\big) \cdot u(x_\Sigma, t). \tag{7.8}$$

Lemma 7.5. *Let P_ε be defined as in (7.4), P_ε^* as in (7.5), $\mathcal{I}_{\varepsilon,\Sigma}$ as in (7.6), S_ε as in (7.7), $\mathcal{I}_{\varepsilon,\Sigma}^{-1}$ and S_ε^* as in Proposition 7.4 and u and v as in (7.8). Moreover, let $\lambda \in \mathbb{C} \setminus [0,\infty)$ and let*

$$G_\lambda(x-y) = \frac{1}{(2\pi)^{d/2}} \left(\frac{|x-y|}{-i\sqrt{\lambda}} \right)^{1-d/2} K_{d/2-1}\big(-i\sqrt{\lambda}|x-y|\big)$$

be the integral kernel of $(-\Delta-\lambda)^{-1}$, cf. Theorem 5.9. Finally, let $\varepsilon \geq 0$ be sufficiently small, let W be the Weingarten map corresponding to Σ, cf. Definition 3.10, and regard $\det(1-\varepsilon s W(y_\Sigma))$ *in the sense of Remark 3.18. Then the following assertions are true:*

(i) *Define the operator $A_\varepsilon(\lambda)$ as*

$$\begin{aligned} &A_\varepsilon(\lambda) : L^2(\Sigma \times (-1,1), \sigma \times \Lambda_1) \to L^2(\mathbb{R}^d), \\ &(A_\varepsilon(\lambda)\Xi)(x) = \int_\Sigma \int_{-1}^1 G_\lambda(x - y_\Sigma - \varepsilon s \nu(y_\Sigma)) v(y_\Sigma, s) \\ &\qquad\qquad \cdot \det(1-\varepsilon s W(y_\Sigma)) \Xi(y_\Sigma, s) \mathrm{d}s \mathrm{d}\sigma(y_\Sigma). \end{aligned}$$

If $\varepsilon > 0$, then

$$A_\varepsilon(\lambda) = (-\Delta-\lambda)^{-1} M_{v_\varepsilon} P_\varepsilon^* \mathcal{I}_{\varepsilon,\Sigma} S_\varepsilon$$

is fulfilled and $A_\varepsilon(\lambda)$ is bounded and everywhere defined.

(ii) *Define the operator $B_\varepsilon(\lambda)$ as*

$$\begin{aligned} &B_\varepsilon(\lambda) : L^2(\Sigma \times (-1,1), \sigma \times \Lambda_1) \to L^2(\Sigma \times (-1,1), \sigma \times \Lambda_1), \\ &(B_\varepsilon(\lambda)\Xi)(x_\Sigma, t) = u(x_\Sigma, t) \int_\Sigma \int_{-1}^1 G_\lambda(x_\Sigma + \varepsilon t \nu(x_\Sigma) - y_\Sigma - \varepsilon s \nu(y_\Sigma)) \\ &\qquad\qquad \cdot v(y_\Sigma, s) \det(1-\varepsilon s W(y_\Sigma)) \Xi(y_\Sigma, s) \mathrm{d}s \mathrm{d}\sigma(y_\Sigma). \end{aligned}$$

If $\varepsilon > 0$, then

$$B_\varepsilon(\lambda) = S_\varepsilon^{-1} \mathcal{I}_{\varepsilon,\Sigma}^{-1} P_\varepsilon M_{u_\varepsilon} (-\Delta-\lambda)^{-1} M_{v_\varepsilon} P_\varepsilon^* \mathcal{I}_{\varepsilon,\Sigma} S_\varepsilon$$

is fulfilled and $B_\varepsilon(\lambda)$ is bounded and everywhere defined. Moreover, if $\lambda \notin \sigma_{\mathrm{p}}(H_{\varepsilon,\Sigma})$, then the operator $(1 - B_\varepsilon(\lambda))^{-1}$ exists and is bounded and everywhere defined in $L^2(\Sigma \times (-1,1), \sigma \times \Lambda_1)$.

(iii) *Define the operator $C_\varepsilon(\lambda)$ as*

$$\begin{aligned} &C_\varepsilon(\lambda) : L^2(\mathbb{R}^d) \to L^2(\Sigma \times (-1,1), \sigma \times \Lambda_1), \\ &(C_\varepsilon(\lambda) f)(x_\Sigma, t) = u(x_\Sigma, t) \int_{\mathbb{R}^d} G_\lambda(x_\Sigma + \varepsilon t \nu(x_\Sigma) - y) f(y) \mathrm{d}y. \end{aligned}$$

If $\varepsilon > 0$, then

$$C_\varepsilon(\lambda) = S_\varepsilon^{-1}\mathcal{I}_{\varepsilon,\Sigma}^{-1}P_\varepsilon M_{u_\varepsilon}(-\Delta-\lambda)^{-1}$$

is fulfilled and $C_\varepsilon(\lambda)$ is bounded and everywhere defined.

Proof. Let $0 < \varepsilon \leq \beta$ be sufficiently small, such that $\mathcal{I}_{\varepsilon,\Sigma}$ is boundedly invertible. Note that such ε exist by Proposition 7.4. Moreover, let $\lambda \in \mathbb{C}\setminus[0,\infty)$ be fixed.

(i) We prove the statement $A_\varepsilon(\lambda) = (-\Delta-\lambda)^{-1}M_{v_\varepsilon}P_\varepsilon^*\mathcal{I}_{\varepsilon,\Sigma}S_\varepsilon$, which implies then that $A_\varepsilon(\lambda)$ is bounded and everywhere defined, as $(-\Delta-\lambda)^{-1}, M_{v_\varepsilon}, P_\varepsilon^*, \mathcal{I}_{\varepsilon,\Sigma}$ and S_ε have this property. Let $\Xi \in L^2(\Sigma\times(-1,1), \sigma\times\Lambda_1)$. By the definition of $\mathcal{I}_{\varepsilon,\Sigma}$, S_ε, v_ε and v, it holds

$$(\mathcal{I}_{\varepsilon,\Sigma}S_\varepsilon\Xi)(y_\Sigma + s\nu(y_\Sigma)) = \frac{1}{\sqrt{\varepsilon}}\Xi\left(y_\Sigma, \frac{s}{\varepsilon}\right)$$

and

$$v_\varepsilon\left(y_\Sigma + s\nu(y_\Sigma)\right) = \frac{1}{\sqrt{\varepsilon}}\operatorname{sign} V\left(y_\Sigma + \frac{\beta}{\varepsilon}s\nu(y_\Sigma)\right)\left|V_\varepsilon\left(y_\Sigma + \frac{\beta}{\varepsilon}s\nu(y_\Sigma)\right)\right|^{1/2} = \frac{1}{\sqrt{\varepsilon}}v\left(y_\Sigma, \frac{s}{\varepsilon}\right)$$

for almost all $y_\Sigma \in \Sigma$ and $s \in (-\varepsilon,\varepsilon)$. Using the transformation $\Omega_\varepsilon \ni y = y_\Sigma + s\nu(y_\Sigma) \mapsto (y_\Sigma, s) \in \Sigma\times(-\varepsilon,\varepsilon)$, that is bijective, since $\varepsilon \leq \beta$, and the transformation formula from Corollary 3.21 we find

$$\begin{aligned}
\big((-\Delta-\lambda)^{-1}&M_{v_\varepsilon}P_\varepsilon^*\mathcal{I}_{\varepsilon,\Sigma}S_\varepsilon\Xi\big)(x) = \int_{\mathbb{R}^d} G_\lambda(x-y)v_\varepsilon(y)P_\varepsilon^*(\mathcal{I}_{\varepsilon,\Sigma}S_\varepsilon\Xi)(y)\mathrm{d}y \\
&= \int_{\Omega_\varepsilon} G_\lambda(x-y)v_\varepsilon(y)(\mathcal{I}_{\varepsilon,\Sigma}S_\varepsilon\Xi)(y)\mathrm{d}y \\
&= \int_\Sigma\int_{-\varepsilon}^{\varepsilon} G_\lambda(x-y_\Sigma-s\nu(y_\Sigma))\frac{1}{\sqrt{\varepsilon}}v\left(y_\Sigma,\frac{s}{\varepsilon}\right)\frac{1}{\sqrt{\varepsilon}}\Xi\left(y_\Sigma,\frac{s}{\varepsilon}\right)\det(1-sW(y_\Sigma))\mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \\
&= \int_\Sigma\int_{-1}^{1} G_\lambda(x-y_\Sigma-\varepsilon r\nu(y_\Sigma))v\left(y_\Sigma,r\right)\det(1-\varepsilon rW(y_\Sigma))\Xi\left(y_\Sigma,r\right)\mathrm{d}r\mathrm{d}\sigma(y_\Sigma) \\
&= (A_\varepsilon(\lambda)\Xi)(x),
\end{aligned}$$

where we used the substitution $r = \frac{s}{\varepsilon}$ in the last step. Since this is true for almost all $x \in \mathbb{R}^d$, the statement of assertion (i) is shown.

(ii) Let $\Xi \in L^2(\Sigma\times(-1,1), \sigma\times\Lambda_1)$. First, we mention that it holds analogously as in (i) for almost all $y_\Sigma \in \Sigma$ and $s \in (-\varepsilon,\varepsilon)$

$$(\mathcal{I}_{\varepsilon,\Sigma}S_\varepsilon\Xi)(y_\Sigma + s\nu(y_\Sigma)) = \frac{1}{\sqrt{\varepsilon}}\Xi\left(y_\Sigma,\frac{s}{\varepsilon}\right) \quad \text{and} \quad v_\varepsilon\left(y_\Sigma + s\nu(y_\Sigma)\right) = \frac{1}{\sqrt{\varepsilon}}v\left(y_\Sigma,\frac{s}{\varepsilon}\right).$$

Moreover, it holds for any $g \in L^2(\mathbb{R}^d)$ and almost all $x_\Sigma \in \Sigma$ and $t \in (-1,1)$

$$\begin{aligned}
(S_\varepsilon^{-1}\mathcal{I}_{\varepsilon,\Sigma}^{-1}P_\varepsilon M_{u_\varepsilon}g)(x_\Sigma,t) &= \sqrt{\varepsilon}(\mathcal{I}_{\varepsilon,\Sigma}^{-1}P_\varepsilon M_{u_\varepsilon}g)(x_\Sigma,\varepsilon t) \\
&= \sqrt{\varepsilon}u_\varepsilon(x_\Sigma+\varepsilon t\nu(x_\Sigma))g(x_\Sigma+\varepsilon t\nu(x_\Sigma)) \\
&= \sqrt{\varepsilon}\cdot\frac{1}{\sqrt{\varepsilon}}\left|V\left(x_\Sigma+\frac{\beta}{\varepsilon}\varepsilon t\nu(y_\Sigma)\right)\right|^{1/2} g(x_\Sigma+\varepsilon t\nu(x_\Sigma)) \\
&= u(x_\Sigma,t)g(x_\Sigma+\varepsilon t\nu(x_\Sigma)).
\end{aligned}$$

Using these facts, the transformation $y = y_\Sigma + s\nu(y_\Sigma)$, Corollary 3.21 and the substitution $r = \frac{s}{\varepsilon}$, we get

$$\begin{aligned}
&\big(S_\varepsilon^{-1}\mathcal{I}_{\varepsilon,\Sigma}^{-1}P_\varepsilon M_{u_\varepsilon}(-\Delta-\lambda)^{-1}M_{v_\varepsilon}P_\varepsilon^*\mathcal{I}_{\varepsilon,\Sigma}S_\varepsilon\Xi\big)(x_\Sigma,t)\\
&\qquad= (S_\varepsilon^{-1}\mathcal{I}_{\varepsilon,\Sigma}^{-1}P_\varepsilon)\left[u_\varepsilon(\cdot)\int_{\mathbb{R}^d}G_\lambda(\cdot-y)v_\varepsilon(y)P_\varepsilon^*(\mathcal{I}_{\varepsilon,\Sigma}S_\varepsilon\Xi)(y)\mathrm{d}y\right](x_\Sigma,t)\\
&\qquad= u(x_\Sigma,t)\int_{\Omega_\varepsilon}G_\lambda(x_\Sigma+\varepsilon t\nu(x_\Sigma)-y)v_\varepsilon(y)(\mathcal{I}_{\varepsilon,\Sigma}S_\varepsilon\Xi)(y)\mathrm{d}y\\
&\qquad= u(x_\Sigma,t)\int_\Sigma\int_{-\varepsilon}^{\varepsilon}G_\lambda(x_\Sigma+\varepsilon t\nu(x_\Sigma)-y_\Sigma-s\nu(y_\Sigma))\frac{1}{\sqrt{\varepsilon}}v\left(y_\Sigma,\frac{s}{\varepsilon}\right)\\
&\qquad\qquad\qquad\cdot\frac{1}{\sqrt{\varepsilon}}\Xi\left(y_\Sigma,\frac{s}{\varepsilon}\right)\det(1-sW(y_\Sigma))\mathrm{d}s\mathrm{d}\sigma(y_\Sigma)\\
&\qquad= u(x_\Sigma,t)\int_\Sigma\int_{-1}^{1}G_\lambda(x_\Sigma+\varepsilon t\nu(x_\Sigma)-y_\Sigma-\varepsilon r\nu(y_\Sigma))v\left(y_\Sigma,r\right)\\
&\qquad\qquad\qquad\cdot\det(1-\varepsilon rW(y_\Sigma))\Xi\left(y_\Sigma,r\right)\mathrm{d}r\mathrm{d}\sigma(y_\Sigma)\\
&\qquad= (B_\varepsilon(\lambda)\Xi)(x_\Sigma,t).
\end{aligned}$$

Therefore, we obtain the desired formula

$$B_\varepsilon(\lambda) = S_\varepsilon^{-1}\mathcal{I}_{\varepsilon,\Sigma}^{-1}P_\varepsilon M_{u_\varepsilon}(-\Delta-\lambda)^{-1}M_{v_\varepsilon}P_\varepsilon^*\mathcal{I}_{\varepsilon,\Sigma}S_\varepsilon. \tag{7.9}$$

This implies that $B_\varepsilon(\lambda)$ is bounded and everywhere defined. Assume now additionally $\lambda \notin \sigma_{\mathrm{p}}(H_{\varepsilon,\Sigma})$. Note that (7.9) is equivalent to

$$\mathcal{I}_{\varepsilon,\Sigma}S_\varepsilon B_\varepsilon(\lambda)S_\varepsilon^{-1}\mathcal{I}_{\varepsilon,\Sigma}^{-1} = P_\varepsilon M_{u_\varepsilon}(-\Delta-\lambda)^{-1}M_{v_\varepsilon}P_\varepsilon^*.$$

This, together with Proposition 7.3, implies that the operator

$$\big(1-P_\varepsilon M_{u_\varepsilon}(-\Delta-\lambda)^{-1}M_{v_\varepsilon}P_\varepsilon^*\big)^{-1} = \big(1-\mathcal{I}_{\varepsilon,\Sigma}S_\varepsilon B_\varepsilon(\lambda)S_\varepsilon^{-1}\mathcal{I}_{\varepsilon,\Sigma}^{-1}\big)^{-1}$$

exists and is bounded and everywhere defined. Hence, also the operator

$$S_\varepsilon^{-1}\mathcal{I}_{\varepsilon,\Sigma}^{-1}\big(1-\mathcal{I}_{\varepsilon,\Sigma}S_\varepsilon B_\varepsilon(\lambda)S_\varepsilon^{-1}\mathcal{I}_{\varepsilon,\Sigma}^{-1}\big)^{-1}\mathcal{I}_{\varepsilon,\Sigma}S_\varepsilon = (1-B_\varepsilon(\lambda))^{-1}$$

is bounded and everywhere defined, which finishes the proof of statement (ii).

(iii) In order to prove the last statement of this lemma, we mention first that it holds

$$\begin{aligned}
(S_\varepsilon^{-1}\mathcal{I}_{\varepsilon,\Sigma}^{-1}P_\varepsilon M_{u_\varepsilon}g)(x_\Sigma,t) &= \sqrt{\varepsilon}\cdot\frac{1}{\sqrt{\varepsilon}}\left|V\left(x_\Sigma+\frac{\beta}{\varepsilon}\varepsilon t\nu(y_\Sigma)\right)\right|^{1/2}g(x_\Sigma+\varepsilon t\nu(x_\Sigma))\\
&= u(x_\Sigma,t)g(x_\Sigma+\varepsilon t\nu(x_\Sigma))
\end{aligned}$$

for any $g \in L^2(\mathbb{R}^d)$ and for almost all $x_\Sigma \in \Sigma$ and $t \in (-1,1)$. Therefore, we find

$$(S_\varepsilon^{-1}\mathcal{I}_{\varepsilon,\Sigma}^{-1}P_\varepsilon M_{u_\varepsilon}(-\Delta-\lambda)^{-1}f)(x_\Sigma,t) = u(x_\Sigma,t)\int_{\mathbb{R}^d}G_\lambda(x_\Sigma+\varepsilon t\nu(x_\Sigma)-y)f(y)\mathrm{d}y$$

for all $f \in L^2(\mathbb{R}^d)$ and almost all $x_\Sigma \in \Sigma$ and $t \in (-1,1)$ which shows assertion (iii). □

After all these preparations, it is easy to transform the resolvent formula for $H_{\varepsilon,\Sigma}$ from Proposition 7.3 into another one which is more convenient for an investigation of its convergence:

Theorem 7.6. *Let $H_{\varepsilon,\Sigma}$ be defined as in (7.2), let $\lambda \in \rho(H_{\varepsilon,\Sigma})$ and let $A_\varepsilon(\lambda), B_\varepsilon(\lambda)$ and $C_\varepsilon(\lambda)$ be defined as in Lemma 7.5. Then it holds for ε sufficiently small*

$$(H_{\varepsilon,\Sigma} - \lambda)^{-1} = (-\Delta - \lambda)^{-1} + A_\varepsilon(\lambda)(1 - B_\varepsilon(\lambda))^{-1} C_\varepsilon(\lambda).$$

Proof. Let $\lambda \in \rho(H_{\varepsilon,\Sigma}) \subset \mathbb{C} \setminus [0,\infty)$, set $R(\lambda) := (-\Delta - \lambda)^{-1}$ and recall the results and the notations from Proposition 7.3 and Lemma 7.5. Thus, $(1 - P_\varepsilon M_{u_\varepsilon} R(\lambda) M_{v_\varepsilon} P_\varepsilon^*)^{-1}$ and $(1 - B_\varepsilon(\lambda))^{-1}$ exist and are bounded and everywhere defined. Moreover, we find that

$$\begin{aligned}(H_{\varepsilon,\Sigma} - \lambda)^{-1} &= R(\lambda) + R(\lambda) M_{v_\varepsilon} P_\varepsilon^* \left(1 - P_\varepsilon M_{u_\varepsilon} R(\lambda) M_{v_\varepsilon} P_\varepsilon^*\right)^{-1} P_\varepsilon M_{v_\varepsilon} R(\lambda) \\ &= R(\lambda) + A_\varepsilon(\lambda) S_\varepsilon^{-1} \mathcal{I}_{\varepsilon,\Sigma}^{-1} \left(1 - \mathcal{I}_{\varepsilon,\Sigma} S_\varepsilon B_\varepsilon(\lambda) S_\varepsilon^{-1} \mathcal{I}_{\varepsilon,\Sigma}^{-1}\right)^{-1} \mathcal{I}_{\varepsilon,\Sigma} S_\varepsilon C_\varepsilon(\lambda) \\ &= R(\lambda) + A_\varepsilon(\lambda) \left(1 - B_\varepsilon(\lambda)\right)^{-1} C_\varepsilon(\lambda)\end{aligned}$$

holds, which proves the statement of this theorem. □

Since we have derived a suitable resolvent formula for $H_{\varepsilon,\Sigma}$, we are prepared to investigate its convergence, which is done in the following proposition:

Proposition 7.7. *Let $\lambda \in \mathbb{C} \setminus [0,\infty)$, let $\varepsilon \geq 0$ be sufficiently small and let $A_\varepsilon(\lambda)$, $B_\varepsilon(\lambda)$ and $C_\varepsilon(\lambda)$ be defined as in Lemma 7.5. Then there exist constants $c_A, c_B, c_C > 0$ depending on λ, the dimension d, the potential V and the geometry of Σ such that*

$$\|A_\varepsilon(\lambda) - A_0(\lambda)\| \leq c_A \varepsilon^{\frac{d+1}{2d}}, \quad \|B_\varepsilon(\lambda) - B_0(\lambda)\| \leq c_B \varepsilon^{\frac{1}{2d}}$$

and

$$\|C_\varepsilon(\lambda) - C_0(\lambda)\| \leq c_C \varepsilon^{\frac{d+1}{2d}}$$

hold as $\varepsilon \to 0+$.

Proof. Let $\lambda \in \mathbb{C} \setminus [0,\infty)$ be fixed. In order to find an estimate for $\|A_\varepsilon(\lambda) - A_0(\lambda)\|$, we introduce for $\varepsilon > 0$ the auxiliary operator $\hat{A}_\varepsilon(\lambda)$ via

$$\begin{aligned}&\hat{A}_\varepsilon(\lambda) : L^2(\Sigma \times (-1,1), \sigma \times \Lambda_1) \to L^2(\mathbb{R}^d), \\ &\hat{A}_\varepsilon(\lambda)\Xi(x) = \int_\Sigma \int_{-1}^1 G_\lambda(x - y_\Sigma - \varepsilon s \nu(y_\Sigma)) v(y_\Sigma, s) \Xi(y_\Sigma, s) \mathrm{d}s \mathrm{d}\sigma(y_\Sigma).\end{aligned}$$

We show that $\left\|\hat{A}_\varepsilon(\lambda) - A_\varepsilon(\lambda)\right\| \leq \tilde{c}_A \varepsilon^{(d+1)/(2d)}$ and $\left\|\hat{A}_\varepsilon(\lambda) - A_0(\lambda)\right\| \leq \hat{c}_A \varepsilon^{(d+1)/(2d)}$ hold, which imply then the claimed result.

Let us start with the estimate of $\left\|\hat{A}_\varepsilon(\lambda) - A_\varepsilon(\lambda)\right\|$. According to Theorem 5.2, it holds

$$\begin{aligned}
&\left\|\hat{A}_\varepsilon(\lambda) - A_\varepsilon(\lambda)\right\|^2 \\
&\quad \leq \sup_{x\in\mathbb{R}^d} \int_\Sigma \int_{-1}^{1} \left|G_\lambda(x - y_\Sigma - \varepsilon s\nu(y_\Sigma))v(y_\Sigma, s)\left(1 - \det(1 - \varepsilon sW(y_\Sigma))\right)\right| \mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \\
&\qquad \cdot \sup_{(y_\Sigma,s)\in\Sigma\times(-1,1)} \int_{\mathbb{R}^d} \left|G_\lambda(x - y_\Sigma - \varepsilon s\nu(y_\Sigma))v(y_\Sigma, s)\left(1 - \det(1 - \varepsilon sW(y_\Sigma))\right)\right| \mathrm{d}x.
\end{aligned}$$

Let $\mu_1(y_\Sigma), \dots, \mu_{d-1}(y_\Sigma)$ be the eigenvalues of the matrix of the Weingarten map $W(y_\Sigma)$, which are independent from the parametrization of Σ by Proposition 3.11. Then again by Proposition 3.11 these eigenvalues are uniformly bounded in $y_\Sigma \in \Sigma$ which implies for $s \in (-1, 1)$ and $\varepsilon \in (0, 1)$ the existence of a constant $c_1 > 0$ such that

$$|1 - \det(1 - \varepsilon sW(y_\Sigma))| = \left|1 - \prod_{k=1}^{d-1}(1 - \varepsilon s\mu_k(y_\Sigma))\right| \leq c_1\varepsilon \tag{7.10}$$

holds for any $y_\Sigma \in \Sigma$. Using this, Proposition A.2 and Proposition A.4, we find

$$\begin{aligned}
&\left\|\hat{A}_\varepsilon(\lambda) - A_\varepsilon(\lambda)\right\|^2 \\
&\quad \leq \sup_{x\in\mathbb{R}^d} \int_\Sigma \int_{-1}^{1} \left|G_\lambda(x - y_\Sigma - \varepsilon s\nu(y_\Sigma))v(y_\Sigma, s)\left(1 - \det(1 - \varepsilon sW(y_\Sigma))\right)\right| \mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \\
&\qquad \cdot \sup_{(y_\Sigma,s)\in\Sigma\times(-1,1)} \int_{\mathbb{R}^d} \left|G_\lambda(x - y_\Sigma - \varepsilon s\nu(y_\Sigma))v(y_\Sigma, s)\left(1 - \det(1 - \varepsilon sW(y_\Sigma))\right)\right| \mathrm{d}x \\
&\quad \leq (c_1\varepsilon)^2 \sup_{x\in\mathbb{R}^d} \int_\Sigma \int_{-1}^{1} \left|G_\lambda(x - y_\Sigma - \varepsilon s\nu(y_\Sigma))v(y_\Sigma, s)\right| \mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \\
&\qquad \cdot \sup_{(y_\Sigma,s)\in\Sigma\times(-1,1)} \int_{\mathbb{R}^d} \left|G_\lambda(x - y_\Sigma - \varepsilon s\nu(y_\Sigma))v(y_\Sigma, s)\right| \mathrm{d}x \\
&\quad \leq \tilde{c}_A^2\varepsilon^2 \cdot \varepsilon^{1/d-1} = \tilde{c}_A^2\varepsilon^{1/d+1}
\end{aligned}$$

with a constant $\tilde{c}_A$ which depends on λ, v, d and Σ and thus, the claimed estimate holds.

In order to show $\hat{A}_\varepsilon(\lambda) \to A_0(\lambda)$, we use again an estimate with the Schur-Holmgren-bound, cf. Theorem 5.2, of the following form:

$$\begin{aligned}
\left\|\hat{A}_\varepsilon(\lambda) - A_0(\lambda)\right\|^2 \leq &\sup_{x\in\mathbb{R}^d} \int_\Sigma \int_{-1}^{1} \left|\big(G_\lambda(x - y_\Sigma - \varepsilon s\nu(y_\Sigma)) - G_\lambda(x - y_\Sigma)\big)v(y_\Sigma, s)\right| \mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \\
&\cdot \sup_{(y_\Sigma,s)\in\Sigma\times(-1,1)} \int_{\mathbb{R}^d} \left|\big(G_\lambda(x - y_\Sigma - \varepsilon s\nu(y_\Sigma)) - G_\lambda(x - y_\Sigma)\big)v(y_\Sigma, s)\right| \mathrm{d}x.
\end{aligned}$$

Using the result from Proposition A.7 we get

$$\sup_{x\in\mathbb{R}^d} \int_\Sigma \int_{-1}^{1} \left|\big(G_\lambda(x - y_\Sigma - \varepsilon s\nu(y_\Sigma)) - G_\lambda(x - y_\Sigma)\big)v(y_\Sigma, s)\right| \mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \leq c_{A,1}\varepsilon^{1/d}$$

with a constant $c_{A,1}$ which depends on λ, d, Σ and v. Analogously, it follows from Proposition A.5

$$\sup_{(y_\Sigma,s)\in\Sigma\times(-1,1)} \int_{\mathbb{R}^d} \big|\big(G_\lambda(x-y_\Sigma-\varepsilon s\nu(y_\Sigma)) - G_\lambda(x-y_\Sigma)\big)v(y_\Sigma,s)\big|\,\mathrm{d}x \le c_{A,2}\varepsilon,$$

where $c_{A,2}$ depends on d, λ and v. Thus, we find

$$\begin{aligned}
\big\|\hat{A}_\varepsilon(\lambda)-A_0(\lambda)\big\|^2 \le{}& \sup_{x\in\mathbb{R}^d}\int_\Sigma\int_{-1}^1 \big|\big(G_\lambda(x-y_\Sigma-\varepsilon s\nu(y_\Sigma)) - G_\lambda(x-y_\Sigma)\big)v(y_\Sigma,s)\big|\,\mathrm{d}s\mathrm{d}\sigma(y_\Sigma)\\
&\cdot \sup_{(y_\Sigma,s)\in\Sigma\times(-1,1)} \int_{\mathbb{R}^d} \big|\big(G_\lambda(x-y_\Sigma-\varepsilon s\nu(y_\Sigma)) - G_\lambda(x-y_\Sigma)\big)v(y_\Sigma,s)\big|\,\mathrm{d}x\\
\le{}& c_{A,1}\varepsilon^{1/d}c_{A,2}\varepsilon = \underbrace{c_{A,1}c_{A,2}}_{=:\hat{c}_A^2}\varepsilon^{(d+1)/d}.
\end{aligned}$$

Thus, we have shown the estimates $\big\|\hat{A}_\varepsilon(\lambda)-A_\varepsilon(\lambda)\big\| \le \tilde{c}_A\varepsilon^{(d+1)/(2d)}$ and $\big\|\hat{A}_\varepsilon(\lambda)-A_0(\lambda)\big\| \le \hat{c}_A\varepsilon^{(d+1)/(2d)}$ and using the triangle inequality, we get

$$\big\|A_\varepsilon(\lambda)-A_0(\lambda)\big\| \le \big\|\hat{A}_\varepsilon(\lambda)-A_\varepsilon(\lambda)\big\| + \big\|\hat{A}_\varepsilon(\lambda)-A_0(\lambda)\big\| \le c_A\varepsilon^{(d+1)/(2d)}.$$

Next, we show the statement on the convergence of $B_\varepsilon(\lambda)$. For this, we introduce for $\varepsilon \ge 0$ the linear operators $\hat{B}_\varepsilon(\lambda)$ and $\tilde{B}_\varepsilon(\lambda)$ as

$$\begin{aligned}
&\hat{B}_\varepsilon(\lambda) : L^2(\Sigma\times(-1,1),\sigma\times\Lambda_1) \to L^2(\Sigma\times(-1,1),\sigma\times\Lambda_1),\\
&\hat{B}_\varepsilon(\lambda)\Xi(x_\Sigma,t) = u(x_\Sigma,t)\int_\Sigma\int_{-1}^1 G_\lambda(x_\Sigma+\varepsilon t\nu(x_\Sigma)-y_\Sigma-\varepsilon s\nu(y_\Sigma))v(y_\Sigma,s)\Xi(y_\Sigma,s)\mathrm{d}s\mathrm{d}\sigma(y_\Sigma)
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{B}_\varepsilon(\lambda) : L^2(\Sigma\times(-1,1),\sigma\times\Lambda_1) \to L^2(\Sigma\times(-1,1),\sigma\times\Lambda_1),\\
&\tilde{B}_\varepsilon(\lambda)\Xi(x_\Sigma,t) = u(x_\Sigma,t)\int_\Sigma\int_{-1}^1 G_\lambda(x_\Sigma-y_\Sigma-\varepsilon s\nu(y_\Sigma))v(y_\Sigma,s)\Xi(y_\Sigma,s)\mathrm{d}s\mathrm{d}\sigma(y_\Sigma).
\end{aligned}$$

We are going to prove the estimates $\big\|\hat{B}_\varepsilon(\lambda)-B_\varepsilon(\lambda)\big\| = \mathcal{O}\big(\varepsilon^{1/d}\big)$, $\big\|\hat{B}_\varepsilon(\lambda)-\tilde{B}_\varepsilon(\lambda)\big\| = \mathcal{O}\big(\varepsilon^{1/d}\big)$ and $\big\|\tilde{B}_\varepsilon(\lambda)-B_0(\lambda)\big\| = \mathcal{O}\big(\varepsilon^{1/(2d)}\big)$, which yield then the claimed convergence result.

First, we look for an estimate of $\big\|\hat{B}_\varepsilon(\lambda)-B_\varepsilon(\lambda)\big\|$. Using a Schur-Holmgren bound and equation (7.10), we find

$$\begin{aligned}
&\big\|\hat{B}_\varepsilon(\lambda) - B_\varepsilon(\lambda)\big\|^2 \\
&\leq \sup_{(x_\Sigma,t)\in\Sigma\times(-1,1)} \int_\Sigma \int_{-1}^1 \big|u(x_\Sigma,t) G_\lambda(x_\Sigma + \varepsilon t\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma)) v(y_\Sigma,s) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot (1 - \det(1 - \varepsilon s W(y_\Sigma)))\,\big| \mathrm{d}s \mathrm{d}\sigma(y_\Sigma) \\
&\quad \cdot \sup_{(y_\Sigma,s)\in\Sigma\times(-1,1)} \int_\Sigma \int_{-1}^1 \big|u(x_\Sigma,t) G_\lambda(x_\Sigma + \varepsilon t\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma)) v(y_\Sigma,s) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot (1 - \det(1 - \varepsilon s W(y_\Sigma)))\,\big| \mathrm{d}t \mathrm{d}\sigma(x_\Sigma) \\
&\leq (c_1\varepsilon)^2 \sup_{(x_\Sigma,t)\in\Sigma\times(-1,1)} \int_\Sigma \int_{-1}^1 \big|u(x_\Sigma,t) G_\lambda(x_\Sigma + \varepsilon t\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma)) v(y_\Sigma,s)\big| \mathrm{d}s \mathrm{d}\sigma(y_\Sigma) \\
&\quad \cdot \sup_{(y_\Sigma,s)\in\Sigma\times(-1,1)} \int_\Sigma \int_{-1}^1 \big|u(x_\Sigma,t) G_\lambda(x_\Sigma + \varepsilon t\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma)) v(y_\Sigma,s)\big| \mathrm{d}t \mathrm{d}\sigma(x_\Sigma).
\end{aligned}$$

Applying now Proposition A.4, we find that

$$\int_\Sigma \int_{-1}^1 \big|u(x_\Sigma,t) G_\lambda(x_\Sigma + \varepsilon t\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma)) v(y_\Sigma,s)\big| \mathrm{d}s \mathrm{d}\sigma(y_\Sigma) \leq c_{B,1}\varepsilon^{1/d-1}$$

holds independent from x_Σ and t, where $c_{B,1}$ depends on λ, d, u, v and Σ. By symmetry a similar estimate is true for the second integral. Thus, we find

$$\begin{aligned}
&\big\|\hat{B}_\varepsilon(\lambda) - B_\varepsilon(\lambda)\big\|^2 \\
&\leq (c_1\varepsilon)^2 \sup_{(x_\Sigma,t)\in\Sigma\times(-1,1)} \int_\Sigma \int_{-1}^1 \big|u(x_\Sigma,t) G_\lambda(x_\Sigma + \varepsilon t\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma)) v(y_\Sigma,s)\big| \mathrm{d}s \mathrm{d}\sigma(y_\Sigma) \\
&\quad \cdot \sup_{(y_\Sigma,s)\in\Sigma\times(-1,1)} \int_\Sigma \int_{-1}^1 \big|u(x_\Sigma,t) G_\lambda(x_\Sigma + \varepsilon t\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma)) v(y_\Sigma,s)\big| \mathrm{d}t \mathrm{d}\sigma(x_\Sigma). \\
&\leq c_{B,2}^2 \varepsilon^{2/d}.
\end{aligned}$$

Next, we look for an estimate for $\big\|\hat{B}_\varepsilon(\lambda) - \tilde{B}_\varepsilon(\lambda)\big\|$. Applying Proposition A.7, we find

$$\begin{aligned}
\sup_{(y_\Sigma,s)\in\Sigma\times(-1,1)} \int_\Sigma \int_{-1}^1 \big|u(x_\Sigma,t)\big(&G_\lambda(x_\Sigma + \varepsilon t\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma)) \\
&- G_\lambda(x_\Sigma - y_\Sigma - \varepsilon s\nu(y_\Sigma))\big) v(y_\Sigma,s)\big| \mathrm{d}t \mathrm{d}\sigma(x_\Sigma) \leq c_{B,3}\varepsilon^{1/d}.
\end{aligned}$$

Moreover, Proposition A.8 gives us

$$\begin{aligned}
\sup_{(x_\Sigma,t)\in\Sigma\times(-1,1)} \int_\Sigma \int_{-1}^1 \big|u(x_\Sigma,t)\big(&G_\lambda(x_\Sigma + \varepsilon t\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma)) \\
&- G_\lambda(x_\Sigma - y_\Sigma - \varepsilon s\nu(y_\Sigma))\big) v(y_\Sigma,s)\big| \mathrm{d}s \mathrm{d}\sigma(y_\Sigma) \leq c_{B,4}\varepsilon^{1/d}
\end{aligned}$$

for ε sufficiently small. Note that the constants $c_{B,3}$ and $c_{B,4}$ depend again on λ, d, u, v and Σ. Hence, using an estimate with a Schur-Holmgren bound, we find

$$\begin{aligned}
\big\|\hat{B}_\varepsilon(\lambda) - \tilde{B}_\varepsilon(\lambda)\big\|^2 \leq \sup_{(x_\Sigma,t)\in\Sigma\times(-1,1)} \int_\Sigma \int_{-1}^1 \big|u(x_\Sigma,t)\big(&G_\lambda(x_\Sigma + \varepsilon t\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma)) \\
&- G_\lambda(x_\Sigma - y_\Sigma - \varepsilon s\nu(y_\Sigma))\big)v(y_\Sigma,s)\big|\mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \\
\cdot \sup_{(y_\Sigma,s)\in\Sigma\times(-1,1)} \int_\Sigma \int_{-1}^1 \big|u(x_\Sigma,t)\big(&G_\lambda(x_\Sigma + \varepsilon t\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma)) \\
&- G_\lambda(x_\Sigma - y_\Sigma - \varepsilon s\nu(y_\Sigma))\big)v(y_\Sigma,s)\big|\mathrm{d}t\mathrm{d}\sigma(x_\Sigma) \\
\leq c_{B,3}\varepsilon^{1/d} c_{B,4}\varepsilon^{1/d} = \underbrace{c_{B,3}c_{B,4}}_{=:c_{B,5}^2}\varepsilon^{2/d}.&
\end{aligned}$$

Analogously, using Proposition A.7 and Proposition A.9, we find

$$\begin{aligned}
\big\|\tilde{B}_\varepsilon(\lambda) - B_0(\lambda)\big\|^2 \leq \sup_{(x_\Sigma,t)\in\Sigma\times(-1,1)} \int_\Sigma \int_{-1}^1 \big|u(x_\Sigma,t)\big(&G_\lambda(x_\Sigma - y_\Sigma - \varepsilon s\nu(y_\Sigma)) \\
&- G_\lambda(x_\Sigma - y_\Sigma)\big)v(y_\Sigma,s)\big|\mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \\
\cdot \sup_{(y_\Sigma,s)\in\Sigma\times(-1,1)} \int_\Sigma \int_{-1}^1 \big|u(x_\Sigma,t)\big(&G_\lambda(x_\Sigma - y_\Sigma - \varepsilon s\nu(y_\Sigma)) \\
&- G_\lambda(x_\Sigma - y_\Sigma)\big)v(y_\Sigma,s)\big|\mathrm{d}t\mathrm{d}\sigma(x_\Sigma) \\
\leq c_{B,6}\varepsilon^{1/d} c_{B,7} = c_{B,8}^2\varepsilon^{1/d}.&
\end{aligned}$$

Here, the constants $c_{B,6}$ and $c_{B,7}$ depend on λ, d, u, v and Σ. Putting together all previous estimates, we get eventually

$$\begin{aligned}
\|B_\varepsilon(\lambda) - B_0(\lambda)\| &\leq \big\|B_\varepsilon(\lambda) - \hat{B}_\varepsilon(\lambda)\big\| + \big\|\hat{B}_\varepsilon(\lambda) - \tilde{B}_\varepsilon(\lambda)\big\| + \big\|\tilde{B}_\varepsilon(\lambda) - B_0(\lambda)\big\| \\
&\leq c_{B,2}\varepsilon^{1/d} + c_{B,5}\varepsilon^{1/d} + c_{B,8}\varepsilon^{1/(2d)} \leq c_B\varepsilon^{1/(2d)}.
\end{aligned}$$

Finally, we prove the statement on the convergence of $C_\varepsilon(\lambda)$. Using again an estimate with a Schur-Holmgren bound and applying Proposition A.5 and Proposition A.7, we find

$$\begin{aligned}
\big\|C_\varepsilon(\lambda) - C_0(\lambda)\big\|^2 \leq \sup_{y\in\mathbb{R}^d} \int_\Sigma \int_{-1}^1 &\big|u(x_\Sigma,t)\big(G_\lambda(x_\Sigma + \varepsilon t\nu(x_\Sigma) - y) - G_\lambda(x_\Sigma - y)\big)\big|\,\mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \\
\cdot \sup_{(x_\Sigma,t)\in\Sigma\times(-1,1)} \int_{\mathbb{R}^d} &\big|u(x_\Sigma,t)\big(G_\lambda(x_\Sigma + \varepsilon t\nu(x_\Sigma) - y) - G_\lambda(x_\Sigma - y)\big)\big|\,\mathrm{d}y \\
\leq c_{C,1}\varepsilon^{1/d}c_{C,2}\varepsilon = \underbrace{c_{C,1}c_{C,2}}_{=:c_C^2}&\varepsilon^{(d+1)/d}
\end{aligned}$$

with a constant c_C that depends on λ, d, u and Σ. This is already the desired estimate. □

So far, we know from Theorem 7.6 and Proposition 7.7 that the sequence of resolvents $\big((H_{\varepsilon,\Sigma}-\lambda)^{-1}\big)_{\varepsilon>0}$ converges for any $\lambda \in \mathbb{C}\setminus\mathbb{R}$ as $\varepsilon \to 0+$ and we have a first representation of the limit operator. In order to show that this limit operator is equal to the resolvent of $A_{\delta,\alpha}$, we need the following technical lemma:

Lemma 7.8. *Let v be defined as in* (7.8). *Then the following assertions are true:*

(i) *The embedding operator $\mathcal{J} : L^2(\Sigma) \to L^2(\Sigma \times (-1,1), \sigma \times \Lambda_1)$, which acts as*

$$(\mathcal{J}\xi)(x_\Sigma, t) := \xi(x_\Sigma)$$

for almost all $x_\Sigma \in \Sigma$ and all $t \in (-1,1)$, is well-defined and bounded.

(ii) *The operator $\hat{V} : L^2(\Sigma \times (-1,1), \sigma \times \Lambda_1) \to L^2(\Sigma)$ that acts as*

$$\big(\hat{V}\Xi\big)(x_\Sigma) := \int_{-1}^{1} v(x_\Sigma, s)\Xi(x_\Sigma, s)\mathrm{d}s$$

for almost all $x_\Sigma \in \Sigma$, is well-defined and continuous.

Proof. (i) First, we prove that $\mathcal{J}$ is well-defined. For this, let ξ_1 and ξ_2 be two representatives of $\xi \in L^2(\Sigma)$. Then there exists a set $\mathcal{N} \subset \Sigma$ with Hausdorff measure zero such that ξ_1 and ξ_2 coincide on $\Sigma \setminus \mathcal{N}$. Hence, $\mathcal{J}\xi_1$ and $\mathcal{J}\xi_2$ coincide on $\Sigma \times (-1,1)$ except the zero set $\mathcal{N} \times (-1,1)$ and thus, $\mathcal{J}$ is well-defined.

In order to prove the boundedness of $\mathcal{J}$, we compute for $\xi \in L^2(\Sigma)$

$$\|\mathcal{J}\xi\|^2_{L^2(\Sigma\times(-1,1))} = \int_\Sigma \int_{-1}^{1} |(\mathcal{J}\xi)(x_\Sigma, s)|^2 \mathrm{d}s\mathrm{d}\sigma = \int_\Sigma \int_{-1}^{1} |\xi(x_\Sigma)|^2 \mathrm{d}s\mathrm{d}\sigma = 2\|\xi\|^2_{L^2(\Sigma)},$$

which shows that $\mathcal{J}$ is continuous.

(ii) In order to show that $\hat{V}$ is well-defined, let v_1 and v_2 be two representatives of v and let Ξ_1 and Ξ_2 be two representatives of $\Xi \in L^2(\Sigma \times (-1,1), \sigma \times \Lambda_1)$. Then the measurable functions $v_1\Xi_1$ and $v_2\Xi_2$ coincide on $\Sigma \times (-1,1)$ except a zero set. Defining for $i \in \{1,2\}$ the function $\hat{V}_i(x_\Sigma) := \int_{-1}^{1} v_i(x_\Sigma, s)\Xi_i(x_\Sigma, s)\mathrm{d}s$ and using the Cauchy-Schwarz inequality, we find

$$\begin{aligned}\int_\Sigma \left|\hat{V}_1 - \hat{V}_2\right|^2 \mathrm{d}\sigma &= \int_\Sigma \left|\int_{-1}^{1} \big(v_1(x_\Sigma, s)\Xi_1(x_\Sigma, s) - v_2(x_\Sigma, s)\Xi_2(x_\Sigma, s)\big)\mathrm{d}s\right|^2 \mathrm{d}\sigma \\ &\leq 2\int_\Sigma \int_{-1}^{1} |v_1(x_\Sigma, s)\Xi_1(x_\Sigma, s) - v_2(x_\Sigma, s)\Xi_2(x_\Sigma, s)|^2 \mathrm{d}s\mathrm{d}\sigma = 0.\end{aligned}$$

Thus, it holds $\hat{V}_1 = \hat{V}_2$ almost everywhere in Σ, which implies that the operator $\hat{V}$ is well-defined.

It remains to show the boundedness of $\hat{V}$. For this, we consider for $\Xi \in L^2(\Sigma \times (-1,1), \sigma \times \Lambda_1)$

$$\begin{aligned}\|\hat{V}\Xi\|^2_{L^2(\Sigma)} &= \int_\Sigma \left| \int_{-1}^1 v(x_\Sigma, s)\Xi(x_\Sigma, s)\mathrm{d}s \right|^2 \mathrm{d}\sigma \leq 2 \int_\Sigma \int_{-1}^1 |v(x_\Sigma, s)\Xi(x_\Sigma, s)|^2 \mathrm{d}s\mathrm{d}\sigma \\ &\leq 2\|v\|^2_{L^\infty} \|\Xi\|^2_{L^2(\Sigma\times(-1,1))},\end{aligned}$$

where we used the Cauchy-Schwarz inequality. Thus, $\hat{V}$ is bounded. □

Finally, we are prepared to prove the main result of this thesis, namely Theorem 7.1:

Proof of Theorem 7.1. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and let for $\varepsilon \geq 0$ the operators $A_\varepsilon(\lambda)$, $B_\varepsilon(\lambda)$ and $C_\varepsilon(\lambda)$ be defined as in Lemma 7.5. Then, according to Theorem 7.6, Proposition 7.7 and Proposition 2.18, it holds

$$\begin{aligned}&\big\|(H_{\varepsilon,\Sigma} - \lambda)^{-1} - \big((-\Delta - \lambda)^{-1} + A_0(\lambda)(1 - B_0(\lambda))^{-1}C_0(\lambda)\big)\big\| \\ &\qquad = \big\|A_\varepsilon(\lambda)(1 - B_\varepsilon(\lambda))^{-1}C_\varepsilon(\lambda) - A_0(\lambda)(1 - B_0(\lambda))^{-1}C_0(\lambda)\big\| \leq c\varepsilon^{1/(2d)}\end{aligned}$$

with a constant $c > 0$ depending on λ, d, V and Σ. Hence, it remains to verify

$$(A_{\delta,\alpha} - \lambda)^{-1} = (-\Delta - \lambda)^{-1} + A_0(\lambda)(1 - B_0(\lambda))^{-1}C_0(\lambda)$$

for a suitable strength $\alpha \in L^\infty(\Sigma)$. In order to prove this, recall the definition of the bounded operators $\mathcal{J}$ and $\hat{V}$ from Lemma 7.8 and define $\hat{U} := M_u\mathcal{J}$, where M_u is the multiplication operator associated to $u \in L^\infty(\Sigma \times (-1,1), \sigma \times \Lambda_1)$ given by (7.8). Note that M_u and hence also $\hat{U}$ is bounded and everywhere defined. Furthermore, recall the definition of the bounded operators $\gamma(\lambda)$, $\gamma(\overline{\lambda})^*$ and $M(\lambda)$ from Lemma 6.5. Then we see that

$$\begin{aligned}(A_0(\lambda)\Xi)(x) &= \int_\Sigma \int_{-1}^1 G_\lambda(x - y_\Sigma)v(y_\Sigma, s)\Xi(y_\Sigma, s)\mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \\ &= \int_\Sigma G_\lambda(x - y_\Sigma) \left(\int_{-1}^1 v(y_\Sigma, s)\Xi(y_\Sigma, s)\mathrm{d}s \right) \mathrm{d}\sigma(y_\Sigma) = \big(\gamma(\lambda)\hat{V}\Xi\big)(x)\end{aligned}$$

holds for any $\Xi \in L^2(\Sigma \times (-1,1), \sigma \times \Lambda_1)$ and almost all $x \in \mathbb{R}^d$. Thus, we conclude $A_0(\lambda) = \gamma(\lambda)\hat{V}$. In a similar way, one finds

$$\begin{aligned}(B_0(\lambda)\Xi)(x_\Sigma, t) &= u(x_\Sigma, t) \int_\Sigma \int_{-1}^1 G_\lambda(x_\Sigma - y_\Sigma)v(y_\Sigma, s)\Xi(y_\Sigma, s)\mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \\ &= u(x_\Sigma, t) \int_\Sigma G_\lambda(x_\Sigma - y_\Sigma) \left(\int_{-1}^1 v(y_\Sigma, s)\Xi(y_\Sigma, s)\mathrm{d}s \right) \mathrm{d}\sigma(y_\Sigma) \\ &= \big(\hat{U}M(\lambda)\hat{V}\Xi\big)(x_\Sigma, t)\end{aligned}$$

for all $\Xi \in L^2(\Sigma \times (-1,1), \sigma \times \Lambda_1)$ and almost all $(x_\Sigma, t) \in \Sigma \times (-1,1)$, which implies $B_0(\lambda) = \hat{U} M(\lambda) \hat{V}$, and

$$(C_0(\lambda) f)(x) = u(x_\Sigma, t) \int_{\mathbb{R}^d} G_\lambda(x_\Sigma - y) f(y) \mathrm{d}y = \left(\hat{U} \gamma(\overline{\lambda})^* \right)(x_\Sigma, t)$$

for all $f \in L^2(\mathbb{R}^d)$ and almost all $(x_\Sigma, t) \in \Sigma \times (-1,1)$ implying $C_0(\lambda) = \hat{U}\gamma(\overline{\lambda})^*$. Thus, we get

$$A_0(\lambda)(1 - B_0(\lambda))^{-1} C_0(\lambda) = \gamma(\lambda) \hat{V} \left(1 - \hat{U} M(\lambda) \hat{V} \right)^{-1} \hat{U} \gamma(\lambda)^*.$$

Because of

$$\begin{aligned} &\hat{V} \left(1 - \hat{U} M(\lambda) \hat{V} \right)^{-1} - \left(1 - \hat{V} \hat{U} M(\lambda) \right)^{-1} \hat{V} \\ &\quad = \left(1 - \hat{V} \hat{U} M(\lambda) \right)^{-1} \left(\left(1 - \hat{V} \hat{U} M(\lambda) \right) \hat{V} - \hat{V} \left(1 - \hat{U} M(\lambda) \hat{V} \right) \right) \left(1 - \hat{U} M(\lambda) \hat{V} \right)^{-1} = 0, \end{aligned}$$

we find

$$A_0(\lambda)(1 - B_0(\lambda))^{-1} C_0(\lambda) = \gamma(\lambda) \left(1 - \hat{V} \hat{U} M(\lambda) \right)^{-1} \hat{V} \hat{U} \gamma(\overline{\lambda})^*.$$

Setting $\alpha(x_\Sigma) := \int_{-1}^1 V(x_\Sigma + \beta s \nu(x_\Sigma)) \mathrm{d}s$ in the sense of $L^\infty(\Sigma)$, we find $\hat{V}\hat{U} M(\lambda) = M_\alpha M(\lambda)$ and $\hat{V} \hat{U} \gamma(\overline{\lambda})^* = M_\alpha \gamma(\overline{\lambda})^*$ and hence, using Theorem 6.6,

$$\begin{aligned} \lim_{\varepsilon \to 0+} (H_{\varepsilon,\Sigma} - \lambda)^{-1} &= (-\Delta - \lambda)^{-1} + A_0(\lambda)(1 - B_0(\lambda))^{-1} C_0(\lambda) \\ &= (-\Delta - \lambda)^{-1} + \gamma(\lambda)(1 - M_\alpha M(\lambda))^{-1} M_\alpha \gamma(\overline{\lambda})^* = (A_{\delta,\alpha} - \lambda)^{-1}. \end{aligned}$$

□

A Estimates for integrals containing modified Bessel functions

In this appendix, we prove some estimates for integrals, which are essential to show convergence of a sequence of Schrödinger operators with local scaled short-range potentials to a Hamiltonian with a δ-interaction supported on a hypersurface $\Sigma \subset \mathbb{R}^d$.

Let $d \geq 2$. Recall that for $\lambda \in \mathbb{C} \setminus [0, \infty)$ the function G_λ, which is the integral kernel of the resolvent of the free Laplacian in $\mathbb{R}^d$, is given by

$$G_\lambda(x-y) = \frac{1}{(2\pi)^{d/2}} \left(\frac{|x-y|}{-i\sqrt{\lambda}} \right)^{1-d/2} K_{d/2-1}\left(-i\sqrt{\lambda}|x-y| \right),$$

where $K_{d/2-1}$ is a modified Bessel function of the second kind, cf. Section 5.3. It is our goal in this appendix to prove some estimates for integrals that contain this function G_λ.

Let us formulate several general assumptions on the hypersurface Σ that should be fulfilled for all results in this appendix:

Hypothesis A.1. *Let $d \geq 2$. We assume that $\Sigma \subset \mathbb{R}^d$ is a closed C^2-smooth hypersurface in the sense of Definition 3.2. Moreover, we denote by $\nu(x_\Sigma)$ the unit normal vector of Σ at $x_\Sigma \in \Sigma$ that points outwards of the bounded part of $\mathbb{R}^d$ with boundary Σ.*

Let us start with a rather simple estimate:

Proposition A.2. *Let $d \geq 2$, assume that $\Sigma \subset \mathbb{R}^d$ fulfills Hypothesis A.1, let $\psi \in L^\infty(\Sigma \times (-1,1), \sigma \times \Lambda_1)$ and let $\lambda \in \mathbb{C} \setminus [0, \infty)$. Then there exists a constant $C > 0$ depending on d, λ and ψ such that*

$$\sup_{(y_\Sigma, s) \in \Sigma \times (-1,1)} \int_{\mathbb{R}^d} |G_\lambda(x - y_\Sigma - \varepsilon s \nu(y_\Sigma)) \psi(y_\Sigma, s)| \,\mathrm{d}x \leq C$$

is true.

Proof. Let $y_\Sigma \in \Sigma$ and $s \in (-1,1)$ be fixed. Using the translation invariance of the Lebesgue measure, we find

$$\int_{\mathbb{R}^d} |G_\lambda(x - y_\Sigma - \varepsilon s \nu(y_\Sigma)) \psi(y_\Sigma, s)| \,\mathrm{d}x \leq \|\psi\|_{L^\infty} \int_{\mathbb{R}^d} |G_\lambda(x)| \,\mathrm{d}x = \|\psi\|_{L^\infty} \|G_\lambda\|_{L^1}.$$

The last integral is finite, as $G_\lambda \in L^1(\mathbb{R}^d)$ by Lemma 5.8. This is already the claimed result. $\square$

The following estimate is needed for several results in this appendix:

Lemma A.3. *Let $d \geq 2$, let Σ be a closed hypersurface in the sense of Definition 3.2 and let $\nu(x_\Sigma)$ be the unit normal vector of Σ at $x_\Sigma \in \Sigma$ that points outwards of the bounded part of $\mathbb{R}^d$ with boundary Σ. Moreover, let*

$$\Omega_\varepsilon := \left\{ x_\Sigma + t\nu(x_\Sigma) : x_\Sigma \in \Sigma, t \in (-\varepsilon, \varepsilon) \right\}$$

and let $\sigma(\Sigma)$ be the Hausdorff measure of Σ. Then there exists a constant $C > 0$ such that

$$\int_{\Omega_\varepsilon} \mathrm{d}x \leq C\varepsilon\sigma(\Sigma)$$

holds for all sufficiently small $\varepsilon > 0$.

Proof. Let $\varepsilon > 0$ be sufficiently small. Then, according to Corollary 3.21, it holds

$$\int_{\Omega_\varepsilon} \mathrm{d}x = \int_\Sigma \int_{-\varepsilon}^{\varepsilon} \det(1 - tW(x_\Sigma))\mathrm{d}t\mathrm{d}\sigma,$$

where W is the Weingarten-map associated to Σ and $\det(1 - tW(x_\Sigma))$ is understood in the sense of Remark 3.18. Since the eigenvalues of the matrix of the Weingarten-map are bounded by Proposition 3.11, it follows that there exists a constant $C > 0$ such that

$$\det(1 - tW(x_\Sigma)) \leq \frac{C}{2}$$

holds for all $x_\Sigma \in \Sigma$ and all $t \in (-\varepsilon, \varepsilon)$, if ε is small enough. Hence, we get

$$\int_{\Omega_\varepsilon} \mathrm{d}x = \int_\Sigma \int_{-\varepsilon}^{\varepsilon} \det(1 - tW(x_\Sigma))\mathrm{d}t\mathrm{d}\sigma \leq \frac{C}{2} \int_\Sigma \int_{-\varepsilon}^{\varepsilon} \mathrm{d}t\mathrm{d}\sigma = C\varepsilon\sigma(\Sigma)$$

and thus the claimed result. □

Proposition A.4. *Let $d \geq 2$, assume that $\Sigma \subset \mathbb{R}^d$ fulfills Hypothesis A.1, let $X \subset \mathbb{R}^d$ and let μ be a measure on X. Furthermore, let $\psi \in L^\infty(\Sigma \times (-1, 1), \sigma \times \Lambda_1)$, let $\varphi \in L^\infty(X, \mu)$ and let $\lambda \in \mathbb{C} \setminus [0, \infty)$. Then it holds for any sufficiently small $\varepsilon > 0$*

$$\sup_{x\in X} \int_\Sigma \int_{-1}^{1} |\varphi(x)G_\lambda(x - y_\Sigma - \varepsilon s\nu(y_\Sigma))\psi(y_\Sigma, s)|\, \mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \leq \begin{cases} -C_1 \ln \varepsilon, & \text{if } d = 2, \\ C_2\varepsilon^{2/d-1}, & \text{if } d \geq 3, \end{cases}$$

with constants $C_1, C_2 > 0$ that depend on d, λ, ψ and φ. In particular, the estimate

$$\sup_{x\in X} \int_\Sigma \int_{-1}^{1} |\varphi(x)G_\lambda(x - y_\Sigma - \varepsilon s\nu(y_\Sigma))\psi(y_\Sigma, s)|\, \mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \leq C\varepsilon^{1/d-1}$$

is true for any space dimension $d \geq 2$.

Proof. Let $x \in X$ be fixed and assume that ε is sufficiently small. Since the eigenvalues of the matrix of the Weingarten map W associated to Σ are bounded by Proposition 3.11, it follows that there exists a constant $c_1 > 0$ such that $1 \leq c_1 \det(1 - \varepsilon sW(y_\Sigma))$ holds for all $s \in (-1, 1)$ and all $y_\Sigma \in \Sigma$. Here, the term $\det(1 - \varepsilon sW(y_\Sigma))$ is regarded in the sense of Remark 3.18. Hence, we get

$$\begin{aligned}
&\int_\Sigma \int_{-1}^{1} |\varphi(x)G_\lambda(x - y_\Sigma - \varepsilon s\nu(y_\Sigma))\psi(y_\Sigma, s)|\, \mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \\
&\quad \leq c_1 \int_\Sigma \int_{-1}^{1} |\varphi(x)G_\lambda(x - y_\Sigma - \varepsilon s\nu(y_\Sigma))\psi(y_\Sigma, s)| \det(1 - \varepsilon sW(y_\Sigma))\mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \\
&\quad \leq \frac{c_2}{\varepsilon} \int_\Sigma \int_{-\varepsilon}^{\varepsilon} |G_\lambda(x - y_\Sigma - r\nu(y_\Sigma))| \det(1 - rW(y_\Sigma))\mathrm{d}r\mathrm{d}\sigma(y_\Sigma),
\end{aligned}$$

where we used the substitution $r = \varepsilon s$. Thus, by applying the transformation formula from Corollary 3.21, we get

$$\int_\Sigma \int_{-1}^{1} |\varphi(x) G_\lambda(x - y_\Sigma - \varepsilon s \nu(y_\Sigma)) \psi(y_\Sigma, s)| \, \mathrm{d}s \mathrm{d}\sigma(y_\Sigma) \leq \frac{c_2}{\varepsilon} \int_{\Omega_\varepsilon} |G_\lambda(x-y)| \, \mathrm{d}y$$

with the tube Ω_ε given as

$$\Omega_\varepsilon = \big\{ y_\Sigma + r\nu(y_\Sigma) : y_\Sigma \in \Sigma, r \in (-\varepsilon, \varepsilon) \big\}.$$

In order to get an estimate for the last integral, we split the integration area into $\tilde\Omega_1 := B\big(x, \varepsilon^{1/d}\big) \cap \Omega_\varepsilon$ and $\tilde\Omega_2 := \Omega_\varepsilon \setminus B\big(x, \varepsilon^{1/d}\big)$, so $\Omega_\varepsilon = \tilde\Omega_1 \dot\cup \tilde\Omega_2$, and we distinguish the cases $d = 2$ and $d > 2$. Recall that G_λ is defined as

$$G_\lambda(x-y) = \frac{1}{(2\pi)^{d/2}} \left(\frac{|x-y|}{-i\sqrt{\lambda}} \right)^{1-d/2} K_{d/2-1}\left(-i\sqrt{\lambda}|x-y| \right).$$

Let us start with the case $d = 2$. Due to the asymptotics of the modified Bessel function K_0, see Proposition 5.6, there exists a constants $c > 0$ such that

$$\left| K_0\big(-i\sqrt{\lambda}|x-y| \big) \right| \leq -c \ln|x-y|$$

holds for $y \to x$. Hence, we find

$$|G_\lambda(x-y)| \leq -c_3 \ln|x-y|$$

for all $y \in B\big(x, \varepsilon^{1/2}\big)$ and we conclude that the integral over $\tilde\Omega_1$ can be estimated as

$$\begin{aligned} \int_{\tilde\Omega_1} |G_\lambda(x-y)| \mathrm{d}y &\leq -c_3 \int_{B(x,\varepsilon^{1/2})} \ln|x-y| \mathrm{d}y = -2\pi c_3 \int_0^{\varepsilon^{1/2}} \ln r \cdot r \mathrm{d}r \\ &= -2\pi c_3 \frac{\varepsilon}{4} (2\ln\varepsilon - 1) \leq -c_4 \varepsilon \ln\varepsilon \end{aligned}$$

independent from x, where we used a substitution to polar coordinates. Moreover, since K_0 is differentiable and bounded, if the argument is not in a neighborhood of zero, we find for $y \in \Omega_\varepsilon \setminus B\big(x, \varepsilon^{1/2}\big)$

$$\big|G_\lambda(x-y)\big| \leq \max\{\hat{c}, -c\ln|x-y|\} \leq -c_5 \ln\varepsilon$$

because of the asymptotic behavior of K_0 for small arguments. Hence, using Lemma A.3 we get

$$\int_{\tilde\Omega_2} \big|G_\lambda(x-y)\big| \mathrm{d}y \leq -c_5 \int_{\Omega_\varepsilon \setminus B\left(x,\varepsilon^{1/2}\right)} \ln\varepsilon \mathrm{d}y \leq -c_6 \varepsilon \ln\varepsilon\, \sigma(\Sigma)$$

with the Hausdorff-measure $\sigma(\Sigma)$ of Σ, again independent from x. Therefore, we find

$$\int_{\Omega_\varepsilon} |G_\lambda(x-y)|\,\mathrm{d}y = \int_{\tilde{\Omega}_1} |G_\lambda(x-y)|\,\mathrm{d}y + \int_{\tilde{\Omega}_2} |G_\lambda(x-y)|\,\mathrm{d}y \leq -c_7\varepsilon\ln\varepsilon.$$

This implies

$$\begin{aligned}\int_\Sigma\int_{-1}^1 |\varphi(x)G_\lambda(x-y_\Sigma-\varepsilon s\nu(y_\Sigma))\psi(y_\Sigma,s)|\,\mathrm{d}s\mathrm{d}\sigma(y_\Sigma) &\leq \frac{c_2}{\varepsilon}\int_{\Omega_\varepsilon}|G_\lambda(x-y)|\,\mathrm{d}y\\ &\leq -\underbrace{c_2c_7}_{=:C_1}\ln\varepsilon,\end{aligned}$$

which is the claimed result in the case $d=2$.

In the case $d>2$, it holds by Proposition 5.6 (iii)

$$\Big|K_{d/2-1}\big(-i\sqrt{\lambda}|x-y|\big)\Big| \leq c|x-y|^{1-d/2}$$

for $y\to x$. Hence, we find

$$|G_\lambda(x-y)| = \left|\frac{1}{(2\pi)^{d/2}}\left(\frac{|x-y|}{-i\sqrt{\lambda}}\right)^{1-d/2} K_{d/2-1}\left(-i\sqrt{\lambda}|x-y|\right)\right| \leq c_8|x-y|^{2-d}$$

for all $y\in\tilde{\Omega}_1$ and we conclude that the integral over $\tilde{\Omega}_1$ can be estimated as

$$\int_{\tilde{\Omega}_1}|G_\lambda(x-y)|\mathrm{d}y \leq c_8\int_{B(x,\varepsilon^{1/d})}|x-y|^{2-d}\mathrm{d}y = c_9\int_0^{\varepsilon^{1/d}} r^{2-d}r^{d-1}\mathrm{d}r = c_{10}\varepsilon^{2/d}$$

independent from x, where we used a substitution to spherical coordinates. Moreover, since $K_{d/2-1}$ is differentiable and bounded, if the argument is not in a neighborhood of zero, we find for $y\notin B\big(x,\varepsilon^{1/d}\big)$

$$\big|G_\lambda(x-y)\big| \leq c_{11}|x-y|^{2-d} \leq c_{11}\varepsilon^{(2-d)/d}$$

because of the asymptotic behavior of $K_{d/2-1}$. Hence, using Lemma A.3 we conclude

$$\int_{\tilde{\Omega}_2}\big|G_\lambda(x-y)\big|\mathrm{d}y \leq c_{11}\int_{\Omega_\varepsilon\setminus B\left(x,\varepsilon^{1/d}\right)}\varepsilon^{(2-d)/d}\mathrm{d}y \leq c_{12}\varepsilon^{2/d}\sigma(\Sigma)$$

with the Hausdorff-measure $\sigma(\Sigma)$ of Σ, again independent from x. Thus, we find

$$\int_{\Omega_\varepsilon} |G_\lambda(x-y)|\,\mathrm{d}y = \int_{\tilde{\Omega}_1} |G_\lambda(x-y)|\,\mathrm{d}y + \int_{\tilde{\Omega}_2} |G_\lambda(x-y)|\,\mathrm{d}y \leq c_{13}\varepsilon^{2/d}.$$

Therefore, we proved

$$\int_\Sigma\int_{-1}^1 |\varphi(x)G_\lambda(x-y_\Sigma-\varepsilon s\nu(y_\Sigma))\psi(y_\Sigma,s)|\,\mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \leq \frac{c_2}{\varepsilon}\int_{\Omega_\varepsilon}|G_\lambda(x-y)|\,\mathrm{d}y \leq \underbrace{c_2c_{13}}_{=:C_2}\varepsilon^{2/d-1}$$

for $d>2$ and thus, the claimed assertion is true. $\square$

The following estimates are slightly more involved as the previous ones, as here also the derivatives of G_λ have to be considered:

Proposition A.5. *Let $d \geq 2$, assume that $\Sigma \subset \mathbb{R}^d$ fulfills Hypothesis A.1, let $\psi \in L^\infty(\Sigma \times (-1,1), \sigma \times \Lambda_1)$ and let $\lambda \in \mathbb{C} \setminus [0,\infty)$. Then there exists a constant $C > 0$ depending on d, λ and ψ such that*

$$\sup_{(y_\Sigma, s) \in \Sigma \times (-1,1)} \int_{\mathbb{R}^d} \big| \big(G_\lambda(x - y_\Sigma - \varepsilon s \nu(y_\Sigma)) - G_\lambda(x - y_\Sigma)\big)\psi(y_\Sigma, s)\big| \, \mathrm{d}x \leq C\varepsilon$$

holds as $\varepsilon \to 0+$.

Proof. Let $y_\Sigma \in \Sigma$ and $s \in (-1,1)$ be fixed. Since the mapping $\mathbb{C} \backslash (-\infty, 0] \ni z \mapsto K_{d/2-1}(z)$ is analytic by Proposition 5.6, it follows that

$$\begin{aligned}
&[0,1] \ni \theta \mapsto G_\lambda(x - y_\Sigma - \varepsilon s \theta \nu(y_\Sigma)) \\
&\qquad = \frac{1}{(2\pi)^{d/2}} \left(\frac{|x - y_\Sigma - \varepsilon s \theta \nu(y_\Sigma)|}{-i\sqrt{\lambda}} \right)^{1-d/2} K_{d/2-1}\left(-i\sqrt{\lambda}|x - y_\Sigma - \varepsilon s \theta \nu(y_\Sigma)|\right)
\end{aligned}$$

is differentiable for almost all $x \in \mathbb{R}^d$. For these x it holds

$$\begin{aligned}
G_\lambda(x - &y_\Sigma - \varepsilon s \nu(y_\Sigma)) - G_\lambda(x - y_\Sigma) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta} G_\lambda(x - y_\Sigma - \varepsilon s \theta \nu(y_\Sigma)) \mathrm{d}\theta \\
&= \int_0^1 \frac{1}{(2\pi)^{d/2}} \frac{\mathrm{d}}{\mathrm{d}\theta} |x - y_\Sigma - \varepsilon s \theta \nu(y_\Sigma)| \\
&\quad \cdot \Bigg(\left(1 - \frac{d}{2}\right) \frac{|x - y_\Sigma - \varepsilon s \theta \nu(y_\Sigma)|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}} K_{d/2-1}\left(-i\sqrt{\lambda}|x - y_\Sigma - \varepsilon s \theta \nu(y_\Sigma)|\right) \\
&\quad - \left(\frac{|x - y_\Sigma - \varepsilon s \theta \nu(y_\Sigma)|}{-i\sqrt{\lambda}} \right)^{1-d/2} i\sqrt{\lambda} K_{d/2-1}'\left(-i\sqrt{\lambda}|x - y_\Sigma - \varepsilon s \theta \nu(y_\Sigma)|\right) \Bigg) \mathrm{d}\theta.
\end{aligned}$$

Note that a simple calculation yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta} |x - y_\Sigma - \varepsilon s \theta \nu(y_\Sigma)| &= \frac{\mathrm{d}}{\mathrm{d}\theta} \left(\sum_{k=1}^d (x_k - y_{\Sigma,k} - \varepsilon s \theta \nu_k(y_\Sigma))^2 \right)^{1/2} \\
&= -|x - y_\Sigma - \varepsilon s \theta \nu(y_\Sigma)|^{-1} \varepsilon s \big(\nu(y_\Sigma), x - y_\Sigma - \varepsilon s \theta \nu(y_\Sigma)\big)
\end{aligned}$$

and using the Cauchy-Schwarz inequality and $s \in (-1,1)$, it follows

$$\left| \frac{\mathrm{d}}{\mathrm{d}\theta} |x - y_\Sigma - \varepsilon s \theta \nu(y_\Sigma)| \right| \leq |\varepsilon s| < \varepsilon.$$

Hence, we find

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \left|\big(G_\lambda(x-y_\Sigma-\varepsilon s\nu(y_\Sigma)) - G_\lambda(x-y_\Sigma)\big)\psi(y_\Sigma,s)\right| \mathrm{d}x \\
&\leq \|\psi\|_{L^\infty} \int_{\mathbb{R}^d}\int_0^1 \frac{1}{(2\pi)^{d/2}} \bigg| \frac{\mathrm{d}}{\mathrm{d}\theta} |x-y_\Sigma-\varepsilon s\theta\nu(y_\Sigma)| \\
&\qquad \cdot \Bigg(\left(1-\frac{d}{2}\right) \frac{|x-y_\Sigma-\varepsilon s\theta\nu(y_\Sigma)|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}} K_{d/2-1}\left(-i\sqrt{\lambda}|x-y_\Sigma-\varepsilon s\theta\nu(y_\Sigma)|\right) \\
&\qquad - \left(\frac{|x-y_\Sigma-\varepsilon s\theta\nu(y_\Sigma)|}{-i\sqrt{\lambda}}\right)^{1-d/2} i\sqrt{\lambda} K'_{d/2-1}\left(-i\sqrt{\lambda}|x-y_\Sigma-\varepsilon s\theta\nu(y_\Sigma)|\right) \Bigg) \bigg| \mathrm{d}\theta \mathrm{d}x \\
&\leq c_1\varepsilon \int_{\mathbb{R}^d}\int_0^1 \Bigg| \Bigg(\left(1-\frac{d}{2}\right) \frac{|x-y_\Sigma-\varepsilon s\theta\nu(y_\Sigma)|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}} K_{d/2-1}\left(-i\sqrt{\lambda}|x-y_\Sigma-\varepsilon s\theta\nu(y_\Sigma)|\right) \\
&\qquad - \left(\frac{|x-y_\Sigma-\varepsilon s\theta\nu(y_\Sigma)|}{-i\sqrt{\lambda}}\right)^{1-d/2} i\sqrt{\lambda} K'_{d/2-1}\left(-i\sqrt{\lambda}|x-y_\Sigma-\varepsilon s\theta\nu(y_\Sigma)|\right) \Bigg) \Bigg| \mathrm{d}\theta \mathrm{d}x.
\end{aligned}
$$

In order to become independent from y_Σ and s, we consider the bijective transformation $T : \mathbb{R}^d \times (0,1) \to \mathbb{R}^d \times (0,1)$ which acts as

$$
\begin{pmatrix} \xi \\ \phi \end{pmatrix} = T \begin{pmatrix} x \\ \theta \end{pmatrix} := \begin{pmatrix} x-y_\Sigma-\varepsilon s\theta\nu(y_\Sigma) \\ \theta \end{pmatrix}.
$$

Note that T is differentiable and that its Jacobian is given by

$$
DT = \begin{pmatrix} I & -\varepsilon s\nu(y_\Sigma) \\ 0 & 1 \end{pmatrix},
$$

where I is the identity matrix in $\mathbb{R}^{d\times d}$. Hence, it holds $|\det DT| = 1$ and we conclude

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \left|\big(G_\lambda(x-y_\Sigma-\varepsilon s\nu(y_\Sigma)) - G_\lambda(x-y_\Sigma)\big)\psi(y_\Sigma,s)\right| \mathrm{d}x \\
&\leq c_1\varepsilon \int_{\mathbb{R}^d}\int_0^1 \Bigg| \Bigg(\left(1-\frac{d}{2}\right) \frac{|x-y_\Sigma-\varepsilon s\theta\nu(y_\Sigma)|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}} K_{d/2-1}\left(-i\sqrt{\lambda}|x-y_\Sigma-\varepsilon s\theta\nu(y_\Sigma)|\right) \\
&\qquad - \left(\frac{|x-y_\Sigma-\varepsilon s\theta\nu(y_\Sigma)|}{-i\sqrt{\lambda}}\right)^{1-d/2} i\sqrt{\lambda} K'_{d/2-1}\left(-i\sqrt{\lambda}|x-y_\Sigma-\varepsilon s\theta\nu(y_\Sigma)|\right) \Bigg) \Bigg| \mathrm{d}\theta \mathrm{d}x \\
&= c_1\varepsilon \int_{\mathbb{R}^d}\int_0^1 \Bigg| \left(1-\frac{d}{2}\right) \frac{|\xi|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}} K_{d/2-1}\left(-i\sqrt{\lambda}|\xi|\right) \\
&\qquad - \left(\frac{|\xi|}{-i\sqrt{\lambda}}\right)^{1-d/2} i\sqrt{\lambda} K'_{d/2-1}\left(-i\sqrt{\lambda}|\xi|\right) \Bigg| \mathrm{d}\phi \mathrm{d}\xi \\
&= c_1\varepsilon \int_{\mathbb{R}^d} \Bigg| \left(1-\frac{d}{2}\right) \frac{|\xi|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}} K_{d/2-1}\left(-i\sqrt{\lambda}|\xi|\right) \\
&\qquad - \left(\frac{|\xi|}{-i\sqrt{\lambda}}\right)^{1-d/2} i\sqrt{\lambda} K'_{d/2-1}\left(-i\sqrt{\lambda}|\xi|\right) \Bigg| \mathrm{d}\xi,
\end{aligned}
$$

where we used in the last step that the integrand was independent from ϕ. It remains to show that the last integral is finite. For this, we decompose the area of integration in the following way: $\mathbb{R}^d = B(0,1) \dot{\cup} (\mathbb{R}^d \setminus B(0,1))$. Since the mapping

$$\left(1-\frac{d}{2}\right)\frac{|\xi|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}}K_{d/2-1}\left(-i\sqrt{\lambda}|\xi|\right)-\left(\frac{|\xi|}{-i\sqrt{\lambda}}\right)^{1-d/2}i\sqrt{\lambda}K'_{d/2-1}\left(-i\sqrt{\lambda}|\xi|\right)$$

is differentiable and bounded, if ξ is not in a neighborhood of zero, and due to the asymptotic behavior

$$\left|K_{d/2-1}\big(-i\sqrt{\lambda}|\xi|\big)\right| \leq ce^{-\mathrm{Im}\sqrt{\lambda}|\xi|} \quad \text{and} \quad \left|K'_{d/2-1}\big(-i\sqrt{\lambda}|\xi|\big)\right| \leq ce^{-\mathrm{Im}\sqrt{\lambda}|\xi|}$$

for $|\xi| \to \infty$, cf. Proposition 5.6 (iv), it follows

$$\begin{aligned}\int_{\mathbb{R}^d\setminus B(0,1)}\Bigg|&\left(1-\frac{d}{2}\right)\frac{|\xi|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}}K_{d/2-1}\left(-i\sqrt{\lambda}|\xi|\right)\\&-\left(\frac{|\xi|}{-i\sqrt{\lambda}}\right)^{1-d/2}i\sqrt{\lambda}K'_{d/2-1}\left(-i\sqrt{\lambda}|\xi|\right)\Bigg|\mathrm{d}\xi < \infty.\end{aligned}$$

In order to prove the boundedness of

$$\begin{aligned}\int_{B(0,1)}\Bigg|&\left(1-\frac{d}{2}\right)\frac{|\xi|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}}K_{d/2-1}\left(-i\sqrt{\lambda}|\xi|\right)\\&-\left(\frac{|\xi|}{-i\sqrt{\lambda}}\right)^{1-d/2}i\sqrt{\lambda}K'_{d/2-1}\left(-i\sqrt{\lambda}|\xi|\right)\Bigg|\mathrm{d}\xi,\end{aligned}$$

we mention

$$\left|K_{d/2-1}\big(-i\sqrt{\lambda}|\xi|\big)\right| \leq c|\xi|^{1-d/2} \quad \text{for} \quad |\xi| \to 0,$$

if $d \geq 3$, and

$$\left|K'_{d/2-1}\big(-i\sqrt{\lambda}|\xi|\big)\right| \leq c|\xi|^{-d/2} \quad \text{for} \quad |\xi| \to 0,$$

see Proposition 5.6 (iii). Hence, we find

$$\begin{aligned}\Bigg|&\left(1-\frac{d}{2}\right)\frac{|\xi|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}}K_{d/2-1}\left(-i\sqrt{\lambda}|\xi|\right)\\&-\left(\frac{|\xi|}{-i\sqrt{\lambda}}\right)^{1-d/2}i\sqrt{\lambda}K'_{d/2-1}\left(-i\sqrt{\lambda}|\xi|\right)\Bigg| \leq c_2|\xi|^{1-d}.\end{aligned}$$

Therefore, using a substitution to spherical coordinates, we get

$$\begin{aligned}\int_{B(0,1)}\Bigg|&\left(1-\frac{d}{2}\right)\frac{|\xi|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}}K_{d/2-1}\left(-i\sqrt{\lambda}|\xi|\right)\\&-\left(\frac{|\xi|}{-i\sqrt{\lambda}}\right)^{1-d/2}i\sqrt{\lambda}K'_{d/2-1}\left(-i\sqrt{\lambda}|\xi|\right)\Bigg|\mathrm{d}\xi\\&\leq c_2\int_{B(0,1)}|\xi|^{1-d}\mathrm{d}\xi = c_3\int_0^1 r^{1-d}\cdot r^{d-1}\mathrm{d}r < \infty.\end{aligned}$$

Thus, it follows finally

$$\int_{\mathbb{R}^d} \Bigg| \left(1-\frac{d}{2}\right) \frac{|\xi|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}} K_{d/2-1}\left(-i\sqrt{\lambda}|\xi|\right) \\ - \left(\frac{|\xi|}{-i\sqrt{\lambda}}\right)^{1-d/2} i\sqrt{\lambda} K'_{d/2-1}\left(-i\sqrt{\lambda}|\xi|\right)\Bigg| \mathrm{d}\xi < \infty,$$

which yields the claimed result due to our preliminary considerations. $\square$

The following lemma contains the main estimate for the next two results:

Lemma A.6. *Let $d \geq 2$, assume that $\Sigma \subset \mathbb{R}^d$ fulfills Hypothesis A.1 and let $\lambda \in \mathbb{C} \setminus [0, \infty)$. Then there exists a constant $C > 0$ depending on d, λ and Σ such that*

$$\int_\Sigma \int_{-\varepsilon}^{\varepsilon} \Bigg| \left(1-\frac{d}{2}\right) \frac{|x-y_\Sigma - s\nu(y_\Sigma)|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}} K_{d/2-1}\left(-i\sqrt{\lambda}|x-y_\Sigma - s\nu(y_\Sigma)|\right) \\ - \left(\frac{|x-y_\Sigma - s\nu(y_\Sigma)|}{-i\sqrt{\lambda}}\right)^{1-d/2} i\sqrt{\lambda} K'_{d/2-1}\left(-i\sqrt{\lambda}|x-y_\Sigma - s\nu(y_\Sigma)|\right)\Bigg| \mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \\ \leq C\varepsilon^{1/d}$$

holds independent from $x \in \mathbb{R}^d$ for sufficiently small $\varepsilon > 0$.

Proof. Let $x \in \mathbb{R}^d$ be fixed. Since the eigenvalues of the matrix of the Weingarten map W associated to the hypersurface Σ are bounded by Proposition 3.11, it follows that there exists a constant $c_1 > 0$ independent from ε such that $1 \leq c_1 \det(1 - sW(y_\Sigma))$ holds for all $s \in (-\varepsilon, \varepsilon)$ and all $y_\Sigma \in \Sigma$, if ε is sufficiently small. Here, $\det(1 - sW(y_\Sigma))$ is regarded in the sense of Remark 3.18. Hence, we get

$$\int_\Sigma \int_{-\varepsilon}^{\varepsilon} \Bigg| \left(1-\frac{d}{2}\right) \frac{|x-y_\Sigma - s\nu(y_\Sigma)|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}} K_{d/2-1}\left(-i\sqrt{\lambda}|x-y_\Sigma - s\nu(y_\Sigma)|\right) \\ - \left(\frac{|x-y_\Sigma - s\nu(y_\Sigma)|}{-i\sqrt{\lambda}}\right)^{1-d/2} i\sqrt{\lambda} K'_{d/2-1}\left(-i\sqrt{\lambda}|x-y_\Sigma - s\nu(y_\Sigma)|\right)\Bigg| \mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \\ \leq c_1 \int_\Sigma \int_{-\varepsilon}^{\varepsilon} \Bigg| \left(1-\frac{d}{2}\right) \frac{|x-y_\Sigma - s\nu(y_\Sigma)|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}} K_{d/2-1}\left(-i\sqrt{\lambda}|x-y_\Sigma - s\nu(y_\Sigma)|\right) \\ - \left(\frac{|x-y_\Sigma - s\nu(y_\Sigma)|}{-i\sqrt{\lambda}}\right)^{1-d/2} i\sqrt{\lambda} K'_{d/2-1}\left(-i\sqrt{\lambda}|x-y_\Sigma - s\nu(y_\Sigma)|\right)\Bigg| \\ \cdot \det(1 - sW(y_\Sigma))\mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \\ \leq c_2 \int_{\Omega_\varepsilon} \Bigg(\Bigg| \left(1-\frac{d}{2}\right) |x-y|^{-d/2} K_{d/2-1}\left(-i\sqrt{\lambda}|x-y|\right)\Bigg| \\ + \Bigg| |x-y|^{1-d/2} K'_{d/2-1}\left(-i\sqrt{\lambda}|x-y|\right)\Bigg|\Bigg)\mathrm{d}y,$$

where we used the transformation formula from Corollary 3.21 for integrals over the tube Ω_ε that is given as

$$\Omega_\varepsilon = \{y_\Sigma + s\nu(y_\Sigma) : y_\Sigma \in \Sigma, s \in (-\varepsilon, \varepsilon)\}.$$

In order to get an estimate for the last integral, we split the integration area into $\tilde{\Omega}_1 := B\big(x, \varepsilon^{1/d}\big) \cap \Omega_\varepsilon$ and $\tilde{\Omega}_2 := \Omega_\varepsilon \setminus B\big(x, \varepsilon^{1/d}\big)$, so $\Omega_\varepsilon = \tilde{\Omega}_1 \,\dot{\cup}\, \tilde{\Omega}_2$. Due to the asymptotic behavior of the modified Bessel function and its derivative, see Proposition 5.6, there exists a constant $c > 0$ such that

$$\left|K_{d/2-1}\big(i\sqrt{\lambda}|x-y|\big)\right| \leq c|x-y|^{1-d/2} \quad \text{for } d > 2 \text{ and } y \to x$$

and

$$\left|K'_{d/2-1}\big(i\sqrt{\lambda}|x-y|\big)\right| \leq c|x-y|^{-d/2} \quad \text{for} \quad y \to x$$

hold. Hence, there exists a constant $c_3 > 0$ such that

$$\begin{aligned} &\left| \left(1 - \frac{d}{2}\right) |x-y|^{-d/2} K_{d/2-1}\left(-i\sqrt{\lambda}|x-y|\right) \right| \\ &\qquad + \left| |x-y|^{1-d/2} K'_{d/2-1}\left(-i\sqrt{\lambda}|x-y|\right) \right| \leq c_3 |x-y|^{1-d} \end{aligned}$$

holds for all $y \in \tilde{\Omega}_1$. Therefore, we conclude that the integral over $\tilde{\Omega}_1$ can be estimated as

$$\begin{aligned} &\int_{\tilde{\Omega}_1} \Bigg(\left| \left(1 - \frac{d}{2}\right) |x-y|^{-d/2} K_{d/2-1}\left(-i\sqrt{\lambda}|x-y|\right) \right| \\ &\qquad + \left| |x-y|^{1-d/2} K'_{d/2-1}\left(-i\sqrt{\lambda}|x-y|\right) \right| \Bigg) \mathrm{d}y \\ &\leq c_3 \int_{B(0,\varepsilon^{1/d})} |y|^{1-d} \mathrm{d}y = c_4 \int_0^{\varepsilon^{1/d}} \mathrm{d}r = c_4 \varepsilon^{1/d} \end{aligned}$$

independent from x, where we used a substitution to spherical coordinates. Moreover, since $K_{d/2-1}$ and $K'_{d/2-1}$ are differentiable and bounded, if the argument is not in a neighborhood of zero, we find for $y \notin B\big(x, \varepsilon^{1/d}\big)$

$$\begin{aligned} &\left| \left(1 - \frac{d}{2}\right) |x-y|^{-d/2} K_{d/2-1}\left(-i\sqrt{\lambda}|x-y|\right) \right| \\ &\qquad + \left| |x-y|^{1-d/2} K'_{d/2-1}\left(-i\sqrt{\lambda}|x-y|\right) \right| \leq c_5 |x-y|^{1-d} \leq c_5 \varepsilon^{(1-d)/d} \end{aligned}$$

because of the asymptotic behavior of $K_{d/2-1}$ and $K'_{d/2-1}$ for small arguments. Thus, using Lemma A.3 we get

$$\begin{aligned} &\int_{\tilde{\Omega}_2} \Bigg(\left| \left(1 - \frac{d}{2}\right) |x-y|^{-d/2} K_{d/2-1}\left(-i\sqrt{\lambda}|x-y|\right) \right| \\ &\qquad + \left| |x-y|^{1-d/2} K'_{d/2-1}\left(-i\sqrt{\lambda}|x-y|\right) \right| \Bigg) \mathrm{d}y \\ &\leq c_5 \int_{\Omega_\varepsilon \setminus B\left(x, \varepsilon^{1/d}\right)} \varepsilon^{(1-d)/d} \mathrm{d}y \leq c_6 \varepsilon^{(1-d)/d} \cdot \varepsilon \sigma(\Sigma) = c_6 \sigma(\Sigma) \varepsilon^{1/d} \end{aligned}$$

with the Hausdorff-measure $\sigma(\Sigma)$ of Σ, again independent from x. Hence, we find finally

$$\begin{aligned}
&\int_\Sigma\int_{-\varepsilon}^{\varepsilon}\bigg|\left(1-\frac{d}{2}\right)\frac{|x-y_\Sigma-s\nu(y_\Sigma)|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}}K_{d/2-1}\left(-i\sqrt{\lambda}|x-y_\Sigma-s\nu(y_\Sigma)|\right)\\
&\qquad-\left(\frac{|x-y_\Sigma-s\nu(y_\Sigma)|}{-i\sqrt{\lambda}}\right)^{1-d/2}i\sqrt{\lambda}K'_{d/2-1}\left(-i\sqrt{\lambda}|x-y_\Sigma-s\nu(y_\Sigma)|\right)\bigg|\mathrm{d}s\mathrm{d}\sigma(y_\Sigma)\\
&\qquad\leq c_2\int_{\Omega_\varepsilon}\left(\left|\left(1-\frac{d}{2}\right)|x-y|^{-d/2}K_{d/2-1}\left(-i\sqrt{\lambda}|x-y|\right)\right|\right.\\
&\qquad\qquad\qquad\left.+\left||x-y|^{1-d/2}K'_{d/2-1}\left(-i\sqrt{\lambda}|x-y|\right)\right|\right)\mathrm{d}y\\
&\qquad= c_2\int_{\tilde{\Omega}_1}\left(\left|\left(1-\frac{d}{2}\right)|x-y|^{-d/2}K_{d/2-1}\left(-i\sqrt{\lambda}|x-y|\right)\right|\right.\\
&\qquad\qquad\qquad\left.+\left||x-y|^{1-d/2}K'_{d/2-1}\left(-i\sqrt{\lambda}|x-y|\right)\right|\right)\mathrm{d}y\\
&\qquad\quad+ c_2\int_{\tilde{\Omega}_2}\left(\left|\left(1-\frac{d}{2}\right)|x-y|^{-d/2}K_{d/2-1}\left(-i\sqrt{\lambda}|x-y|\right)\right|\right.\\
&\qquad\qquad\qquad\left.+\left||x-y|^{1-d/2}K'_{d/2-1}\left(-i\sqrt{\lambda}|x-y|\right)\right|\right)\mathrm{d}y\\
&\qquad\leq c_7\varepsilon^{1/d}
\end{aligned}$$

and thus, the statement of this lemma. □

Using Lemma A.6, we can prove the next two estimates that are needed in Chapter 7 to show the main results of this thesis:

Proposition A.7. *Let $d\geq 2$, let $\Sigma\subset\mathbb{R}^d$ be such that Hypothesis A.1 is fulfilled, let $X\subset\mathbb{R}^d$ and let μ be a measure on X. Furthermore, let $\psi\in L^\infty(\Sigma\times(-1,1),\sigma\times\Lambda_1)$, let $\varphi\in L^\infty(X,\mu)$ and let $\lambda\in\mathbb{C}\setminus[0,\infty)$. Then there exists a constant $C>0$ depending on d,λ,φ,ψ and Σ such that*

$$\sup_{x\in X}\int_\Sigma\int_{-1}^{1}\left|\varphi(x)\big(G_\lambda(x-y_\Sigma-\varepsilon s\nu(y_\Sigma))-G_\lambda(x-y_\Sigma)\big)\psi(y_\Sigma,s)\right|\mathrm{d}s\mathrm{d}\sigma\leq C\varepsilon^{1/d}$$

holds as $\varepsilon\to 0+$.

Proof. Let $x\in X$ be fixed. Since the mapping $\mathbb{C}\setminus(-\infty,0]\ni z\mapsto K_{d/2-1}(z)$ is analytic by Proposition 5.6 (i), it follows that

$$\begin{aligned}
[0,1]\ni\theta\mapsto\ & G_\lambda(x-y_\Sigma-\varepsilon s\theta\nu(y_\Sigma))\\
&=\frac{1}{(2\pi)^{d/2}}\left(\frac{|x-y_\Sigma-\varepsilon s\theta\nu(y_\Sigma)|}{-i\sqrt{\lambda}}\right)^{1-d/2}K_{d/2-1}\left(-i\sqrt{\lambda}|x-y_\Sigma-\varepsilon s\theta\nu(y_\Sigma)|\right)
\end{aligned}$$

is differentiable for almost all $y_\Sigma \in \Sigma$. Hence, using the main theorem of calculus, we find

$$
\begin{aligned}
G_\lambda(x - y_\Sigma - \varepsilon s\nu(y_\Sigma)) - G_\lambda(x - y_\Sigma) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta} G_\lambda(x - y_\Sigma - \varepsilon s\theta\nu(y_\Sigma))\mathrm{d}\theta \\
= \int_0^1 \frac{1}{(2\pi)^{d/2}} \frac{\mathrm{d}}{\mathrm{d}\theta} |x - y_\Sigma - \varepsilon s\theta\nu(y_\Sigma)| \\
\cdot \Bigg(\left(1 - \frac{d}{2}\right) \frac{|x - y_\Sigma - \varepsilon s\theta\nu(y_\Sigma)|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}} K_{d/2-1}\left(-i\sqrt{\lambda}|x - y_\Sigma - \varepsilon s\theta\nu(y_\Sigma)|\right) \\
- \left(\frac{|x - y_\Sigma - \varepsilon s\theta\nu(y_\Sigma)|}{-i\sqrt{\lambda}}\right)^{1-d/2} i\sqrt{\lambda}K'_{d/2-1}\left(-i\sqrt{\lambda}|x - y_\Sigma - \varepsilon s\theta\nu(y_\Sigma)|\right)\Bigg)\mathrm{d}\theta.
\end{aligned}
$$

Note that a simple calculation yields

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}|x - y_\Sigma - \varepsilon s\theta\nu(y_\Sigma)| &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sum_{k=1}^d (x_k - y_{\Sigma,k} - \varepsilon s\theta\nu_k(y_\Sigma))^2\right)^{1/2} \\
&= -|x - y_\Sigma - \varepsilon s\theta\nu(y_\Sigma)|^{-1}\varepsilon s\big(\nu(y_\Sigma), x - y_\Sigma - \varepsilon s\theta\nu(y_\Sigma)\big)\big)
\end{aligned}
$$

and using the Cauchy-Schwarz inequality, it follows

$$
\left|\frac{\mathrm{d}}{\mathrm{d}\theta}|x - y_\Sigma - \varepsilon s\theta\nu(y_\Sigma)|\right| \leq \varepsilon|s|.
$$

Hence, we find

$$
\begin{aligned}
&\int_\Sigma \int_{-1}^1 \left|\varphi(x)\big(G_\lambda(x - y_\Sigma - \varepsilon s\nu(y_\Sigma)) - G_\lambda(x - y_\Sigma)\big)\psi(y_\Sigma, s)\right| \mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \\
&\leq \|\varphi\|_{L^\infty}\|\psi\|_{L^\infty} \int_\Sigma \int_{-1}^1 \Bigg| \int_0^1 \frac{1}{(2\pi)^{d/2}} \frac{\mathrm{d}}{\mathrm{d}\theta}|x - y_\Sigma - \varepsilon s\theta\nu(y_\Sigma)| \\
&\cdot \Bigg(\left(1 - \frac{d}{2}\right) \frac{|x - y_\Sigma - \varepsilon s\theta\nu(y_\Sigma)|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}} K_{d/2-1}\left(-i\sqrt{\lambda}|x - y_\Sigma - \varepsilon s\theta\nu(y_\Sigma)|\right) \\
&- \left(\frac{|x - y_\Sigma - \varepsilon s\theta\nu(y_\Sigma)|}{-i\sqrt{\lambda}}\right)^{1-d/2} i\sqrt{\lambda}K'_{d/2-1}\left(-i\sqrt{\lambda}|x - y_\Sigma - \varepsilon s\theta\nu(y_\Sigma)|\right)\Bigg)\mathrm{d}\theta\Bigg|\mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \\
&\leq c_1 \int_\Sigma \Bigg(\int_{-1}^1 \int_0^1 \Bigg| \left(1 - \frac{d}{2}\right) \frac{|x - y_\Sigma - \varepsilon s\theta\nu(y_\Sigma)|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}} K_{d/2-1}\left(-i\sqrt{\lambda}|x - y_\Sigma - \varepsilon s\theta\nu(y_\Sigma)|\right) \\
&- \left(\frac{|x - y_\Sigma - \varepsilon s\theta\nu(y_\Sigma)|}{-i\sqrt{\lambda}}\right)^{1-d/2} i\sqrt{\lambda}K'_{d/2-1}\left(-i\sqrt{\lambda}|x - y_\Sigma - \varepsilon s\theta\nu(y_\Sigma)|\right)\Bigg|\varepsilon|s|\mathrm{d}\theta\mathrm{d}s\Bigg)\mathrm{d}\sigma(y_\Sigma).
\end{aligned}
$$

In order to find a suitable estimate for the double integral in the large brackets in the last line above, we introduce the transformation $T : (-1,1) \times (0,1) \to \mathcal{M} := \{(r,\phi) \in \mathbb{R}^2 : r \in (-1,1), \phi = \varepsilon r\theta \text{ for } \theta \in (0,1)\}$ via

$$
\begin{pmatrix} r \\ \phi \end{pmatrix} = T\begin{pmatrix} s \\ \theta \end{pmatrix} := \begin{pmatrix} s \\ \varepsilon s\theta \end{pmatrix}
$$

and we note that the Jacobian of T is given by

$$DT = \begin{pmatrix} 1 & 0 \\ \varepsilon\theta & \varepsilon s \end{pmatrix}.$$

Thus, it holds $|\det DT| = \varepsilon|s|$ which implies

$$\begin{aligned}
&\int_\Sigma \int_{-1}^{1} \left|\varphi(x)\big(G_\lambda(x - y_\Sigma - \varepsilon s\nu(y_\Sigma)) - G_\lambda(x - y_\Sigma)\big)\psi(y_\Sigma, s)\right| \mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \\
&\leq c_1 \int_\Sigma \Bigg(\int_{-1}^{1}\int_0^1 \Bigg| \left(1 - \frac{d}{2}\right) \frac{|x - y_\Sigma - \varepsilon s\theta\nu(y_\Sigma)|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}} K_{d/2-1}\left(-i\sqrt{\lambda}|x - y_\Sigma - \varepsilon s\theta\nu(y_\Sigma)|\right) \\
&- \left(\frac{|x - y_\Sigma - \varepsilon s\theta\nu(y_\Sigma)|}{-i\sqrt{\lambda}}\right)^{1-d/2} i\sqrt{\lambda}K'_{d/2-1}\left(-i\sqrt{\lambda}|x - y_\Sigma - \varepsilon s\theta\nu(y_\Sigma)|\right) \Bigg| \varepsilon|s|\mathrm{d}\theta\mathrm{d}s \Bigg) \mathrm{d}\sigma(y_\Sigma) \\
&= c_1 \int_\Sigma \Bigg(\int_{\mathcal{M}} \Bigg| \left(1 - \frac{d}{2}\right) \frac{|x - y_\Sigma - \phi\nu(y_\Sigma)|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}} K_{d/2-1}\left(-i\sqrt{\lambda}|x - y_\Sigma - \phi\nu(y_\Sigma)|\right) \\
&- \left(\frac{|x - y_\Sigma - \phi\nu(y_\Sigma)|}{-i\sqrt{\lambda}}\right)^{1-d/2} i\sqrt{\lambda}K'_{d/2-1}\left(-i\sqrt{\lambda}|x - y_\Sigma - \phi\nu(y_\Sigma)|\right) \Bigg| \mathrm{d}\phi\mathrm{d}r \Bigg) \mathrm{d}\sigma(y_\Sigma).
\end{aligned}$$

Note that it holds $\mathcal{M} \subset (-1,1) \times (-\varepsilon, \varepsilon)$. Hence, going on with the above computation, we get

$$\begin{aligned}
&\int_\Sigma \int_{-1}^{1} \left|\varphi(x)\big(G_\lambda(x - y_\Sigma - \varepsilon s\nu(y_\Sigma)) - G_\lambda(x - y_\Sigma)\big)\psi(y_\Sigma, s)\right| \mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \\
&\quad\leq c_1 \int_\Sigma \int_{-1}^{1}\int_{-\varepsilon}^{\varepsilon} \Bigg| \left(1 - \frac{d}{2}\right) \frac{|x - y_\Sigma - \phi\nu(y_\Sigma)|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}} K_{d/2-1}\left(-i\sqrt{\lambda}|x - y_\Sigma - \phi\nu(y_\Sigma)|\right) \\
&\qquad - \left(\frac{|x - y_\Sigma - \phi\nu(y_\Sigma)|}{-i\sqrt{\lambda}}\right)^{1-d/2} i\sqrt{\lambda}K'_{d/2-1}\left(-i\sqrt{\lambda}|x - y_\Sigma - \phi\nu(y_\Sigma)|\right) \Bigg| \mathrm{d}\phi\mathrm{d}r\mathrm{d}\sigma(y_\Sigma) \\
&\quad= c_2 \int_\Sigma \int_{-\varepsilon}^{\varepsilon} \Bigg| \left(1 - \frac{d}{2}\right) \frac{|x - y_\Sigma - \phi\nu(y_\Sigma)|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}} K_{d/2-1}\left(-i\sqrt{\lambda}|x - y_\Sigma - \phi\nu(y_\Sigma)|\right) \\
&\qquad - \left(\frac{|x - y_\Sigma - \phi\nu(y_\Sigma)|}{-i\sqrt{\lambda}}\right)^{1-d/2} i\sqrt{\lambda}K'_{d/2-1}\left(-i\sqrt{\lambda}|x - y_\Sigma - \phi\nu(y_\Sigma)|\right) \Bigg| \mathrm{d}\phi\mathrm{d}\sigma(y_\Sigma),
\end{aligned}$$

where we used in the last step that the integrand was independent from r. Using now the statement of Lemma A.6, we find

$$\begin{aligned}
&\int_\Sigma \int_{-1}^{1} \big|\varphi(x)\big(G_\lambda(x - y_\Sigma - \varepsilon s\nu(y_\Sigma)) - G_\lambda(x - y_\Sigma)\big)\psi(y_\Sigma, s)\big| \,\mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \\
&\leq c_2 \int_\Sigma \int_{-\varepsilon}^{\varepsilon} \bigg| \left(1 - \frac{d}{2}\right) \frac{|x - y_\Sigma - \phi\nu(y_\Sigma)|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}} K_{d/2-1}\left(-i\sqrt{\lambda}|x - y_\Sigma - \phi\nu(y_\Sigma)|\right) \\
&\qquad - \left(\frac{|x - y_\Sigma - \phi\nu(y_\Sigma)|}{-i\sqrt{\lambda}}\right)^{1-d/2} i\sqrt{\lambda} K'_{d/2-1}\left(-i\sqrt{\lambda}|x - y_\Sigma - \phi\nu(y_\Sigma)|\right) \bigg| \mathrm{d}\phi\mathrm{d}\sigma(y_\Sigma) \\
&\leq C\varepsilon^{1/d}
\end{aligned}$$

with a constant C which depends on d, λ, φ, ψ and Σ. Thus, the assertion of this proposition is true. $\square$

The next estimate is quite similar as the one of Proposition A.7:

Proposition A.8. *Let $d \geq 2$, assume that $\Sigma \subset \mathbb{R}^d$ fulfills Hypothesis A.1, let $\psi, \varphi \in L^\infty(\Sigma \times (-1, 1), \sigma \times \Lambda_1)$ and let $\lambda \in \mathbb{C} \setminus [0, \infty)$. Then there exists a constant $C > 0$ depending on $d, \lambda, \varphi, \psi$ and Σ such that*

$$\begin{aligned}
\sup_{(x_\Sigma, t) \in \Sigma \times (-1,1)} \int_\Sigma \int_{-1}^{1} \big|\varphi(x_\Sigma, t)\big(&G_\lambda(x_\Sigma + \varepsilon t\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma)) \\
&- G_\lambda(x_\Sigma - y_\Sigma - \varepsilon s\nu(y_\Sigma))\big)\psi(y_\Sigma, s)\big| \mathrm{d}s\mathrm{d}\sigma(y_\Sigma) \leq C\varepsilon^{1/d}
\end{aligned}$$

holds as $\varepsilon \to 0+$.

Proof. Let $x_\Sigma \in \Sigma$ and $t \in (-1, 1)$ be fixed. Since the mapping $\mathbb{C}\setminus(-\infty, 0] \ni z \mapsto K_{d/2-1}(z)$ is analytic, cf. Proposition 5.6, it follows that

$$\begin{aligned}
[0, 1] \ni \theta \mapsto\ & G_\lambda(x_\Sigma + \varepsilon t\theta\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma)) \\
&= \frac{1}{(2\pi)^{d/2}} \left(\frac{|x_\Sigma + \varepsilon t\theta\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma)|}{-i\sqrt{\lambda}}\right)^{1-d/2} \\
&\qquad \cdot K_{d/2-1}\left(-i\sqrt{\lambda}|x_\Sigma + \varepsilon t\theta\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma)|\right)
\end{aligned}$$

is differentiable for almost all $y_\Sigma \in \Sigma$. Hence, using the main theorem of calculus, we get

$$\begin{aligned}
G_\lambda(x_\Sigma + \varepsilon t\nu(x_\Sigma) &- y_\Sigma - \varepsilon s\nu(y_\Sigma)) - G_\lambda(x_\Sigma - y_\Sigma - \varepsilon s\nu(y_\Sigma)) \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta} G_\lambda(x_\Sigma + \varepsilon t\theta\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma))\mathrm{d}\theta \\
&= \int_0^1 \frac{1}{(2\pi)^{d/2}} \frac{\mathrm{d}}{\mathrm{d}\theta}|x_\Sigma + \varepsilon t\theta\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma)| \\
&\quad \cdot \Bigg(\left(1 - \frac{d}{2}\right) \frac{|x_\Sigma + \varepsilon t\theta\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma)|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}} \\
&\qquad \cdot K_{d/2-1}\left(-i\sqrt{\lambda}|x_\Sigma + \varepsilon t\theta\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma)|\right) \\
&\quad - \left(\frac{|x_\Sigma + \varepsilon t\theta\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma)|}{-i\sqrt{\lambda}}\right)^{1-d/2} \\
&\qquad \cdot i\sqrt{\lambda} K'_{d/2-1}\left(-i\sqrt{\lambda}|x_\Sigma + \varepsilon t\theta\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma)|\right) \Bigg) \mathrm{d}\theta.
\end{aligned}$$

Now, a straightforward calculation shows

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}|x_\Sigma + \varepsilon t\theta\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma)| &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sum_{k=1}^d (x_k + \varepsilon t\theta\nu_k(x_\Sigma) - y_{\Sigma,k} - \varepsilon s\nu_k(y_\Sigma))^2\right)^{1/2} \\
= |x_\Sigma + \varepsilon t\theta\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma)|^{-1} &\varepsilon t\big(\nu(x_\Sigma), x_\Sigma + \varepsilon t\theta\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma))\big)
\end{aligned}$$

and using the Cauchy-Schwarz inequality and $t \in (-1, 1)$ we find

$$\left|\frac{\mathrm{d}}{\mathrm{d}\theta}|x_\Sigma + \varepsilon t\theta\nu(x_\Sigma) - y_\Sigma - \varepsilon s\nu(y_\Sigma)|\right| \le \varepsilon|t| \le \varepsilon.$$

Hence, it follows

$$
\begin{aligned}
&\int_\Sigma \int_{-1}^{1} \big| \varphi(x_\Sigma, t) \big(G_\lambda(x_\Sigma + \varepsilon t \nu(x_\Sigma) - y_\Sigma - \varepsilon s \nu(y_\Sigma)) \\
&\qquad\qquad\qquad - G_\lambda(x_\Sigma - y_\Sigma - \varepsilon s \nu(y_\Sigma)) \big) \psi(y_\Sigma, s) \big| \mathrm{d}s \mathrm{d}\sigma(y_\Sigma) \\
&\quad \le \|\varphi\|_{L^\infty} \|\psi\|_{L^\infty} \int_\Sigma \int_{-1}^{1} \bigg| \int_0^1 \frac{1}{(2\pi)^{d/2}} \frac{\mathrm{d}}{\mathrm{d}\theta} |x_\Sigma + \varepsilon t\theta \nu(x_\Sigma) - y_\Sigma - \varepsilon s \nu(y_\Sigma)| \\
&\qquad \cdot \Bigg(\left(1 - \frac{d}{2}\right) \frac{|x_\Sigma + \varepsilon t\theta \nu(x_\Sigma) - y_\Sigma - \varepsilon s \nu(y_\Sigma)|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}} \\
&\qquad\quad \cdot K_{d/2-1} \left(-i\sqrt{\lambda} |x_\Sigma + \varepsilon t\theta \nu(x_\Sigma) - y_\Sigma - \varepsilon s \nu(y_\Sigma)| \right) \\
&\qquad - \left(\frac{|x_\Sigma + \varepsilon t\theta \nu(x_\Sigma) - y_\Sigma - \varepsilon s \nu(y_\Sigma)|}{-i\sqrt{\lambda}} \right)^{1-d/2} \\
&\qquad\quad \cdot i\sqrt{\lambda} K'_{d/2-1} \left(-i\sqrt{\lambda} |x_\Sigma + \varepsilon t\theta \nu(x_\Sigma) - y_\Sigma - \varepsilon s \nu(y_\Sigma)| \right) \Bigg) \mathrm{d}\theta \bigg| \mathrm{d}s \mathrm{d}\sigma(y_\Sigma) \\
&\quad \le c_1 \int_\Sigma \int_{-\varepsilon}^{\varepsilon} \int_0^1 \bigg| \frac{\varepsilon}{\varepsilon} \Bigg(\left(1 - \frac{d}{2}\right) \frac{|x_\Sigma + \varepsilon t\theta \nu(x_\Sigma) - y_\Sigma - r \nu(y_\Sigma)|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}} \\
&\qquad\qquad\qquad \cdot K_{d/2-1} \left(-i\sqrt{\lambda} |x_\Sigma + \varepsilon t\theta \nu(x_\Sigma) - y_\Sigma - r \nu(y_\Sigma)| \right) \\
&\qquad\qquad - \left(\frac{|x_\Sigma + \varepsilon t\theta \nu(x_\Sigma) - y_\Sigma - r \nu(y_\Sigma)|}{-i\sqrt{\lambda}} \right)^{1-d/2} \\
&\qquad\qquad\qquad \cdot i\sqrt{\lambda} K'_{d/2-1} \left(-i\sqrt{\lambda} |x_\Sigma + \varepsilon t\theta \nu(x_\Sigma) - y_\Sigma - r \nu(y_\Sigma)| \right) \Bigg) \bigg| \mathrm{d}\theta \mathrm{d}r \mathrm{d}\sigma(y_\Sigma),
\end{aligned}
$$

where we used the substitution $r = \varepsilon s$. Therefore, using Lemma A.6 we find

$$
\begin{aligned}
&\int_\Sigma \int_{-1}^{1} \big| \varphi(x_\Sigma, t) \big(G_\lambda(x_\Sigma + \varepsilon t \nu(x_\Sigma) - y_\Sigma - \varepsilon s \nu(y_\Sigma)) \\
&\qquad\qquad\qquad - G_\lambda(x_\Sigma - y_\Sigma - \varepsilon s \nu(y_\Sigma)) \big) \psi(y_\Sigma, s) \big| \mathrm{d}s \mathrm{d}\sigma(y_\Sigma) \\
&\quad \le c_1 \int_0^1 \int_\Sigma \int_{-\varepsilon}^{\varepsilon} \bigg| \frac{\varepsilon}{\varepsilon} \Bigg(\left(1 - \frac{d}{2}\right) \frac{|x_\Sigma + \varepsilon t\theta \nu(x_\Sigma) - y_\Sigma - r \nu(y_\Sigma)|^{-d/2}}{(-i\sqrt{\lambda})^{1-d/2}} \\
&\qquad\qquad\qquad \cdot K_{d/2-1} \left(-i\sqrt{\lambda} |x_\Sigma + \varepsilon t\theta \nu(x_\Sigma) - y_\Sigma - r \nu(y_\Sigma)| \right) \\
&\qquad\qquad - \left(\frac{|x_\Sigma + \varepsilon t\theta \nu(x_\Sigma) - y_\Sigma - r \nu(y_\Sigma)|}{-i\sqrt{\lambda}} \right)^{1-d/2} \\
&\qquad\qquad\qquad \cdot i\sqrt{\lambda} K'_{d/2-1} \left(-i\sqrt{\lambda} |x_\Sigma + \varepsilon t\theta \nu(x_\Sigma) - y_\Sigma - r \nu(y_\Sigma)| \right) \Bigg) \bigg| \mathrm{d}r \mathrm{d}\sigma(y_\Sigma) \mathrm{d}\theta \\
&\quad \le c_2 \int_0^1 \varepsilon^{1/d} \mathrm{d}\theta = C \varepsilon^{1/d}
\end{aligned}
$$

with a constant $C > 0$ that depends on d, λ, Σ, φ and ψ. This is the claimed result of the above proposition. $\square$

The following proposition contains the last estimate that is needed for our approximation procedure:

Proposition A.9. *Let $d \geq 2$, assume that $\Sigma \subset \mathbb{R}^d$ fulfills Hypothesis A.1, let $\psi, \varphi \in L^\infty(\Sigma \times (-1,1), \sigma \times \Lambda_1)$ and let $\lambda \in \mathbb{C} \setminus [0,\infty)$. Then there exists a constant $C > 0$ depending on $d, \lambda, \varphi, \psi$ and Σ such that*

$$\sup_{(y_\Sigma,s)\in\Sigma\times(-1,1)} \int_\Sigma \int_{-1}^1 \big|\varphi(x_\Sigma,t)\big(G_\lambda(x_\Sigma - y_\Sigma - \varepsilon s\nu(y_\Sigma)) \\ - G_\lambda(x_\Sigma - y_\Sigma)\big)\psi(y_\Sigma,s)\big|\mathrm{d}t\mathrm{d}\sigma(x_\Sigma) \leq C$$

holds for all sufficiently small $\varepsilon > 0$.

Proof. Let $y_\Sigma \in \Sigma$ and $s \in (-1,1)$ be fixed. First, it follows from

$$\begin{aligned}
&\int_\Sigma \int_{-1}^1 \big|\varphi(x_\Sigma,t)\big(G_\lambda(x_\Sigma - y_\Sigma - \varepsilon s\nu(y_\Sigma)) - G_\lambda(x_\Sigma - y_\Sigma)\big)\psi(y_\Sigma,s)\big|\mathrm{d}t\mathrm{d}\sigma(x_\Sigma) \\
&\qquad \leq \|\varphi\|_{L^\infty}\|\psi\|_{L^\infty} \int_\Sigma \int_{-1}^1 \big(\big|G_\lambda(x_\Sigma - y_\Sigma - \varepsilon s\nu(y_\Sigma))\big| + \big|G_\lambda(x_\Sigma - y_\Sigma)\big|\big)\mathrm{d}t\mathrm{d}\sigma(x_\Sigma) \\
&\qquad = 2\|\varphi\|_{L^\infty}\|\psi\|_{L^\infty} \int_\Sigma \big(\big|G_\lambda(x_\Sigma - y_\Sigma - \varepsilon s\nu(y_\Sigma))\big| + \big|G_\lambda(x_\Sigma - y_\Sigma)\big|\mathrm{d}\big)\sigma(x_\Sigma),
\end{aligned}$$

that it is sufficient to prove that there exists a constant $\tilde{C} > 0$ such that

$$\int_\Sigma \big|G_\lambda(x_\Sigma - y_\Sigma - r\varepsilon\nu(y_\Sigma))\big|\mathrm{d}\sigma(x_\Sigma) \leq \tilde{C}$$

holds for $r \in \{0, s\}$ independent from y_Σ and s. In order to show this estimate, let $\{\varphi_i, U_i, V_i\}_{i\in I}$ be a parametrization of Σ in the sense of Definition 3.2 and let $\{\chi_i\}_{i\in I}$ be a partition of unity for $\{V_i\}_{i\in I}$ as in Lemma 3.13. Then by Definition 3.14 the above integral is given by

$$\begin{aligned}
&\int_\Sigma \big|G_\lambda(x_\Sigma - y_\Sigma - r\varepsilon\nu(y_\Sigma))\big|\mathrm{d}\sigma(x_\Sigma) \\
&\qquad = \sum_{i\in I} \int_{U_i} \chi_i(\varphi_i(u))\big|G_\lambda(\varphi_i(u) - y_\Sigma - r\varepsilon\nu(y_\Sigma))\big|\sqrt{\det G_i(u)}\mathrm{d}u,
\end{aligned}$$

where $G_i(u)$ is the matrix of the first fundamental form associated to Σ. Thus, it is sufficient to prove that for any $i \in I$ there exists a constant $C_i > 0$ such that

$$\int_{U_i} \chi_i(\varphi_i(u))\big|G_\lambda(\varphi_i(u) - y_\Sigma - r\varepsilon\nu(y_\Sigma))\big|\sqrt{\det G_i(u)}\mathrm{d}u \leq C_i$$

holds. Let $i \in I$ be fixed. Note that $\det G_i$ is bounded on the compact set $\varphi_i^{-1}(\operatorname{supp} \chi_i)$, as the mapping $u \mapsto G_i(u)$ is continuous due to our assumptions in Σ. Moreover, as $\operatorname{supp} \chi_i$ is compact in $V_i \cap \Sigma$, there exists a constant $c_i > 0$ such that $\operatorname{dist}(\operatorname{supp} \chi_i, \partial V_i \cap \Sigma) \geq 2c_i$. We define

$$K_i := \{x_\Sigma \in \Sigma : \operatorname{dist}(x_\Sigma, \operatorname{supp} \chi_i) \leq c_i\}$$

and note that K_i is compact in $\Sigma \cap V_i$. We distinguish two cases, namely $y_\Sigma \in K_i$ and $y_\Sigma \notin K_i$.

If $y_\Sigma \notin K_i$, it holds for all $x_\Sigma \in \operatorname{supp} \chi_i$

$$|x_\Sigma - y_\Sigma - r\varepsilon\nu(y_\Sigma)| \geq |x_\Sigma - y_\Sigma| - \varepsilon|r| \geq \frac{c_i}{2},$$

if ε is chosen sufficiently small. Therefore, since G_λ has only a singularity at zero, cf. Theorem 5.9 and Proposition 5.6, $G_\lambda(x_\Sigma - y_\Sigma - r\varepsilon\nu(y_\Sigma))$ is uniformly bounded in $x_\Sigma \in \operatorname{supp} \chi_i$ independent from y_Σ and s and we find

$$\int_{U_i} \chi_i(\varphi_i(u))\big|G_\lambda(\varphi_i(u) - y_\Sigma - r\varepsilon\nu(y_\Sigma))\big|\sqrt{\det G_i(u)}\mathrm{d}u \leq C_{i,1}.$$

Let now $y_\Sigma \in K_i$ and set $v := \varphi_i^{-1}(y_\Sigma)$. Note that v belongs to the compact set $\varphi_i^{-1}(K_i)$. Then by Proposition 3.20 there exists a constant $\tilde{C}_i > 0$ such that

$$|\varphi_i(u) - y_\Sigma - r\varepsilon\nu(y_\Sigma)| = |\varphi_i(u) - \varphi_i(v) - r\varepsilon\nu(\varphi_i(v))| \geq \tilde{C}_i \left(|u-v|^2 + |\varepsilon r|^2\right)^{1/2} \geq \tilde{C}_i|u-v|$$

holds for all $u \in \varphi_i^{-1}(K_i)$.

We consider first the case $d = 2$. Here, the differentiable function G_λ has the asymptotics

$$G_\lambda(x) \sim -\frac{1}{2\pi}\ln\big(-i\sqrt{\lambda}|x|\big) \quad \text{for} \quad x \to 0$$

and

$$G_\lambda(x) \sim \sqrt{\frac{1}{-8\pi i\sqrt{\lambda}|x|}}e^{i\sqrt{\lambda}|x|}\left(1 + \mathcal{O}\left(\frac{1}{-i\sqrt{\lambda}|x|}\right)\right) \quad \text{for} \quad |x| \to \infty,$$

cf. Proposition 5.6. So, if $|\varphi_i(u) - \varphi_i(v) - \varepsilon r\nu(\varphi_i(v))| \geq \frac{1}{2}$, then $\big|G_\lambda(\varphi_i(u) - \varphi_i(v) - \varepsilon r\nu(\varphi_i(v)))\big|$ is uniformly bounded and we find

$$\begin{aligned}\int_{\left\{u\in U_i : |\varphi_i(u)-\varphi_i(v)-\varepsilon r\nu(\varphi_i(v))|\geq\frac{1}{2}\right\}} &\chi_i(\varphi_i(u))\big|G_\lambda(\varphi_i(u) - y_\Sigma - r\varepsilon\nu(y_\Sigma))\big|\sqrt{\det G_i(u)}\mathrm{d}u \\ &\leq c\int_{\left\{u\in\varphi_i^{-1}(K_i) : |\varphi_i(u)-\varphi_i(v)-\varepsilon r\nu(\varphi_i(v))|\geq\frac{1}{2}\right\}} \mathrm{d}u \leq c\Lambda_{d-1}\big(\varphi_i^{-1}(K_i)\big),\end{aligned}$$

where $\Lambda_{d-1}\big(\varphi_i^{-1}(K_i)\big)$ is the Lebesgue measure of $\varphi_i^{-1}(K_i)$. On the other hand, if $|\varphi_i(u) - \varphi_i(v) - \varepsilon r\nu(\varphi_i(v))| < \frac{1}{2}$, it holds

$$\big|G_\lambda(\varphi_i(u) - y_\Sigma - r\varepsilon\nu(y_\Sigma))\big| \leq c\big|\ln|\varphi_i(u) - y_\Sigma - r\varepsilon\nu(y_\Sigma)|\big| \leq c\big|\ln(\tilde{C}_i|u-v|)\big|$$

by the above considerations. Now, choose $R > 0$ such that the bounded set $\big\{u \in \varphi_i^{-1}(K_i) : |\varphi_i(u) - \varphi_i(v) - \varepsilon r\nu(\varphi_i(v))| < \frac{1}{2}\big\}$ is contained in $B(v, R)$. Then using a transformation to polar coordinates we get

$$\begin{aligned}
\int_{\big\{u\in U_i:|\varphi_i(u)-\varphi_i(v)-\varepsilon r\nu(\varphi_i(v))|<\frac{1}{2}\big\}} &\chi_i(\varphi_i(u))\big|G_\lambda(\varphi_i(u) - y_\Sigma - r\varepsilon\nu(y_\Sigma))\big|\sqrt{\det G_i(u)}\mathrm{d}u \\
&\leq c\int_{\big\{u\in\varphi_i^{-1}(K_i):|\varphi_i(u)-\varphi_i(v)-\varepsilon r\nu(\varphi_i(v))|<\frac{1}{2}\big\}} \big|\ln(\tilde{C}_i|u-v|)\big|\mathrm{d}u \\
&\leq c\int_{B(v,R)} \big|\ln(\tilde{C}_i|u-v|)\big|\mathrm{d}u = 2\pi c\int_0^R \big|\ln(\tilde{C}_i r)\big| r\mathrm{d}r \\
&\leq C_{i,2}
\end{aligned}$$

independent from y_Σ and s. Thus, we find

$$\begin{aligned}
&\int_{U_i} \chi_i(\varphi_i(u))\big|G_\lambda(\varphi_i(u) - y_\Sigma - r\varepsilon\nu(y_\Sigma))\big|\sqrt{\det G_i(u)}\mathrm{d}u \\
&= \int_{\big\{u\in U_i:|\varphi_i(u)-\varphi_i(v)-\varepsilon r\nu(\varphi_i(v))|\geq\frac{1}{2}\big\}} \chi_i(\varphi_i(u))\big|G_\lambda(\varphi_i(u) - y_\Sigma - r\varepsilon\nu(y_\Sigma))\big|\sqrt{\det G_i(u)}\mathrm{d}u \\
&\quad + \int_{\big\{u\in U_i:|\varphi_i(u)-\varphi_i(v)-\varepsilon r\nu(\varphi_i(v))|<\frac{1}{2}\big\}} \chi_i(\varphi_i(u))\big|G_\lambda(\varphi_i(u) - y_\Sigma - r\varepsilon\nu(y_\Sigma))\big|\sqrt{\det G_i(u)}\mathrm{d}u \\
&\leq C_{i,3}.
\end{aligned}$$

In the case $d \geq 3$, the asymptotics of G_λ are

$$G_\lambda(x) \sim \frac{1}{2(2\pi)^{d/2}}\Gamma\left(1 - \frac{d}{2}\right)\left(\frac{|x|}{-i\sqrt{\lambda}}\right)^{1-d/2}\left(\frac{-i\sqrt{\lambda}|x|}{2}\right)^{1-d/2} \qquad \text{for} \quad x \to 0$$

and

$$G_\lambda(x) \sim \frac{1}{(2\pi)^{d/2}}\left(\frac{|x|}{-i\sqrt{\lambda}}\right)^{1-d/2}\sqrt{\frac{\pi}{-2i\sqrt{\lambda}|x|}}e^{i\sqrt{\lambda}|x|}\left(1 + \mathcal{O}\left(\frac{1}{-i\sqrt{\lambda}|x|}\right)\right) \quad \text{for} \quad |x| \to \infty,$$

see Proposition 5.6. So, if $|\varphi_i(u) - \varphi_i(v) - \varepsilon r\nu(\varphi_i(v))| \geq \frac{1}{2}$, then $\big|G_\lambda(\varphi_i(u) - \varphi_i(v) - \varepsilon r\nu(\varphi_i(v)))\big|$ is uniformly bounded and we find

$$\begin{aligned}
\int_{\big\{u\in U_i:|\varphi_i(u)-\varphi_i(v)-\varepsilon r\nu(\varphi_i(v))|\geq\frac{1}{2}\big\}} &\chi_i(\varphi_i(u))\big|G_\lambda(\varphi_i(u) - y_\Sigma - r\varepsilon\nu(y_\Sigma))\big|\sqrt{\det G_i(u)}\mathrm{d}u \\
&\leq c\int_{\big\{u\in\varphi_i^{-1}(K_i):|\varphi_i(u)-\varphi_i(v)-\varepsilon r\nu(\varphi_i(v))|\geq\frac{1}{2}\big\}} \mathrm{d}u \leq c\Lambda_{d-1}\big(\varphi_i^{-1}(K_i)\big),
\end{aligned}$$

where again $\Lambda_{d-1}\big(\varphi_i^{-1}(K_i)\big)$ is the Lebesgue measure of the compact set $\varphi_i^{-1}(K_i)$. On the other hand, if $|\varphi_i(u) - \varphi_i(v) - \varepsilon r\nu(\varphi_i(v))| < \frac{1}{2}$, it holds

$$\big|G_\lambda(\varphi_i(u) - y_\Sigma - r\varepsilon\nu(y_\Sigma))\big| \leq c|\varphi_i(u) - y_\Sigma - r\varepsilon\nu(y_\Sigma)|^{1-d} \leq c\big(\tilde{C}_i|u-v|\big)^{1-d}$$

and thus, using a transformation to spherical coordinates, we find

$$\begin{aligned}
&\int_{\{u\in U_i:|\varphi_i(u)-\varphi_i(v)-\varepsilon r\nu(\varphi_i(v))|<\frac{1}{2}\}}\chi_i(\varphi_i(u))\big|G_\lambda(\varphi_i(u)-y_\Sigma-r\varepsilon\nu(y_\Sigma))\big|\sqrt{\det G_i(u)}\mathrm{d}u\\
&\qquad\qquad\leq c\int_{\left\{u\in\varphi_i^{-1}(K_i):|\varphi_i(u)-\varphi_i(v)-\varepsilon r\nu(\varphi_i(v))|<\frac{1}{2}\right\}}\big(\tilde{C}_i|u-v|\big)^{1-d}\mathrm{d}u\\
&\qquad\qquad\leq\tilde{c}\int_0^R r^{1-d}\cdot r^{d-1}\mathrm{d}r\leq C_{i,4}
\end{aligned}$$

independent from y_Σ and s, where $R>0$ is chosen in such a way that $\big\{u\in\varphi_i^{-1}(K_i):|\varphi_i(u)-\varphi_i(v)-\varepsilon r\nu(\varphi_i(v))|<\frac{1}{2}\big\}\subset B(v,R)$. Thus, we find

$$\begin{aligned}
&\int_{U_i}\chi_i(\varphi_i(u))\big|G_\lambda(\varphi_i(u)-y_\Sigma-r\varepsilon\nu(y_\Sigma))\big|\sqrt{\det G_i(u)}\mathrm{d}u\\
&=\int_{\left\{u\in U_i:|\varphi_i(u)-\varphi_i(v)-\varepsilon r\nu(\varphi_i(v))|\geq\frac{1}{2}\right\}}\chi_i(\varphi_i(u))\big|G_\lambda(\varphi_i(u)-y_\Sigma-r\varepsilon\nu(y_\Sigma))\big|\sqrt{\det G_i(u)}\mathrm{d}u\\
&\quad+\int_{\left\{u\in U_i:|\varphi_i(u)-\varphi_i(v)-\varepsilon r\nu(\varphi_i(v))|<\frac{1}{2}\right\}}\chi_i(\varphi_i(u))\big|G_\lambda(\varphi_i(u)-y_\Sigma-r\varepsilon\nu(y_\Sigma))\big|\sqrt{\det G_i(u)}\mathrm{d}u\\
&\leq C_{i,5}.
\end{aligned}$$

Hence, we have shown in all cases

$$\int_{U_i}\chi_i(\varphi_i(u))\big|G_\lambda(\varphi_i(u)-y_\Sigma-r\varepsilon\nu(y_\Sigma))\big|\sqrt{\det G_i(u)}\mathrm{d}u\leq\max\{C_{i,1},C_{i,3},C_{i,5}\}=:C_i$$

and thus, the statement of this proposition follows from the considerations at the beginning of this proof. □

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