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Model-based control of a switching linear multibody system

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Zusammenfassung

Die Automatisierung von Gangschaltungen in komplexen Antriebsstrangtopologien stellt die zentrale Aufgabenstellung dar, die dieser Arbeit zugrunde liegt. Durch das Schließen beziehungsweise Öffnen von Kupplungen muss ein Antriebsstrang als schaltendes Mehrkörpersystem betrachtet werden. Die Umschaltungen sind aus mechanischer Sicht durch das Einbringen oder Entfernen von holonomen Zwangsbedingungen beschreibbar.

Die vorliegende Arbeit gliedert sich in einen theoretischen und einen praktischen Teil. Der theoretische Teil widmet sich der Modellierung von Mehrkörpersystemen, unter der Einbringung von holonomen Zwangsbedingungen. Es werden die Auswirkungen dieser eingebrachten Zwangsbedingungen auf Zustandstransformationen, beziehungsweise auf die Erhaltung der Stabilität und Optimalität einer entworfenen Zustandsregelung oder Zustandsbeobachtung untersucht. Das Ziel dieser Untersuchungen ist der Entwurf einer optimalen Zustandsregelung oder eines optimalen Zustandsbeobachters für das Mehrkörpersystem, unabhängig von zusätzlichen mechanischen Zwangsbedingungen.

Im praktischen Teil wird zuerst die theoretische Vorarbeit des ersten Teiles verwendet, um eine systematische Modellierung für allgemeine Antriebsstrangtopologien zu formulieren. Im Speziellen beschäftigt sich dieser zweite Teil mit der Modellierung und modellbasierten Regelung eines hybriden Antriebsstranges. Es wird eine Regelungsstrategie bestehend aus einer flachheitsbasierten Antriebsmomentensteuerung und einer LQ-Regelung entworfen. Zusätzlich wird eine modellfreie Kupplungsmomentensteuerung verwendet. Diese Kombination ermöglicht die Regelung von Gangschaltungen unter Einhaltung einer geforderten Fahrzeuggeschwindigkeit. Die entworfene Steuerung und Regelung nützt sowohl die Vorteile der vorgestellten Modellierungsart, als auch die mechanischen Eigenheiten des betrachteten Antriebsstranges.

Schlagwörter: beschränkte Mehrkörpersysteme, Antriebsstrangmodellierung, flachheitsbasierte Vorsteuerung

Abstract

The central task of this work is to automate gear shifts in complex transmission drivetrain topologies. Due to the opening and closing of clutches drivetrains have to be regarded as switching multibody systems. From the mechanical point of view the switching corresponds to adding or releasing holonomic constraints.

The presented work is divided into a theoretical and a practical part. The theoretical part is dedicated to the linear modeling of multibody systems with respect to additional holonomic constraints. The impact of these additional constraints on state transformations and the conservation of stability and optimality of designed state controllers and state observers is investigated. These investigations target on designing an optimal state control or an optimal state observer for the multibody system regardless of any additional mechanical constraint.

In the practical part in a first step the work of the first part is used to formulate a systematic modeling approach for general drivetrain topologies. In particular the second part covers modeling and model-based control of gear shifts in a hybrid drivetrain topology. The proposed control strategy consists of a flatness-based feedforward propulsion torque control and a LQR feedback control. Additionally a model-free clutch torque actuation is used. This combination enables control of gear shifts while tracking a required vehicle speed trajectory. Feedforward and feedback control take advantage of both the presented modeling approach and the mechanical peculiarities of the considered drivetrain topology.

Keywords: constrained multibody systems, drivetrain modeling, flatness-based feedforward control

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Contents

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Introduction and overview

One of the most challenging fact in automating gear shifts for complex transmission drivetrain topologies is the consideration of clutches. On the one hand they can be used to dissipate energy in slipping state, on the other hand they reduce to simple rigid shafts when they are locked. Consequently to achieve the required objective first of all the impact of a locked clutch on the mathematical model of a drivetrain has to be investigated. According to the necessary condition, that the differential angular velocity between the plates of a locked clutch is zero, its impact is equal to the impact of adding a linear holonomic constraint to the underlying mechanical problem.

Therefore chapter 2 deals with the issue from the mechanical point of view, in particular with constrained linear multibody systems. A specific type of system reduction is introduced and in a next step this idea is applied to state space notation, which is conventionally used in control theory. Common state transformation with respect to additional mechanical constraints is considered in chapter 2.6.3. Chapter 3 contains investigations on the stability of linear multibody systems with respect to linear constraints. Furthermore this issue is extended to the problem of stability of linear state control or observation designed for the considered systems. The end of the general mechanical resp. mathematical investigations is marked by chapter 4. It deals with the conservation of some kind of optimality of linear state feedback and state observation designed on the considered systems. The motivation behind these investigations concerning stability and optimality is to design a linear state feedback for optimal or at least stable control of a multibody system regardless of any additional linear constraints. In order to meet the demanded task of investigating the impact of a locked clutch on the mathematical model of a drivetrain, the general approach of modeling linear multibody systems under additional constraints is applied to general drivetrain topologies in chapter 5.

To facilitate the overview on the theoretical part of this work figure 1.1 illustratively summarizes the treated subjects and their connections.

The practical part of this work is the actual application of this approach to a hybrid automatic transmission drivetrain. The advantages of the presented modeling approach and the mechanical peculiarities of the considered drivetrain are used to design a control system automating gear shifts while tracking a required vehicle speed trajectory. The control strategy consists of a flatness-based feedforward propulsion torque control with LQR feedback control and a model-free clutch torque actuation. Functionality of the control system is shown based on several simulation experiments in chapter 6.5. Finally several approaches to improve the control performance and to generalize the presented approaches give an outlook for future work (see chapter 7).

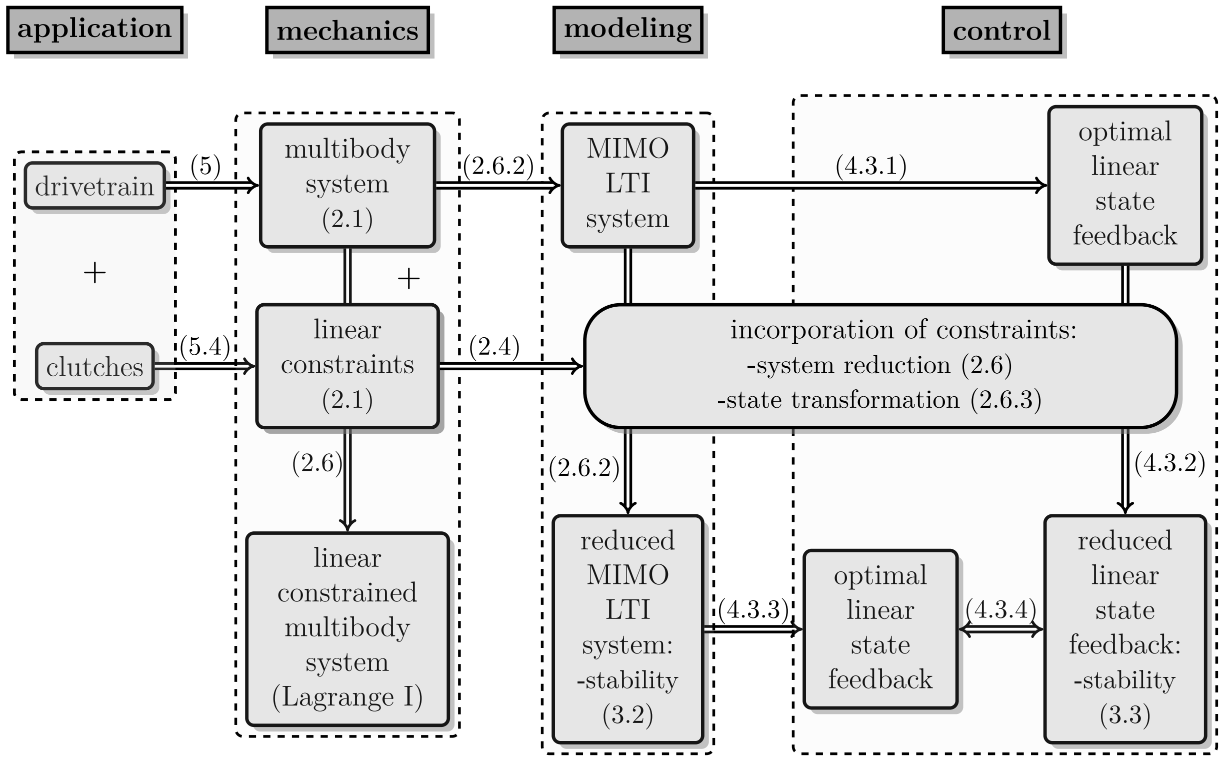

Figure 1.1: Graphical overview on the subjects of the theoretical part

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Constrained linear multibody systems - system reduction

The initial situation is constituted by a mechanical multibody systems consisting of N lumped inertias and k holonomic constraints. This chapter shows a standard approach to achieve a minimal set of linear differential equations describing the motion of inertias. It is similarly documented in [1], [2], [3] and [4]. Based on Newton's laws of motion, the additional compliance of the holonomic constraints is ensured by constraining forces with unknown magnitudes in the Lagrange equation of the first kind. The directions of the constraining forces are stated in d'Alembert's principle. Elimination of the Lagrangian multipliers and transformation to generalized coordinates reduces the system of differential equations to a minimal system. Therefore in the content of this work this approach will be called system reduction. This system reduction focuses on a compact notation for the mathematical description of the considered physical problems to facilitate the later application of common control approaches. Subsequently follows the restriction to linear constraints, with respect to positions, and the formulation of the system reduction in state space notation. Finally this chapter investigates application of common state transformation in consistency to additional linear constraints. Thereby the correct transformation of the mass matrix plays a significant role.

2.1 Newton's laws of motion

The motion of inertias under influence of acting forces is basically defined in Newton's laws of motion that provide foundation for classical mechanics. In particular Newton's first and second law of motion state the existence of at least one frame of reference, called inertial frame of reference, in which net force $\mathbf{F}_i$ acting on inertia m_i at position $\mathbf{r}_i$ equals the rate of change of its momentum $\mathbf{p}_i$:

$$\mathbf{F}_i = \dot{\mathbf{p}}_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(m_i\dot{\mathbf{r}}_i\right) = m_I\ddot{\mathbf{r}}_i \quad | \quad i = 1, \ldots, N; \quad m_i \in \mathbb{R} \tag{2.1}$$

In the considered problems every frame of reference fixed in earth shall be considered to be an inertial frame of reference. Usage of Cartesian coordinates in $\mathbb{R}^3$ defines the vectors $\mathbf{F}_i$, $\mathbf{r}_i$ and $\mathbf{p}_i$ as follows:

$$\mathbf{F}_i = \begin{bmatrix} F_{\mathrm{x}i} \\ F_{\mathrm{y}i} \\ F_{\mathrm{z}i} \end{bmatrix}, \quad \mathbf{r}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}, \quad \mathbf{p}_i = \begin{bmatrix} p_{\mathrm{x}i} \\ p_{\mathrm{y}i} \\ p_{\mathrm{z}i} \end{bmatrix} \tag{2.2}$$

Compact matrix notation eventuates in a system of $3N$ differential equations represented in equation 2.3.

$$\begin{aligned} &\mathbf{M}_i = \begin{bmatrix} m_i & m_i & m_i \end{bmatrix}^T \quad | \quad i = 1, \dots, N \\ &\underbrace{\mathrm{diag}\left(\begin{bmatrix} \mathbf{M}_1^T & \mathbf{M}_2^T & \dots & \mathbf{M}_N^T \end{bmatrix}\right)}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{\mathbf{r}}_1^T & \ddot{\mathbf{r}}_2^T & \dots & \ddot{\mathbf{r}}_N^T \end{bmatrix}^T}_{\ddot{\mathbf{r}}} = \underbrace{\begin{bmatrix} \mathbf{F}_1^T & \mathbf{F}_2^T & \dots & \mathbf{F}_N^T \end{bmatrix}^T}_{\mathbf{F}} \end{aligned} \tag{2.3}$$

$$\mathbf{M}\ddot{\mathbf{r}} = \mathbf{F} \tag{2.4}$$

Considering linear multibody systems, net force can be split up into linear velocity dependent, linear position dependent and coordinate independent forces:

$$\mathbf{M}\ddot{\mathbf{r}} = -\mathbf{D}\dot{\mathbf{r}} - \mathbf{K}\mathbf{r} + \mathbf{F} \tag{2.5}$$

In contrast to equation 2.4, here and within this work vector $\mathbf{F}$ represents all external forces that do not depend on coordinates $\mathbf{r}$. Diagonal matrix $\mathbf{M}$ is called mass matrix or inertia matrix, $\mathbf{D}$ damping matrix and $\mathbf{K}$ stiffness matrix. Due to Newton's third law of motion, stating that an inertias reactive force is equal in magnitude and opposite in direction to the force exerted by another inertia, $\mathbf{D}$ and $\mathbf{K}$ have to be symmetric matrices. Further due to defined positivity of energy term $\mathbf{M}$ is a positive definite matrix and $\mathbf{D}$ and $\mathbf{K}$ are positive semidefinite matrices. Restriction to real problems also ensures $\mathbf{D}$ being a positive definite matrix, since there exists no undamped system.

$$\begin{aligned} \mathbf{M} &= \mathbf{M}^T > 0 \\ \mathbf{D} &= \mathbf{D}^T > 0 \\ \mathbf{K} &= \mathbf{K}^T \geqslant 0 \end{aligned} \tag{2.6}$$

Expanding the unconstrained system by $k < 3N$ holonomic constraints $\mathbf{f}(\mathbf{r}, t) = 0$ results in a so called differential-algebraic equation (DAE) system consisting of $3N$ differential and k algebraic equations:

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{r}} &= \mathbf{F} + \mathbf{Z} \\ \mathbf{f}(\mathbf{r}, t) &= \begin{bmatrix} f_1(\mathbf{r}, t) \\ \vdots \\ f_k(\mathbf{r}, t) \end{bmatrix} = \mathbf{0} \end{aligned} \tag{2.7}$$

$\mathbf{Z}$ denotes the *unknown* constraining forces acting on inertias to ensure constraints are satisfied:

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Z}_1^T & \dots & \mathbf{Z}_N^T \end{bmatrix}^T = \begin{bmatrix} Z_{1\mathrm{x}} & Z_{1\mathrm{y}} & Z_{1\mathrm{z}} & \dots & Z_{N\mathrm{x}} & Z_{N\mathrm{y}} & Z_{N\mathrm{z}} \end{bmatrix}^T \tag{2.8}$$

Adding constraints has two significant effects on the multibody system:

1. loss of k degrees of freedom,
2. the resulting system is under-determined: $3N \cdot 2$ variables ($\mathbf{r}$ and $\mathbf{Z}$), $3N + k$ equations

Consequently formulation 2.7 does not use all available information respecting constraining forces.

2.2 D‘Alembert’s principle

Additional information concerning the direction of constraining forces can be received by introducing term of **virtual displacement**.

Definition 1.

Virtual displacement $\delta\mathbf{r}$ *of a system is an arbitrary, infinitesimal, instant (*$\delta t = 0$*) displacement of the inertias satisfying the constraints.*

D‘Alembert principle postulates that constraining forces do not perform virtual work:

$$\mathbf{Z}^T \delta\mathbf{r} = 0 \tag{2.9}$$

This axiom is an autonomous axiom in mechanics and does not succeed Newton’s laws of motion.

Notation of constraints in differential form and considering virtual displacement ($d\mathbf{r} = \delta\mathbf{r}$, $dt = \delta t = 0$) achieves:

$$\begin{aligned} \mathrm{d}\mathbf{f}(\mathbf{r}, t) &= \frac{\partial \mathbf{f}}{\partial \mathbf{r}} \mathrm{d}\mathbf{r} + \frac{\partial \mathbf{f}}{\partial t} \mathrm{d}t = \mathbf{0} \\ &= \frac{\partial \mathbf{f}}{\partial \mathbf{r}} \delta\mathbf{r} + \frac{\partial \mathbf{f}}{\partial t} \delta t = \frac{\partial \mathbf{f}}{\partial \mathbf{r}} \delta\mathbf{r} = \mathbf{J}_\mathrm{f} \delta\mathbf{r} = \mathbf{0} \end{aligned} \tag{2.10}$$

$\mathbf{J}_\mathrm{f}$ is the constraint’s Jacobian matrix (also called constraining matrix):

$$\mathbf{J}_\mathrm{f} = \frac{\partial \mathbf{f}}{\partial \mathbf{r}} = \begin{bmatrix} \frac{\partial f_1}{\partial \mathbf{r}} \\ \vdots \\ \frac{\partial f_k}{\partial \mathbf{r}} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial \mathbf{r}_1} & \dots & \frac{\partial f_1}{\partial \mathbf{r}_N} \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial \mathbf{r}_1} & \dots & \frac{\partial f_k}{\partial \mathbf{r}_N} \end{bmatrix} \tag{2.11}$$

Equations 2.9 and 2.10 demand a *necessary* and *sufficient* condition on direction of constraining forces summarized in Lemma 1.

Lemma 1.

$$\mathbf{Z} = \mathbf{J}_\mathrm{f}^T \boldsymbol{\lambda} \tag{2.12}$$

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_k \end{bmatrix}^T$$

$\boldsymbol{\lambda}$ *are so called Lagrangian multipliers, which define the unknown magnitudes of the constraining forces.*

Proof.

- Equation 2.10: virtual displacement is element of the null space [1] of $\mathbf{J}_\mathrm{f}$:

$$\delta\mathbf{r} \in \mathcal{V} = \mathcal{N}\left(\mathbf{J}_\mathrm{f}\right) \tag{2.13}$$

- Equation 2.9: constraining force is element of the orthogonal complement of virtual displacement $\delta\mathbf{r}$:

$$\mathbf{Z} \in \mathcal{Z} = \mathcal{V}^\perp \tag{2.14}$$

- A transformation's null space is equal to the orthogonal complement of the transposed transformation (see [5]):

$$\mathcal{N}\left(\mathbf{J}_\mathrm{f}\right) = \left(\mathbf{J}_\mathrm{f}^T\right)^\perp \tag{2.15}$$

Consequently the following holds:

$$\mathcal{Z} = \mathcal{V}^\perp = \left(\mathcal{N}\left(\mathbf{J}_\mathrm{f}\right)\right)^\perp = \left(\left(\mathbf{J}_\mathrm{f}^T\right)^\perp\right)^\perp = \mathbf{J}_\mathrm{f}^T \tag{2.16}$$

In consequence constraining forces are represented by:

$$\mathbf{Z} = \mathbf{J}_\mathrm{f}^T \boldsymbol{\lambda} \tag{2.17}$$

□

2.3 Lagrange equations of the first kind

Usage of equation 2.12, containing information about direction of constraining forces, in equation 2.7 leads to Lagrange equations of the first kind:

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{r}} &= \mathbf{F} + \mathbf{J}_\mathrm{f}^T \boldsymbol{\lambda} \\ \mathbf{f}(\mathbf{r}, t) &= \mathbf{0} \end{aligned} \tag{2.18}$$

The system is still differential-algebraic but now determined: $3N$ differential and k algebraic equations resp. $3N + k$ variables ($\mathbf{r}$ and $\boldsymbol{\lambda}$).

Twice done differentiation of holonomic constraints achieves a system of differential equation consisting of $3N + k$ ordinary differential equations second order.

$$\begin{aligned} &\mathbf{f}(\mathbf{r}, t) = \mathbf{0} \quad \Big/ \frac{\mathrm{d}}{\mathrm{d}t} \\ &\frac{\partial \mathbf{f}}{\partial \mathbf{r}}\dot{\mathbf{r}} + \frac{\partial \mathbf{f}}{\partial t} = \mathbf{J}_\mathrm{f}\dot{\mathbf{r}} + \frac{\partial \mathbf{f}}{\partial t} = \mathbf{0} \quad \Big/ \frac{\mathrm{d}}{\mathrm{d}t} \\ &\mathbf{J}_\mathrm{f}\ddot{\mathbf{r}} + \dot{\mathbf{J}}_\mathrm{f}\dot{\mathbf{r}} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{f}}{\partial t}\right) = \mathbf{0} \end{aligned} \tag{2.19}$$

[1] The null space of a matrix is the kernel of the linear map defined by the matrix.

Restriction to *scleronomic* constraints reduces equation 2.19, due to the fact that there is no explicit time dependency in constraints:

$$\mathbf{J}_\mathrm{f}\ddot{\mathbf{r}} + \dot{\mathbf{J}}_\mathrm{f}\dot{\mathbf{r}} = \mathbf{0} \tag{2.20}$$

Combination of equations 2.18 and 2.20 completes the system of differential equations:

$$\begin{bmatrix} \mathbf{M} & \mathbf{J}_\mathrm{f}^T \\ \mathbf{J}_\mathrm{f} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ -\dot{\mathbf{J}}_f\dot{\mathbf{r}} \end{bmatrix} \tag{2.21}$$

Main disadvantage of this formulation considering positions $\mathbf{r}$ is the need of $3N + k$ equations to describe a system of $3N - k$ mechanical degrees of freedom. To get rid of this drawback the term of generalized coordinates is introduced.

2.4 Generalized coordinates

Provided a $(k \times 3N)$-matrix $\mathbf{J}_\mathrm{f}$ does not contain linear dependent rows, i. e. there are no redundant constraints, dimension of null space of $\mathbf{J}_\mathrm{f}$ satisfies:

$$\dim\left(\mathcal{N}\left(\mathbf{J}_\mathrm{f}\right)\right) = \mathrm{def}\left(\mathbf{J}_\mathrm{f}\right) = 3N - k =: f \tag{2.22}$$

In words: Dimension of $\mathbf{J}_\mathrm{f}$'s null space, i. e. the defect of $\mathbf{J}_\mathrm{f}$, equals the number of mechanical degrees of freedom f.

If the columns of an introduced $(3N \times f)$-matrix $\mathbf{J}_\mathrm{r}$ are a basis of $\mathbf{J}_\mathrm{f}$'s null space,

$$\mathbf{J}_\mathrm{f}\underbrace{\mathbf{J}_\mathrm{r}\delta\mathbf{q}}_{\delta\bar{\mathbf{r}}} = \mathbf{0} \tag{2.23}$$

holds for arbitrary $\delta\mathbf{q}$, because of

$$\mathbf{J}_\mathrm{f}\mathbf{J}_\mathrm{r} = \mathbf{0}\ . \tag{2.24}$$

Coordinates $\mathbf{q}$ appearing in virtual displacement $\delta\mathbf{q}$ are called generalized coordinates:

$$\mathbf{q} = \begin{bmatrix} q_1 & q_2 & \dots & q_f \end{bmatrix}^T \tag{2.25}$$

The number of generalized coordinates equals number of mechanical degrees of freedom f. It is unique and a system parameter. Hence generalized coordinates are a minimal set of coordinates describing motion of inertias. However the choice of generalized coordinates is not unique due to the fact that equation 2.23 only offers a *necessary* condition.

But equation 2.23 insures that, if $\mathbf{q}$ is a set of generalized coordinates,

$$\mathbf{J}_\mathrm{r} = \frac{\partial\mathbf{r}}{\partial\mathbf{q}} \tag{2.26}$$

is a basis of $\mathbf{J}_\mathrm{f}$'s null space, because

$$\mathbf{J}_\mathrm{f}\underbrace{\frac{\partial\mathbf{r}}{\partial\mathbf{q}}\delta\mathbf{q}}_{\delta\mathbf{r}} = \mathbf{0}\ . \tag{2.27}$$

The approach, determination of a set of $\mathbf{q}$ and calculation of respecting $\mathbf{J}_\mathrm{r}$ is important for practical use, because it allows a mechanical meaningful interpretation of generalized coordinates and further the reduced set of differential equations.

2.4.1 Transformation to generalized coordinates

In order to achieve a minimal set of differential equations transformation into generalized coordinates is required. Relationship between coordinates $\mathbf{r}$ and generalized coordinates $\mathbf{q}$ is given by:

$$\begin{aligned}\mathbf{r} &= \mathbf{r}(\mathbf{q}, t)\\ \dot{\mathbf{r}} &= \frac{\partial \mathbf{r}}{\partial \mathbf{q}}\dot{\mathbf{q}} + \underbrace{\frac{\partial \mathbf{r}}{\partial t}}_{=0} = \mathbf{J}_{\mathrm{r}}\dot{\mathbf{q}}\\ \ddot{\mathbf{r}} &= \mathbf{J}_{\mathrm{r}}\ddot{\mathbf{q}} + \dot{\mathbf{J}}_{\mathrm{r}}\dot{\mathbf{q}}\end{aligned} \tag{2.28}$$

Second line in equation 2.28 provides that mechanical system is *scleronomous*, meaning there is actually no explicit time dependency in $\mathbf{r}$. Inserting equation 2.28 to Lagrange equation of the first kind (2.18) transforms the system into generalized coordinates:

$$\mathbf{M}\mathbf{J}_{\mathrm{r}}\ddot{\mathbf{q}} = \mathbf{F} + \mathbf{J}_{\mathrm{f}}^T\boldsymbol{\lambda} - \mathbf{M}\dot{\mathbf{J}}_{\mathrm{r}}\dot{\mathbf{q}} \tag{2.29}$$

2.5 Elimination of Lagrangian multipliers

Multiplying equation 2.29 by $\mathbf{J}_{\mathrm{r}}^T$ delivers

$$\mathbf{J}_{\mathrm{r}}^T\mathbf{M}\mathbf{J}_{\mathrm{r}}\ddot{\mathbf{q}} = \mathbf{J}_{\mathrm{r}}^T\mathbf{F} + \underbrace{\mathbf{J}_{\mathrm{r}}^T\mathbf{J}_{\mathrm{f}}^T}_{=\mathbf{0}}\boldsymbol{\lambda} - \mathbf{J}_{\mathrm{r}}^T\mathbf{M}\dot{\mathbf{J}}_{\mathrm{r}}\dot{\mathbf{q}} \tag{2.30}$$

$$\underbrace{\mathbf{J}_{\mathrm{r}}^T\mathbf{M}\mathbf{J}_{\mathrm{r}}}_{\tilde{\mathbf{M}}}\ddot{\mathbf{q}} = \underbrace{\mathbf{J}_{\mathrm{r}}^T\mathbf{F}}_{\mathbf{Q}} - \underbrace{\mathbf{J}_{\mathrm{r}}^T\mathbf{M}\dot{\mathbf{J}}_{\mathrm{r}}\dot{\mathbf{q}}}_{\mathbf{b}}$$

Vector $\mathbf{Q}$ represents generalized forces acting on the system. Due to precondition to $\mathbf{J}_{\mathrm{r}}$ multiplication eliminates Lagrangian multipliers. The remaining system consists of $3N - k$ differential equations second order resp. $3N - k$ variables.

2.6 Linear constrained linear multibody systems

In the later important case of *holonomic*, *scleronomic* and *linear resp.* $\mathbf{r}$ constraints, corresponding Jacobian matrix $\mathbf{J}_{\mathrm{f}}$ is constant and therefore every basis of its null space is constant ($\Rightarrow \dot{\mathbf{J}}_{\mathrm{r}} = \mathbf{0}$). Equation 2.30 reduces to:

$$\begin{aligned}\mathbf{J}_{\mathrm{r}}^T\mathbf{M}\mathbf{J}_{\mathrm{r}}\ddot{\mathbf{q}} &= \mathbf{J}_{\mathrm{r}}^T\mathbf{F}\\ \tilde{\mathbf{M}}\ddot{\mathbf{q}} &= \mathbf{Q}\end{aligned} \tag{2.31}$$

Assuming linearity of the considered multibody system (see equation 2.5) achieves:

$$\underbrace{\mathbf{J}_{\mathrm{r}}^T\mathbf{M}\mathbf{J}_{\mathrm{r}}}_{\tilde{\mathbf{M}}}\ddot{\mathbf{q}} = -\underbrace{\mathbf{J}_{\mathrm{r}}^T\mathbf{D}\mathbf{J}_{\mathrm{r}}}_{\tilde{\mathbf{D}}}\dot{\mathbf{q}} - \underbrace{\mathbf{J}_{\mathrm{r}}^T\mathbf{K}\mathbf{J}_{\mathrm{r}}}_{\tilde{\mathbf{K}}}\mathbf{q} + \underbrace{\mathbf{J}_{\mathrm{r}}^T\mathbf{F}}_{\mathbf{Q}} \tag{2.32}$$

Definition 2. ***System Reduction***
The process of reducing a system of differential equations describing motion of a linear multibody system to a minimal differential equation system in the case of added linear constraints shall be defined as ***system reduction*** *for the scope of this work. The term is a shortcut for the transformation to generalized coordinates and the elimination of Lagrangian multipliers. The resulting system is further called* ***reduced system****.*

It's an important fact that $\mathbf{J}_\mathrm{f} = \mathrm{const.}$ implies that choosing f specific components of $\mathbf{r}$ obtains a set of generalized coordinates $\mathbf{q}$. This choice is not unique but it exists. It's advantage is the conservation of the physical interpretability of the coordinates.

To apply this approach, it is sufficient to find a basis of $\mathbf{J}_\mathrm{f}$'s null space containing all rows of f-dimensional identity matrix, insuring $\mathbf{q}$ contains f components of $\mathbf{r}$. Following section presents a method to generate a matrix $\mathbf{J}_\mathrm{r}$ based on this consideration. Central point is the usage $\mathbf{J}_\mathrm{f}$'s reduced row echelon form $\bar{\mathbf{J}}_\mathrm{f}$ to generate a requested basis of $\mathbf{J}_\mathrm{f}$'s null space.

Definition 3. ***Reduced row echelon form (see [5])***
A matrix has reduced row echelon form, if it satisfies following two conditions:

- *The leading coefficient (a row's first nonzero element), called pivot, is 1 in all rows and is the only nonzero coefficient in its column.*
- *The pivot's column index, called pivot index* $\mathbf{p}_\mathrm{i}$*, has to increase strictly for increasing row index (* $\rightarrow$ *echelon form).*

Remark 1.

- *Reduced row echelon form of a matrix does exist, is unique and can be achieved by a finite sequence of elementary operations (Gauss elimination, see [5]).*
- $\mathrm{rank}(\mathbf{J}_\mathrm{f}) = n < k$ *leads to* $k - n$ *zero rows at the bottom of* $\bar{\mathbf{J}}_\mathrm{f}$*.*
- *Provided* $\mathrm{rank}(\mathbf{J}_\mathrm{f}) = k$*, all columns of* k*-dimensional identity matrix appear in* $\bar{\mathbf{J}}_\mathrm{f}$*. Pivot indices* $\mathbf{p}_\mathrm{i}$ *declare position of respecting columns. Therefore* $\bar{\mathbf{J}}_\mathrm{f}$ *is a fusion of* k*-dimensional identity matrix* $\mathbf{I}_k$ *and* f *remaining columns* $\tilde{\mathbf{J}}_\mathrm{f}$ *.*

Using reduced row echelon form of a matrix offers a simple possibility to find a basis of its null space: Starting with f-dimensional identity matrix $\mathbf{I}_f$ the rows of $\tilde{\mathbf{J}}_\mathrm{f}$ have to be inserted with switched signs. Position to insert is stated in the associated pivot index $\mathbf{p}_\mathrm{i}$.

$$\mathbf{J}_\mathrm{r} = \mathbf{I}_{f,\mathrm{piv}} - \tilde{\mathbf{J}}_{\mathrm{f},f,\mathrm{piv}} \tag{2.33}$$

$\mathbf{I}_{f,\mathrm{piv}}$ contains additional zero rows on positions stated by $\bar{\mathbf{J}}_\mathrm{f}$'s pivot indices $\mathbf{p}_\mathrm{i}$. $\tilde{\mathbf{J}}_{\mathrm{f},f,\mathrm{piv}}$ has only zero entries except of $\tilde{\mathbf{J}}_\mathrm{f}$'s columns as rows on positions of resp. pivot indices.

Due to this construction method of $\mathbf{J}_\mathrm{r}$, transformation to generalized coordinates has two properties worth mentioning:

Remark 2.

- *Pivot indices* $\mathbf{p}_\mathrm{i}$ *of matrix* $\bar{\mathbf{J}}_\mathrm{f}$ *declares the components of* $\mathbf{r}$ *that do not appear in the set of generalized coordinates* $\mathbf{q}$*.*
- $\mathbf{q}$ *is the minimal set of components of* $\mathbf{r}$ *with the highest possible row indices.*

2.6.1 Example

$$\mathbf{f}(\mathbf{r}) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{bmatrix} \mathbf{r} \quad \Rightarrow \quad \mathbf{J}_\mathrm{f} = \frac{\partial \mathbf{f}}{\partial \mathbf{r}} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{bmatrix}$$

$$n = 5, k = 2 \quad \Rightarrow \quad f = n - k = 3$$

$$\bar{\mathbf{J}}_\mathrm{f} = \left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \quad \begin{matrix} -2 & 0 & 0 \\ 2 & 0 & 0 \end{matrix} \right] = \left[\mathbf{I}_k \ \tilde{\mathbf{J}}_\mathrm{f} \right] \quad \Rightarrow \quad \mathbf{p}_\mathrm{i} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\mathbf{J}_\mathrm{r} = \mathbf{I}_{3,\mathrm{piv}} - \tilde{\mathbf{J}}_{\mathrm{f},3,\mathrm{piv}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{bmatrix} \quad \Rightarrow \quad \mathbf{q} = \begin{bmatrix} r_3 \\ r_4 \\ r_5 \end{bmatrix}$$

2.6.2 State space model

In general a system of $3N$ linear differential equations second order can be transformed in a system of $2 \cdot 3N$ linear differential equations first order. This transition to so called state space is useful in order to apply common methods of control theory.
Application on equation 2.5 and definitions $\mathbf{x} = \begin{bmatrix} \mathbf{r}^T & \dot{\mathbf{r}}^T \end{bmatrix}$ and $\mathbf{u} = \mathbf{F}$ leads to:

$$\mathbf{M}\ddot{\mathbf{r}} = -\mathbf{D}\dot{\mathbf{r}} - \mathbf{K}\mathbf{r} + \mathbf{F} \tag{2.34}$$

$$\Updownarrow$$

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{r}} \\ \ddot{\mathbf{r}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \dot{\mathbf{r}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \mathbf{F} \quad \Rightarrow \quad \bar{\mathbf{M}}\dot{\mathbf{x}} = \bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{B}}\mathbf{u} \tag{2.35}$$

$$\begin{bmatrix} \dot{\mathbf{r}} \\ \ddot{\mathbf{r}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \dot{\mathbf{r}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \end{bmatrix} \mathbf{F} \quad \Rightarrow \quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{2.36}$$

Addition of linear constraints $\mathbf{f}(\mathbf{x})$ is considered by applying system reduction analogously to equation 2.31. Therefore basis $\mathbf{J}_\mathrm{x}$ to $\mathbf{J}_\mathrm{f}$'s null-space has to be found.

$$\mathbf{J}_\mathrm{x}^T \bar{\mathbf{M}} \mathbf{J}_\mathrm{x} \dot{\mathbf{q}} = \mathbf{J}_\mathrm{x}^T \bar{\mathbf{A}} \mathbf{J}_\mathrm{x} \mathbf{q} + \mathbf{J}_\mathrm{x}^T \bar{\mathbf{B}} \mathbf{u} \tag{2.37}$$

$$\tilde{\mathbf{M}} \dot{\mathbf{q}} = \mathbf{J}_\mathrm{x}^T \bar{\mathbf{A}} \mathbf{J}_\mathrm{x} \mathbf{q} + \mathbf{J}_\mathrm{x}^T \bar{\mathbf{B}} \mathbf{u}$$

$$\dot{\mathbf{q}} = \underbrace{\tilde{\mathbf{M}}^{-1} \mathbf{J}_\mathrm{x}^T \bar{\mathbf{A}} \mathbf{J}_\mathrm{x}}_{\tilde{\mathbf{A}}} \mathbf{q} + \underbrace{\tilde{\mathbf{M}}^{-1} \mathbf{J}_\mathrm{x}^T \bar{\mathbf{B}}}_{\tilde{\mathbf{B}}} \mathbf{u}$$

$$\dot{\mathbf{q}} = \tilde{\mathbf{A}} \mathbf{q} + \tilde{\mathbf{B}} \mathbf{u} \tag{2.38}$$

Remark 3.

- *Assumption* $\bar{\mathbf{M}}$ *is positive definite guarantees the existence of* $\tilde{\mathbf{M}}^{-1}$ *(see chapter 3).*
- *Although arbitrary linear constraints* $\mathbf{f}(\mathbf{r})$ *are holonomic, arbitrary linear constraints* $\mathbf{f}(\mathbf{x})$ *are not, since arbitrary linear* $\mathbf{f}(\mathbf{x})$ *enables coupling positions and velocities. Such a constraint can not be formulated either in terms of positions or velocities therefore it is non holonomic. Consideration of non holonomic constraints is not covered in the approach presented in this chapter. This restriction to* $\mathbf{f}(\mathbf{x})$ *entails block structure of constraints Jacobian matrix* $\mathbf{J}_{\mathrm{f}}$ *and consequently of* $\mathbf{J}_{\mathrm{x}}$*.*

2.6.3 System reduction and state transformation

Applying state transformations on state space models in order to achieve special structures of system parameters, e. g. canonical forms or diagonal form, while conserving system's dynamic, is a common task in control theory. This chapter provides a consistent approach of applying system reduction on state transformed systems and finally back transforming.

To simplify following considerations the analyzed system is assumed to be autonomous ($\mathbf{B} = \mathbf{0}$). The indices x and z denote affiliation of system parameters to the corresponding coordinates.

The regular transformation $\mathbf{x} = \mathbf{T}\mathbf{z}$ transforms the considered state space model from $\mathbf{x}$- into $\mathbf{z}$-coordinates. Dynamic matrix $\mathbf{A}_{\mathrm{z}}$ is consequently defined by a similarity transformation (see equation 2.39) applied on the original system's dynamic matrix $\mathbf{A}_{\mathrm{x}}$. Therefore the matrices $\mathbf{A}_{\mathrm{x}}$ and $\mathbf{A}_{\mathrm{z}}$ are called similar, i. e. their eigenvalues are identical.

$$\dot{\mathbf{x}} = \mathbf{A}_{\mathrm{x}}\mathbf{x} \quad \overset{\mathbf{x}=\mathbf{T}\mathbf{z}}{\longleftrightarrow} \quad \dot{\mathbf{z}} = \underbrace{\mathbf{T}^{-1}\mathbf{A}_{\mathrm{x}}\mathbf{T}}_{\mathbf{A}_{\mathrm{z}}}\mathbf{z} = \mathbf{A}_{\mathrm{z}}\mathbf{z} \tag{2.39}$$

Further there exist relations between the coordinates $\mathbf{x}$ and $\mathbf{z}$ and the corresponding reduced coordinates $\mathbf{q}_{\mathrm{x}}$ and $\mathbf{q}_{\mathrm{z}}$:

$$\mathbf{x} = \mathbf{J}_{\mathrm{x}}\mathbf{q}_{\mathrm{x}} \tag{2.40}$$

$$\mathbf{z} = \mathbf{J}_{\mathrm{z}}\mathbf{q}_{\mathrm{z}} \tag{2.41}$$

Still unknown at this point is the relation between the reduced coordinates $\mathbf{q}_{\mathrm{x}}$ and $\mathbf{q}_{\mathrm{z}}$ (see figure 2.1).

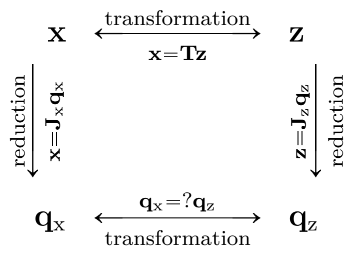

Figure 2.1: Incomplete transformation between coordinates $\mathbf{x}$, $\mathbf{z}$, $\mathbf{q}_\mathrm{x}$ and $\mathbf{q}_\mathrm{z}$

The combination of the introduced transformations determines this relation:

$$\left.\begin{array}{l}\left.\begin{array}{rcl}\mathbf{x} & = & \mathbf{Tz}\\ \mathbf{x} & = & \mathbf{J}_\mathrm{x}\mathbf{q}_\mathrm{x}\end{array}\right\} \ \mathbf{Tz} = \mathbf{J}_\mathrm{x}\mathbf{q}_\mathrm{x}\\ \qquad\qquad\qquad \mathbf{z} = \mathbf{J}_\mathrm{z}\mathbf{q}_\mathrm{z}\end{array}\right\} \mathbf{TJ}_\mathrm{z}\mathbf{q}_\mathrm{z} = \mathbf{J}_\mathrm{x}\mathbf{q}_\mathrm{x} \quad \Rightarrow \quad \mathbf{q}_\mathrm{x} = \left[\mathbf{J}_\mathrm{x}^T\mathbf{J}_\mathrm{x}\right]^{-1}\mathbf{J}_\mathrm{x}^T\mathbf{TJ}_\mathrm{z}\mathbf{q}_\mathrm{z} \tag{2.42}$$

Transformation of constraints further offers a relation between the Jacobian matrix in $\mathbf{x}$- and $\mathbf{z}$- coordinates:

$$\mathbf{J}_{\mathrm{f,x}}\mathbf{x} = \mathbf{0} \ \Leftrightarrow \ \underbrace{\mathbf{J}_{\mathrm{f,x}}\mathbf{T}}_{\mathbf{J}_{\mathrm{f,z}}}\,\mathbf{z} = \mathbf{0} \tag{2.43}$$

Since $\mathbf{J}_\mathrm{x}$ is a basis of $\mathbf{J}_{\mathrm{f,x}}$'s (Jacobian matrix of the constraints in $\mathbf{x}$-coordinates) null space and $\mathbf{J}_\mathrm{z}$ of $\mathbf{J}_{\mathrm{f,z}}$'s (Jacobian matrix of the constraints in $\mathbf{z}$-coordinates) null space, the two matrices are not independent:

$$\left.\begin{array}{rcl}\mathbf{J}_{\mathrm{f,x}}\mathbf{J}_\mathrm{x} & = & \mathbf{0}\\ \mathbf{J}_{\mathrm{f,z}}\mathbf{J}_\mathrm{z} & = & \mathbf{0}\\ \mathbf{J}_{\mathrm{f,z}} & = & \mathbf{J}_{\mathrm{f,x}}\mathbf{T}\end{array}\right\}\mathbf{J}_\mathrm{z} = \mathbf{T}^{-1}\mathbf{J}_\mathrm{x}\mathbf{R} \tag{2.44}$$

$\mathbf{R}$ is an arbitrary q x q full column rank matrix that generalizes the choice of the basis of $\mathbf{J}_{\mathrm{f,z}}$'s null-space. It performs compressions, extensions and rotations on the basis of $\mathbf{J}_{\mathrm{f,x}}$'s null-space. In combination with equation 2.42 follows:

$$\mathbf{q}_\mathrm{x} = \left[\mathbf{J}_\mathrm{x}^T\mathbf{J}_\mathrm{x}\right]^{-1}\mathbf{J}_\mathrm{x}^T\mathbf{TT}^{-1}\mathbf{J}_\mathrm{x}\mathbf{Rq}_\mathrm{z} = \left[\mathbf{J}_\mathrm{x}^T\mathbf{J}_\mathrm{x}\right]^{-1}\mathbf{J}_\mathrm{x}^T\mathbf{J}_\mathrm{x}\mathbf{Rq}_\mathrm{z} = \mathbf{Rq}_\mathrm{z} \tag{2.45}$$

Remark 4.
The mapping $\mathbf{R}$ of constraining matrix $\mathbf{J}_\mathrm{f}$'s null-space defines the transformation between the corresponding sets of reduced coordinates.

In the special choice of the in general arbitrary full rank matrix $\mathbf{R} = \mathbf{I}$ reduced $\mathbf{x}$-coordinates $\mathbf{q}_\mathrm{x}$ are identically the reduced $\mathbf{z}$-coordinates $\mathbf{q}_\mathrm{z}$.

Using equation 2.45 respectively 2.44 delivers entire relations between the coordinates $\mathbf{x}$, $\mathbf{z}$, $\mathbf{q}_\mathrm{x}$ and $\mathbf{q}_\mathrm{z}$ (see figure 2.2).

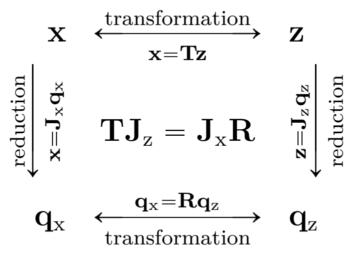

Figure 2.2: Complete transformation between coordinates $\mathbf{x}$, $\mathbf{z}$, $\mathbf{q}_\mathrm{x}$ and $\mathbf{q}_\mathrm{z}$

In extension to this interim result the application of the system reduction in $\mathbf{z}$ coordinates has to be determined.

Linear constraining of the mechanical system, stated in equation 2.39, needs adaption of notation in $\mathbf{z}$-coordinates:

$$\bar{\mathbf{M}}_\mathrm{x}\dot{\mathbf{x}} = \bar{\mathbf{A}}_\mathrm{x}\mathbf{x} \quad \overset{\mathbf{x}=\mathbf{T}\mathbf{z}}{\longleftrightarrow} \quad \bar{\mathbf{M}}_\mathrm{z}\dot{\mathbf{z}} = \underbrace{\bar{\mathbf{M}}_\mathrm{z}\mathbf{T}^{-1}\bar{\mathbf{M}}_\mathrm{x}^{-1}\bar{\mathbf{A}}_\mathrm{x}\mathbf{T}}_{\bar{\mathbf{A}}_\mathrm{z}}\,\mathbf{z} = \bar{\mathbf{A}}_\mathrm{z}\mathbf{z} \tag{2.46}$$

The question now is how to choose matrix $\bar{\mathbf{M}}_\mathrm{z}$ to close the following equation problem:

$$\begin{array}{ccccc}
\mathbf{x} & \Leftrightarrow & \mathbf{x} = \mathbf{T}\mathbf{z} & \Leftrightarrow & \mathbf{z} \\
\Downarrow & & \boxed{\bar{\mathrm{M}}_\mathrm{z} = ?} & & \Downarrow \\
\begin{array}{c}\bar{\mathbf{M}}_\mathrm{x}\dot{\mathbf{x}} = \bar{\mathbf{A}}_\mathrm{x}\mathbf{x} \\ \mathbf{J}_\mathrm{f,x}\mathbf{x} = \mathbf{0}\end{array} & \Leftrightarrow & \begin{array}{c}\bar{\mathbf{A}}_\mathrm{z} = \bar{\mathbf{M}}_\mathrm{z}\mathbf{T}^{-1}\bar{\mathbf{M}}_\mathrm{x}^{-1}\bar{\mathbf{A}}_\mathrm{x}\mathbf{T} \\ \mathbf{A}_\mathrm{z} = \mathbf{T}^{-1}\mathbf{A}_\mathrm{x}\mathbf{T} \\ \mathbf{J}_\mathrm{f,z} = \mathbf{J}_\mathrm{f,x}\mathbf{T}\end{array} & \Leftrightarrow & \begin{array}{c}\bar{\mathbf{M}}_\mathrm{z}\dot{\mathbf{z}} = \bar{\mathbf{A}}_\mathrm{z}\mathbf{z} \\ \mathbf{J}_\mathrm{f,z}\mathbf{z} = \mathbf{0}\end{array} \\
\Downarrow & & & & \Downarrow \\
\bar{\mathbf{M}}_\mathrm{x}\dot{\mathbf{x}} = \bar{\mathbf{A}}_\mathrm{x} + \mathbf{J}_\mathrm{f,x}^T\boldsymbol{\lambda}_\mathrm{x} & & \Downarrow & & \bar{\mathbf{M}}_\mathrm{z}\dot{\mathbf{z}} = \bar{\mathbf{A}}_\mathrm{z} + \mathbf{J}_\mathrm{f,z}^T\boldsymbol{\lambda}_\mathrm{z} \\
\Downarrow & & & & \Downarrow \\
\begin{array}{c}\mathbf{x} = \mathbf{J}_\mathrm{x}\mathbf{q}_\mathrm{x} \\ \mathbf{J}_\mathrm{f,x}\mathbf{J}_\mathrm{x} \overset{!}{=} \mathbf{0}\end{array} & \Leftrightarrow & \mathbf{J}_\mathrm{z} = \mathbf{T}^{-1}\mathbf{J}_\mathrm{x}\mathbf{R} & \Leftrightarrow & \begin{array}{c}\mathbf{z} = \mathbf{J}_\mathrm{z}\mathbf{q}_\mathrm{z} \\ \mathbf{J}_\mathrm{f,z}\mathbf{J}_\mathrm{z} \overset{!}{=} \mathbf{0}\end{array} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\begin{array}{c}\dot{\mathbf{q}}_\mathrm{x} = \left[\mathbf{J}_\mathrm{x}^T\bar{\mathbf{M}}_\mathrm{x}\mathbf{J}_\mathrm{x}\right]^{-1}\mathbf{J}_\mathrm{x}^T\bar{\mathbf{A}}_\mathrm{x}\mathbf{J}_\mathrm{x}\mathbf{q}_\mathrm{x} \\ \dot{\mathbf{q}}_\mathrm{x} = \tilde{\mathbf{A}}_\mathrm{x}\mathbf{q}_\mathrm{x}\end{array} & \Leftrightarrow & \begin{array}{c}\mathbf{q}_\mathrm{x} = \mathbf{R}\mathbf{q}_\mathrm{z} \\ \tilde{\mathbf{A}}_\mathrm{z} = \mathbf{R}^{-1}\tilde{\mathbf{A}}_\mathrm{x}\mathbf{R}\end{array} & \Leftrightarrow & \begin{array}{c}\dot{\mathbf{q}}_\mathrm{z} = \left[\mathbf{J}_\mathrm{z}^T\bar{\mathbf{M}}_\mathrm{z}\mathbf{J}_\mathrm{z}\right]^{-1}\mathbf{J}_\mathrm{z}^T\bar{\mathbf{A}}_\mathrm{z}\mathbf{J}_\mathrm{z}\mathbf{q}_\mathrm{z} \\ \dot{\mathbf{q}}_\mathrm{z} = \tilde{\mathbf{A}}_\mathrm{z}\mathbf{q}_\mathrm{z}\end{array}
\end{array} \tag{2.47}$$

Starting at bottom in equation 2.47 and using of relations on the right side (**z**-coordinates) above achieves:

$$\begin{aligned}\tilde{\mathbf{A}}_{\mathrm{x}} &= \mathbf{R}\tilde{\mathbf{A}}_{\mathrm{z}}\mathbf{R}^{-1} = \mathbf{R}\left[\mathbf{J}_{\mathrm{z}}^{T}\bar{\mathbf{M}}_{\mathrm{z}}\mathbf{J}_{\mathrm{z}}\right]^{-1}\mathbf{J}_{\mathrm{z}}^{T}\bar{\mathbf{A}}_{\mathrm{z}}\mathbf{J}_{\mathrm{z}}\mathbf{R}^{-1} \\ &= \mathbf{R}\left[\mathbf{R}^{T}\mathbf{J}_{\mathrm{x}}^{T}\mathbf{T}^{-T}\bar{\mathbf{M}}_{\mathrm{z}}\mathbf{T}^{-1}\mathbf{J}_{\mathrm{x}}\mathbf{R}\right]^{-1}\mathbf{R}^{T}\mathbf{J}_{\mathrm{x}}^{T}\mathbf{T}^{-T}\bar{\mathbf{M}}_{\mathrm{z}}\mathbf{A}_{\mathrm{z}}\mathbf{T}^{-1}\mathbf{J}_{\mathrm{x}}\mathbf{R}\mathbf{R}^{-1} \\ &= \mathbf{R}\mathbf{R}^{-1}\left[\mathbf{J}_{\mathrm{x}}^{T}\mathbf{T}^{-T}\bar{\mathbf{M}}_{\mathrm{z}}\mathbf{T}^{-1}\mathbf{J}_{\mathrm{x}}\right]^{-1}\mathbf{R}^{-T}\mathbf{R}^{T}\mathbf{J}_{\mathrm{x}}^{T}\mathbf{T}^{-T}\bar{\mathbf{M}}_{\mathrm{z}}\mathbf{T}^{-1}\mathbf{A}_{\mathrm{x}}\mathbf{T}\mathbf{T}^{-1}\mathbf{J}_{\mathrm{x}}\mathbf{R}\mathbf{R}^{-1} \\ &= \left[\mathbf{J}_{\mathrm{x}}^{T}\mathbf{T}^{-T}\bar{\mathbf{M}}_{\mathrm{z}}\mathbf{T}^{-1}\mathbf{J}_{\mathrm{x}}\right]^{-1}\mathbf{J}_{\mathrm{x}}^{T}\mathbf{T}^{-T}\bar{\mathbf{M}}_{\mathrm{z}}\mathbf{T}^{-1}\mathbf{A}_{\mathrm{x}}\mathbf{J}_{\mathrm{x}}\end{aligned} \tag{2.48}$$

As stated in the equation at the bottom left also holds:

$$\tilde{\mathbf{A}}_{\mathrm{x}} = \left[\mathbf{J}_{\mathrm{x}}^{T}\bar{\mathbf{M}}_{\mathrm{x}}\mathbf{J}_{\mathrm{x}}\right]^{-1}\mathbf{J}_{\mathrm{x}}^{T}\bar{\mathbf{M}}_{\mathrm{x}}\mathbf{A}_{\mathrm{x}}\mathbf{J}_{\mathrm{x}} \tag{2.49}$$

Comparing equations 2.48 and 2.49 enables relation between matrices $\bar{\mathbf{M}}_{\mathrm{x}}$ and $\bar{\mathbf{M}}_{\mathrm{z}}$:

$$\bar{\mathbf{M}}_{\mathrm{x}} = \mathbf{T}^{-T}\bar{\mathbf{M}}_{\mathrm{z}}\mathbf{T}^{-1} \quad \Rightarrow \boxed{\bar{\mathbf{M}}_{\mathrm{z}} = \mathbf{T}^{T}\bar{\mathbf{M}}_{\mathrm{x}}\mathbf{T}} \tag{2.50}$$

Remark 5.

The definition of matrix $\bar{\mathbf{M}}_{\mathrm{z}}$*,*

$$\bar{\mathbf{M}}_{\mathrm{z}} = \mathbf{T}^{T}\bar{\mathbf{M}}_{\mathrm{x}}\mathbf{T}, \tag{2.51}$$

enables consistency of state transformation $\mathbf{x} = \mathbf{T}\mathbf{z}$ *with respect to constraints on the original system. Note that contrary to the similarity transformation applied on the dynamic matrix* $\bar{\mathbf{A}}_{\mathrm{x}}$*, a congruence transformation has to be applied on* $\bar{\mathbf{M}}_{\mathrm{x}}$*.*

Physical interpretation

Consideration of system energy $V(\mathbf{x})$ offers a physical interpretation of the above result:

$$V(\mathbf{x}) = \mathbf{x}^{T}\begin{bmatrix}\mathbf{K} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}_{\mathrm{x}}\end{bmatrix}\mathbf{x} \quad \overset{\mathbf{x}=\mathbf{T}\mathbf{z}}{\Longrightarrow} \quad V(\mathbf{z}) = \mathbf{z}^{T}\mathbf{T}^{T}\begin{bmatrix}\mathbf{K} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}_{\mathrm{x}}\end{bmatrix}\mathbf{T}\mathbf{z} \tag{2.52}$$

Compare to equation 2.50:

$$\bar{\mathbf{M}}_{\mathrm{z}} = \mathbf{T}^{T}\bar{\mathbf{M}}_{\mathrm{x}}\mathbf{T} = \mathbf{T}^{T}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}_{\mathrm{x}}\end{bmatrix}\mathbf{T} \tag{2.53}$$

Due to definition of $\bar{\mathbf{M}}_{\mathrm{x}}$, $\mathbf{x}^{T}\bar{\mathbf{M}}_{\mathrm{x}}\mathbf{x}$ is related to term of system energy.
By modification of equation 2.35 this relationship can be fixed:

$$\begin{aligned}\begin{bmatrix}\mathbf{K} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{r}}\\ \ddot{\mathbf{r}}\end{bmatrix} &= \begin{bmatrix}\mathbf{K} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{K} & -\mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{r}\\ \dot{\mathbf{r}}\end{bmatrix} + \begin{bmatrix}\mathbf{K} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{0}\\ \mathbf{I}\end{bmatrix}\mathbf{F} \\ \underbrace{\begin{bmatrix}\mathbf{K} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}\end{bmatrix}}_{\bar{\mathbf{M}}_{\mathrm{x}}}\begin{bmatrix}\dot{\mathbf{r}}\\ \ddot{\mathbf{r}}\end{bmatrix} &= \begin{bmatrix}\mathbf{0} & \mathbf{K}\\ -\mathbf{K} & -\mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{r}\\ \dot{\mathbf{r}}\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ \mathbf{I}\end{bmatrix}\mathbf{F}\end{aligned} \tag{2.54}$$

$$V(\mathbf{x}) = \mathbf{x}^T \tilde{\mathbf{M}}_\mathrm{x} \mathbf{x} \quad \overset{\mathbf{x}=\mathbf{Tz}}{\Longrightarrow} \quad V(\mathbf{z}) = \mathbf{z}^T \underbrace{\mathbf{T}^T \tilde{\mathbf{M}}_\mathrm{x} \mathbf{T}}_{\tilde{\mathbf{M}}_\mathrm{z}} \mathbf{z} \tag{2.55}$$

This consideration offers a physical plausibility of remark 5:

Remark 6.

- *Considering equation 2.34 mass matrix* $\mathbf{M}$ *defines some kind of weighting between the differential equations. Although this weighting is not essential for describing systems dynamic, it plays a significant roll, when applying mechanical constraints, since it influences the fusion of inertias coupled through a constraint.*
- *In order to ensure correct involvement of mechanical constraints, system energy has to be conserved during state transformation. This condition requires transformation of the mass matrix as stated in remark 5.*
- *In state space formulation the system decomposes into two parts. The first part describing the relation between positions and velocities, is simply the realization of a chain of integrators. State transformation of states referring to this chain of integrators, i. e. positions, has no effect either on the systems dynamic nor on the later addition of mechanical constraints, since it does not influence the original system of differential equations second order. Consequently such state transformations can be considered as common state transformations and applied as usual. The second part is the realization of the multibody dynamic. State transformations including states referring to this part, i. e. velocities, have to be considered as defined above to ensure consistency with respect to the later addition of mechanical constraints.*

2.7 Conclusion

This chapter shows the consideration of additional holonomic constraints to a mechanical multibody problems. Thereby the constraints Jacobian matrix plays an essential role for reducing the system of differential equations to a minimal number. This system reduction can be applied with an arbitrary basis to the null space of this Jacobian matrix. Transition to state space formulation enables application of common control methods. Another important conclusion to this chapter is, that performing state transformations and afterwards applying additional constraints to the system needs a congruence transformation of the mass matrix to conserve the original mechanical impact of the constraints.

3

System reduction and stability

The objective of this chapter is to investigate the impact of linear constraints on the stability of linear multibody systems. The proof of stability is done by using the theorem of Lyapunov. Furthermore the issue, if stability of a linearly state controlled system is endangered by linear constraints, needs to be clarified. The motivation behind this task is to investigate the possibility to use only one state feedback matrix to control a linear multibody system regardless of any additional linear constraint applied to the system. In this case fictitious mechanical systems are used to evaluate stability. The same considerations are valid for stability of a state observer's observation error dynamic.

3.1 Stability of linear constrained linear multibody systems

Theorem 1. ***Stability of linear constrained linear multibody systems***
Application of linear constraints to a linear multibody system can not destabilize the system.

This theorem is a physical necessity due to d'Alembert's principle (see chapter 2.2) and can also be proofed mathematically.

Proof.
In a first step conditions for stability of a linear multibody system have to be developed.

Theorem 2. ***Theorem of Lyapunov*** *[6]*
Let $\mathbf{x} = \mathbf{0}$ *be an equilibrium point for* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *and* $\mathcal{D} \subset \mathcal{R}^n$ *be a domain containing* $\mathbf{x} = \mathbf{0}$. *Let* $V : \mathcal{D} \rightarrow \mathcal{R}$ *be a continuously differentiable (scalar) function such that,*

$$V(\mathbf{0}) = 0 \quad and \quad V(\mathbf{x}) > 0 \quad in \quad \mathcal{D} - \{\mathbf{0}\} \tag{3.1}$$

$$\dot{V}(\mathbf{x}) \leqslant 0 \quad in \quad \mathcal{D} \tag{3.2}$$

Then, $\mathbf{x} = \mathbf{0}$ *is stable (in the sense of Lyapunov). Moreover, if*

$$\dot{V}(\mathbf{x}) < 0 \quad in \quad \mathcal{D} - \{\mathbf{0}\} \tag{3.3}$$

then $\mathbf{x} = \mathbf{0}$ *is asymptotically stable.*

Further $V(\mathbf{x})$ is called Lyapunov function.

Adaption of notation of equation 2.36 with respect to Lyapunov theorem leads to:

$$\begin{bmatrix}\mathbf{r}^T & \dot{\mathbf{r}}^T\end{bmatrix} = \begin{bmatrix}\mathbf{x}^T & \dot{\mathbf{x}}^T\end{bmatrix} \tag{3.4}$$

$$\Downarrow$$

$$\begin{bmatrix}\dot{\mathbf{x}}\\ \ddot{\mathbf{x}}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ \dot{\mathbf{x}}\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ \mathbf{M}^{-1}\mathbf{F}\end{bmatrix} \tag{3.5}$$

Evaluation of stability considers autonomous system ($\mathbf{F} = \mathbf{0}$).
Due to its positivity system energy is a candidate to be a Lyapunov function:

$$V(\mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2}\dot{\mathbf{x}}^T\mathbf{M}\dot{\mathbf{x}} + \frac{1}{2}\mathbf{x}^T\mathbf{K}\mathbf{x} > 0, \quad \forall \mathbf{x}, \dot{\mathbf{x}} \neq \mathbf{0} \tag{3.6}$$

Using symmetry properties of $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{K}$ and equation 3.4 enables (see also [7]):

$$\begin{aligned}\dot{V}(\mathbf{x}, \dot{\mathbf{x}}) &= \frac{1}{2}\left(\ddot{\mathbf{x}}^T\mathbf{M}\dot{\mathbf{x}} + \dot{\mathbf{x}}^T\mathbf{M}\ddot{\mathbf{x}} + \dot{\mathbf{x}}^T\mathbf{K}\mathbf{x} + \mathbf{x}^T\mathbf{K}\dot{\mathbf{x}}\right)\\ &= \frac{1}{2}\left(\left[\mathbf{M}\ddot{\mathbf{x}}\right]^T\dot{\mathbf{x}} + \dot{\mathbf{x}}^T\mathbf{M}\ddot{\mathbf{x}} + \dot{\mathbf{x}}^T\mathbf{K}\mathbf{x} + \mathbf{x}^T\mathbf{K}\dot{\mathbf{x}}\right)\\ &= \frac{1}{2}\left(\left[-\dot{\mathbf{x}}^T\mathbf{D} - \mathbf{x}^T\mathbf{K}\right]\dot{\mathbf{x}} + \dot{\mathbf{x}}^T\left[-\mathbf{D}\dot{\mathbf{x}} - \mathbf{K}\mathbf{x}\right] + \dot{\mathbf{x}}^T\mathbf{K}\mathbf{x} + \mathbf{x}^T\mathbf{K}\dot{\mathbf{x}}\right)\\ &= -\dot{\mathbf{x}}^T\mathbf{D}\dot{\mathbf{x}} < 0, \quad \forall \dot{\mathbf{x}} \neq \mathbf{0}\end{aligned} \tag{3.7}$$

According theorem of Lyapunov matrix $\mathbf{D}$ positive definite (see equation 2.6) ensures asymptotic stability of the system.

The addition of linear constraints to the system results in a reduced system (see equation 2.32):

$$\underbrace{\mathbf{J}_\mathrm{r}^T\mathbf{M}\mathbf{J}_\mathrm{r}}_{\tilde{\mathbf{M}}}\ddot{\mathbf{q}} = -\underbrace{\mathbf{J}_\mathrm{r}^T\mathbf{D}\mathbf{J}_\mathrm{r}}_{\tilde{\mathbf{D}}}\dot{\mathbf{q}} - \underbrace{\mathbf{J}_\mathrm{r}^T\mathbf{K}\mathbf{J}_\mathrm{r}}_{\tilde{\mathbf{K}}}\mathbf{q} + \underbrace{\mathbf{J}_\mathrm{r}^T\mathbf{F}}_{\mathbf{Q}} \tag{3.8}$$

If

$$\mathbf{x}^T\mathbf{M}\mathbf{x} > 0 \;\; \forall \mathbf{x} \neq \mathbf{0} \tag{3.9}$$

the symmetric matrix $\mathbf{M}$ is called positive definite. All eigenvalues of a symmetric real matrix are real. The eigenvalues of a positive definite matrix are true positive. Due to equation 3.9 and the choice $\mathbf{x} = \mathbf{J}_\mathrm{r}\mathbf{q}$,

$$\mathbf{q}^T\mathbf{J}_\mathrm{r}^T\mathbf{M}\mathbf{J}_\mathrm{r}\mathbf{q} > 0 \;\; \forall \mathbf{q} \neq \mathbf{0} \tag{3.10}$$

holds, if $\mathbf{J}_\mathrm{r}$ has full column rank (linear independent columns), because $\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{q} = \mathbf{0}$. In consequence symmetric matrix $\mathbf{J}_\mathrm{r}^T\mathbf{M}\mathbf{J}_\mathrm{r}$ is positive definite too and its eigenvalues are true positive. Analogously this fact holds for negative definite matrices $\mathbf{M}$

$$\mathbf{x}^T\mathbf{M}\mathbf{x} < 0 \;\; \forall \mathbf{x} \neq \mathbf{0} \tag{3.11}$$

and its true negative eigenvalues.

Remark 7.
If all eigenvalues of a symmetric matrix $\mathbf{M}$ *have the same sign, the transformation* $\mathbf{J}_\mathrm{r}^T\mathbf{M}\mathbf{J}_\mathrm{r}$ *does not influent this sign for arbitrary full column rank matrices* $\mathbf{J}_\mathrm{r}$*. Although the number of eigenvalues and its values in general are different.*

According to remark 7 matrices $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{D}}$ again are positive definite:

$$\begin{aligned}\tilde{\mathbf{M}} &= \tilde{\mathbf{M}}^T > 0 \\ \tilde{\mathbf{D}} &= \tilde{\mathbf{D}}^T > 0\end{aligned} \tag{3.12}$$

Due to the fact that symmetric matrix $\mathbf{K}$ can contain zero columns and corresponding zero rows it is not a positive definite but a positive semidefinite matrix. Removing those columns and rows achieves a positive definite sub matrix. This sub matrix stays positive definite during the above transformation. The zero columns and rows result in zero columns and rows of the transformed matrix $\tilde{\mathbf{K}}$. Consequently $\tilde{\mathbf{K}}$ is again positive semidefinite:

$$\tilde{\mathbf{K}} = \tilde{\mathbf{K}}^T \geqslant 0 \tag{3.13}$$

In analogy to the unconstrained case system energy can be used to find a Lyapunov function ensuring stability of the constrained system:

$$\tilde{\mathbf{M}}\ddot{\mathbf{q}} = -\tilde{\mathbf{D}}\dot{\mathbf{q}} - \tilde{\mathbf{K}}\mathbf{q} + \mathbf{Q} \tag{3.14}$$

$$\Updownarrow \tag{3.15}$$

$$\begin{bmatrix}\dot{\mathbf{q}} \\ \ddot{\mathbf{q}}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathbf{I} \\ -\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{K}} & -\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{D}}\end{bmatrix}\begin{bmatrix}\mathbf{q} \\ \dot{\mathbf{q}}\end{bmatrix} + \begin{bmatrix}\mathbf{0} \\ \tilde{\mathbf{M}}^{-1}\mathbf{Q}\end{bmatrix} \tag{3.16}$$

$$V(\mathbf{q},\dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^T\tilde{\mathbf{M}}\dot{\mathbf{q}} + \frac{1}{2}\mathbf{q}^T\tilde{\mathbf{K}}\mathbf{q} > 0, \quad \forall \mathbf{q},\dot{\mathbf{q}} \neq \mathbf{0} \tag{3.17}$$

$$\begin{aligned}\dot{V}(\mathbf{q},\dot{\mathbf{q}}) &= \frac{1}{2}\left(\ddot{\mathbf{q}}^T\tilde{\mathbf{M}}\dot{\mathbf{q}} + \dot{\mathbf{q}}^T\tilde{\mathbf{M}}\ddot{\mathbf{q}} + \dot{\mathbf{q}}^T\tilde{\mathbf{K}}\mathbf{q} + \mathbf{q}^T\tilde{\mathbf{K}}\dot{\mathbf{q}}\right) \\ &= \frac{1}{2}\left(\left[\tilde{\mathbf{M}}\ddot{\mathbf{q}}\right]^T\dot{\mathbf{q}} + \dot{\mathbf{q}}^T\tilde{\mathbf{M}}\ddot{\mathbf{q}} + \dot{\mathbf{q}}^T\tilde{\mathbf{K}}\mathbf{q} + \mathbf{q}^T\tilde{\mathbf{K}}\dot{\mathbf{q}}\right) \\ &= \frac{1}{2}\left(\left[-\dot{\mathbf{q}}^T\tilde{\mathbf{D}} - \mathbf{q}^T\tilde{\mathbf{K}}\right]\dot{\mathbf{q}} + \dot{\mathbf{q}}^T\left[-\tilde{\mathbf{D}}\dot{\mathbf{q}} - \tilde{\mathbf{K}}\mathbf{q}\right] + \dot{\mathbf{q}}^T\tilde{\mathbf{K}}\mathbf{q} + \mathbf{q}^T\tilde{\mathbf{K}}\dot{\mathbf{q}}\right) \\ &= -\dot{\mathbf{q}}^T\tilde{\mathbf{D}}\dot{\mathbf{q}} < 0, \quad \forall \dot{\mathbf{q}} \neq \mathbf{0}\end{aligned} \tag{3.18}$$

According theorem of Lyapunov a positive definite matrix $\tilde{\mathbf{D}}$ (remark 7) ensures asymptotic stability of the reduced system.

Remark 8.

- *A linear multibody system is asymptotically stable, if the damping matrix* $\mathbf{D}$ *is positive definite, provided mass matrix* $\mathbf{M}$ *is positive definite and stiffness matrix* $\mathbf{K}$ *is positive semidefinite.*
- *Linear constraining does not influence stability, due to the fact that transformed damping matrix* $\tilde{\mathbf{D}}$ *is positive definite if* $\mathbf{D}$ *is.*

□

3.2 Stability in state space formulation

In state space formulation asymptotic stability is defined by the negative signs of the real parts of the eigenvalues of its dynamic matrix $\mathbf{A}$. Considering state space formulation of the constrained system (see equation 2.37) remark 8 enables important rationale.

$$\begin{aligned}
\bar{\mathbf{M}}\dot{\mathbf{x}} &= \bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{B}}\mathbf{u} \\
\dot{\mathbf{x}} &= \underbrace{\bar{\mathbf{M}}^{-1}\bar{\mathbf{A}}}_{\mathbf{A}}\,\mathbf{x} + \underbrace{\bar{\mathbf{M}}^{-1}\bar{\mathbf{B}}}_{\mathbf{B}}\,\mathbf{u} \qquad (3.19)\\
&\Downarrow \\
\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}\mathbf{J}_{\mathrm{x}}\dot{\mathbf{q}} &= \mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{A}}\mathbf{J}_{\mathrm{x}}\mathbf{q} + \mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{B}}\mathbf{u} \\
\tilde{\mathbf{M}}\dot{\mathbf{q}} &= \mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{A}}\mathbf{J}_{\mathrm{x}}\mathbf{q} + \mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{B}}\mathbf{u} \\
\dot{\mathbf{q}} &= \underbrace{\tilde{\mathbf{M}}^{-1}\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{A}}\mathbf{J}_{\mathrm{x}}\mathbf{q}}_{\tilde{\mathbf{A}}} + \underbrace{\tilde{\mathbf{M}}^{-1}\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{B}}}_{\tilde{\mathbf{B}}}\,\mathbf{u} \\
\dot{\mathbf{q}} &= \tilde{\mathbf{A}}\mathbf{q} + \tilde{\mathbf{B}}\mathbf{u} \qquad (3.20)
\end{aligned}$$

Remark 9.
According to theorem 1 stability of the unconstrained system (equation 3.19) guarantees stability of the reduced system (equation 3.20). This fact is manifested in the linked properties of the matrices $\bar{\mathbf{M}}$, $\bar{\mathbf{A}}$ and $\mathbf{J}_{\mathrm{x}}$. Evaluating stability of the reduced system in state space formulation by considering eigenvalues of $\tilde{\mathbf{A}}$ is nontrivial.

3.3 Stability of controlled subsystems

This section answers the question if the stability of the dynamic of an controlled system 3.21 in general is conserved while applying linear constraints (equation 3.22), by mechanical interpretation.
Considering a system controlled by linear state feedback $\mathbf{u} = \mathbf{K}\mathbf{x}$,

$$\dot{\mathbf{x}} = [\mathbf{A} - \mathbf{B}\mathbf{K}]\,\mathbf{x}, \qquad (3.21)$$

application of linear constraints results in:

$$\dot{\mathbf{q}} = \left[\underbrace{\left[\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}_{\mathrm{x}}\mathbf{J}_{\mathrm{x}}\right]^{-1}\mathbf{J}_{\mathrm{x}}^T\mathbf{A}\mathbf{J}_{\mathrm{x}}}_{\tilde{\mathbf{A}}} - \underbrace{\left[\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}_{\mathrm{x}}\mathbf{J}_{\mathrm{x}}\right]^{-1}\mathbf{J}_{\mathrm{x}}^T\mathbf{B}}_{\tilde{\mathbf{B}}}\ \underbrace{\mathbf{K}\mathbf{J}_{\mathrm{x}}}_{\tilde{\mathbf{K}}}\right]\mathbf{q} \qquad (3.22)$$

This system can be interpreted as linear state controlled system with feedback matrix $\tilde{\mathbf{K}} = \mathbf{K}\mathbf{J}_{\mathrm{x}}$. In order to evaluate stability of this system with respect to additional linear constraints a state transformation has to be applied. This transformation targets on achieving a diagonal form of the system's dynamic matrix for later mechanical interpretation.

Definition 4. ***Diagonalizable matrix***

A quadratic matrix $\mathbf{A} \in \mathbb{R}^{nxn}$ *is called diagonalizable if it is similar to a diagonal matrix* $\mathbf{D} \in \mathbb{C}^{nxn}$*. Therefore exists a regular matrix* $\mathbf{T} \in \mathbb{C}^{nxn}$ *such that* $\mathbf{D} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$*.*

Remark 10.

All eigenvalues of a diagonalizable matrix have equal algebraic and geometric multiplicity. Consequently dimension of eigenspace is n, i. e. there exist n linear independent eigenvectors.

Corollary 1. ***Stability of constrained controlled systems (case 1)***

If the stable controlled system's dynamic $\mathbf{A} - \mathbf{BK} \in \mathbb{R}^{nxn}$ *is diagonalizable and all eigenvalues* λ_i *to* $\mathbf{A} - \mathbf{BK}$ *fulfill* $\lambda_i \in \mathbb{R}, \quad i = 1, \ldots, n$*, linear constraints can not destabilize it.*

According to the conditions in corollary 1 the controlled system can be regular transformed to a similar diagonal system with dynamic $\mathbf{D} \in \mathbb{R}^{nxn}$. This similar system can easily be interpreted to describe the dynamical behavior of a mechanical system consisting of damped inertias ($\rightarrow \mathbf{K} = \mathbf{0}$). Therefore arbitrary linear constraints are physically consistent to this fictive system. According to theorem 1 linear constraints can not destabilize the fictitious system and consequently also the controlled system $\mathbf{A} - \mathbf{BK}$.

Corollary 2. ***Stability of constrained controlled systems (case 2)***

If the stable controlled system's dynamic $\mathbf{A} - \mathbf{BK} \in \mathbb{R}^{nxn}$ *is diagonalizable and linear constraints in general can destabilize it.*

According to the conditions in corollary 2 the eigenvalues of the controlled system can be either real or complex conjugate pairs. The controlled systems now can be regularly transformed to a similar but tridiagonal system with dynamic $\mathbf{D} \in \mathbb{R}^{nxn}$. $\mathbf{D}$ consists of diagonal parts corresponding to the real eigenvalues and 2x2 blocks corresponding to complex conjugate eigenvalue pairs. Mechanical interpretation of this system dynamic requires damped inertias **and** elementary mass-spring-damper configurations. Therefore state vector contains in general velocities **and** positions. Consequently arbitrary linear constraints are non holonomic and system can be destabilized in general.

Stability can be evaluated by transforming possible linear constraints of the original mechanical system from $\mathbf{x}$- to $\mathbf{z}$-coordinates (see chapter 2.6.3) and assessing their physical consistency to the fictive system.

Remark 11.

- *If the stable controlled system's dynamic* $\mathbf{A} - \mathbf{BK} \in \mathbb{R}^{nxn}$ *is not diagonalizable, stability evaluation due to linear constraints is not possible by mechanical interpretation of* $\mathbf{A} - \mathbf{BK}$*.*
- *All above statements hold analogously for an observer error dynamic and application of linear constraints.*
- *The feedback matrix* $\mathbf{K}$ *and consequently the choice of eigenvalues in the controlled system are design parameters. Therefore it is possible to design* $\mathbf{K}$ *with respect to achieving stable subsystems for given linear constraints.*

- *In the case of specific constraints consideration of the eigenvalues of the controlled subsystems enables immediate stability evaluation.*

3.4 Stability under arbitrary switching

Due to the fact that arbitrary switching between asymptotically stable systems can result in an unstable behavior, stability of arbitrary switching systems is an actual research topic in control theory (see for example [8]).

Source of this fact are complex conjugate eigenvalue pairs. Negative sign of their real parts ensures that any norm of the state vector x tends to zero (equation 3.23), i. e. the system is asymptotically stable, but it does not certainly decrease at any time instant (equation 3.24).

$$\lim_{t\to\infty} ||\mathbf{x}(t)|| = 0 \tag{3.23}$$

$$\exists t_0, \Delta t > 0 : \quad ||\mathbf{x}(t_0)|| < ||\mathbf{x}(t_0 + \Delta t)|| \tag{3.24}$$

Considering two systems with complex conjugate eigenvalue pairs and different eigenvectors, switching between those systems at those instants of time $t_0 + \Delta t$ defined in equation 3.24 enables possibly increasing norm. In consequence the total system's dynamic is possibly unstable for arbitrary switching between the two asymptotically stable systems.

Nevertheless there exist stability criteria for stable arbitrary switched systems for example the development of a common Lyapunov function (see for example [9] and [10]).

Considering linearly constrained linear multibody systems it is obvious that arbitrary switching between constraints is not feasible since the transition from one constraint to another can not occur instantly. Feasible transitions need unconstrained interims, where the first constraint is not valid any more and the second not valid so far. In this interim time states $\mathbf{x}$ have to be adapted to suffice the new constraint (denoted in $\mathbf{x}_\mathrm{q}$). This condition can be mathematically formulated by use of generalized inverse $\mathbf{J}_\mathrm{x}^+$:

$$\begin{aligned}
\mathbf{x} &= \mathbf{J}_\mathrm{x}\mathbf{q} \\
\mathbf{J}_\mathrm{x}^T\mathbf{x} &= \mathbf{J}_\mathrm{x}^T\mathbf{J}_\mathrm{x}\mathbf{q} \\
\underbrace{\left[\mathbf{J}_\mathrm{x}^T\mathbf{J}_\mathrm{x}\right]^{-1}\mathbf{J}_\mathrm{x}^T}_{\mathbf{J}_\mathrm{x}^+}\mathbf{x} &= \mathbf{q} \\
\mathbf{J}_\mathrm{x}\mathbf{J}_\mathrm{x}^+\mathbf{x} &= \mathbf{x}_\mathrm{q}
\end{aligned} \tag{3.25}$$

Equation 3.25 can be interpreted as a projection of an arbitrary vector $\mathbf{x}$ into a subspace that fulfills the constraints. As already stated in theorem 1 transition can not destabilize the system. Considering equations 3.6 and 3.17 describing systems energy at switching time and equation 3.25 it is obvious that the Lyapunov functions are equal at switching time.

Remark 12.
$\forall t: \ \mathbf{x}(t) \equiv \mathbf{x}_{\mathrm{q}}(t)$ *switching between the unconstrained system and the constrained system is physical feasible and arbitrary switching can not destabilize the system.*

Remark 12 holds for stability of arbitrary switching between controlled system and controlled subsystem if they are physically interpretable (see chapter 3.3).

3.5 Conclusion

The target of this chapter was to clarify possible stability problems of a multibody system due to linear constraints. From the mechanical point of view stability is ensured by d'Alembert's principle. This can be mathematically proofed by usage of Lyapunov theorem.

The chapter further offers conditions for conservation of stability of state controlled and later linearly constrained multibody systems. The motivation of this is the possibility to design only one state feedback matrix to achieve a stable control of the multibody system regardless of any additional linear constraint. Thereby physical interpretation of the constraints acting on a fictitious mechanical system represented by the controlled system's dynamic plays an decisive role. The same considerations are valid for the conservation of the stability of a state observer's observation error dynamic.

4

System reduction and optimality

Optimality in a specific definition is a common way to design feedback control laws or state estimators in control theory. The LQR problem *and the* Kalman filter problem *(see [11]) are famous examples to this approach. The task of this chapter is to apply this control design approach, respectively state estimator design approach, to linear constrained multibody systems. Therefore the impact of system reduction, introduced in chapter 2, on the optimality of a linear state feedback has to be analyzed. This task is an extension to the stability evaluation in chapter 3. Similarly the motivation is to design one optimal state feedback or one optimal Kalman filter system for optimal control or optimal observation of a linear multibody system regardless of any additional linear constraints applied to the system.*

Therefore at the beginning of this chapter the algebraic matrix Riccati equation (see [11]), which provides the solutions of such problems, and its solvability is summarized. Then a short overview on the optimal state estimation problem follows. The system reduction enables the calculation of a reduced state estimator. The question is, if this reduced state estimator still is optimal considering the reduced system. In a similar way optimality of a reduced optimal state feedback is investigated, after introducing the LQ optimal control problem in general.

4.1 Algebraic Matrix Riccati equation

The algebraic matrix Riccati equation plays a decisive role in optimal control and state estimation problems, because its solution provides the solution of these problems. Due to its importance the algebraic matrix Riccati equation is summarized shortly.

Definition 5. ***The algebraic matrix Riccati equation*** *(see [12])*
The equation

$$\mathbf{A}^T\mathbf{X} + \mathbf{X}\mathbf{A} + \mathbf{X}\mathbf{R}\mathbf{X} + \mathbf{Q} = \mathbf{0} \tag{4.1}$$

is called algebraic matrix Riccati equation in respect of the symmetric $(n \times n)$ matrix variable $\mathbf{X} = \mathbf{X}^T$ and with $\mathbf{A}, \mathbf{R}, \mathbf{Q} \in \mathbb{R}^{nxn}$.

Definition 6. ***Stabilizing solution of the algebraic matrix Riccati equation*** *(see [12])*
There exists at maximum one solution $\bar{\mathbf{X}}$, which stabilizes the system $[\mathbf{A} + \mathbf{R}\bar{\mathbf{X}}]$. The existence of this stabilizing solution is guaranteed by following conditions:

1. *The pair $(\mathbf{A}, \mathbf{R})$ is stabilizable (see definition 7).*
2. *The pair $(\mathbf{Q}, \mathbf{A})$ does not have not observable modes on the imaginary axis.*
3. $\mathbf{R} \leqslant 0$

Remark 13.

- *Satisfying conditions in definition 6 algebraic matrix Riccati equation has a unique positive semi-definite solution $\bar{\mathbf{X}} = \bar{\mathbf{X}}^T \geqslant 0$.*
- *First condition in definition 6 ensures the existence of a solution of the algebraic matrix Riccati equation. Conditions two and three ensure $\bar{\mathbf{X}} \geqslant 0$ and in consequence stability of $[\mathbf{A} + \mathbf{R}\bar{\mathbf{X}}]$.*
- *Condition "Pair $(\mathbf{Q}, \mathbf{A})$ **detectable**" implicates second condition in definition 6.*
- *Condition "Pair $(\mathbf{Q}, \mathbf{A})$ **observable**" implicates second condition in definition 6 and $\bar{\mathbf{X}} > 0$ ([13]).*

4.1.1 Considerations on stabilizability and detectability

This section is a short excursus on stabilizability and detectability of a system:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned} \tag{4.2}$$

Definition 7. ***Stabilizability and Detectability*** *(see [11])*
A system is stabilizable if all unstable modes are state controllable[1]. A system is detectable if all unstable modes are observable[2].

Lemma 2. *For full rank matrix $\mathbf{A} \in \mathbb{R}^{mxn}$ $(m < n)$ and symmetric positive definite matrix $\mathbf{T} > 0 \in \mathbb{R}^{nxn}$ holds*

$$\mathcal{N}(\mathbf{A}) = \mathcal{N}(\mathbf{A}^T\mathbf{T}\mathbf{A}) \tag{4.3}$$

Proof.

$$\dim\left(\mathcal{N}(\mathbf{A}^T)\right) = m - \operatorname{rank}\left(\mathbf{A}^T\right) = 0 \tag{4.4}$$

$$\mathbf{A}^T\mathbf{y} = \mathbf{0} \iff \mathbf{y} \in \mathcal{N}(\mathbf{A}^T) \iff \mathbf{y} = \mathbf{0} \tag{4.5}$$

$$\mathbf{A}^T \underbrace{\mathbf{A}\mathbf{x}}_{\mathbf{y}} = \mathbf{0} \iff \underbrace{\mathbf{A}\mathbf{x}}_{\mathbf{y}} = \mathbf{0} \iff \mathbf{x} \in \mathcal{N}(\mathbf{A}) \tag{4.6}$$

[1] For controllability criteria see [11].
[2] For observability criteria see [11].

Consequently holds:

$$\mathcal{N}(\mathbf{A}) = \mathcal{N}(\mathbf{A}^T\mathbf{A}) \tag{4.7}$$

$$\mathbf{A}^T\mathbf{T}\mathbf{A} = \underbrace{\mathbf{A}^T\left[\mathbf{T}^{\frac{1}{2}}\right]^T}_{\tilde{\mathbf{A}}^T}\underbrace{\left[\mathbf{T}^{\frac{1}{2}}\right]\mathbf{A}}_{\tilde{\mathbf{A}}} \tag{4.8}$$

$\mathbf{T}^{\frac{1}{2}}$ is called principal square root. It is unique and positive definite due to the fact that $\mathbf{T}$ is positive definite. In consequence $\tilde{\mathbf{A}}$ is again a full rank matrix, due to:

$$\operatorname{rank}\left(\tilde{\mathbf{A}}^T\right) = \operatorname{rank}\left(\tilde{\mathbf{A}}\right) = \operatorname{rank}\left(\mathbf{T}^{\frac{1}{2}}\mathbf{A}\right) = \operatorname{rank}\left(\mathbf{A}\right) \tag{4.9}$$

According to equations 4.4 to 4.6 follows:

$$\tilde{\mathbf{A}}^T\tilde{\mathbf{A}}\mathbf{x} = \mathbf{0} \iff \tilde{\mathbf{A}}\mathbf{x} = \mathbf{0} \iff \mathbf{x} \in \mathcal{N}(\tilde{\mathbf{A}}) = \mathcal{N}(\mathbf{T}^{\frac{1}{2}}\mathbf{A}) \tag{4.10}$$

$$\mathbf{T}^{\frac{1}{2}} > 0 \implies \dim\left(\mathcal{N}\left(\mathbf{T}^{\frac{1}{2}}\right)\right) = 0 \tag{4.11}$$

$$\mathbf{T}^{\frac{1}{2}}\mathbf{A}\mathbf{x} = \mathbf{0} \iff \mathbf{A}\mathbf{x} = \mathbf{0} \iff \mathbf{x} \in \mathcal{N}(\mathbf{A}) \tag{4.12}$$

Consequently holds:

$$\mathcal{N}(\mathbf{A}) = \mathcal{N}(\mathbf{A}^T\mathbf{T}\mathbf{A}) \tag{4.13}$$

□

Definition 8. ***Popov-Belevitch-Hautus criterion for stabilizability*** *(see for example [14])*

λ is an arbitrary eigenvalue of matrix $\mathbf{A}$ where $\mathrm{Re}\{\lambda\} \geqslant 0$. The pair $(\mathbf{A}, \mathbf{B})$ is stabilizable if and only if for every vector $\mathbf{p}$ generated from $\mathbf{p}^*\mathbf{A} = \mathbf{p}^*\lambda$ holds $\mathbf{p}^*\mathbf{B} \neq \mathbf{0}$.

The condition $\mathbf{p}^*\mathbf{B} \neq \mathbf{0}$ is equivalent to

$$[\mathbf{p}^*]^T \notin \mathcal{N}\left(\mathbf{B}^T\right). \tag{4.14}$$

Definition 9. ***Popov-Belevitch-Hautus criterion for detectability*** *(see for example [14])*

λ is an arbitrary eigenvalue of matrix $\mathbf{A}$ where $\mathrm{Re}\{\lambda\} \geqslant 0$. The pair $(\mathbf{C}, \mathbf{A})$ is detectable if and only if for every vector $\mathbf{q}$ generated from $\mathbf{A}\mathbf{q} = \lambda\mathbf{q}$ holds $\mathbf{C}\mathbf{q} \neq \mathbf{0}$.

The condition $\mathbf{C}\mathbf{q} \neq \mathbf{0}$ is equivalent to

$$\mathbf{q} \notin \mathcal{N}\left(\mathbf{C}\right). \tag{4.15}$$

Consideration of Popov-Belevitch-Hautus criterion (definition 8 and 9), Lemma 2 and duality between state control and state observation enables following remark:

Remark 14.

- *Pair $(\mathbf{A}, \mathbf{B}\mathbf{T}\mathbf{B}^T)$ is stabilizable if and only if pair $(\mathbf{A}, \mathbf{B})$ is stabilizable.*
- *Pair $(\mathbf{C}^T\mathbf{T}\mathbf{C}, \mathbf{A})$ is detectable if and only if pair $(\mathbf{C}, \mathbf{A})$ is detectable.*
- *Pair $(\mathbf{A}^T, \mathbf{C}^T)$ is stabilizable if and only if pair $(\mathbf{C}, \mathbf{A})$ is detectable.*
- *Pair $(\mathbf{B}^T, \mathbf{A}^T)$ is detectable if and only if pair $(\mathbf{A}, \mathbf{B})$ is stabilizable.*

4.2 System reduction and optimality of state estimation

In many control applications system states are not directly measurable, although their knowledge is essential for the control system design. This problem can be solved by designing a state observer, that observes the system states employing the inputs and outputs. In order to consider model and measurement uncertainties from the stochastic point of view, the system can be interpreted as noised system. Consequently, the system states are stochastic quantities. To achieve an optimal design of a state observer (in this context also called state estimator), the variance of the estimation error can be minimized. In the following sections this task is formulated mathematically and its solution the so called Kalman filter (see [11]) is presented.

Afterwards system reduction is applied on a general Kalman filter system. It turns out that the reduced system again has the form of a common state observer, employing a reduced feedback matrix $\tilde{\mathbf{K}}_{\mathrm{f},1}$. Since it is possible to design a Kalman filter system on the reduced problem (feedback matrix $\tilde{\mathbf{K}}_{\mathrm{f},2}$) the question is, if there is any equivalence between these two feedback matrices $\tilde{\mathbf{K}}_{\mathrm{f},1}$ and $\tilde{\mathbf{K}}_{\mathrm{f},2}$. If they are equal optimality of a Kalman filter system would be conserved during application of system reduction on the original system. In consequence it would be possible to design only one optimal Kalman filter system for optimal observation of the linear multibody system regardless of any additional linear constraints applied on the system. The last section of this chapter tries to answer this question.

4.2.1 Optimal state estimation problem

As suggested in [11] state space model is extended by stochastic measurement inputs $\mathbf{w}_\mathrm{n}$ and disturbance signals $\mathbf{w}_\mathrm{d}$ (process noise):

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{w}_\mathrm{d} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{w}_\mathrm{n} \\ \mathbf{x}(0) &= \mathbf{x}_0\end{aligned} \tag{4.16}$$

Stochastic signals are assumed to be uncorrelated zero-mean ergodic Gaussian stochastic processes:

$$\begin{aligned}E\{\mathbf{w}_\mathrm{d}(t)\} &= E\{\mathbf{w}_\mathrm{n}(t)\} = 0 \\ E\{\mathbf{w}_\mathrm{d}(t)\mathbf{w}_\mathrm{n}^T(\tau)\} &= E\{\mathbf{w}_\mathrm{n}(t)\mathbf{w}_\mathrm{d}^T(\tau)\} = \mathbf{0} \\ E\{\mathbf{w}_\mathrm{d}(t)\mathbf{w}_\mathrm{d}^T(\tau)\} &= \mathbf{W}\delta(t-\tau) \\ E\{\mathbf{w}_\mathrm{n}(t)\mathbf{w}_\mathrm{n}^T(\tau)\} &= \mathbf{V}\delta(t-\tau) \\ \\ \mathbf{W} &= \mathbf{W}^T \geqslant 0 \\ \mathbf{V} &= \mathbf{V}^T > 0\end{aligned} \tag{4.17}$$

$\mathbf{W}$ and $\mathbf{V}$ are the covariance matrices of respectively $\mathbf{w}_\mathrm{d}$ and $\mathbf{w}_\mathrm{n}$.

The task is to find a dynamic system that estimates the states $\mathbf{x}$ of the system in 4.16 employing inputs $\mathbf{u}$ and measurements $\mathbf{y}$, in a way that the estimation error's dynamic is asymptotically stable and the expected value of the estimation error $E\{(\mathbf{x}-\hat{\mathbf{x}})^T(\mathbf{x}-\hat{\mathbf{x}})\}$ is minimal.

The solution of optimal state estimation problem is called Kalman filter:

Definition 10. ***Kalman filter*** *(see [11])*
The Kalman filter has the structure of an ordinary state estimator or observer with

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{K}_\mathrm{f}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}) \tag{4.18}$$

The optimal choice of $\mathbf{K}_\mathrm{f}$*, which minimizes* $E\{(\mathbf{x}-\hat{\mathbf{x}})^T(\mathbf{x}-\hat{\mathbf{x}})\}$*, is given by*

$$\mathbf{K}_\mathrm{f} = \mathbf{Y}\mathbf{C}^T\mathbf{V}^{-1} \tag{4.19}$$

where $\mathbf{Y} = \mathbf{Y}^T \geqslant 0$ *is the unique positive semi-definite solution of the algebraic Riccati equation*

$$\mathbf{Y}\mathbf{A}^T + \mathbf{A}\mathbf{Y} - \mathbf{Y}\mathbf{C}^T\mathbf{V}^{-1}\mathbf{C}\mathbf{Y} + \mathbf{W} = \mathbf{0} \tag{4.20}$$

Kalman filter - preconditions

According to chapter 4.1 preconditions for existence of unique positive semi-definite solution $\mathbf{Y}$ of the algebraic Riccati equation (4.20) have to be adapted. Comparison to equation 4.1 shows the parameter equivalences:

$$\begin{aligned} \mathbf{A} &\rightarrow \mathbf{A}^T \\ \mathbf{R} &\rightarrow -\mathbf{C}^T\mathbf{V}^{-1}\mathbf{C} \\ \mathbf{Q} &\rightarrow \mathbf{W} \end{aligned} \tag{4.21}$$

Due to $\mathbf{V} > 0$ the inverse $\mathbf{V}^{-1}$ exists and is positive definite too ($\mathbf{V}^{-1} > 0$). In consequence of remark 7 and provided that $\mathbf{C}$ has full column rank, $\mathbf{C}^T\mathbf{V}^{-1}\mathbf{C}$ is positive definite and $-\mathbf{C}^T\mathbf{V}^{-1}\mathbf{C}$ is negative definite. Therefore the condition $\mathbf{R} \leqslant 0$ holds due to definition of $\mathbf{V}$.

According to remark 14 following two conditions ensure a positive semi-definite solution of the algebraic Riccati equation in 4.20:

1. Pair $(\mathbf{A}, \mathbf{W})$ has to be stabilizable.
2. Pair $(\mathbf{C}, \mathbf{A})$ has to be detectable.

4.2.2 Reduced Kalman filter

In this section system reduction is applied on a Kalman filter system:

$$\dot{\hat{\mathbf{x}}} = [\mathbf{A} - \mathbf{K}_\mathrm{f}\mathbf{C}]\,\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{K}_\mathrm{f}\mathbf{y} \tag{4.22}$$

Although it does not appear in equation 4.22 mass matrix $\bar{\mathbf{M}}$ plays a significant roll in system reduction.

$$\dot{\hat{\mathbf{q}}} = \underbrace{\left[\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}\mathbf{J}_{\mathrm{x}}\right]^{-1}\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}}_{\tilde{\mathbf{J}}_{\mathrm{x}}^T}\left[\mathbf{A}-\mathbf{K}_{\mathrm{f}}\mathbf{C}\right]\mathbf{J}_{\mathrm{x}}\hat{\mathbf{q}}\dots$$

$$\dots + \underbrace{\left[\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}\mathbf{J}_{\mathrm{x}}\right]^{-1}\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}}_{\tilde{\mathbf{J}}_{\mathrm{x}}^T}\mathbf{B}\mathbf{u} + \underbrace{\left[\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}\mathbf{J}_{\mathrm{x}}\right]^{-1}\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}}_{\tilde{\mathbf{J}}_{\mathrm{x}}^T}\mathbf{K}_{\mathrm{f}}\mathbf{y}$$

$$\dot{\hat{\mathbf{q}}} = \underbrace{\tilde{\mathbf{J}}_{\mathrm{x}}^T\mathbf{A}\mathbf{J}_{\mathrm{x}}}_{\tilde{\mathbf{A}}}\hat{\mathbf{q}} - \underbrace{\tilde{\mathbf{J}}_{\mathrm{x}}^T\mathbf{K}_{\mathrm{f}}}_{\tilde{\mathbf{K}}_{\mathrm{f},1}}\underbrace{\mathbf{C}\mathbf{J}_{\mathrm{x}}}_{\tilde{\mathbf{C}}}\hat{\mathbf{q}} + \underbrace{\tilde{\mathbf{J}}_{\mathrm{x}}^T\mathbf{B}}_{\tilde{\mathbf{B}}}\mathbf{u} + \underbrace{\tilde{\mathbf{J}}_{\mathrm{x}}^T\mathbf{K}_{\mathrm{f}}}_{\tilde{\mathbf{K}}_{\mathrm{f},1}}\mathbf{y}$$

$$\dot{\hat{\mathbf{q}}} = \left[\tilde{\mathbf{A}} - \tilde{\mathbf{K}}_{\mathrm{f},1}\tilde{\mathbf{C}}\right]\hat{\mathbf{q}} + \tilde{\mathbf{B}}\mathbf{u} + \tilde{\mathbf{K}}_{\mathrm{f},1}\mathbf{y} \tag{4.23}$$

As shown in equations 4.23 system reduction applied on the Kalman filter leads to a reduced Kalman filter. Nevertheless this does not allow any conclusions about its optimality in terms of chapter 4.2.1. Therefore the reduced Kalman filter has to be compared to a Kalman filter applied on the reduced system.

4.2.3 Kalman filter on reduced system

Reducing the noised system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{w}_{\mathrm{d}} \tag{4.24}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{w}_{\mathrm{n}}$$

according to equation 2.37, leads to:

$$\dot{\mathbf{q}} = \tilde{\mathbf{A}}\mathbf{q} + \tilde{\mathbf{B}}\mathbf{u} + \tilde{\mathbf{J}}_{\mathrm{x}}^T\mathbf{w}_{\mathrm{d}} \tag{4.25}$$
$$\mathbf{y} = \tilde{\mathbf{C}}\mathbf{q} + \mathbf{w}_{\mathrm{n}}$$

Hence covariance of process noise has to be recalculated, whereas system reduction has no influence on the measurement noise covariance matrix:

$$E\{\tilde{\mathbf{J}}_{\mathrm{x}}^T\mathbf{w}_{\mathrm{d}}(t)\mathbf{w}_{\mathrm{d}}^T(t)\tilde{\mathbf{J}}_{\mathrm{x}}\} = \tilde{\mathbf{J}}_{\mathrm{x}}^T E\{\mathbf{w}_{\mathrm{d}}(t)\mathbf{w}_{\mathrm{d}}^T(t)\}\tilde{\mathbf{J}}_{\mathrm{x}} = \underbrace{\tilde{\mathbf{J}}_{\mathrm{x}}^T\mathbf{W}\tilde{\mathbf{J}}_{\mathrm{x}}}_{\tilde{\mathbf{W}}}\delta(t-\tau) \tag{4.26}$$

$$\mathbf{V} = \tilde{\mathbf{V}}$$

According to chapter 4.2.1 the Kalman filter on the reduced system is given by

$$\dot{\hat{\mathbf{q}}} = \left[\tilde{\mathbf{A}} - \tilde{\mathbf{K}}_{\mathrm{f},2}\tilde{\mathbf{C}}\right]\hat{\mathbf{q}} + \tilde{\mathbf{B}}\mathbf{u} + \tilde{\mathbf{K}}_{\mathrm{f},2}\mathbf{y} \tag{4.27}$$
$$\tilde{\mathbf{K}}_{\mathrm{f},2} = \tilde{\mathbf{Y}}\tilde{\mathbf{C}}^T\tilde{\mathbf{V}}^{-1} \tag{4.28}$$

where $\tilde{\mathbf{Y}} = \tilde{\mathbf{Y}}^T \geqslant 0$ is the unique positive semi-definite solution of the algebraic Riccati equation

$$\tilde{\mathbf{Y}}\tilde{\mathbf{A}}^T + \tilde{\mathbf{A}}\tilde{\mathbf{Y}} - \tilde{\mathbf{Y}}\tilde{\mathbf{C}}^T\tilde{\mathbf{V}}^{-1}\tilde{\mathbf{C}}\tilde{\mathbf{Y}} + \tilde{\mathbf{W}} = \mathbf{0} \tag{4.29}$$

According to chapter 4.2.1 two preconditions have to be mentioned:

1. Pair $(\tilde{\mathbf{A}}, \tilde{\mathbf{W}})$ has to be stabilizable.
2. Pair $(\tilde{\mathbf{C}}, \tilde{\mathbf{A}})$ has to be detectable.

4.2.4 Evaluation of the conservation of optimality

The point of interest is, if the reduced Kalman filter equals the Kalman filter designed for the reduced system. The answer can be achieved by comparing the two feedback matrices $\tilde{\mathbf{K}}_{\mathrm{f},1}$ and $\tilde{\mathbf{K}}_{\mathrm{f},2}$ and the corresponding algebraic Riccati equations:

$$\begin{aligned} \textbf{I:} \quad & \mathbf{Y}\mathbf{A}^T + \mathbf{A}\mathbf{Y} - \mathbf{Y}\mathbf{C}^T\mathbf{V}^{-1}\mathbf{C}\mathbf{Y} + \mathbf{W} = \mathbf{0} \\ & \tilde{\mathbf{K}}_{\mathrm{f},1} = \tilde{\mathbf{J}}_{\mathrm{x}}^T\mathbf{Y}\mathbf{C}^T\mathbf{V}^{-1} \end{aligned} \tag{4.30}$$

$$\begin{aligned} \textbf{II:} \quad & \tilde{\mathbf{Y}}\tilde{\mathbf{A}}^T + \tilde{\mathbf{A}}\tilde{\mathbf{Y}} - \tilde{\mathbf{Y}}\tilde{\mathbf{C}}^T\tilde{\mathbf{V}}^{-1}\tilde{\mathbf{C}}\tilde{\mathbf{Y}} + \tilde{\mathbf{W}} = \mathbf{0} \\ & \tilde{\mathbf{K}}_{\mathrm{f},2} = \tilde{\mathbf{Y}}\tilde{\mathbf{C}}^T\tilde{\mathbf{V}}^{-1} \end{aligned} \tag{4.31}$$

Further exist the already known relations between the parameters of the original and the reduced systems:

$$\begin{aligned} \tilde{\mathbf{A}} &= \tilde{\mathbf{J}}_{\mathrm{x}}^T\mathbf{A}\mathbf{J}_{\mathrm{x}} = \left[\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}\mathbf{J}_{\mathrm{x}}\right]^{-1}\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}\mathbf{A}\mathbf{J}_{\mathrm{x}} \\ \tilde{\mathbf{C}} &= \mathbf{C}\mathbf{J}_{\mathrm{x}} \\ \tilde{\mathbf{W}} &= \tilde{\mathbf{J}}_{\mathrm{x}}^T\mathbf{W}\tilde{\mathbf{J}}_{\mathrm{x}} = \left[\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}\mathbf{J}_{\mathrm{x}}\right]^{-1}\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}\mathbf{W}\bar{\mathbf{M}}^T\mathbf{J}_{\mathrm{x}}\left(\left[\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}\mathbf{J}_{\mathrm{x}}\right]^{-1}\right)^T \\ \tilde{\mathbf{V}} &= \mathbf{V} \end{aligned} \tag{4.32}$$

The assumption that feedback matrices are equal leads to a relation between $\mathbf{Y}$ and $\tilde{\mathbf{Y}}$:

$$\begin{aligned} \tilde{\mathbf{K}}_{\mathrm{f},1} &\stackrel{!}{=} \tilde{\mathbf{K}}_{\mathrm{f},2} \\ \tilde{\mathbf{J}}_{\mathrm{x}}^T\mathbf{Y}\mathbf{C}^T\mathbf{V}^{-1} &\stackrel{!}{=} \tilde{\mathbf{Y}}\tilde{\mathbf{C}}^T\tilde{\mathbf{V}}^{-1} \\ \tilde{\mathbf{J}}_{\mathrm{x}}^T\mathbf{Y}\mathbf{C}^T &\stackrel{!}{=} \tilde{\mathbf{Y}}\mathbf{J}_{\mathrm{x}}^T\mathbf{C}^T \\ \left[\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}\mathbf{J}_{\mathrm{x}}\right]^{-1}\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}\mathbf{Y}\mathbf{C}^T &\stackrel{!}{=} \tilde{\mathbf{Y}}\mathbf{J}_{\mathrm{x}}^T\mathbf{C}^T \\ \mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}\mathbf{Y}\mathbf{C}^T &\stackrel{!}{=} \mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}\mathbf{J}_{\mathrm{x}}\tilde{\mathbf{Y}}\mathbf{J}_{\mathrm{x}}^T\mathbf{C}^T \\ \left[\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}\right]\cdot\mathbf{Y}\cdot\mathbf{C}^T &\stackrel{!}{=} \left[\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}\right]\cdot\left[\mathbf{J}_{\mathrm{x}}\tilde{\mathbf{Y}}\mathbf{J}_{\mathrm{x}}^T\right]\cdot\mathbf{C}^T \end{aligned} \tag{4.33}$$

$$\left[\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}\right]\left[\mathbf{Y} - \mathbf{J}_{\mathrm{x}}\tilde{\mathbf{Y}}\mathbf{J}_{\mathrm{x}}^T\right]\mathbf{C}^T \stackrel{!}{=} \mathbf{0} \tag{4.34}$$

Equation 4.34 can be transfered into eight sufficient conditions, assuming $\bar{\mathbf{M}}$ is a diagonal, positive definite matrix:

$$\mathbf{1)} \quad \mathbf{C} \overset{!}{=} \mathbf{0} \tag{4.35}$$

$$\mathbf{2)} \quad \mathbf{J}_{\mathrm{x}} \overset{!}{=} \mathbf{0} \tag{4.36}$$

$$\mathbf{3)} \quad \bar{\mathbf{M}} \overset{!}{=} \mathbf{0} \tag{4.37}$$

$$\mathbf{4)} \quad \mathbf{Y} \overset{!}{=} \mathbf{0}, \quad \tilde{\mathbf{Y}} \overset{!}{=} \mathbf{0} \tag{4.38}$$

$$\mathbf{5)} \quad \mathbf{Y} - \mathbf{J}_{\mathrm{x}}\tilde{\mathbf{Y}}\mathbf{J}_{\mathrm{x}}^T \overset{!}{=} \mathbf{0} \tag{4.39}$$

$$\mathbf{6)} \quad \mathbf{Y} - \mathbf{J}_{\mathrm{x}}\tilde{\mathbf{Y}}\mathbf{J}_{\mathrm{x}}^T \in \mathcal{N}(\mathbf{C}) \tag{4.40}$$

$$\mathbf{7)} \quad \mathcal{C}\left(\left[\mathbf{Y} - \mathbf{J}_{\mathrm{x}}\tilde{\mathbf{Y}}\mathbf{J}_{\mathrm{x}}^T\right]\right) \in \mathcal{C}\left(\mathbf{J}_{\mathrm{f}}^T\right) = \mathcal{N}\left(\mathbf{J}_{\mathrm{x}}^T\right) \tag{4.41}$$

$$\mathbf{8)} \quad \mathcal{C}\left(\left[\mathbf{Y} - \mathbf{J}_{\mathrm{x}}\tilde{\mathbf{Y}}\mathbf{J}_{\mathrm{x}}^T\right]\mathbf{C}^T\right) \in \mathcal{C}\left(\mathbf{J}_{\mathrm{f}}^T\right) = \mathcal{N}\left(\mathbf{J}_{\mathrm{x}}^T\right) \tag{4.42}$$

Where $\mathcal{C}(\mathbf{Z})$ denotes the column space of a matrix $\mathbf{Z}$.

At least one of these sufficient conditions has to hold in order to ensure that equation 4.34 holds. Due to the fact that equations 4.35 to 4.38 are obviously trivial, it is useful to further investigate conditions 4.39 to 4.42.

The effect of condition 4.39 can be investigated by considering the respective Riccati equations. Applying relations between system parameters in the second Riccati equation (4.31) achieves:

$$\tilde{\mathbf{Y}}\mathbf{J}_{\mathrm{x}}^T\mathbf{A}^T\tilde{\mathbf{J}}_{\mathrm{x}} + \tilde{\mathbf{J}}_{\mathrm{x}}^T\mathbf{A}\mathbf{J}_{\mathrm{x}}\tilde{\mathbf{Y}} + \tilde{\mathbf{J}}_{\mathrm{x}}^T\mathbf{W}\tilde{\mathbf{J}}_{\mathrm{x}} - \tilde{\mathbf{Y}}\mathbf{J}_{\mathrm{x}}^T\mathbf{C}^T\mathbf{V}^{-1}\mathbf{C}\mathbf{J}_{\mathrm{x}}\tilde{\mathbf{Y}} = \mathbf{0} \tag{4.43}$$

Since $\mathbf{J}_{\mathrm{x}}$ is a full column rank $(n \times q)$ matrix, with n>q, the following equation is necessary and sufficient to equation 4.43.

$$\mathbf{J}_{\mathrm{x}}\tilde{\mathbf{Y}}\mathbf{J}_{\mathrm{x}}^T\mathbf{A}^T\tilde{\mathbf{J}}_{\mathrm{x}}\mathbf{J}_{\mathrm{x}}^T + \mathbf{J}_{\mathrm{x}}\tilde{\mathbf{J}}_{\mathrm{x}}^T\mathbf{A}\mathbf{J}_{\mathrm{x}}\tilde{\mathbf{Y}}\mathbf{J}_{\mathrm{x}}^T + \mathbf{J}_{\mathrm{x}}\tilde{\mathbf{J}}_{\mathrm{x}}^T\mathbf{W}\tilde{\mathbf{J}}_{\mathrm{x}}\mathbf{J}_{\mathrm{x}}^T - \mathbf{J}_{\mathrm{x}}\tilde{\mathbf{Y}}\mathbf{J}_{\mathrm{x}}^T\mathbf{C}^T\mathbf{V}^{-1}\mathbf{C}\mathbf{J}_{\mathrm{x}}\tilde{\mathbf{Y}}\mathbf{J}_{\mathrm{x}}^T = \mathbf{0} \tag{4.44}$$

Using the sufficient relation between the two algebraic Riccati equations (equation 4.39) leads to:

$$\mathbf{Y}\mathbf{A}^T\left[\mathbf{J}_{\mathrm{x}}\tilde{\mathbf{J}}_{\mathrm{x}}^T\right]^T + \left[\mathbf{J}_{\mathrm{x}}\tilde{\mathbf{J}}_{\mathrm{x}}^T\right]\mathbf{A}\mathbf{Y} + \left[\mathbf{J}_{\mathrm{x}}\tilde{\mathbf{J}}_{\mathrm{x}}^T\right]\mathbf{W}\left[\mathbf{J}_{\mathrm{x}}\tilde{\mathbf{J}}_{\mathrm{x}}^T\right]^T - \mathbf{Y}\mathbf{C}^T\mathbf{V}^{-1}\mathbf{C}\mathbf{Y} = \mathbf{0} \tag{4.45}$$

Consequently comparison to the first Riccati (4.30) equation gives a sufficient condition for the equality of $\tilde{\mathbf{K}}_{\mathrm{f},1}$ and $\tilde{\mathbf{K}}_{\mathrm{f},2}$:

$$\begin{aligned} &\mathbf{Y}\mathbf{A}^T\left[\mathbf{J}_{\mathrm{x}}\tilde{\mathbf{J}}_{\mathrm{x}}^T\right]^T + \left[\mathbf{J}_{\mathrm{x}}\tilde{\mathbf{J}}_{\mathrm{x}}^T\right]\mathbf{A}\mathbf{Y} + \left[\mathbf{J}_{\mathrm{x}}\tilde{\mathbf{J}}_{\mathrm{x}}^T\right]\mathbf{W}\left[\mathbf{J}_{\mathrm{x}}\tilde{\mathbf{J}}_{\mathrm{x}}^T\right]^T - \mathbf{Y}\mathbf{C}^T\mathbf{V}^{-1}\mathbf{C}\mathbf{Y} = \mathbf{0} \\ &\mathbf{J}_{\mathrm{x}}\tilde{\mathbf{J}}_{\mathrm{x}}^T = \mathbf{J}_{\mathrm{x}}\left[\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}\mathbf{J}_{\mathrm{x}}\right]^{-1}\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}} \overset{!}{=} \mathbf{I} \end{aligned} \tag{4.46}$$

Remark 15.

The Matrix $\mathbf{J}_{\mathrm{x}}\tilde{\mathbf{J}}_{\mathrm{x}}^T$ *is idempotent for arbitrary matrices* $\bar{\mathbf{M}}$ *and* $\mathbf{J}_{\mathrm{x}}$*. In consequence it has eigenvalues at* $\lambda_i = 1 \ / \ 0, \quad i = 1, ..., n$*.*

Condition 4.46 holds for arbitrary invertible matrices $\mathbf{J}_\mathrm{x}$, but generally not for non quadratic matrices $\mathbf{J}_\mathrm{x}$ used in system reduction (see chapter 2.4).

In general also conditions 4.40, 4.41 and 4.42 do not hold.

Remark 16.

System reduction in general does not conserve optimality of an ordinary state estimator.

4.2.5 Conclusion on system reduction and Kalman filtering

According to remark 16 it is in general not possible to design an optimal Kalman filter in order to optimally estimate the states of a linear multibody system regardless of any additional linear constraint. Although stability of the estimation error dynamic can be possibly ensured (see chapter 3), performance of the state estimation in dependency of additional constraining of the mechanical system can not be evaluated in general.

4.3 System reduction and optimality of LQ optimal control

The *linear quadratic regulator* (LQR) is a linear state feedback, that minimizes a quadratic cost function. A similar approach to the problem of optimal state estimation can be used to consider the influence of system reduction on a LQ optimal control. As already stated in the introduction of this chapter, the motivation of this task is to design only one optimal state feedback for optimal control of a linear multibody system regardless of any additional linear constraint applied to the system.

At the beginning of this section the LQR problem is introduced in general and its solvability and solution is presented. Application of system reduction on a system controlled by an LQ optimal feedback matrix delivers a reduced LQ optimal feedback matrix. Condition for conservation of optimality is the equivalence of this reduced LQ optimal feedback matrix $\tilde{\mathbf{K}}_{\mathrm{r},1}$ and the LQ optimal feedback matrix $\tilde{\mathbf{K}}_{\mathrm{r},2}$ achieved by solving the LQR problem on the reduced system.

4.3.1 LQR problem

Definition 11. ***Optimal state feedback*** *(see [11])*

The LQR problem, is a deterministic initial value problem: given the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ *with a non-zero initial state* $\mathbf{x}(0)$, *find the input signal u(t) which takes the system to the zero state (*$\mathbf{x} = 0$*) in an optimal manner, i.e. by minimizing the deterministic cost*

$$J = \int_0^\infty \left(\mathbf{x}(t)^T\mathbf{Q}\mathbf{x}(t) + \mathbf{u}(t)^T\mathbf{R}\mathbf{u}(t)\right)\mathrm{d}t, \tag{4.47}$$

with $\mathbf{Q} = \mathbf{Q}^T \geqslant 0$ *and* $\mathbf{R} = \mathbf{R}^T > 0$.

The optimal solution (for any initial state) is $\mathbf{u}(t) = -\mathbf{K}_\mathrm{r}\mathbf{x}(t)$*, where*

$$\mathbf{K}_\mathrm{r} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{X} \tag{4.48}$$

and $\mathbf{X} = \mathbf{X}^T \geqslant 0$ *is the unique positive semi-definite solution of the algebraic Riccati equation*

$$\mathbf{A}^T\mathbf{X} + \mathbf{X}\mathbf{A} - \mathbf{X}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{X} + \mathbf{Q} = \mathbf{0} \tag{4.49}$$

Optimal state feedback - preconditions

According to chapter 4.1 preconditions for existence of unique positive semi-definite solution $\mathbf{X}$ of the algebraic Riccati equation (4.49) have to be adapted. Comparison to equation 4.1 shows the parameter equivalences:

$$\mathbf{R} = \mathbf{R}' \quad \rightarrow \quad -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T \tag{4.50}$$

Due to $\mathbf{R} > 0$ the inverse $\mathbf{R}^{-1}$ exists and is positive definite. In consequence of remark 7 and provide matrix $\mathbf{B}$ has full column rank, $\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T$ is positive definite and therefore $-\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T$ is negative definite. In consequence condition three ($\mathbf{R}' \leqslant 0$) holds due to definition of $\mathbf{R}$.

According to remark 14 following two conditions ensure a positive semi-definite solution of the algebraic Riccati equation in 4.20:

1. Pair $(\mathbf{A}, \mathbf{B})$ has to be stabilizable.
2. Pair $(\mathbf{Q}, \mathbf{A})$ has to be detectable.

Remark 17.
According to remark 13 the stricter condition, pair $(\mathbf{Q}, \mathbf{A})$ *observable implicates condition 2. and further guarantees a unique positive definite solution* $\mathbf{X} > 0$ *of the algebraic Riccati equation in 4.49.*

4.3.2 Reduced LQ optimal control

System reduction can be applied on the LQ optimal controlled system:

$$\dot{\mathbf{x}} = [\mathbf{A} - \mathbf{B}\mathbf{K}_\mathrm{r}]\,\mathbf{x} \tag{4.51}$$

$$\begin{aligned}\dot{\mathbf{q}} &= \underbrace{\tilde{\mathbf{J}}_\mathrm{x}^T\mathbf{A}\mathbf{J}_\mathrm{x}}_{\tilde{\mathbf{A}}}\,\mathbf{q} - \underbrace{\tilde{\mathbf{J}}_\mathrm{x}^T\mathbf{B}}_{\tilde{\mathbf{B}}}\,\underbrace{\mathbf{K}_\mathrm{r}\mathbf{J}_\mathrm{x}}_{\tilde{\mathbf{K}}_{\mathrm{r},1}}\,\mathbf{q} \\ \dot{\mathbf{q}} &= \left[\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\tilde{\mathbf{K}}_{\mathrm{r},1}\right]\mathbf{q}\end{aligned} \tag{4.52}$$

Equation 4.52 shows that the resulting system can again be interpreted as controlled system with state feedback matrix $\tilde{\mathbf{K}}_\mathrm{r}$.

4.3.3 LQ optimal control of reduced system

In order to analyze optimality of this state feedback, the LQ optimal control has to be applied on the reduced system in equation 4.53.

$$\dot{\mathbf{q}} = \tilde{\mathbf{A}}\mathbf{q} + \tilde{\mathbf{B}}\mathbf{u} \tag{4.53}$$

According to the coordinate transformation $\mathbf{x} = \mathbf{J}_\mathrm{x}\mathbf{q}$ the deterministic cost has to be adapted:

$$J = \int_0^\infty (\mathbf{q}(t)^T \underbrace{\mathbf{J}_\mathrm{x}^T\mathbf{Q}\mathbf{J}_\mathrm{x}}_{\tilde{\mathbf{Q}}} \mathbf{q}(t) + \mathbf{u}(t)^T\mathbf{R}\mathbf{u}(t))\mathrm{d}t \tag{4.54}$$

$$\tilde{\mathbf{Q}} = \mathbf{J}_\mathrm{x}^T\mathbf{Q}\mathbf{J}_\mathrm{x}$$
$$\tilde{\mathbf{R}} = \mathbf{R}$$

The solution of the LQR problem on the reduced system therefore is determined by:

$$\mathbf{u}(t) = \tilde{\mathbf{K}}_{\mathrm{r},2}\mathbf{q}(t) \tag{4.55}$$
$$\tilde{\mathbf{K}}_{\mathrm{r},2} = \tilde{\mathbf{R}}^{-1}\tilde{\mathbf{B}}^T\tilde{\mathbf{X}}$$

where $\tilde{\mathbf{X}} = \tilde{\mathbf{X}}^T \geqslant 0$ is the unique positive semi-definite solution of the algebraic Riccati equation

$$\tilde{\mathbf{A}}^T\tilde{\mathbf{X}} + \tilde{\mathbf{X}}\tilde{\mathbf{A}} - \tilde{\mathbf{X}}\tilde{\mathbf{B}}\tilde{\mathbf{R}}^{-1}\tilde{\mathbf{B}}^T\tilde{\mathbf{X}} + \tilde{\mathbf{Q}} = \mathbf{0} \tag{4.56}$$

According to chapter 4.3.1 two preconditions have to be mentioned:

1. Pair $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}})$ has to be stabilizable.
2. Pair $(\tilde{\mathbf{Q}}, \tilde{\mathbf{A}})$ has to be detectable.

4.3.4 Evaluation of the conservation of optimality

Comparing the reduced LQ optimal control and the LQ optimal control of the reduced system represented by its feedback matrices and their corresponding algebraic Riccati equations provides information about the conservation of optimality with respect to system reduction.

$$\mathbf{I:}\quad \mathbf{A}^T\mathbf{X} + \mathbf{X}\mathbf{A} - \mathbf{X}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{X} + \mathbf{Q} = \mathbf{0} \tag{4.57}$$
$$\tilde{\mathbf{K}}_{\mathrm{r},1} = \mathbf{R}^{-1}\mathbf{B}^T\mathbf{X}$$

$$\mathbf{II:}\quad \tilde{\mathbf{A}}^T\tilde{\mathbf{X}} + \tilde{\mathbf{X}}\tilde{\mathbf{A}} - \tilde{\mathbf{X}}\tilde{\mathbf{B}}\tilde{\mathbf{R}}^{-1}\tilde{\mathbf{B}}^T\tilde{\mathbf{X}} + \tilde{\mathbf{Q}} = \mathbf{0} \tag{4.58}$$
$$\tilde{\mathbf{K}}_{\mathrm{r},2} = \tilde{\mathbf{R}}^{-1}\tilde{\mathbf{B}}^T\tilde{\mathbf{X}}$$

Stating the equality of feedback matrices achieves a relation between the inverse solution of the algebraic Riccati equations. These inverses $\mathbf{X}^{-1}$ and $\tilde{\mathbf{X}}^{-1}$ only exist assuming $\mathbf{X}$ and $\tilde{\mathbf{X}}$ are positive definite. Therefore the pairs $(\mathbf{Q}, \mathbf{A})$ and $(\tilde{\mathbf{Q}}, \tilde{\mathbf{A}})$ have to be observable (see remark 13).

$$\tilde{\mathbf{K}}_{\mathrm{r},1} \overset{!}{=} \tilde{\mathbf{K}}_{\mathrm{r},2} \tag{4.59}$$

$$\mathbf{R}^{-1}\mathbf{B}^T\mathbf{X}\mathbf{J}_{\mathrm{x}} \overset{!}{=} \tilde{\mathbf{R}}^{-1}\tilde{\mathbf{B}}^T\tilde{\mathbf{X}}$$

$$\mathbf{B}^T\mathbf{X}\mathbf{J}_{\mathrm{x}} \overset{!}{=} \mathbf{B}^T\tilde{\mathbf{J}}_{\mathrm{x}}\tilde{\mathbf{X}} \tag{4.60}$$

$$\mathbf{X}\mathbf{J}_{\mathrm{x}} \overset{!}{=} \tilde{\mathbf{J}}_{\mathrm{x}}\tilde{\mathbf{X}} \qquad *$$

$$\mathbf{J}_{\mathrm{x}}\tilde{\mathbf{X}}^{-1} \overset{!}{=} \mathbf{X}^{-1}\tilde{\mathbf{J}}_{\mathrm{x}}$$

$$\mathbf{J}_{\mathrm{x}}\tilde{\mathbf{X}}^{-1} \overset{!}{=} \mathbf{X}^{-1}\bar{\mathbf{M}}^T\mathbf{J}_{\mathrm{x}}\left[\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}\mathbf{J}_{\mathrm{x}}\right]^{-T}$$

$$\mathbf{J}_{\mathrm{x}}\tilde{\mathbf{X}}^{-1}\left[\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}\mathbf{J}_{\mathrm{x}}\right]^{T} \overset{!}{=} \mathbf{X}^{-1}\bar{\mathbf{M}}^T\mathbf{J}_{\mathrm{x}}$$

$$\mathbf{J}_{\mathrm{x}}\tilde{\mathbf{X}}^{-1}\mathbf{J}_{\mathrm{x}}^T\bar{\mathbf{M}}^T\mathbf{J}_{\mathrm{x}} \overset{!}{=} \mathbf{X}^{-1}\bar{\mathbf{M}}^T\mathbf{J}_{\mathrm{x}}$$

$$\mathbf{J}_{\mathrm{x}}\tilde{\mathbf{X}}^{-1}\mathbf{J}_{\mathrm{x}}^T \overset{!}{=} \mathbf{X}^{-1} \qquad * \tag{4.61}$$

The *-marked equations are only sufficient each with respect to the above equation, but not necessary.

To apply relation 4.61 the algebraic Riccati equations in 4.57 have to be formulated in terms of the inverse solutions:

$$\mathbf{A}^T\mathbf{X} + \mathbf{X}\mathbf{A} - \mathbf{X}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{X} + \mathbf{Q} = \mathbf{0}$$

$$\mathbf{X}^{-1}\mathbf{A}^T\mathbf{X} + \mathbf{X}^{-1}\mathbf{X}\mathbf{A} - \mathbf{X}^{-1}\mathbf{X}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{X} + \mathbf{X}^{-1}\mathbf{Q} = \mathbf{0}$$

$$\mathbf{X}^{-1}\mathbf{A}^T\mathbf{X}\mathbf{X}^{-1} + \mathbf{X}^{-1}\mathbf{X}\mathbf{A}\mathbf{X}^{-1} - \mathbf{X}^{-1}\mathbf{X}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{X}\mathbf{X}^{-1} + \mathbf{X}^{-1}\mathbf{Q}\mathbf{X}^{-1} = \mathbf{0}$$

$$\mathbf{X}^{-1}\mathbf{A}^T + \mathbf{A}\mathbf{X}^{-1} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T + \mathbf{X}^{-1}\mathbf{Q}\mathbf{X}^{-1} = \mathbf{0}$$

$$\mathbf{X}^{-1}\mathbf{A}^T + \mathbf{A}\mathbf{X}^{-1} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T + \mathbf{X}^{-1}\mathbf{Q}\mathbf{X}^{-1} = \mathbf{0} \tag{4.62}$$

$$\tilde{\mathbf{X}}^{-1}\tilde{\mathbf{A}}^T + \tilde{\mathbf{A}}\tilde{\mathbf{X}}^{-1} - \tilde{\mathbf{B}}\tilde{\mathbf{R}}^{-1}\tilde{\mathbf{B}}^T + \tilde{\mathbf{X}}^{-1}\tilde{\mathbf{Q}}\tilde{\mathbf{X}}^{-1} = \mathbf{0} \tag{4.63}$$

According to chapter 4.3.1 the solutions of the algebraic Riccati equations in terms of the inverse solutions (4.62 and 4.63) for sure have unique positive definite solutions if

1. Pair $(-\mathbf{A}^T, -\mathbf{Q})$ is stabilizable
2. Pair $(-\tilde{\mathbf{A}}^T, -\tilde{\mathbf{Q}})$ is stabilizable
3. Pair $(\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T, -\mathbf{A}^T)$ is observable
4. Pair $(\tilde{\mathbf{B}}\tilde{\mathbf{R}}^{-1}\tilde{\mathbf{B}}^T, -\tilde{\mathbf{A}}^T)$ is observable

These conditions can be reformulated be use of remark 14:

1. Pair $(-\mathbf{Q}, -\mathbf{A})$ is detectable
2. Pair $(-\tilde{\mathbf{Q}}, -\tilde{\mathbf{A}})$ is detectable
3. Pair $(-\mathbf{A}^T, \mathbf{B})$ is controllable
4. Pair $(-\tilde{\mathbf{A}}^T, \tilde{\mathbf{B}})$ is controllable

The relation between the two solutions in 4.61, the algebraic Riccati equations in terms of the inverse solutions (4.62 and 4.63) and the relations between the parameters of the original and the reduced system in 4.64 can be used to formulate a sufficient condition (4.67) for equality of feedback matrices (4.59).

$$\begin{aligned}
\tilde{\mathbf{A}} &= \tilde{\mathbf{J}}_{\mathrm{x}}^T \mathbf{A} \mathbf{J}_{\mathrm{x}} = \left[\mathbf{J}_{\mathrm{x}}^T \bar{\mathbf{M}} \mathbf{J}_{\mathrm{x}}\right]^{-1} \mathbf{J}_{\mathrm{x}}^T \bar{\mathbf{M}} \mathbf{A} \mathbf{J}_{\mathrm{x}} \\
\tilde{\mathbf{B}} &= \tilde{\mathbf{J}}_{\mathrm{x}}^T \mathbf{B} \\
\tilde{\mathbf{Q}} &= \mathbf{J}_{\mathrm{x}}^T \mathbf{Q} \mathbf{J}_{\mathrm{x}} \\
\tilde{\mathbf{R}} &= \mathbf{R}
\end{aligned} \tag{4.64}$$

In the first step relations between system parameters were applied in the inverse formulation of the second Riccati equation (4.63):

$$\begin{aligned}
&\tilde{\mathbf{X}}^{-1} \mathbf{J}_{\mathrm{x}}^T \mathbf{A}^T \tilde{\mathbf{J}}_{\mathrm{x}} + \tilde{\mathbf{J}}_{\mathrm{x}}^T \mathbf{A} \mathbf{J}_{\mathrm{x}} \tilde{\mathbf{X}}^{-1} - \tilde{\mathbf{J}}_{\mathrm{x}}^T \mathbf{B} \tilde{\mathbf{R}}^{-1} \mathbf{B}^T \tilde{\mathbf{J}}_{\mathrm{x}} + \tilde{\mathbf{X}}^{-1} \mathbf{J}_{\mathrm{x}}^T \mathbf{Q} \mathbf{J}_{\mathrm{x}} \tilde{\mathbf{X}}^{-1} = \mathbf{0} \\
&\mathbf{J}_{\mathrm{x}} \tilde{\mathbf{X}}^{-1} \mathbf{J}_{\mathrm{x}}^T \mathbf{A}^T \tilde{\mathbf{J}}_{\mathrm{x}} \mathbf{J}_{\mathrm{x}}^T + \mathbf{J}_{\mathrm{x}} \tilde{\mathbf{J}}_{\mathrm{x}}^T \mathbf{A} \mathbf{J}_{\mathrm{x}} \tilde{\mathbf{X}}^{-1} \mathbf{J}_{\mathrm{x}}^T - \mathbf{J}_{\mathrm{x}} \tilde{\mathbf{J}}_{\mathrm{x}}^T \mathbf{B} \tilde{\mathbf{R}}^{-1} \mathbf{B}^T \tilde{\mathbf{J}}_{\mathrm{x}} \mathbf{J}_{\mathrm{x}}^T \dots \\
&\qquad \dots + \mathbf{J}_{\mathrm{x}} \tilde{\mathbf{X}}^{-1} \mathbf{J}_{\mathrm{x}}^T \mathbf{Q} \mathbf{J}_{\mathrm{x}} \tilde{\mathbf{X}}^{-1} \mathbf{J}_{\mathrm{x}}^T = \mathbf{0}
\end{aligned} \tag{4.65}$$

Then the sufficient relations between the solutions of the inverse formulated algebraic Riccati equations can be used:

$$\mathbf{X}^{-1} \mathbf{A}^T \left[\mathbf{J}_{\mathrm{x}} \tilde{\mathbf{J}}_{\mathrm{x}}^T\right]^T + \left[\mathbf{J}_{\mathrm{x}} \tilde{\mathbf{J}}_{\mathrm{x}}^T\right] \mathbf{A} \mathbf{X}^{-1} - \left[\mathbf{J}_{\mathrm{x}} \tilde{\mathbf{J}}_{\mathrm{x}}^T\right] \mathbf{B} \tilde{\mathbf{R}}^{-1} \mathbf{B}^T \left[\mathbf{J}_{\mathrm{x}} \tilde{\mathbf{J}}_{\mathrm{x}}^T\right]^T + \mathbf{X}^{-1} \mathbf{Q} \mathbf{X}^{-1} = \mathbf{0} \tag{4.66}$$

Finally the sufficient condition for equality of the feedback matrices $\tilde{\mathbf{K}}_{\mathrm{r},1}$ and $\tilde{\mathbf{K}}_{\mathrm{r},2}$ can be stated by comparing 4.66 to the inverse formulation of the first Riccati equation (4.62).

$$\begin{aligned}
&\mathbf{X}^{-1} \mathbf{A}^T \left[\mathbf{J}_{\mathrm{x}} \tilde{\mathbf{J}}_{\mathrm{x}}^T\right]^T + \left[\mathbf{J}_{\mathrm{x}} \tilde{\mathbf{J}}_{\mathrm{x}}^T\right] \mathbf{A} \mathbf{X}^{-1} - \left[\mathbf{J}_{\mathrm{x}} \tilde{\mathbf{J}}_{\mathrm{x}}^T\right] \mathbf{B} \tilde{\mathbf{R}}^{-1} \mathbf{B}^T \left[\mathbf{J}_{\mathrm{x}} \tilde{\mathbf{J}}_{\mathrm{x}}^T\right]^T + \mathbf{X}^{-1} \mathbf{Q} \mathbf{X}^{-1} = \mathbf{0} \qquad (4.67) \\
&\mathbf{J}_{\mathrm{x}} \tilde{\mathbf{J}}_{\mathrm{x}}^T = \mathbf{J}_{\mathrm{x}} \left[\mathbf{J}_{\mathrm{x}}^T \bar{\mathbf{M}} \mathbf{J}_{\mathrm{x}}\right]^{-1} \mathbf{J}_{\mathrm{x}}^T \bar{\mathbf{M}} \overset{!}{=} \mathbf{I}
\end{aligned}$$

Analogous to the optimal state estimator condition 4.67 holds for arbitrary invertible matrices $\mathbf{J}_{\mathrm{x}}$, but generally not for non quadratic matrices $\mathbf{J}_{\mathrm{x}}$ used in system reduction (see chapter 2.4).

Remark 18.

System reduction in general does not conserve optimality of a state feedback.

4.3.5 Conclusion on system reduction and optimality of LQ optimal control

The motivation of this chapter was to design an optimal LQ optimal control to a linear multibody system regardless of any additional linear constraint. As already stated in remark 18 and expected due to section 4.2.5 this is not possible in general. Consequently although stability of all reduced controlled subsystems can be ensured (see chapter 3), loss of optimality can in general cause bad control performance.

Nevertheless exemplary application in chapter 6 shows that for certain problems loss of optimality has hardly no influence on the control performance (see section 6.5.3). Therefore, for these problems it is indeed possible to solve the LQR problem on the unconstrained multibody system and use reduced controllers in constrained case. Calculation of these suboptimal (in the sense of: not optimal anymore) reduced controllers is determined by an arbitrary basis to the null space of the Jacobian matrix of the constraints.

5

Drivetrain modeling employing system reduction

From the mechanical point of view general drivetrain topologies have to be considered as multibody systems. Further locked clutches correspond to the application of additional linear holonomic constraints. Therefore the task of this chapter is a practical application of the methods discussed in chapter 2 on general drivetrain modeling.

First, linear modeling of general drivetrain topologies is discussed. In a next step the constraints due to locked clutches are investigated employing unconstrained system parameters. Applying system reduction finally achieves the transformation between the drivetrain model considering all clutches to be slipping, i. e. not locked, and the drivetrain model considering certain clutches to be locked.

5.1 Typical drivetrain configuration

Dynamic of a drivetrain topology is uniquely determined by the motions of its inertias. Since the motion of inertias are restricted to rotations, they can be described employing moments of inertias $J_i > 0$ $(i = 1, \ldots, N)$ and corresponding rotational coordinates φ_i $(i = 1, \ldots, N)$.

$$\begin{aligned} \mathbf{M} &= \operatorname{diag}\begin{pmatrix} J_1 & \ldots & J_N \end{pmatrix} = \mathbf{M}^T > 0 \\ \boldsymbol{\varphi} &= \begin{bmatrix} \varphi_1 & \ldots & \varphi_N \end{bmatrix} \end{aligned} \tag{5.1}$$

Net forces acting on inertias consequently are net torques. These torques $\boldsymbol{\tau}$ (see equation 5.2) represent the summation of internal torques $\boldsymbol{\tau}_{\text{in}}$ and external torques $\boldsymbol{\tau}_{\text{ex}}$. Internal torques $\boldsymbol{\tau}_{\text{in}}$ (see equation 5.3) describe on the one hand interaction between inertias over flexible shafts modeled by spring torques $\boldsymbol{\tau}_{\text{k}}$ and damper torques $\boldsymbol{\tau}_{\text{d}}$, and on the other hand torques due to kinetic friction $\boldsymbol{\tau}_{\text{kin}}$ within the drivetrain. External torques $\boldsymbol{\tau}_{\text{ex}}$ (see equation 5.4) acting on inertias in the drivetrain appear as controllable torques including propulsion torques applied by combustion engines $\boldsymbol{\tau}_{\text{E}}$ and electric motors $\boldsymbol{\tau}_{\text{M}}$ and torques impressed in slipping clutches $(\bar{\boldsymbol{\tau}}_{\text{C}})$. Additionally external torques $\boldsymbol{\tau}_{\text{ex}}$ consider non controllable torques $\boldsymbol{\tau}_{\text{V}}$ applied due to gravity (influence of road slope), rolling resistances and air drag.

$$\boldsymbol{\tau} = \boldsymbol{\tau}_{\text{in}} + \boldsymbol{\tau}_{\text{ex}} \tag{5.2}$$
$$\boldsymbol{\tau}_{\text{in}} = \boldsymbol{\tau}_{\text{kin}} + \boldsymbol{\tau}_{\text{k}} + \boldsymbol{\tau}_{\text{d}} \tag{5.3}$$
$$\boldsymbol{\tau}_{\text{ex}} = \boldsymbol{\tau}_{\text{E}} + \boldsymbol{\tau}_{\text{M}} + \bar{\boldsymbol{\tau}}_{\text{C}} + \boldsymbol{\tau}_{\text{V}} \tag{5.4}$$

Clutches can be considered as elements switching between a torque impressing state in slipping case and rigid shaft state in locked case. In a first step all clutches are considered to be slipping. Transmitted torque of a slipping clutch ($\boldsymbol{\tau}_{\text{C}}$) is a function of pressure applied on clutch plates. In practice actuation of the clutch plates is done by controlling the current in a coil acting on a valve. This valve converts a given constant volume flow to pressure on clutch plates by cross section modification. A simplification in modeling clutch actuation is to enable direct torque impression between clutch plates. This torque impression has to be strictly dissipative, i. e. there is no way to impress energy on a clutch. Therefore its sign is defined by the sign of differential angular velocity between the clutch plates.

To simplify the later consideration of locking clutches it is useful to split the external torques into clutch torques $\bar{\boldsymbol{\tau}}_{\text{C}}$ and remaining torques $\bar{\boldsymbol{\tau}}_{\text{R}}$ acting on inertias:

$$\boldsymbol{\tau}_{\text{ex}} = \underbrace{\boldsymbol{\tau}_{\text{E}} + \boldsymbol{\tau}_{\text{M}} + \boldsymbol{\tau}_{\text{V}}}_{\bar{\boldsymbol{\tau}}_{\text{R}}} + \bar{\boldsymbol{\tau}}_{\text{C}} \tag{5.5}$$

In order to achieve a linear behavior of the multibody system (see chapter 2) internal torques $\boldsymbol{\tau}_{\text{in}}$ have to be linear functions of either angular positions or angular velocities (see equation 5.6).

$$\begin{aligned} \boldsymbol{\tau}_{\text{kin}} &= -\mathbf{D}_{\text{kin}}\dot{\boldsymbol{\varphi}} \\ \boldsymbol{\tau}_{\text{d}} &= -\mathbf{D}_{\text{d}}\dot{\boldsymbol{\varphi}} \\ \boldsymbol{\tau}_{\text{k}} &= -\mathbf{K}\boldsymbol{\varphi} \end{aligned} \tag{5.6}$$

$$\boldsymbol{\tau}_{\text{in}} = -\mathbf{D}_{\text{kin}}\dot{\boldsymbol{\varphi}} - \mathbf{D}_{\text{d}}\dot{\boldsymbol{\varphi}} - \mathbf{K}\boldsymbol{\varphi} = -\mathbf{D}\dot{\boldsymbol{\varphi}} - \mathbf{K}\boldsymbol{\varphi} \tag{5.7}$$

As already stated in equation 2.6 matrices $\mathbf{K}$, $\mathbf{D}_{\text{d}}$ and $\mathbf{D}_{\text{kin}}$ have the following properties:

$$\begin{aligned} &\mathbf{K} = \mathbf{K}^T \geqslant 0 \\ &\mathbf{D}_{\text{kin}} = \mathbf{D}_{\text{kin}}^T > 0 \\ &\mathbf{D}_{\text{d}} = \mathbf{D}_{\text{d}}^T \geqslant 0 \\ &\mathbf{D} = \mathbf{D}_{\text{d}} + \mathbf{D}_{\text{kin}} > 0 \end{aligned} \tag{5.8}$$

Also action of external torques $\boldsymbol{\tau}_{\text{ex}}$ on inertias is considered to be linear:

$$\boldsymbol{\tau}_{\text{ex}} = \bar{\boldsymbol{\tau}}_{\text{R}} + \bar{\boldsymbol{\tau}}_{\text{C}} = \begin{bmatrix} \mathbf{B}_{\text{R}} & \mathbf{B}_{\text{C}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\tau}_{\text{R}} \\ \boldsymbol{\tau}_{\text{C}} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{\text{E}} & \mathbf{B}_{\text{M}} & \mathbf{B}_{\text{V}} & \mathbf{B}_{\text{C}} \end{bmatrix} \begin{bmatrix} \tau_{\text{E}} \\ \tau_{\text{M}} \\ \tau_{\text{V}} \\ \boldsymbol{\tau}_{\text{C}} \end{bmatrix} \tag{5.9}$$

Other common drivetrain elements are gear ratios and planetary gear sets. They can be incorporated into stiffness and damping matrices ($\mathbf{K}$, $\mathbf{D}$) and input matrices $\mathbf{B}_{\mathrm{R}}$ and $\mathbf{B}_{\mathrm{C}}$.

5.2 Unconstrained equations of motion

Applying Newton's laws of motion delivers a set of N linear differential equations second order (see also equation 2.5):

$$\begin{aligned} \mathbf{M}\ddot{\boldsymbol{\varphi}} &= \boldsymbol{\tau} \\ \mathbf{M}\ddot{\boldsymbol{\varphi}} &= -\mathbf{D}\dot{\boldsymbol{\varphi}} - \mathbf{K}\boldsymbol{\varphi} + \mathbf{B}_{\mathrm{R}}\boldsymbol{\tau}_{\mathrm{R}} + \mathbf{B}_{\mathrm{C}}\boldsymbol{\tau}_{\mathrm{C}} \end{aligned} \tag{5.10}$$

5.3 Unconstrained state space model

Substitution

$$\tilde{\mathbf{x}} = \begin{bmatrix} \boldsymbol{\varphi}^T & \dot{\boldsymbol{\varphi}}^T \end{bmatrix}^T \tag{5.11}$$

allows to create a state space model consisting of $2N$ differential equations of first order (see chapter 2.6.2).

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\varphi}} \\ \ddot{\boldsymbol{\varphi}} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varphi} \\ \dot{\boldsymbol{\varphi}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_{\mathrm{R}} \end{bmatrix} \boldsymbol{\tau}_{\mathrm{R}} + \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_{\mathrm{C}} \end{bmatrix} \boldsymbol{\tau}_{\mathrm{C}} \tag{5.12}$$

Actually in most transmission use-cases rotational speeds represent the task of interest, therefore it is sufficient to use only speeds $\dot{\boldsymbol{\varphi}}$ and necessary components of $\boldsymbol{\varphi}$ in state vector $\mathbf{x}$. The necessary components of $\boldsymbol{\varphi}$ are those which describe spring and damper forces due to flexible shafts. The simplest choice are the angular differences describing the tension of the flexible shafts in the drivetrain topology. Definition of a new state vector $\mathbf{x}$ (equation 5.13) reduces the system to $N+g$ linear differential equations first order assuming a number of g flexible shafts.

$$\mathbf{x} = \begin{bmatrix} \Delta\varphi_1 & \dots & \Delta\varphi_g & \dot{\varphi}_1 & \dots & \dot{\varphi}_N \end{bmatrix}^T = \mathbf{T} \begin{bmatrix} \boldsymbol{\varphi} \\ \dot{\boldsymbol{\varphi}} \end{bmatrix} \tag{5.13}$$

Introduced transformation $\mathbf{T}$ is not a regular state transformation, since matrix $\mathbf{T}$ is not quadratic. In consequence there exists no inverse. Nevertheless the transformation can be partitioned into a regular transformation $\mathbf{T}_1$ changing g angular coordinates into angular difference coordinates (see equations 5.14 and 5.15) and a non-regular transformation $\mathbf{T}_2$ eliminating the remaining angular coordinates from state vector $\mathbf{x}$ (see equations 5.16 and 5.18). For second transformation columns resp. rows in every matrix resp. vectors referring to angular coordinates have to be eliminated. Therefore generalized right inverse $\mathbf{T}_2^+$ (see equation 5.17) can be used.

$$\begin{bmatrix} \Delta\varphi_1 & \dots & \Delta\varphi_g & \varphi_{g+1} & \dots & \varphi_N & \dot{\varphi}_1 & \dots & \dot{\varphi}_N \end{bmatrix}^T = \mathbf{T}_1 \begin{bmatrix} \boldsymbol{\varphi} \\ \dot{\boldsymbol{\varphi}} \end{bmatrix} \tag{5.14}$$

$$\mathbf{T}_1 \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \mathbf{T}_1^{-1} \mathbf{T}_1 \begin{bmatrix} \dot{\boldsymbol{\varphi}} \\ \ddot{\boldsymbol{\varphi}} \end{bmatrix} = \mathbf{T}_1 \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix} \mathbf{T}_1^{-1} \mathbf{T}_1 \begin{bmatrix} \boldsymbol{\varphi} \\ \dot{\boldsymbol{\varphi}} \end{bmatrix} + \mathbf{T}_1 \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_\mathrm{R} \end{bmatrix} \boldsymbol{\tau}_\mathrm{R} + \mathbf{T}_1 \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_\mathrm{C} \end{bmatrix} \boldsymbol{\tau}_\mathrm{C} \tag{5.15}$$

$$\mathbf{x} = \begin{bmatrix} \Delta\varphi_1 & \dots & \Delta\varphi_g & \dot{\varphi}_1 & \dots & \dot{\varphi}_N \end{bmatrix}^T = \mathbf{T}_2 \mathbf{T}_1 \begin{bmatrix} \boldsymbol{\varphi} \\ \dot{\boldsymbol{\varphi}} \end{bmatrix} \tag{5.16}$$

$$\mathbf{T}_2^+ = \mathbf{T}_2^T \left[\mathbf{T}_2 \mathbf{T}_2^T \right]^{-1} \tag{5.17}$$

$$\underbrace{\mathbf{T}_2 \mathbf{T}_1 \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \mathbf{T}_1^{-1} \mathbf{T}_2^+}_{\bar{\mathbf{M}}} \underbrace{\mathbf{T}_2 \mathbf{T}_1 \begin{bmatrix} \dot{\boldsymbol{\varphi}} \\ \ddot{\boldsymbol{\varphi}} \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\mathbf{T}_2 \mathbf{T}_1 \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K} & -\mathbf{D} \end{bmatrix} \mathbf{T}_1^{-1} \mathbf{T}_2^+}_{\bar{\mathbf{A}}} \underbrace{\mathbf{T}_2 \mathbf{T}_1 \begin{bmatrix} \boldsymbol{\varphi} \\ \dot{\boldsymbol{\varphi}} \end{bmatrix}}_{\mathbf{x}} \dots$$
$$\dots + \underbrace{\mathbf{T}_2 \mathbf{T}_1 \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_\mathrm{R} \end{bmatrix}}_{\bar{\mathbf{B}}_\mathrm{R}} \boldsymbol{\tau}_\mathrm{R} + \underbrace{\mathbf{T}_2 \mathbf{T}_1 \begin{bmatrix} \mathbf{0} \\ \mathbf{B}_\mathrm{C} \end{bmatrix}}_{\bar{\mathbf{B}}_\mathrm{C}} \boldsymbol{\tau}_\mathrm{C} \tag{5.18}$$

$$\bar{\mathbf{M}} \dot{\mathbf{x}} = \bar{\mathbf{A}} \mathbf{x} + \bar{\mathbf{B}}_\mathrm{R} \boldsymbol{\tau}_\mathrm{R} + \bar{\mathbf{B}}_\mathrm{C} \boldsymbol{\tau}_\mathrm{C} = \bar{\mathbf{A}} \mathbf{x} + \underbrace{\begin{bmatrix} \bar{\mathbf{B}}_\mathrm{R} & \bar{\mathbf{B}}_\mathrm{C} \end{bmatrix}}_{\bar{\mathbf{B}}} \underbrace{\begin{bmatrix} \boldsymbol{\tau}_\mathrm{R} \\ \boldsymbol{\tau}_\mathrm{C} \end{bmatrix}}_{\mathbf{u}} \tag{5.19}$$

$$\dot{\mathbf{x}} = \underbrace{\bar{\mathbf{M}}^{-1} \bar{\mathbf{A}}}_{\mathbf{A}} \mathbf{x} + \underbrace{\bar{\mathbf{M}}^{-1} \bar{\mathbf{B}}}_{\mathbf{B}} \mathbf{u} \tag{5.20}$$

Remark 19.

- *The state vector* $\mathbf{x}$ *consists of* g *angular differences and* N *rotational speeds. Where* g *is the number of flexible shafts and* N *is the number of inertias in the drivetrain topology.*
- *Matrix* $\bar{\mathbf{M}}$ *is diagonal. It includes moments of inertias and unity masses, due to* g *additional entries in* $\mathbf{x}$*. Therefore it is positive definite and in consequence invertible:*

$$\bar{\mathbf{M}} = \bar{\mathbf{M}}^T > 0 \tag{5.21}$$

- *Input matrix* $\bar{\mathbf{B}}$ *contains zero rows ensuring that external torques and clutch torques can only act on inertias.*
- *In the special case* $\mathbf{K} = \mathbf{0}$ *the angular difference states can be removed. Therefore* $\bar{\mathbf{A}}$ *and consequently* $\mathbf{A}$ *become symmetric matrices:*

$$\bar{\mathbf{A}} = \bar{\mathbf{A}}^T \quad \Rightarrow \quad \mathbf{A} = \mathbf{A}^T \tag{5.22}$$

- *Since state transformations* $\mathbf{T}_1$ *and* $\mathbf{T}_2$ *do not change angular velocities (see remark 6) it is valid to use common state transformation without regarding conservation of system energy and further mechanical constraints (see chapter 2.6.3).*

5.4 System reduction due to locked clutches

Locking one or several clutches corresponds to introducing holonomic, scleronomic and linear resp. $\boldsymbol{\varphi}$ constraints into the mechanical system. $\boldsymbol{\tau}_{\mathrm{C}}$ can be separated into slipping and locking clutches:

$$\mathbf{M}\ddot{\boldsymbol{\varphi}} = \underbrace{-\mathbf{D}\dot{\boldsymbol{\varphi}} - \mathbf{K}\boldsymbol{\varphi} + \mathbf{B}_{\mathrm{R}}\boldsymbol{\tau}_{\mathrm{R}} + \mathbf{B}_{\mathrm{C,sl}}\boldsymbol{\tau}_{\mathrm{C,sl}}}_{\mathbf{F}} + \mathbf{B}_{\mathrm{C,lk}}\boldsymbol{\tau}_{\mathrm{C,lk}} \tag{5.23}$$

$$\mathbf{M}\ddot{\boldsymbol{\varphi}} = \mathbf{F} + \mathbf{B}_{\mathrm{C,lk}}\boldsymbol{\tau}_{\mathrm{C,lk}} \tag{5.24}$$

5.4.1 Dual Space of linear mapping

For proofing relation between $\mathbf{B}_{\mathrm{C,lk}}$ and constraints due to locked clutches the concept of **dual space of linear mapping** is used.

Definition 12. ***Dual space of linear mapping (see [5])***
To every linear mapping:

$$\mathbf{A} : \mathcal{V} \to \mathcal{V}' \tag{5.25}$$

*there exists a **dual mapping**:*

$$\mathbf{A}^T : \mathcal{V}^* \to \mathcal{V}'^* \tag{5.26}$$

satisfying:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}', \mathbf{y}' \rangle \quad | \quad \mathbf{x} \in \mathcal{V},\ \mathbf{x}' \in \mathcal{V}',\ \mathbf{y} \in \mathcal{V}^* \textit{ and } \mathbf{y}' \in \mathcal{V}'^* \ . \tag{5.27}$$

*$\mathcal{V}^*$ and $\mathcal{V}'^*$ are called **dual spaces** of $\mathcal{V}$ and $\mathcal{V}'$*

Figure 5.1 summarizes the relations used in definition 12.

$$\begin{array}{ccc} \mathbf{x} \in \mathcal{V} & \xrightarrow{\mathbf{A}} & \mathbf{x}' \in \mathcal{V}' \\ \downarrow & & \downarrow \\ \langle \mathbf{x}, \mathbf{y} \rangle & = & \langle \mathbf{x}', \mathbf{y}' \rangle \\ \uparrow & & \uparrow \\ \mathbf{y} \in \mathcal{V}^* & \xleftarrow{\mathbf{A}^T} & \mathbf{y}' \in \mathcal{V}'^* \end{array}$$

Figure 5.1: Illustration to the dual space of linear mapping

Due to conservation of energy in mechanics transformation of force and velocity, resp. torque and angular velocity, form a dual mapping (see definition 12). According to equation 5.23, $\mathbf{B}_{\mathrm{C}}$ maps clutch torques $\boldsymbol{\tau}_{\mathrm{C}}$ to resulting torques $\bar{\boldsymbol{\tau}}_{\mathrm{C}}$ acting directly on inertias:

$$\bar{\boldsymbol{\tau}}_{\mathrm{C}} = \mathbf{B}_{\mathrm{C}}\boldsymbol{\tau}_{\mathrm{C}} \tag{5.28}$$

Due to the equilibrium of power the following equation holds:

$$\langle \bar{\boldsymbol{\tau}}_{\mathrm{C}}, \dot{\boldsymbol{\varphi}} \rangle = \langle \boldsymbol{\tau}_{\mathrm{C}}, \Delta\dot{\boldsymbol{\varphi}}_{\mathrm{C}} \rangle, \tag{5.29}$$

Consequently mapping of inertia's velocities $\dot{\boldsymbol{\varphi}}$ to differential speeds on clutches $\Delta\dot{\boldsymbol{\varphi}}_{\mathrm{C}}$ must fulfill:

$$\Delta\dot{\boldsymbol{\varphi}}_{\mathrm{C}} = \mathbf{B}_{\mathrm{C}}^{T}\dot{\boldsymbol{\varphi}} \tag{5.30}$$

5.4.2 Interpretation of locked clutches' torques

One condition to enable clutch locking is:

$$\Delta\dot{\boldsymbol{\varphi}}_{\mathrm{C,lk}} = \mathbf{0} \tag{5.31}$$

According to equation 5.30 constraints due to locking clutches can also be formulated in terms of inertias' angular velocities:

$$\mathbf{B}_{\mathrm{C,lk}}^{T}\dot{\boldsymbol{\varphi}} = \tilde{\mathbf{f}}(\dot{\boldsymbol{\varphi}}) = \mathbf{0} \tag{5.32}$$

Since this constraints are holonomic, their integrability is ensured. Consequently they have to hold also in terms of angular coordinates $\boldsymbol{\varphi}$:

$$\mathbf{B}_{\mathrm{C,lk}}^{T}\boldsymbol{\varphi} = \mathbf{f}(\boldsymbol{\varphi}) = \mathbf{0} \quad \Rightarrow \quad \mathbf{J}_{\mathrm{f}} = \frac{\partial}{\partial \boldsymbol{\varphi}}\left(\mathbf{B}_{\mathrm{C,lk}}^{T}\boldsymbol{\varphi}\right) = \mathbf{B}_{\mathrm{C,lk}}^{T} \tag{5.33}$$

In state space formulation the same considerations are valid:

$$\bar{\mathbf{B}}_{\mathrm{C,lk}}^{T}\mathbf{x} = \mathbf{f}(\mathbf{x}) = \mathbf{0} \quad \Rightarrow \quad \mathbf{J}_{\mathrm{f}} = \frac{\partial}{\partial \mathbf{x}}\left(\bar{\mathbf{B}}_{\mathrm{C,lk}}^{T}\mathbf{x}\right) = \bar{\mathbf{B}}_{\mathrm{C,lk}}^{T} \tag{5.34}$$

$$\begin{aligned}\bar{\mathbf{M}}\dot{\mathbf{x}} &= \underbrace{\bar{\mathbf{A}}\mathbf{x} + \begin{bmatrix}\bar{\mathbf{B}}_{\mathrm{R}} & \bar{\mathbf{B}}_{\mathrm{C,sl}}\end{bmatrix}\begin{bmatrix}\boldsymbol{\tau}_{\mathrm{R}}\\ \boldsymbol{\tau}_{\mathrm{C,sl}}\end{bmatrix}}_{\mathbf{F}} + \bar{\mathbf{B}}_{\mathrm{C,lk}}\boldsymbol{\tau}_{\mathrm{C,lk}}\\ \bar{\mathbf{M}}\dot{\mathbf{x}} &= \mathbf{F} + \bar{\mathbf{B}}_{\mathrm{C,lk}}\boldsymbol{\tau}_{\mathrm{C,lk}}\end{aligned} \tag{5.35}$$

Remark 20.

- *The transposed Jacobian matrix of constraints* $\mathbf{J}_{\mathrm{f}}^{T}$ *equals input matrix of locking clutch torques* $\mathbf{B}_{\mathrm{C,lk}}$ *resp.* $\bar{\mathbf{B}}_{\mathrm{C,lk}}$*, due to conservation of energy.*
- *Torques transmitted on locked clutches* $\boldsymbol{\tau}_{\mathrm{C,lk}}$ *assume role of Lagrangian multipliers (compare equation 2.18 and 5.24 resp. 5.35).*
- *Constraints can be formulated either in terms angular velocities* $\dot{\boldsymbol{\varphi}}$ *or angular positions* $\boldsymbol{\varphi}$*, because they are holonomic and therefore integrable. A constraint coupling angular positions* $\boldsymbol{\varphi}$ *and angular velocities* $\dot{\boldsymbol{\varphi}}$ *is non holonomic although such a constraint would still be linear in* $\mathbf{x}$ *(see also remark 3). Block structure of the matrix* $\bar{\mathbf{B}}_{\mathrm{C}}$ *(see equation 5.18) ensures that constraints due to locking clutches are holonomic.*

5.4.3 Application of system reduction

As shown in chapter 2 it is possible to eliminate Lagrangian multipliers (torques transmitted on locked clutches) and transform the system to a minimal set of differential equations, by finding a fitting basis $\mathbf{J}_\mathrm{x}$ to $\mathbf{J}_\mathrm{f}$'s null space.

$$\mathbf{J}_\mathrm{x}^T\bar{\mathbf{M}}\mathbf{J}_\mathrm{x}\dot{\mathbf{q}} = \mathbf{J}_\mathrm{x}^T\bar{\mathbf{A}}\mathbf{J}_\mathrm{x}\mathbf{q} + \mathbf{J}_\mathrm{x}^T\begin{bmatrix}\bar{\mathbf{B}}_\mathrm{R} & \bar{\mathbf{B}}_\mathrm{C}\end{bmatrix}\begin{bmatrix}\boldsymbol{\tau}_\mathrm{R}\\ \boldsymbol{\tau}_\mathrm{C}\end{bmatrix} \tag{5.36}$$

$$\tilde{\mathbf{M}}\dot{\mathbf{q}} = \mathbf{J}_\mathrm{x}^T\bar{\mathbf{A}}\mathbf{J}_\mathrm{x}\mathbf{q} + \mathbf{J}_\mathrm{x}^T\begin{bmatrix}\bar{\mathbf{B}}_\mathrm{R} & \bar{\mathbf{B}}_\mathrm{C}\end{bmatrix}\begin{bmatrix}\boldsymbol{\tau}_\mathrm{R}\\ \boldsymbol{\tau}_\mathrm{C}\end{bmatrix}$$

$$\dot{\mathbf{q}} = \underbrace{\tilde{\mathbf{M}}^{-1}\mathbf{J}_\mathrm{x}^T\bar{\mathbf{A}}\mathbf{J}_\mathrm{x}}_{\tilde{\mathbf{A}}}\mathbf{q} + \underbrace{\tilde{\mathbf{M}}^{-1}\mathbf{J}_\mathrm{x}^T\begin{bmatrix}\bar{\mathbf{B}}_\mathrm{R} & \bar{\mathbf{B}}_\mathrm{C}\end{bmatrix}}_{\begin{bmatrix}\tilde{\mathbf{B}}_\mathrm{R} & \tilde{\mathbf{B}}_\mathrm{C}\end{bmatrix}}\begin{bmatrix}\boldsymbol{\tau}_\mathrm{R}\\ \boldsymbol{\tau}_\mathrm{C}\end{bmatrix}$$

$$\dot{\mathbf{q}} = \tilde{\mathbf{A}}\mathbf{q} + \tilde{\mathbf{B}}\mathbf{u} \tag{5.37}$$

Remark 21.

- *The unreduced system (equation 5.19) considering all clutches to be in slipping state and the clutch state (containing the information which clutches are slipping resp. locking) form a set of reduced subsystems (equation 5.37) due to additional constraints caused by locked clutches.*
- *Due to chapter 3 stability of the original (unconstrained) system (equation 5.19) guarantees stability of all reduced subsystems (equation 5.37).*
- *Due to construction of $\mathbf{J}_\mathrm{x}$, $\tilde{\mathbf{B}}_\mathrm{C}$ contains zero columns, confirming that it is not possible to impress torques on locked clutches, i. e. for a locked clutch the corresponding input is deactivated.*

5.5 Conclusion

In consequence of considering the clutch state, defining which clutches are slipping resp. locked, to be a time varying quantity, general drivetrain models are switching systems. Standard drivetrain modeling requires calculation of the different systems according to all possible clutch states (two to the power of the number of clutches) in advance.

This chapter however shows that the dynamic of a general drivetrain is fully described by the mathematical model considering all clutches to be slipping, i. e. not locked, and the additional binary information of the clutch state. The essential point thereby is the fact that columns of the clutch torque input matrix, that correspond to locked clutches, define the transformation that is necessary for applying system reduction in order to achieve a minimal set of differential equations.

6

Exemplary application of system reduction in control

So far this work offers a general modeling approach to linear constrained linear multibody systems presented in chapter 2 and more specific to general drivetrain topologies presented in chapter 5. This approach is now applied to an exemplary hybrid drivetrain topology. Further the advantages of this approach and the mechanical peculiarities of the topology are used to design a model-based control system, controlling gear shifts, while tracking a required vehicle speed trajectory.

6.1 Topology and notation

The considered drivetrain topology (see figure 6.1) is hybrid, in terms of it contains both a combustion engine and a electric motor, automatic transmission drivetrain.

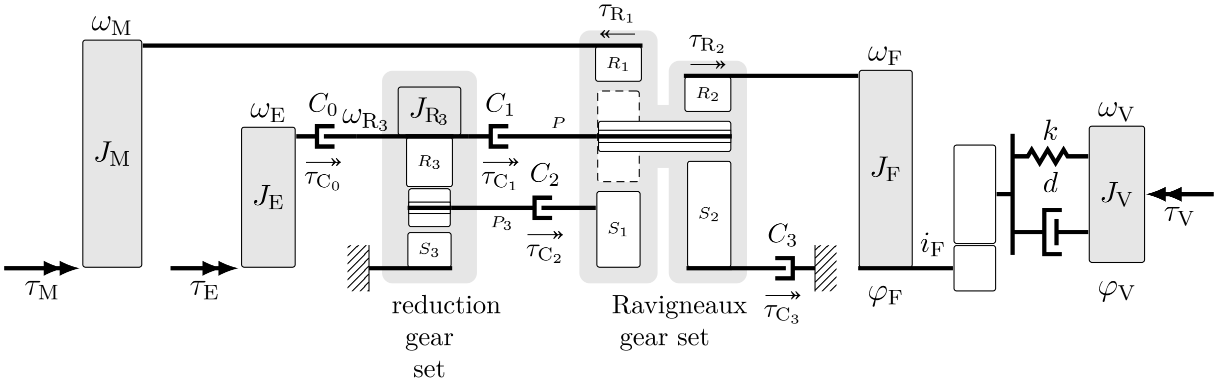

Figure 6.1: Simplified scheme of a hybrid-electric automatic transmission drivetrain [15]. For notation see table 6.1.

Notation			
Variables		Indices	
C	clutch	C	clutch
d	damping constant	E	combustion engine
i	gear ratio	d	damping
J	inertia	F	gearbox output
k	spring constant	J	inertia
τ	torque	kin	kinetic friction
ω	angular velocity	M	electric motor
φ	angular position	P	planet carrier (planetary gear set)
		R	ring (planetary gear set)
		S	sun (planetary gear set)
		V	vehicle

Table 6.1: Notation of the considered drivetrain topology (see figure 6.1)

6.2 Modeling

In this chapter the general method of drivetrain modeling presented in chapter 5 is shown in detail by an exemplary application. To keep track of the single steps here is an short overview:

1. Equations of motion (section 6.2.1)
2. Incorporation of gear ratios (section 6.2.2)
3. State space model (section 6.2.3)
4. State transformation: angular wheel speed $\mapsto$ vehicle speed (section 6.2.4)

6.2.1 Equations of motion

As shown in general in chapter 2 and more specific for drivetrain topologies in chapter 5, the dynamic of the presented drivetrain can be linearly modeled by a set of ordinary differential equations second order:

$$\mathbf{M}\ddot{\boldsymbol{\varphi}} = -\underbrace{\left[\mathbf{D}_{\mathrm{kin}} + \mathbf{D}_{\mathrm{d}}\right]}_{\mathbf{D}}\dot{\boldsymbol{\varphi}} - \mathbf{K}\boldsymbol{\varphi} + \mathbf{B}_{\mathrm{R}}\boldsymbol{\tau}_{\mathrm{R}} + \mathbf{B}_{\mathrm{C}}\boldsymbol{\tau}_{\mathrm{C}} \tag{6.1}$$

$$\begin{aligned} \boldsymbol{\varphi}^T &= \begin{bmatrix}\varphi_{\mathrm{E}} & \varphi_{\mathrm{R}_3} & \varphi_{\mathrm{M}} & \varphi_{\mathrm{F}} & \varphi_{\mathrm{V}}\end{bmatrix} \\ \boldsymbol{\tau}_{\mathrm{R}} &= \begin{bmatrix}\boldsymbol{\tau}_{\mathrm{E}} & \boldsymbol{\tau}_{\mathrm{M}} & \boldsymbol{\tau}_{\mathrm{V}}\end{bmatrix}^T \\ \mathbf{B}_{\mathrm{R}} &= \begin{bmatrix}\mathbf{B}_{\mathrm{E}} & \mathbf{B}_{\mathrm{M}} & \mathbf{B}_{\mathrm{V}}\end{bmatrix} \end{aligned} \tag{6.2}$$

Matrix $\mathbf{D}_{\mathrm{kin}}$ includes additional kinetic friction acting on each inertia. Disregarding gear ratio i_{F} and planetary gear sets in a first step, enables direct definition of matrices

$\mathbf{M}$, $\tilde{\mathbf{K}}$, $\mathbf{D}_{\mathrm{kin}}$, $\tilde{\mathbf{D}}_{\mathrm{d}}$ and $\mathbf{B}_{\mathrm{R}}$:

$$\mathbf{M} = \begin{bmatrix} J_{\mathrm{E}} & 0 & 0 & 0 & 0 \\ 0 & J_{R_3} & 0 & 0 & 0 \\ 0 & 0 & J_{\mathrm{M}} & 0 & 0 \\ 0 & 0 & 0 & J_{\mathrm{F}} & 0 \\ 0 & 0 & 0 & 0 & J_{\mathrm{V}} \end{bmatrix}, \qquad \tilde{\mathbf{K}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & k & -k \\ 0 & 0 & 0 & -k & k \end{bmatrix}$$

$$\mathbf{D}_{\mathrm{kin}} = \begin{bmatrix} d_{\mathrm{J}} & 0 & 0 & 0 & 0 \\ 0 & d_{\mathrm{J}} & 0 & 0 & 0 \\ 0 & 0 & d_{\mathrm{J}} & 0 & 0 \\ 0 & 0 & 0 & d_{\mathrm{J}} & 0 \\ 0 & 0 & 0 & 0 & d_{\mathrm{J}} \end{bmatrix}, \qquad \tilde{\mathbf{D}}_{\mathrm{d}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & -d \\ 0 & 0 & 0 & -d & d \end{bmatrix} \tag{6.3}$$

$$\mathbf{B}_{\mathrm{R}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{6.4}$$

Due to Newton's third law of motion (see chapter 2.1) matrices $\tilde{\mathbf{D}}_{\mathrm{d}}$ and $\tilde{\mathbf{K}}$ have off diagonal elements with converse signs.

6.2.2 Additional gear ratios

Final gear ratio

For correct incorporation of final gear ratio i_{F} in stiffness matrix $\mathbf{K}$, potential energy in the spring has to be considered:

$$V = \frac{k}{2}\left[\varphi_{\mathrm{F}} - \varphi_{\mathrm{V}}\right]^2 = \frac{1}{2}\boldsymbol{\varphi}^T\tilde{\mathbf{K}}\boldsymbol{\varphi} \tag{6.5}$$

After regarding gear ratio i_{F}, potential energy still has to be a quadratic function of angular positions $\boldsymbol{\varphi}$. Therefore matrix $\mathbf{K}$ has to be transformed:

$$\tilde{V} = \frac{k}{2}\left[\frac{\varphi_{\mathrm{F}}}{i_{\mathrm{F}}} - \varphi_{\mathrm{V}}\right]^2 \stackrel{!}{=} \frac{1}{2}\boldsymbol{\varphi}^T \underbrace{\mathbf{T}^T\tilde{\mathbf{K}}\mathbf{T}}_{\mathbf{K}} \boldsymbol{\varphi} \tag{6.6}$$

$$\Rightarrow \mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{i_{\mathrm{F}}} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.7}$$

The same consideration can be made on damping matrix $\tilde{\mathbf{D}}_{\mathrm{d}}$ and dissipation in the damper. Using the introduced transformation $\mathbf{T}$, stiffness matrix $\tilde{\mathbf{K}}$ and damping matrix $\tilde{\mathbf{D}}_\mathrm{d}$, can be transformed ($\rightarrow \mathbf{K}, \mathbf{D}_\mathrm{d}$), with respect to the gear ratio i_F:

$$\mathbf{K} = \mathbf{T}^T\tilde{\mathbf{K}}\mathbf{T} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{k}{i_\mathrm{F}^2} & -\frac{k}{i_\mathrm{F}} \\ 0 & 0 & 0 & -\frac{k}{i_\mathrm{F}} & k \end{bmatrix} \tag{6.8}$$

$$\mathbf{D}_\mathrm{d} = \mathbf{T}^T\tilde{\mathbf{D}}_\mathrm{d}\mathbf{T} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{d}{i_\mathrm{F}^2} & -\frac{d}{i_\mathrm{F}} \\ 0 & 0 & 0 & -\frac{d}{i_\mathrm{F}} & d \end{bmatrix} \tag{6.9}$$

Damping matrix $\mathbf{D}$ assemblies the damping due to kinetic friction $\mathbf{D}_\mathrm{kin}$ and the damping due to additional damping elements $\mathbf{D}_\mathrm{d}$:

$$\mathbf{D} = \mathbf{D}_\mathrm{kin} + \mathbf{D}_\mathrm{d} = \begin{bmatrix} d_\mathrm{J} & 0 & 0 & 0 & 0 \\ 0 & d_\mathrm{J} & 0 & 0 & 0 \\ 0 & 0 & d_\mathrm{J} & 0 & 0 \\ 0 & 0 & 0 & d_\mathrm{J} + \frac{d}{i_\mathrm{F}^2} & -\frac{d}{i_\mathrm{F}} \\ 0 & 0 & 0 & -\frac{d}{i_\mathrm{F}} & d_\mathrm{J} + d \end{bmatrix} \tag{6.10}$$

This is a simple approach to regard one additional gear ratio in drivetrain topology. It can be generalized to several gear ratios. Construction of transformation matrix $\mathbf{T}$ in that case becomes more complicate due to the fact that possibly one coordinate is affected by more than one gear ratio.

Planetary gear set

Gear ratios in planetary gear sets are described by the *Willis*-equation (see for example [16]). It introduces linear dependencies between angular velocities (resp. angular positions) of the connected shafts and constant stationary gear ratios.

In this example (reduction gear set and massless Ravigneaux gear set) the stationary planetary gear set ratios $i_\mathrm{S1}^\mathrm{R1}$, $i_\mathrm{S2}^\mathrm{R2}$, $i_\mathrm{S3}^\mathrm{R3}$ and $i_\mathrm{S1}^\mathrm{R2}$ fully describe the gear ratios between the connected shafts (see [16]). Therefore they are considered as given constants. In this example the outputs of the planetary gear set are rigidly connected on the one hand to inertias and on the other hand to clutches. Therefore, these constants can be used to define the mapping of clutch torques to torques acting directly on inertias, i. e. the input matrix $\mathbf{B}_\mathrm{C}$:

$$\mathbf{B}_\mathrm{C} = \mathbf{f}(i_\mathrm{S1}^\mathrm{R1}, i_\mathrm{S2}^\mathrm{R2}, i_\mathrm{S3}^\mathrm{R3}, i_\mathrm{S1}^\mathrm{R2}) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & i_\mathrm{P3}^\mathrm{R3} & 0 \\ 0 & i_\mathrm{P}^\mathrm{R1} & i_\mathrm{S1}^\mathrm{R1} & i_\mathrm{S2}^\mathrm{R1} \\ 0 & i_\mathrm{P}^\mathrm{R2} & i_\mathrm{S1}^\mathrm{R2} & i_\mathrm{S2}^\mathrm{R2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.11}$$

As shown in chapter 5 matrix $\mathbf{B}_\mathrm{C}$ on the one hand maps impressed clutch torques $\boldsymbol{\tau}_\mathrm{C}$ on torques $\bar{\boldsymbol{\tau}}_\mathrm{C}$ acting directly on inertias in slipping case and on the other hand defines the additional holonomic constraints in locking case.

6.2.3 State space model

In order to applicate common control methods it is useful to transform the mathematical model into state space form (see also chapters 2.6.2 and 5.3):

$$\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}\end{bmatrix}\begin{bmatrix}\dot{\boldsymbol{\varphi}}\\ \ddot{\boldsymbol{\varphi}}\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{K} & -\mathbf{D}\end{bmatrix}\begin{bmatrix}\boldsymbol{\varphi}\\ \dot{\boldsymbol{\varphi}}\end{bmatrix} + \begin{bmatrix}\mathbf{0}\\ \mathbf{B}_\mathrm{R}\end{bmatrix}\boldsymbol{\tau}_\mathrm{R} + \begin{bmatrix}\mathbf{0}\\ \mathbf{B}_\mathrm{C}\end{bmatrix}\boldsymbol{\tau}_\mathrm{C} \tag{6.12}$$

As already mentioned in chapter 5 the number of differential equations can be reduced due to the fact that angular velocities are of interest, but most angular positions are not.

$$\mathbf{x} = \begin{bmatrix}\dot{\varphi}_\mathrm{E} & \dot{\varphi}_{\mathrm{R}_3} & \dot{\varphi}_\mathrm{M} & \dot{\varphi}_\mathrm{F} & \dot{\varphi}_\mathrm{V} & \Delta\varphi_1\end{bmatrix} \tag{6.13}$$

$$\mathbf{x} = \underbrace{\begin{bmatrix}0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&1\\0&0&0&\frac{1}{i_\mathrm{F}}&-1&0&0&0&0&0\end{bmatrix}}_{\mathbf{T}=\mathbf{T}_2\mathbf{T}_1}\begin{bmatrix}\boldsymbol{\varphi}\\ \dot{\boldsymbol{\varphi}}\end{bmatrix}$$

The transformation achieves a set of 6 differential equations first order:

$$\underbrace{\mathbf{T}_2\mathbf{T}_1\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}\end{bmatrix}\mathbf{T}_1^{-1}\mathbf{T}_2^{+}}_{\bar{\mathbf{M}}_\mathrm{x}}\underbrace{\mathbf{T}_2\mathbf{T}_1\begin{bmatrix}\dot{\boldsymbol{\varphi}}\\ \ddot{\boldsymbol{\varphi}}\end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\mathbf{T}_2\mathbf{T}_1\begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{K} & -\mathbf{D}\end{bmatrix}\mathbf{T}_1^{-1}\mathbf{T}_2^{+}}_{\bar{\mathbf{A}}_\mathrm{x}}\underbrace{\mathbf{T}_2\mathbf{T}_1\begin{bmatrix}\boldsymbol{\varphi}\\ \dot{\boldsymbol{\varphi}}\end{bmatrix}}_{\mathbf{x}}\cdots$$

$$\cdots + \underbrace{\mathbf{T}_2\mathbf{T}_1\begin{bmatrix}\mathbf{0}\\ \mathbf{B}_\mathrm{R}\end{bmatrix}}_{\bar{\mathbf{B}}_\mathrm{x,R}}\boldsymbol{\tau}_\mathrm{R} + \underbrace{\mathbf{T}_2\mathbf{T}_1\begin{bmatrix}\mathbf{0}\\ \mathbf{B}_\mathrm{C}\end{bmatrix}}_{\bar{\mathbf{B}}_\mathrm{x,C}}\boldsymbol{\tau}_\mathrm{C} \tag{6.14}$$

$$\bar{\mathbf{M}}_\mathrm{x}\dot{\mathbf{x}} = \bar{\mathbf{A}}_\mathrm{x}\mathbf{x} + \bar{\mathbf{B}}_\mathrm{x,R}\boldsymbol{\tau}_\mathrm{R} + \bar{\mathbf{B}}_\mathrm{x,C}\boldsymbol{\tau}_\mathrm{C} \tag{6.15}$$

$$\bar{\mathbf{M}}_{\mathrm{x}} = \begin{bmatrix} J_{\mathrm{E}} & 0 & 0 & 0 & 0 & 0 \\ 0 & J_{R_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & J_{\mathrm{M}} & 0 & 0 & 0 \\ 0 & 0 & 0 & J_{\mathrm{F}} & 0 & 0 \\ 0 & 0 & 0 & 0 & J_{\mathrm{V}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\bar{\mathbf{A}}_{\mathrm{x}} = \begin{bmatrix} -d_{\mathrm{J}} & 0 & 0 & 0 & 0 & 0 \\ 0 & -d_{\mathrm{J}} & 0 & 0 & 0 & 0 \\ 0 & 0 & -d_{\mathrm{J}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -d_{\mathrm{J}} - \frac{d}{i_{\mathrm{F}}^2} & \frac{d}{i_{\mathrm{F}}} & -\frac{k}{i_{\mathrm{F}}} \\ 0 & 0 & 0 & \frac{d}{i_{\mathrm{F}}} & -d_{\mathrm{J}} - d & k \\ 0 & 0 & 0 & \frac{1}{i_{\mathrm{F}}} & -1 & 0 \end{bmatrix}$$

$$\bar{\mathbf{B}}_{\mathrm{x,R}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\bar{\mathbf{B}}_{\mathrm{x,C}} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & i_{\mathrm{P3}}^{\mathrm{R3}} & 0 \\ 0 & i_{\mathrm{P}}^{\mathrm{R1}} & i_{\mathrm{S1}}^{\mathrm{R1}} & i_{\mathrm{S2}}^{\mathrm{R1}} \\ 0 & i_{\mathrm{P}}^{\mathrm{R2}} & i_{\mathrm{S1}}^{\mathrm{R2}} & i_{\mathrm{S2}}^{\mathrm{R2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.16}$$

6.2.4 vehicle speed - state transformation

Definition of wheel radius r enables transformation of state $\dot{\varphi}_{\mathrm{V}}$ representing angular wheel speed to vehicle speed $v = r \cdot \dot{\varphi}_{\mathrm{V}}$.

$$\underbrace{\begin{bmatrix} \dot{\varphi}_{\mathrm{E}} \\ \dot{\varphi}_{\mathrm{R}_3} \\ \dot{\varphi}_{\mathrm{M}} \\ \dot{\varphi}_{\mathrm{F}} \\ \dot{\varphi}_{\mathrm{V}} \\ \Delta\varphi_1 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{T}} \underbrace{\begin{bmatrix} \dot{\varphi}_{\mathrm{E}} \\ \dot{\varphi}_{\mathrm{R}_3} \\ \dot{\varphi}_{\mathrm{M}} \\ \dot{\varphi}_{\mathrm{F}} \\ v \\ \Delta\varphi_1 \end{bmatrix}}_{\mathbf{z}} \tag{6.17}$$

In contrast to the above transformation of angular position to angular differences, this state transformation concerns angular velocities. In order to ensure consistency with respect to constraints, denoted in columns of $\bar{\mathbf{B}}_{\mathrm{x,C}}$, it is necessary to apply the energy preserving state transformation as defined in chapter 2.6.3:

$$\bar{\mathbf{M}}_\mathrm{z} = \mathbf{T}^T\bar{\mathbf{M}}_\mathrm{x}\mathbf{T} = \begin{bmatrix} J_\mathrm{E} & 0 & 0 & 0 & 0 & 0 \\ 0 & J_{R_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & J_\mathrm{M} & 0 & 0 & 0 \\ 0 & 0 & 0 & J_\mathrm{F} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{J_\mathrm{V}}{r^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.18}$$

Note that weighting of the fifth differential equation, describing vehicle dynamics, is now the vehicle mass ($m_\mathrm{V} = \frac{J_\mathrm{V}}{r^2}$). Chapter 2.6.3 also offers necessary relations between the remaining system parameters in x- and z-coordinates (equation 6.17):

$$\bar{\mathbf{A}}_\mathrm{z} = \bar{\mathbf{M}}_\mathrm{z}\mathbf{T}^{-1}\bar{\mathbf{M}}_\mathrm{x}^{-1}\bar{\mathbf{A}}_\mathrm{x}\mathbf{T} = \begin{bmatrix} -d_\mathrm{J} & 0 & 0 & 0 & 0 & 0 \\ 0 & -d_\mathrm{J} & 0 & 0 & 0 & 0 \\ 0 & 0 & -d_\mathrm{J} & 0 & 0 & 0 \\ 0 & 0 & 0 & -d_\mathrm{J} - \frac{d}{i_\mathrm{F}^2} & \frac{d}{r \cdot i_\mathrm{F}} & -\frac{k}{i_\mathrm{F}} \\ 0 & 0 & 0 & \frac{d}{r \cdot i_\mathrm{F}} & -\frac{d_\mathrm{J}+d}{r^2} & \frac{k}{r} \\ 0 & 0 & 0 & \frac{1}{i_\mathrm{F}} & -\frac{1}{r} & 0 \end{bmatrix} \tag{6.19}$$

$$\bar{\mathbf{B}}_\mathrm{z,R} = \bar{\mathbf{M}}_\mathrm{z}\mathbf{T}^{-1}\bar{\mathbf{M}}_\mathrm{x}^{-1}\bar{\mathbf{B}}_\mathrm{x,R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{r} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\bar{\mathbf{B}}_\mathrm{z,C} = \bar{\mathbf{M}}_\mathrm{z}\mathbf{T}^{-1}\bar{\mathbf{M}}_\mathrm{x}^{-1}\bar{\mathbf{B}}_\mathrm{x,C} = \bar{\mathbf{B}}_\mathrm{x,R}$$

$$\bar{\mathbf{M}}_\mathrm{z}\dot{\mathbf{z}} = \bar{\mathbf{A}}_\mathrm{z}\mathbf{z} + \bar{\mathbf{B}}_\mathrm{z,R}\boldsymbol{\tau}_\mathrm{R} + \bar{\mathbf{B}}_\mathrm{z,C}\boldsymbol{\tau}_\mathrm{C} \tag{6.20}$$

To improve readability from now on the following notation is used to describe the system in equation 6.20:

$$\bar{\mathbf{M}}\dot{\mathbf{x}} = \bar{\mathbf{A}}\mathbf{x} + \bar{\mathbf{B}}_\mathrm{R}\boldsymbol{\tau}_\mathrm{R} + \bar{\mathbf{B}}_\mathrm{C}\boldsymbol{\tau}_\mathrm{C} \tag{6.21}$$

6.3 Drivetrain control task and assumptions

Main ambition in drivetrain control is to provide an output torque at vehicle wheels to achieve a desired vehicle acceleration specified by the driver via accelerator pedal. Based on this torque request and other parameters, e. g. internal angular velocities or state of charge of the battery, the controller has to choose a fitting gear and potential torque split between combustion engine and electric motor in case of a hybrid drivetrain. Choice of gear and performing necessary gear shifts without influencing output torque on the wheels is the main challenge in drivetrain control.

Within this work the focus is on gear shift control. In the following exemplary application (for drivetrain topology see figure 6.1) the actual gear (G_1) and future gear (G_2), as well as potential torque split ratios and also the time when the shifting process is started are considered to be known in advance. Therefore the main task is to perform a defined gear shift while tracking required vehicle velocity.

6.3.1 Considered gear shifts

As a further restriction the following approach considers only *conventional gears* G_1 and G_2. *Conventional gears* are gears that are stationary drivable with respect to the combustion engine. In consequence transmission from the combustion engine to the wheels has to be possible without impressing any torques in slipping clutches. Therefore gearbox output angular velocity ($\omega_F = x_4$) has to define angular velocities of the combustion engine ($\omega_E = x_1$), and consequently of the electric motor ($\omega_M = x_3$) and also $\omega_{R_3} = x_2$. Further there exists a stationary relation between a desired vehicle speed ($v = x_5$) and angular velocity of the combustion engine (see also chapter 6.4.4). To achieve this condition in valid gears G_1 and G_2 clutch C_0 and two more arbitrary clutches have to be locked. The set of all valid gears in this sense is summarized in table 6.2. It is an important fact that it is mechanically not possible to lock all four clutches at the same time, because of opposing constraints. This would end up in destruction of the gear box. Consequently in any valid gear shift first one clutch has to return to slipping state before another one can transition to locking state. Hence the necessary interim gear (G_i) for any valid gear shift is defined in advance. The necessary interim gears are presented in table 6.3.

gear	clutch state			
	C_0	C_1	C_2	C_3
G_a	1	0	1	1
G_b	1	1	0	1
G_c	1	1	1	0

Table 6.2: Conventional gears

gear	clutch state			
	C_0	C_1	C_2	C_3
$G_{i,a\leftrightarrow b}$	1	0	0	1
$G_{i,a\leftrightarrow c}$	1	0	1	0
$G_{i,b\leftrightarrow c}$	1	1	0	0

Table 6.3: Interim gears

6.3.2 State observation

The actual state vector $\mathbf{x}$ is considered to be known at any instant of time, i. e. there exists an ideal state observation. The same assumption is done on the actual clutch state containing the binary informations if clutches are locked or in slipping state, i. e. not locked. It should be noted that knowledge of actual clutch states implies knowledge of gear shift time T_{gs} and consequently gear shift duration ΔT_{gs} afterwards. To approximately realize this assumption a state observer has to be designed. [17] covers the design of a state observer on this exemplary drivetrain layout.

6.4 Control strategy

The basic control strategy to achieve the control task defined in chapter 6.3 is flatness-based propulsion torque feedforward control with underlaid LQR feedback control loop and model free clutch torque feedforward control.

Below is a short introduction into the single tasks covered by the single elements of the used control strategy before going into detail with respect to their mathematical design.

- **Model free clutch torque feedforward control:** Controlling the clutch torques $\boldsymbol{\tau}_{\mathrm{C}}$, enables on the one hand keeping of the current gear and on the other hand the initiation of a required gear shift.
- **Flatness-based propulsion torque feedforward control:** Flatness-based propulsion torque feedforward control calculates the necessary actuation on the system on the one hand the to achieve required vehicle speed and on the other hand to support gear shifts, initiated by clutch torque feedforward control, employing drive torques of combustion engine τ_{E} and electric motor τ_{M}.
- **LQR feedback control:** With respect to possible model uncertainties and external disturbances $\boldsymbol{\tau}_{\mathrm{V}}$ an additional feedback control is indispensable to achieve a robust control system. Employed actuators are again drive torques of combustion engine τ_{E} and electric motor τ_{M}.

In the following the control strategy is summarized beginning with the clutch torque control and the mathematical models of the involved systems, going on to fitting reference signals and finally to the design of the used feedforward and underlaid feedback control system.

6.4.1 Clutch torque control

Due to remark 21, clutch torque τ_{C} is not impressed on a locked clutch, ensured by a zero column in the reduced input matrix. The transmitted torque is defined by the corresponding Lagrangian multiplier ($\bar{\tau}_{\mathrm{C}}$, see remark 20). Nevertheless the value of τ_{C}, defining a certain pressure on clutch plates, is essential even in locking state. If τ_{C} exceeds the value of the corresponding Lagrangian multiplier, the clutch stays locked, otherwise it will start slipping. Note that the value of the Lagrangian multiplier is a function of the states $\mathbf{x}$ and acting torques $\boldsymbol{\tau}$.

Therefore in stationary case keeping a clutch torque on a certain constant level $\tau_{\mathrm{C,max}}$ higher than the value of the corresponding $\bar{\tau}_{\mathrm{C}}$, ensures that a locked clutch stays locked. In order to initiate a specified gear shift at time $T_{\mathrm{gs,init}}$ clutch torque τ_{C} of the clutch, which is desired to start slipping can be decreased from $\tau_{\mathrm{C}} = \tau_{\mathrm{C,max}}$ to zero in a specified time (clutch ramp time ΔT_{cl}). Note that from the moment when locking state is left the slipping clutch torque is actually impressed in the clutch. The opposite approach, increasing impressed clutch torque in a clutch from zero to maximum clutch torque $\tau_{\mathrm{C,max}}$, can be used to force a slipping clutch to lock. To avoid unnecessary dissipation it is useful to delay closing clutch until the first clutch definitely returned to slipping state. Figure 6.2 shows the initiation of a gear shift ($\mathrm{G}_1 \rightarrow \mathrm{G}_2$) in a stationary driving situation ($\bar{\tau}_{\mathrm{C,a}}$ = const., $\bar{\tau}_{\mathrm{C,b}}$ = const.). Thereby clutch C_a returns to slipping state and clutch C_b transitions into locking state.

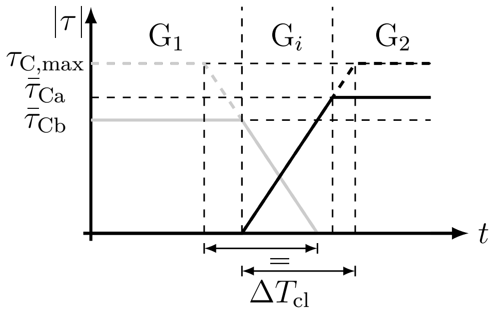

Figure 6.2: Initiation of a gear shift in a stationary driving situation

6.4.2 Switching system

Performing gear shifts as defined in chapter 6.3.1 requires consideration of a system switching between three sub systems (Σ_{G1}, Σ_{G2} and Σ_{Gi}), which are all subsystems of the unconstrained system Σ (see figure 6.3). The difference between the systems is that Σ_{G1} and Σ_{G2} have to satisfy a different additional constraint (locked clutch) compared to Σ_{Gi}. Therefore systems Σ_{G1} and Σ_{G2} can be calculated applying system reduction (see chapters 2 and 5.4.3) on the interim system Σ_{Gi}. The interim system Σ_{Gi} itself can be calculated applying system reduction (two locked clutches) on unconstrained drivetrain state space model (equation 6.21):

$$
\begin{aligned}
\Sigma: \quad & \bar{\mathbf{M}}\dot{\mathbf{q}} = \bar{\mathbf{A}}\mathbf{q} + \begin{bmatrix}\bar{\mathbf{B}}_{\mathrm{R}} & \bar{\mathbf{B}}_{\mathrm{C}}\end{bmatrix}\begin{bmatrix}\boldsymbol{\tau}_{\mathrm{R}}\\ \boldsymbol{\tau}_{\mathrm{C}}\end{bmatrix}\\
& \Downarrow\\
\Sigma_{\mathrm{Gi}}: \quad & \underbrace{\mathbf{J}_{\mathrm{x,Gi}}^T\bar{\mathbf{M}}\mathbf{J}_{\mathrm{x,Gi}}}_{\bar{\mathbf{M}}_{\mathrm{Gi}}}\dot{\mathbf{q}}_{\mathrm{Gi}} = \underbrace{\mathbf{J}_{\mathrm{x,Gi}}^T\bar{\mathbf{A}}\mathbf{J}_{\mathrm{x,Gi}}}_{\bar{\mathbf{A}}_{\mathrm{Gi}}}\mathbf{q}_{\mathrm{Gi}} + \underbrace{\mathbf{J}_{\mathrm{x,Gi}}^T\begin{bmatrix}\bar{\mathbf{B}}_{\mathrm{R}} & \bar{\mathbf{B}}_{\mathrm{C}}\end{bmatrix}}_{\bar{\mathbf{B}}_{\mathrm{Gi}}}\begin{bmatrix}\boldsymbol{\tau}_{\mathrm{R}}\\ \boldsymbol{\tau}_{\mathrm{C}}\end{bmatrix}\\
& \dot{\mathbf{q}}_{\mathrm{Gi}} = \underbrace{\bar{\mathbf{M}}_{\mathrm{Gi}}^{-1}\bar{\mathbf{A}}_{\mathrm{Gi}}\mathbf{q}_{\mathrm{Gi}}}_{\mathbf{A}_{\mathrm{Gi}}} + \underbrace{\bar{\mathbf{M}}_{\mathrm{Gi}}^{-1}\bar{\mathbf{B}}_{\mathrm{Gi}}}_{\mathbf{B}_{\mathrm{Gi}}}\mathbf{u}
\end{aligned}
\tag{6.22}
$$

Definition of transformation matrices $\mathbf{J}_{\mathrm{x,GiG1}}$ and $\mathbf{J}_{\mathrm{x,GiG2}}$ (basis to the null space of the Jacobian matrix of the additionally activated constraints) defines relation between Σ_{Gi} and Σ_{G1}, Σ_{G2}:

$$
\begin{aligned}
\Sigma_{\mathrm{G1}}: \quad & \underbrace{\mathbf{J}_{\mathrm{x,GiG1}}^T\bar{\mathbf{M}}_{\mathrm{Gi}}\mathbf{J}_{\mathrm{x,GiG1}}}_{\bar{\mathbf{M}}_{\mathrm{G1}}}\dot{\mathbf{q}}_{\mathrm{G1}} = \underbrace{\mathbf{J}_{\mathrm{x,GiG1}}^T\bar{\mathbf{A}}_{\mathrm{Gi}}\mathbf{J}_{\mathrm{x,G1}}}_{\bar{\mathbf{A}}_{\mathrm{G1}}}\mathbf{q}_{\mathrm{G1}} + \underbrace{\mathbf{J}_{\mathrm{x,GiG1}}^T\bar{\mathbf{B}}_{\mathrm{Gi}}}_{\bar{\mathbf{B}}_{\mathrm{G1}}}\mathbf{u}\\
& \dot{\mathbf{q}}_{\mathrm{G1}} = \underbrace{\bar{\mathbf{M}}_{\mathrm{G1}}^{-1}\bar{\mathbf{A}}_{\mathrm{G1}}\mathbf{q}_{\mathrm{G1}}}_{\mathbf{A}_{\mathrm{G1}}} + \underbrace{\bar{\mathbf{M}}_{\mathrm{G1}}^{-1}\bar{\mathbf{B}}_{\mathrm{G1}}}_{\mathbf{B}_{\mathrm{G1}}}\mathbf{u}
\end{aligned}
\tag{6.23}
$$

$$\Sigma_{\mathrm{G2}}: \quad \underbrace{\mathbf{J}^T_{\mathrm{x,GiG2}}\bar{\mathbf{M}}_{\mathrm{Gi}}\mathbf{J}_{\mathrm{x,GiG2}}}_{\bar{\mathbf{M}}_{\mathrm{G2}}}\dot{\mathbf{q}}_{\mathrm{G2}} = \underbrace{\mathbf{J}^T_{\mathrm{x,GiG2}}\bar{\mathbf{A}}_{\mathrm{Gi}}\mathbf{J}_{\mathrm{x,G2}}}_{\bar{\mathbf{A}}_{\mathrm{G2}}}\mathbf{q}_{\mathrm{G2}} + \underbrace{\mathbf{J}^T_{\mathrm{x,GiG2}}\bar{\mathbf{B}}_{\mathrm{Gi}}}_{\bar{\mathbf{B}}_{\mathrm{G2}}}\mathbf{u} \tag{6.24}$$

$$\dot{\mathbf{q}}_{\mathrm{G2}} = \underbrace{\bar{\mathbf{M}}^{-1}_{\mathrm{G2}}\bar{\mathbf{A}}_{\mathrm{G2}}}_{\mathbf{A}_{\mathrm{G2}}}\mathbf{q}_{\mathrm{G2}} + \underbrace{\bar{\mathbf{M}}^{-1}_{\mathrm{G2}}\bar{\mathbf{B}}_{\mathrm{G2}}}_{\mathbf{B}_{\mathrm{G2}}}\mathbf{u}$$

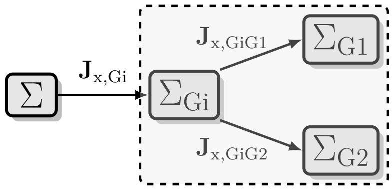

Figure 6.3: Relations between the subsystems involved in a gear shift

The restriction on considered stationary gears G_1, G_2 and consequently G_i (see chapter 6.3.1) and the design of the transformation matrices $\mathbf{J}_{\mathrm{x,Gi}}$, $\mathbf{J}_{\mathrm{x,GiG1}}$ and $\mathbf{J}_{\mathrm{x,GiG2}}$ (see chapter 2.6) enable important relations between the states respectively coordinates and between the system parameters:

Remark 22.

- *The set of generalized coordinates in system Σ_{G1} equals the set of generalized coordinates in system Σ_{G2}. Both are a subset of the set of generalized coordinates in system Σ_{Gi} and further of the unconstrained coordinates in system Σ:*

$$\{\mathbf{q}_{\mathrm{G1}}\} = \{\mathbf{q}_{\mathrm{G2}}\} \subseteq \{\mathbf{q}_{\mathrm{Gi}}\} \subseteq \{\mathbf{x}\} \tag{6.25}$$

- *Consequently holds:*

$$\{\bar{\mathbf{A}}_{\mathrm{G1}}\} = \{\bar{\mathbf{A}}_{\mathrm{G2}}\} \subseteq \{\bar{\mathbf{A}}_{\mathrm{Gi}}\} \subseteq \{\bar{\mathbf{A}}\} \tag{6.26}$$

  In the sense of: Matrices $\bar{\mathbf{A}}_{\mathrm{G1}}$ and $\bar{\mathbf{A}}_{\mathrm{G2}}$ are equal and appear as block in matrix $\bar{\mathbf{A}}_{\mathrm{Gi}}$ and further $\bar{\mathbf{A}}_{\mathrm{Gi}}$ in $\bar{\mathbf{A}}$.

- *The different gear ratios in Σ_{G1} and Σ_{G2} are stated in the first rows of the input matrices $\bar{\mathbf{B}}_{\mathrm{G1}}$ and $\bar{\mathbf{B}}_{\mathrm{G2}}$ (mapping: $\tau_{\mathrm{E}}, \tau_{\mathrm{M}} \mapsto \tau_{x_4}$).*
- *The different system dynamic in the systems Σ_{G1} and Σ_{G2} is a consequence of the different first diagonal elements in matrices $\bar{\mathbf{M}}_{\mathrm{G1}}$ and $\bar{\mathbf{M}}_{\mathrm{G2}}$ (mapping: $J_{\mathrm{E}}, J_{\mathrm{R}_3}, J_{\mathrm{M}} \mapsto J_{\mathrm{F}}$).*

6.4.3 Flatness-based design of linear feedforward control

This chapter discusses the general method of flatness-based linear feedforward control design based on [18] and [19] in SISO and MIMO case. Afterwards the method is applied on the exemplary drivetrain.

The basic idea of linear feedforward control is to calculate a necessary actuation $\mathbf{u}$ to achieve a specific behavior of the linear system specified in $\mathbf{x}$ and $\dot{\mathbf{x}}$.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad \Rightarrow \quad \mathbf{u} = \mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}) \tag{6.27}$$

Thereby two problems occur:

- Input matrix $\mathbf{B}$ in general is a non quadratic (but full column rank) matrix. Therefore there is no unique solution of equation 6.27 with respect to $\mathbf{u}$.
- Due to differential dependencies between the coordinates $\mathbf{x}$, determined in the dynamic matrix $\mathbf{A}$, in general arbitrary specifications of $\mathbf{x}$ and $\dot{\mathbf{x}}$ do not define feasible system dynamic.

Both problems can be solved by usage of flatness property of controllable linear systems.

Providing the considered system (SISO for the moment) is controllable[1], there exists a regular state transformation $\mathbf{z} = \mathbf{T}\mathbf{x}$ that transforms the system into its controllable canonical form:

$$\dot{\mathbf{z}} = \underbrace{\begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \\ -a_0 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix}}_{\mathbf{A}_c = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}} \mathbf{z} + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{B}_c = \mathbf{T}\mathbf{B}} u \tag{6.28}$$

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} z_1 \\ \dot{z}_1 \\ \vdots \\ z_1^{(n-1)} \end{bmatrix}, \quad \dot{\mathbf{z}} = \begin{bmatrix} \dot{z}_1 \\ \ddot{z}_1 \\ \vdots \\ z_1^{(n)} \end{bmatrix} \tag{6.29}$$

According to equations 6.28 and 6.29 entire system dynamic is constituted by one n-times differentiable trajectory $z_1(t)$ and its n derivatives ($z_1(t) \rightarrow \mathbf{z}, \dot{\mathbf{z}}$). Consequently there is no problem in defining feasible system dynamics. Coordinate z_1 is called flat output of the system and the system's controllability ensures its existence. At the same time transformation into controllable canonical form enables direct calculation of the necessary actuation u as function of the desired system dynamic ($z_1(t) \rightarrow \mathbf{z}, \dot{\mathbf{z}}$) by evaluating the last row in equation 6.28:

$$u = z_1^{(n)} + a_{n-1} z_1^{(n-1)} + \dots + a_1 \dot{z}_1 + a_0 z_1 = z_1^{(n)} - \mathbf{a}_{c,n}^T \mathbf{z} \tag{6.30}$$

The n-times differentiability of the trajectory $z_1(t)$ and the calculation of its derivatives can be realized by e. g. filtering. An appropriate filter structure is suggested in figure 6.4 can be used.

[1]For controllability criteria see [11].

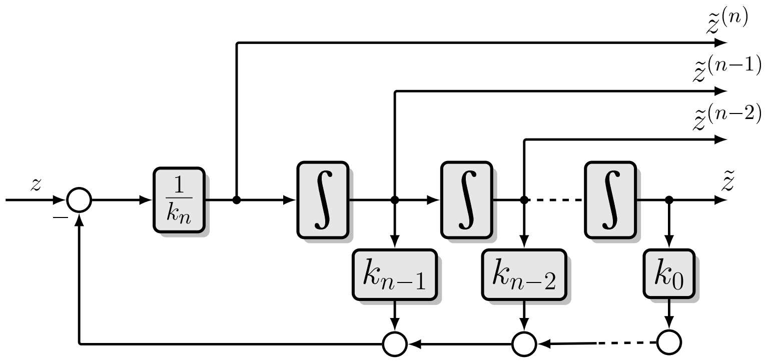

Figure 6.4: Low-pass filter n-th order

Filter coefficients

In order to achieve a common low-pass filter transfer function n-th order, defined by only one parameter τ,

$$F(s) = \frac{1}{(s\tau + 1)^n}, \tag{6.31}$$

the filter coefficients $k_1, \dots, k_n$ (see figure 6.4) are:

$$k_i = \binom{n}{i} \tau^i, \quad i = 0, \dots, n \tag{6.32}$$

Proof.
Transfer function $\tilde{F}(s)$ of the suggested filter structure (figure 6.4) is given by:

$$\begin{aligned} s^n \tilde{z}(s) =& \frac{1}{k_n} \left[z(s) - k_{n-1} s^{n-1} \tilde{z}(s) - \dots - k_0 \tilde{z}(s) \right] \\ \frac{\tilde{z}(s)}{z(s)} =& \tilde{F}(s) = \frac{1}{k_n s^n + k_{n-1} s^{n-1} + \dots + k_0} \end{aligned}$$

Equation 6.32 can be easily proofed equating coefficients of suggested filter $\tilde{F}(s)$ and desired filter $F(s)$:

$$F(s) = \frac{1}{(s\tau + 1)^n} = \frac{1}{\binom{n}{n} s^n \tau^n + \binom{n}{n-1} s^{n-1} \tau^{n-1} + \dots + \binom{n}{1} s\tau + \binom{n}{0}}$$

$$\frac{1}{k_n s^n + k_{n-1} s^{n-1} + \dots + k_0} \stackrel{!}{=} \frac{1}{\binom{n}{n} s^n \tau^n + \binom{n}{n-1} s^{n-1} \tau^{n-1} + \dots + \binom{n}{1} s\tau + \binom{n}{0}}$$

$$\Rightarrow k_i = \binom{n}{i} \tau^i, \quad i = 0, \dots, n$$

□

In MIMO case flatness-based linear feedforward control can be applied similar. Again controllability property guarantees existence of a state transformation $\mathbf{z} = \mathbf{Tx}$ to controllable canonical form:

$$\dot{\mathbf{z}} = \mathbf{A}_\mathrm{c}\mathbf{z} + \mathbf{B}_\mathrm{c}\mathbf{u} \tag{6.33}$$

Equation 6.34 shows the structure of system's dynamic matrix $\mathbf{A}_\mathrm{c}$ and its input matrix $\mathbf{B}_\mathrm{c}$ in MIMO controllable canonical form.

$$\mathbf{A}_\mathrm{c} = \left[\begin{array}{cccc|ccccc}
0 & 1 & \dots & 0 & 0 & & \dots & & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots & & & & \vdots \\
0 & \dots & 0 & 1 & 0 & & \dots & & 0 \\
* & \dots & * & * & * & \dots & \dots & \dots & * \\
\vdots & & \ddots & & \ddots & & \ddots & & \vdots \\
0 & & \dots & & 0 & 0 & 1 & \dots & 0 \\
\vdots & & & & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & & \dots & & 0 & 0 & \dots & 0 & 1 \\
* & & \dots & & * & * & \dots & * & *
\end{array}\right]
\begin{array}{c} \uparrow \\ \mu_1 \\ \downarrow \\ \vdots \\ \uparrow \\ \mu_m \\ \downarrow \end{array}
, \quad
\mathbf{B}_\mathrm{c} = \left[\begin{array}{ccc}
0 & \dots & 0 \\
\vdots & & \vdots \\
0 & \dots & 0 \\
1 & \dots & * \\
\vdots & \ddots & \vdots \\
0 & \dots & 0 \\
\vdots & & \vdots \\
0 & \dots & 0 \\
* & \dots & 1
\end{array}\right]
\begin{array}{c} \uparrow \\ \mu_1 \\ \downarrow \\ \vdots \\ \uparrow \\ \mu_m \\ \downarrow \end{array} \tag{6.34}$$

Both decompose into m parts, where m is the number of inputs. This decomposition also appears in state vector $\mathbf{z}$:

$$\mathbf{z} = \begin{bmatrix} z_1 & \dot{z}_1 & \dots & z_1^{(\mu_1-1)} & \dots & z_m & \dot{z}_m & \dots & z_m^{(\mu_m-1)} \end{bmatrix}^T \tag{6.35}$$

$\mu_1, \dots, \mu_m$ denote the controllability indices of the corresponding inputs $u_1, \dots, u_m$. Due to controllability of the system following condition on the sum of these controllability indices holds:

$$\sum_{i=1}^{m} \mu_i = n \tag{6.36}$$

In consequence feasible entire system dynamic is specified by m μ_i-times differentiable trajectories $\mathbf{z}_i$ $(i = 1, \dots, m)$.

Evaluation of the σ_k-th (see equation 6.38 and dark highlighting in equation 6.34) rows of equation 6.33 enables unique calculation of necessary actuation $\mathbf{u}$ for a required feasible system behavior $(z_1(t), \dots, z_m(t) \rightarrow \mathbf{z}, \dot{\mathbf{z}})$:

$$\mathbf{u} = \mathbf{B}_{c,\sigma_k}^{-1}\left[\dot{\mathbf{z}}_{\sigma_k} - \mathbf{A}_{c,\sigma_k}\mathbf{z}\right], \quad \text{with} \tag{6.37}$$

$$\sigma_k = \sum_{i=1}^{k} \mu_i, \quad k = 1, \dots, m \tag{6.38}$$

$$\boldsymbol{\sigma} = [\sigma_1 \dots \sigma_m]^T \tag{6.39}$$

Application on drivetrain model

Back to the exemplary drivetrain and considered gear shifts, systems Σ_{G1} and Σ_{G2} are controllable with respect to either the combustion engine <u>or</u> the electric motor. Consequently there exist each two transformation matrices ($\mathbf{T}_{\mathrm{G1,E}}$, $\mathbf{T}_{\mathrm{G1,M}}$, $\mathbf{T}_{\mathrm{G2,E}}$, $\mathbf{T}_{\mathrm{G2,M}}$) transforming systems $\Sigma_{\mathrm{G1,E}}$, $\Sigma_{\mathrm{G1,M}}$, $\Sigma_{\mathrm{G2,E}}$ resp. $\Sigma_{\mathrm{G2,M}}$ into SISO controllable canonical forms with respect to engine or motor.

As extension to remark 22 in this example modification of the first diagonal element of matrices $\bar{\mathbf{M}}_{\mathrm{G1}}$ or $\bar{\mathbf{M}}_{\mathrm{G2}}$ has only a scaling effect on the corresponding transformation matrices. Consequently it is possible to use only one transformation $\bar{\mathbf{T}}$ for all four systems, taking the loss of normalization of the input matrices $\mathbf{B}_{\mathrm{c}}$ in controllable canonical form.

The possibility to use only one transformation matrix $\bar{\mathbf{T}}$ to achieve a kind of controllable canonical form of the systems ($\Sigma_{\mathrm{G1,E}}$, $\Sigma_{\mathrm{G1,M}}$, $\Sigma_{\mathrm{G2,E}}$, $\Sigma_{\mathrm{G2,M}}$) implicates that feasible system dynamics of all four systems can be defined by one common flat output trajectory $z_1(t)$ and its derivatives.

Equations 6.40 and 6.41 show calculation of necessary actuation to achieve a desired system behavior ($z_1(t) \rightarrow \bar{\mathbf{z}}, \dot{\bar{\mathbf{z}}}$) in gear G_1 and G_2.

$$\begin{bmatrix} \bar{u}_{\mathrm{G1,E}} \\ \bar{u}_{\mathrm{G1,M}} \end{bmatrix} = \begin{bmatrix} \frac{1}{b_{\mathrm{G1,E}}} \\ \frac{1}{b_{\mathrm{G1,M}}} \end{bmatrix} \cdot \left[z_1^{(n)} - \mathbf{a}_{c,\mathrm{G1},n}^T \bar{\mathbf{z}} \right] \tag{6.40}$$

$$\begin{bmatrix} \bar{u}_{\mathrm{G2,E}} \\ \bar{u}_{\mathrm{G2,M}} \end{bmatrix} = \begin{bmatrix} \frac{1}{b_{\mathrm{G2,E}}} \\ \frac{1}{b_{\mathrm{M,G2}}} \end{bmatrix} \cdot \left[z_1^{(n)} - \mathbf{a}_{c,\mathrm{G2},n}^T \bar{\mathbf{z}} \right] \tag{6.41}$$

$$\bar{\mathbf{z}} = \begin{bmatrix} z_1 & \dot{z}_1 & \dots & z_1^{(n-1)} \end{bmatrix}^T \tag{6.42}$$

Note that parameters $b_{\mathrm{G1,E}}$, $b_{\mathrm{G1,M}}$, $b_{\mathrm{G2,E}}$ and $b_{\mathrm{G2,M}}$ denoting the last elements of the transformed input vectors of the systems $\Sigma_{\mathrm{G1,E}}$, $\Sigma_{\mathrm{G1,M}}$, $\Sigma_{\mathrm{G2,E}}$ and $\Sigma_{\mathrm{G2,M}}$ (see equations 6.43) are in general not equal to one (compare to equation 6.28), due to scaling of the transformation matrices.

$$\begin{aligned} \mathbf{B}_{\mathrm{c,G1,E}} &= \bar{\mathbf{T}}\mathbf{B}_{\mathrm{G1,E}} = \begin{bmatrix} \mathbf{0} \\ b_{\mathrm{G1,E}} \end{bmatrix} \\ \mathbf{B}_{\mathrm{c,G1,M}} &= \bar{\mathbf{T}}\mathbf{B}_{\mathrm{G1,M}} = \begin{bmatrix} \mathbf{0} \\ b_{\mathrm{G1,M}} \end{bmatrix} \\ \mathbf{B}_{\mathrm{c,G2,E}} &= \bar{\mathbf{T}}\mathbf{B}_{\mathrm{G2,E}} = \begin{bmatrix} \mathbf{0} \\ b_{\mathrm{G2,E}} \end{bmatrix} \\ \mathbf{B}_{\mathrm{c,G2,M}} &= \bar{\mathbf{T}}\mathbf{B}_{\mathrm{G2,M}} = \begin{bmatrix} \mathbf{0} \\ b_{\mathrm{G2,M}} \end{bmatrix} \end{aligned} \tag{6.43}$$

Due to their calculation $\bar{u}_{\mathrm{G1,E}}$ or $\bar{u}_{\mathrm{G1,M}}$ resp. $\bar{u}_{\mathrm{G2,E}}$ or $\bar{u}_{\mathrm{G2,M}}$ separately are the necessary actuations in gear G_1 resp. G_2 for the required system behavior with respect to combustion engine or electric motor.

The power P impressed by an rotational actuator is the product between the torque τ, which the actuator applies, and the corresponding angular velocity ω:

$$P = \tau \cdot \omega \tag{6.44}$$

With respect to the fixed angular velocities in stationary gears (see also chapter 6.4.4), power impressed by the actuators is only defined through the torques $\bar{u}_{\mathrm{G1,E}}$ and $\bar{u}_{\mathrm{G1,M}}$ resp. $\bar{u}_{\mathrm{G2,E}}$ and $\bar{u}_{\mathrm{G2,M}}$. Consequently simultaneous usage of both actuations is not going to achieve the required system behavior, since twice the power which is necessary would be impressed into the system. In order to get rid of this excess of power, actuation has to be split up into two parts:

$$\begin{aligned} u_{\mathrm{E}} &= k_{\mathrm{E}}\bar{u}_{\mathrm{E}} \\ u_{\mathrm{M}} &= k_{\mathrm{M}}\bar{u}_{\mathrm{M}}, \quad \text{with } k_{\mathrm{M}} := (1-k_{\mathrm{E}}), \quad \text{and } 0 \leqslant k_{\mathrm{E}} \leqslant 1 \end{aligned} \tag{6.45}$$

This additional degree of freedom k_{E} is called torque split between combustion engine and electric motor. Due to equation 6.44 and fixed angular velocities it is also the power split between the two actuators. According to this consideration equations 6.40 and 6.41 are adapted:

$$\begin{bmatrix} u_{\mathrm{G1,E}} \\ u_{\mathrm{G1,M}} \end{bmatrix} = \begin{bmatrix} \frac{k_{\mathrm{G1,E}}}{b_{\mathrm{G1,E}}} \\ \frac{(1-k_{\mathrm{G1,E}})}{b_{\mathrm{G1,M}}} \end{bmatrix} \cdot \left[z_1^{(n)} - \mathbf{a}_{c,\mathrm{G1},n}^T \bar{\mathbf{z}} \right] \tag{6.46}$$

$$\begin{bmatrix} u_{\mathrm{G2,E}} \\ u_{\mathrm{G2,M}} \end{bmatrix} = \begin{bmatrix} \frac{k_{\mathrm{G2,E}}}{b_{\mathrm{G2,E}}} \\ \frac{(1-k_{\mathrm{G2,E}})}{b_{\mathrm{M,G2}}} \end{bmatrix} \cdot \left[z_1^{(n)} - \mathbf{a}_{c,\mathrm{G2},n}^T \bar{\mathbf{z}} \right] \tag{6.47}$$

Note that the combination of the torques $u_{\mathrm{G1,E}}$ and $u_{\mathrm{G1,M}}$ resp. $u_{\mathrm{G2,E}}$ and $u_{\mathrm{G2,M}}$ calculated in equation 6.46 resp. 6.47 achieves the required system behavior in gear G_1 resp. G_2.

Analysis reveals that the interim system Σ_{Gi} is controllable with respect to both combustion engine <u>and</u> electric motor. Therefore there exists a transformation $\mathbf{T}$ to transform Σ_{Gi} into MIMO controllable canonical form.

The restriction on considered stationary gears G1, G2 and Gi (see chapter 6.3.1) and consequently remark 22 enable following unproved remark:

Remark 23.

- *It is possible to use one transformation* $\bar{\mathbf{T}}$ *to transform the controllable systems* $\Sigma_{\mathrm{G1,E}}$, $\Sigma_{\mathrm{G1,M}}$, $\Sigma_{\mathrm{G2,E}}$ *and* $\Sigma_{\mathrm{G2,M}}$ *into a kind of SISO controllable canonical form (see remark 22):*

$$\mathbf{T}_{\mathrm{G1,E}} = \mathbf{T}_{\mathrm{G1,M}} = \mathbf{T}_{\mathrm{G2,E}} = \mathbf{T}_{\mathrm{G2,m}} = \bar{\mathbf{T}} \tag{6.48}$$

- *Transformation* $\bar{\mathbf{T}}$ *is part of the transformation* $\mathbf{T}$*, which transforms system* Σ_{Gi} *into MIMO controllable canonical form:*

$$\mathbf{T}_{\mathrm{Gi}} = \begin{bmatrix} \mathbf{0} & \bar{\mathbf{T}} \\ * & * \end{bmatrix} = \mathbf{T} \tag{6.49}$$

Considering the exemplary drivetrain, systems Σ_{G1} and Σ_{G2} have system order $n_{\text{G1}} = n_{\text{G2}} = 3$. Feasible system dynamics $(\bar{\mathbf{z}}, \dot{\bar{\mathbf{z}}})$ therefore can be described by one trajectory $z_1(t)$ and its first, second and third derivative (see equations 6.28 and 6.29). Interim system Σ_{Gi} is a system of order $n_{\text{Gi}} = 4$. According to remark 23 corresponding MIMO controllable canonical form decomposes into two parts with controllable indices $\mu_1 = 3$ and $\mu_2 = 1$. Consequently feasible system dynamic $(\mathbf{z}, \dot{\mathbf{z}})$ for system Σ_{Gi} is entirely described by two trajectories $z_1(t)$ and $z_4(t)$:

$$\mathbf{z} = \begin{bmatrix} z_1 & \dot{z}_1 & \ddot{z}_z & z_4 \end{bmatrix}^T = \begin{bmatrix} \bar{\mathbf{z}}^T & z_4 \end{bmatrix}^T \tag{6.50}$$

$$\dot{\mathbf{z}} = \begin{bmatrix} \dot{z}_1 & \ddot{z}_z & \dddot{z}_1 & z_4 & \dot{z}_4 \end{bmatrix}^T = \begin{bmatrix} \dot{\bar{\mathbf{z}}}^T & \dot{z}_4 \end{bmatrix}^T \tag{6.51}$$

$$\tag{6.52}$$

Entire flatness-based propulsion torque feedforward control is defined by summarizing equations 6.40, 6.41 and 6.37:

$$\begin{bmatrix} u_{\text{G1,E}} \\ u_{\text{G1,M}} \end{bmatrix} = \begin{bmatrix} \frac{k_{\text{G1,E}}}{b_{\text{G1,E}}} \\ \frac{(1-k_{\text{G1,E}})}{b_{\text{G1,M}}} \end{bmatrix} \cdot \begin{bmatrix} z_1^{(n)} - \mathbf{a}_{c,\text{G1},n}^T \bar{\mathbf{z}} \end{bmatrix} \tag{6.53}$$
$$\begin{bmatrix} u_{\text{G2,E}} \\ u_{\text{G2,M}} \end{bmatrix} = \begin{bmatrix} \frac{k_{\text{G2,E}}}{b_{\text{G2,E}}} \\ \frac{(1-k_{\text{G2,E}})}{b_{\text{M,G2}}} \end{bmatrix} \cdot \begin{bmatrix} z_1^{(n)} - \mathbf{a}_{c,\text{G2},n}^T \bar{\mathbf{z}} \end{bmatrix}$$
$$\begin{bmatrix} u_{\text{Gi,E}} \\ u_{\text{Gi,M}} \end{bmatrix} = \mathbf{B}_{c,\text{Gi},\sigma_k}^{-1} \left[\dot{\mathbf{z}}_{\sigma_k} - \mathbf{A}_{c,\text{Gi},\sigma_k} \mathbf{z} \right]$$

$$n = 3, \quad \mu_1 = 3, \quad \mu_2 = 1, \quad \boldsymbol{\sigma} = \begin{bmatrix} 3 & 4 \end{bmatrix}^T \tag{6.54}$$
$$0 \leqslant k_{\text{G1,E}} \leqslant 1, \quad 0 \leqslant k_{\text{G2,E}} \leqslant 1$$

Equation 6.55 rewrites equation 6.53 in compact matrix notation.

$$\begin{bmatrix} u_{\text{G1,E}} \\ u_{\text{G1,M}} \\ u_{\text{G2,E}} \\ u_{\text{G2,M}} \\ u_{\text{Gi,E}} \\ u_{\text{Gi,M}} \end{bmatrix} = \begin{bmatrix} \frac{k_{\text{G1,E}}}{b_{\text{G1,E}}} & 0 & \mathbf{0} \\ \frac{(1-k_{\text{G1,E}})}{b_{\text{G1,M}}} & 0 & \mathbf{0} \\ 0 & \frac{k_{\text{G2,E}}}{b_{\text{G2,E}}} & \mathbf{0} \\ 0 & \frac{(1-k_{\text{G2,E}})}{b_{\text{G2,M}}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \begin{bmatrix} \mathbf{b}_{\text{c,Gi,3}}^T \\ \mathbf{b}_{\text{c,Gi,4}}^T \end{bmatrix}^{-1} \end{bmatrix} \begin{bmatrix} -\mathbf{a}_{c\text{G1},n}^T & 0 & 1 & 0 \\ -\mathbf{a}_{c,\text{G2},3}^T & 0 & 1 & 0 \\ -\mathbf{a}_{c,\text{Gi},3}^T & & 1 & 0 \\ -\mathbf{a}_{c,\text{Gi},4}^T & & 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ \dot{z}_1 \\ \ddot{z}_1 \\ z_4 \\ \dddot{z}_1 \\ \dot{z}_4 \end{bmatrix} \tag{6.55}$$

Definition of a 3-times differentiable trajectory $z_1(t)$ and once differentiable trajectory $z_4(t)$ enables continuous calculation rule for necessary actuations $u_{\text{E}}(t)$ and $u_{\text{M}}(t)$ to achieve desired system behavior $(z_1(t), z_4(t) \rightarrow \mathbf{z}, \dot{\mathbf{z}})$:

$$\mathbf{u}_{\mathrm{ff}} = \begin{bmatrix} u_{\mathrm{E}}(t) \\ u_{\mathrm{M}}(t) \end{bmatrix} = \begin{cases} \begin{bmatrix} u_{\mathrm{G1,E}}(t) \\ u_{\mathrm{G1,M}}(t) \end{bmatrix}, & t < T_{\mathrm{gs}} \\ \begin{bmatrix} u_{\mathrm{G2,E}}(t) \\ u_{\mathrm{G2,M}}(t) \end{bmatrix}, & T_{\mathrm{gs}} < t < T_{\mathrm{gs}} + \Delta T_{\mathrm{gs}} \\ \begin{bmatrix} u_{\mathrm{Gi,E}}(t) \\ u_{\mathrm{Gi,M}}(t) \end{bmatrix}, & t > T_{\mathrm{gs}} + \Delta T_{\mathrm{gs}} \end{cases} \tag{6.56}$$

It is a so far undiscussed issue how to plan trajectory $z_1(t)$ in stationary gears G_1 and G_2 to track the required vehicle speed and how to plan trajectories $z_1(t)$ and $z_4(t)$ in interim gear G_i in order to support a required gear shift. Remark 24 states an important fact, to clarify this issue.

Remark 24.

The flat output z_1 of the systems Σ_{G1}, Σ_{G2} and Σ_{Gi} is approximately the vehicle speed v:

$$z_1 \approx x_5 = v \tag{6.57}$$

Actually it is a linear combination of the differential angular velocity of the flexible shaft x_6 and the vehicle speed x_5. The approximation would be exact for damping constant $d = 0$.

According to remark 24 planning trajectory $z_1(t)$ is approximately equivalent to planning vehicle speed trajectory $v(t)$. Consequently every three-times differentiable trajectory is approximately a feasible vehicle velocity profile.

As already mentioned in chapter 6.3.1 in gears G_1 and G_2 definition of a constant reference vehicle speed $v_{\mathrm{r}} = x_{\mathrm{r},5}$ requires stationary angular velocities in the whole drivetrain $\mathbf{x}_{\mathrm{r,G1},\infty}$ resp. $\mathbf{x}_{\mathrm{r,G2},\infty}$:

$$\begin{aligned} \mathbf{x}_{\mathrm{r,G1},\infty} &= \mathbf{f}(x_{\mathrm{r},5}) \\ \mathbf{x}_{\mathrm{r,G2},\infty} &= \mathbf{f}(x_{\mathrm{r},5}) \end{aligned} \tag{6.58}$$

The stationary reference states $\mathbf{x}_{\mathrm{r,G1},\infty}$ resp. $\mathbf{x}_{\mathrm{r,G2},\infty}$ (for calculation see chapter 6.4.4) are consistent with respect to the corresponding constraints. Consequently a smooth change of operating point from $\mathbf{x}_{\mathrm{r,G1},\infty}$ to $\mathbf{x}_{\mathrm{r,G2},\infty}$ is equivalent to a smooth change of constraints. Planning of trajectory $z_4(t)$ with respect to this consideration will tend the combustion engine and the electric motor to support the gear shift, i. e. decreasing the differential angular velocity of the clutch that is supposed to transition into locking state to zero. Therefore the next chapter deals with the determination of the stationary consistent reference states $\mathbf{x}_{\mathrm{r},\infty}$, the smooth change of operating point and the resulting trajectories $\mathbf{x}_{\mathrm{r}}$ resp. $\mathbf{q}_{\mathrm{r}}$. Due to the fact that this approach is based on an approximation (see remark 24) subsequent filtering of the required trajectories, planned in $\mathbf{x}$ resp. $\mathbf{q}$-coordinates and transformed to $\mathbf{z}$-coordinates is necessary ($\rightarrow \tilde{\mathbf{z}}_\mathrm{r}$). For this purpose a filter structure as suggested in figure 6.4 can be used for each trajectory. If combining feedforward control and feedback control (see figure 6.8) the transformed and filtered trajectories have to be back transformed ($\rightarrow \tilde{\mathbf{q}}_\mathrm{r}$), to provide correct reference values for the feedback controller. Since the controller deals with reduced coordinates ($\mathbf{q}_\mathrm{G1}$, $\mathbf{q}_\mathrm{G2}$ resp. $\mathbf{q}_\mathrm{Gi}$) this back transformation $\mathbf{T}^{-1}$ has to be adapted with respect to the current gear, i. e. $\bar{\mathbf{T}}^{-1}$ in gears G_1 and G_2 and $\mathbf{T}^{-1}$ in interim gear G_i.

Figure 6.5 shows the general structure of the flatness-based feedforward control design.

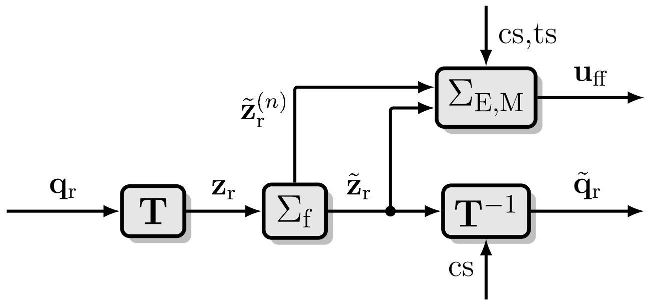

Figure 6.5: Structure of flatness-based feedforward control: $\mathbf{T}$ transformation into controllable canonical form, Σ_f filter (see figure 6.4), $\Sigma_\mathrm{E,M}$ calculation of propulsion torques (see equations 6.55 and 6.56), $\mathbf{T}^{-1}$ back transformation, cs clutch state, ts torque split

6.4.4 Consistent reference trajectories

As already mentioned, in the considered stationary gears (G_1 and G_2) there exist stationary relations between vehicle velocity and angular velocities of remaining inertias. In order to specify stationary consistent sets of reference values $\mathbf{x}_{\mathrm{r,G1},\infty}$ and $\mathbf{x}_{\mathrm{r,G2},\infty}$ it is necessary to calculate these stationary gains of the systems. The following calculations are exemplary done on system Σ_G1.

In a first step defining vehicle speed x_5 to be the system's output ($\rightarrow$ $\mathbf{C}_\mathrm{G1}$) and using **final value theorem of Laplace transformation** (see equation 6.61) enables calculation of stationary necessary input for one actuator (i.e. engine or motor).

$$G_\mathrm{G1}(s) = \frac{x_{\mathrm{r},5}(s)}{u(s)} = \mathbf{C}_\mathrm{G1}\left[s\mathbf{E} - \mathbf{A}_\mathrm{G1}\right]^{-1}\mathbf{B}_\mathrm{G1} \tag{6.59}$$

$$\begin{aligned} \lim_{t\to\infty} x_{\mathrm{r},5}(t) = x_{\mathrm{r},5,\infty} &= \lim_{s\to 0} s \cdot x_{\mathrm{r},5}(s) = \lim_{s\to 0} s \cdot G_\mathrm{G1}(s)u(s) \\ &= \lim_{s\to 0} s \cdot G_\mathrm{G1}(s) \cdot \frac{u_\infty}{s} = G_\mathrm{G1}(0)u_\infty = -\mathbf{C}_\mathrm{G1}\mathbf{A}_\mathrm{G1}^{-1}\mathbf{B}_\mathrm{G1}u_\infty \qquad (6.60) \\ \Rightarrow \quad u_\infty &= -\left[\mathbf{C}_\mathrm{G1}\mathbf{A}_\mathrm{G1}^{-1}\mathbf{B}_\mathrm{G1}\right]^{-1} x_{\mathrm{r},5,\infty} \qquad (6.61) \end{aligned}$$

Equation 6.61 and again **final value theorem** define stationary states $\mathbf{q}_{\mathrm{G1},\infty}$ depending on u_∞:

$$\dot{\mathbf{q}}_{\mathrm{G1}}(t) = \mathbf{A}_{\mathrm{G1}}\mathbf{q}_{\mathrm{G1}}(t) + \mathbf{B}_{\mathrm{G1}}u \quad \Rightarrow \quad \mathbf{q}_{\mathrm{G1}}(s)s = \mathbf{A}_{\mathrm{G1}}\mathbf{q}_{\mathrm{G1}}(s) + \mathbf{B}_{\mathrm{G1}}u(s)$$
$$\mathbf{q}_{\mathrm{G1}}(s) = \left[s\mathbf{I} - \mathbf{A}_{\mathrm{G1}}\right]^{-1}\mathbf{B}_{\mathrm{G1}}u(s) \tag{6.62}$$

$$\begin{aligned}\lim_{t\to\infty}\mathbf{q}_{\mathrm{G1}}(t) = \mathbf{q}_{\mathrm{G1},\infty} &= \lim_{s\to 0} s\cdot\mathbf{q}_{\mathrm{G1}}(s) = \lim_{s\to 0} s\left[s\mathbf{I} - \mathbf{A}_{\mathrm{G1}}\right]^{-1}\mathbf{B}_{\mathrm{G1}}u(s)\\ &= \lim_{s\to 0} s\left[s\mathbf{I} - \mathbf{A}_{\mathrm{G1}}\right]^{-1}\mathbf{B}_{\mathrm{G1}}\frac{u_\infty}{s} = \lim_{s\to 0}\left[s\mathbf{I} - \mathbf{A}_{\mathrm{G1}}\right]^{-1}\mathbf{B}_{\mathrm{G1}}u_\infty\\ &= -\mathbf{A}_{\mathrm{G1}}^{-1}\mathbf{B}_{\mathrm{G1}}u_\infty\end{aligned} \tag{6.63}$$

Equation 6.63 enables final stationary relation between $x_{\mathrm{r},5,\infty}$ and $\mathbf{q}_{\mathrm{G1},\infty}$:

$$\mathbf{q}_{\mathrm{G1},\infty} = \underbrace{\mathbf{A}_{\mathrm{G1}}^{-1}\mathbf{B}_{\mathrm{G1}}\left[\mathbf{C}_{\mathrm{G1}}\mathbf{A}_{\mathrm{G1}}^{-1}\mathbf{B}_{\mathrm{G1}}\right]^{-1}}_{\mathbf{S}_{\mathrm{G1},\infty}} x_{\mathrm{r},5,\infty} = \mathbf{S}_{\mathrm{G1},\infty}x_{\mathrm{r},5,\infty} \tag{6.64}$$

Using transformation matrix $\mathbf{J}_{\mathrm{x,G1}}$ and equation 6.64 achieves a consistent set of stationary reference values $\mathbf{x}_{\mathrm{r},\infty}$ with respect to gear G_1 and in dependency on the required stationary vehicle speed $x_{\mathrm{r},5,\infty}$:

$$\mathbf{q}_{\mathrm{G1},\infty} = \mathbf{S}_{\mathrm{G1},\infty}x_{\mathrm{r},5,\infty} \quad \Rightarrow \quad \mathbf{x}_{\mathrm{r,G1},\infty} = \mathbf{J}_{\mathrm{x,G1}}\mathbf{S}_{\mathrm{G1},\infty}x_{\mathrm{r},5,\infty} \tag{6.65}$$

Analogously in system Σ_{G2} holds:

$$\mathbf{q}_{\mathrm{G2},\infty} = \mathbf{S}_{\mathrm{G2},\infty}x_{\mathrm{r},5,\infty} \quad \Rightarrow \quad \mathbf{x}_{\mathrm{r,G2},\infty} = \mathbf{J}_{\mathrm{x,G2}}\mathbf{S}_{\mathrm{G2},\infty}x_{\mathrm{r},5,\infty} \tag{6.66}$$

As already mentioned, in order to perform a gear shift from G_1 to G_2 it is necessary to provide a smooth transition from $\mathbf{x}_{\mathrm{r,G1},\infty}$ to $\mathbf{x}_{\mathrm{r,G2},\infty}$. This approach includes the requirement of achieving a zero differential angular velocity on the clutch that transitions into locking state, while differential angular velocity on the clutch that returns into slipping state increases from zero.

Following section offers a general method to generate n-times differentiable smooth trajectories for change of operating point using a polynomial approach similar to [18].

Polynomial approach for change of operating point

At the beginning a simplified definition of the problem is useful: The task is to generate an at least n-times differentiable trajectory $x(t)$, fulfilling the following conditions, which define a smooth change of operating point:

1. $x(t=0) = 0$
2. $x(t=1) = 1$
3. $\dot{x}(t)|_{t=0} = \ddot{x}(t)|_{t=0} = \cdots = x^{(n)}(t)|_{t=0} = 0$

4. $\dot{x}(t)|_{t=1} = \ddot{x}(t)|_{t=1} = \dots = x^{(n)}(t)|_{t=1} = 0$

The minimal degree of a polynomial function that satisfies all $2n+2$ conditions is $2n+1$:

$$x(t) = \sum_{i=0}^{2n+1} a_i t^i \tag{6.67}$$

According to conditions 1 and 3 the first $n+1$ coefficients $a_0, \dots, a_n$ have to be zero. Determination of the remaining $n+1$ coefficients $a_{n+1}, \dots, a_{2n+1}$, using the remaining conditions 2 and 4, ends up into following linear equation system:

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ (n+1) & (n+2) & \dots & (2n+1) \\ (n+1)(n) & (n+2)(n+1) & \dots & (2n+1)(2n) \\ \vdots & \vdots & & \vdots \\ (n+1)(n)\dots(2) & (n+2)(n+1)\dots(3) & \dots & (2n+1)(2n)\dots(n+2) \end{bmatrix} \begin{bmatrix} a_{n+1} \\ a_{n+2} \\ \vdots \\ a_{2n+1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{6.68}$$

This approach can now be extended to the following more general conditions:

1. $x(t = T_0) = x_0$
2. $x(t = T_1) = x_1$
3. $\dot{x}(t)|_{t=T_0} = \ddot{x}(t)|_{t=T_0} = \dots = x^{(n)}(t)|_{t=T_0} = 0$
4. $\dot{x}(t)|_{t=T_1} = \ddot{x}(t)|_{t=T_1} = \dots = x^{(n)}(t)|_{t=T_1} = 0$

Scaling of polynomial function 6.67 in magnitude and time enables to fulfill generalized conditions without changing the coefficients a_i ($i = 0, \dots, 2n+1$):

$$x(t) = x_0 + (x_1 - x_0) \sum_{i=0}^{2n+1} a_i \left(\frac{t - T_0}{T_1 - T_0} \right)^i \tag{6.69}$$

According to the fact that polynomial coefficients a_i do not depend on the constants x_0, x_1, T_0 and T_1, table 6.4 shows constant polynomial coefficients for orders $n = 1, \dots, 5$ obtained by solving the linear equation system 6.68.

order	polynomial coefficients											
n	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}
1	0	0	3	-2	0	0	0	0	0	0	0	0
2	0	0	0	10	-15	6	0	0	0	0	0	0
3	0	0	0	0	35	-84	70	-20	0	0	0	0
4	0	0	0	0	0	126	-420	540	-315	70	0	0
5	0	0	0	0	0	0	462	-1980	3465	-3080	1386	-252

Table 6.4: Polynomial coefficients for a n-times differentiable change of operating point

According to equations 6.69, 6.65 and 6.66 consistent gear shift trajectories $\mathbf{x}_\mathrm{r}$ can be formulated in dependency of desired vehicle speed $x_{\mathrm{r},5}(t)$, gear shift time T_gs and gear shift duration ΔT_gs:

$$\mathbf{x}_\mathrm{r}(t) = \begin{cases} \mathbf{x}_{\mathrm{r,G1}}, & t < T_\mathrm{gs} \\ \mathbf{x}_{\mathrm{r,G1}} + (\mathbf{x}_{\mathrm{r,G1}} - \mathbf{x}_{\mathrm{r,G2}}) \sum_{i=0}^{2n+1} a_i \left(\frac{t-T_\mathrm{gs}}{\Delta T_\mathrm{gs}}\right)^i, & T_\mathrm{gs} < t < T_\mathrm{gs} + \Delta T_\mathrm{gs} \\ \mathbf{x}_{\mathrm{r,G2}}, & t > T_\mathrm{gs} + \Delta T_\mathrm{gs} \end{cases} \tag{6.70}$$

$$= \begin{cases} \mathbf{J}_{\mathrm{x,G1}}\mathbf{S}_{\mathrm{G1},\infty} \cdot x_{\mathrm{r},5}(t), & t < T_\mathrm{gs} \\ [\mathbf{J}_{\mathrm{x,G1}}\mathbf{S}_{\mathrm{G1},\infty} + (\mathbf{J}_{\mathrm{x,G1}}\mathbf{S}_{\mathrm{G1},\infty} - \dots & \\ \quad \dots \mathbf{J}_{\mathrm{x,G2}}\mathbf{S}_{\mathrm{G2},\infty}) \sum_{i=0}^{2n+1} a_i \left(\frac{t-T_\mathrm{gs}}{\Delta T_\mathrm{gs}}\right)^i \Bigg] \cdot x_{\mathrm{r},5}(t), & T_\mathrm{gs} < t < T_\mathrm{gs} + \Delta T_\mathrm{gs} \\ \mathbf{J}_{\mathrm{x,G2}}\mathbf{S}_{\mathrm{G2},\infty} \cdot x_{\mathrm{r},5}(t), & t > T_\mathrm{gs} + \Delta T_\mathrm{gs} \end{cases}$$

The set of consistent reduced coordinates can be determined by choosing the last four components of $\mathbf{x}_\mathrm{r}$:

$$\mathbf{q}_\mathrm{r}(t) = \mathbf{D}\mathbf{x}_\mathrm{r}(t) \tag{6.71}$$

Effective gear shift time T_gs and gear shift duration ΔT_gs are not yet known while performing the gear shift. Therefore in calculation 6.70 they are considered to be design parameters ($\rightarrow T_\mathrm{gs,init}$ and $\Delta\bar{T}_\mathrm{gs}$).

Figure 6.6 shows the generation of consistent reference trajectories in block notation.

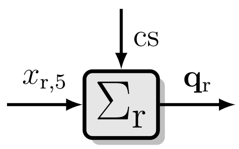

Figure 6.6: Consistent reference trajectory generation: Σ_r (see equation 6.70), cs clutch state

6.4.5 LQR feedback control

Due to the fact that flatness-based feedforward control is quite susceptible to model uncertainties and other disturbances, it is indispensable to extend the control strategy by a feedback loop.

In this exemplary application a linear state feedback is used to cover model uncertainties and other disturbances. The linear state feedback is designed in an optimal manner using LQR approach (see chapter 4.3.1). As investigated in chapter 4.3.2 it is possible to design an optimal linear state feedback on the interim system Σ_Gi and to use reduced controllers for the subsystems Σ_G1 and Σ_G2. Since stability of reduced controllers is not guaranteed in advance (see chapter 3.3), it has to be evaluated by considering eigenvalues of the unreduced controller (see section 6.5.2). According to chapter 4.3 also the performance of the reduced and therefore suboptimal controllers has to be evaluated (see section 6.5.3).

The degree of freedom in design of an LQ optimal linear state feedback is the choice of state weight matrix **Q** and input weight matrix **R**:

- Since the control task of tracking a required vehicle velocity has the highest priority, it is advantageous to increase weight on the difference between required and actual vehicle speed by increasing third diagonal element in state weight matrix **Q**. Since in stationary gears G_1 and G_2 there exist stationary relations between the angular velocities in the drivetrain and the desired vehicle speed, this weighting will mainly take effect in interim gear G_i.
- Manipulation of the diagonal elements of the input weight matrix **R** enables power partition between combustion engine and electric motor with respect to disturbance coverage.

Figure 6.7 shows the general structure of the feedback control.

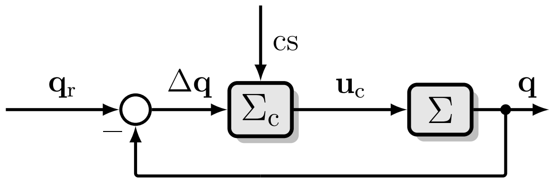

Figure 6.7: Structure of feedback control loop: Σ_c controller, Σ plant

6.4.6 Combined structure

Figure 6.8 shows the suggested combined control structure.

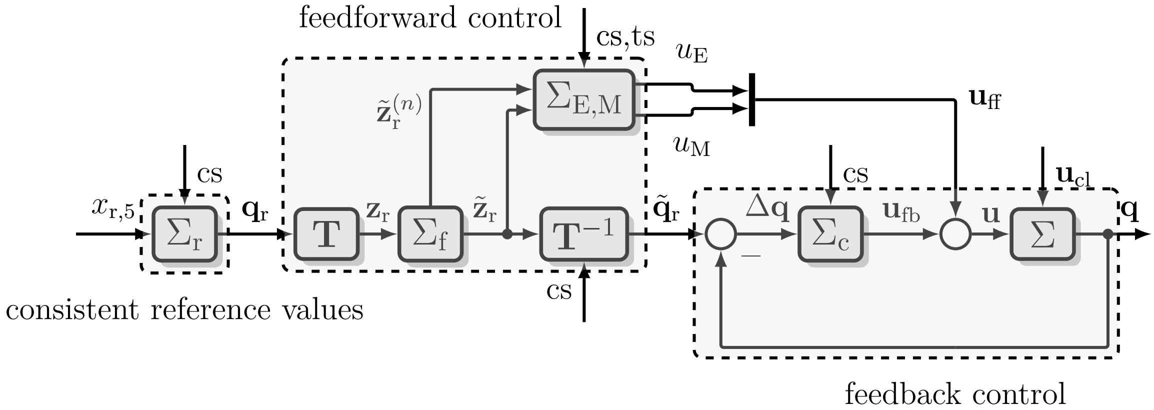

Figure 6.8: Structure of flatness-based feedforward control (see figure 6.5) including a feedback loop (see figure 6.7) and the consistent generation of reference trajectories (see figure 6.6)

6.5 Simulation and results

In order to validate functionality of the used control strategy (see chapter 6.4) it has been implemented in MATLAB®/Simulink®. The modeling approach, discussed in chapter 6.4.2 has been used for the design of the control system. Simulation employs plant model applied in [17]. This model provides current clutch states as assumed in section 6.3.2 and further models influence of air drag, and rolling resistance, which are considered to be external disturbances.

6.5.1 Parametrization

Tables 6.5 and 6.6 list the physical parameters used for modeling (see chapter 6.2) and table 6.7 lists the parameters used for the control system design (see chapter 6.4).

Drivetrain

symbol	value	unit
J_{E}	$64.0 \cdot 10^{-3}$	$\mathrm{kg} \cdot \mathrm{m}^2$
J_{R3}	$10.0 \cdot 10^{-3}$	$\mathrm{kg} \cdot \mathrm{m}^2$
J_{M}	$32.5 \cdot 10^{-3}$	$\mathrm{kg} \cdot \mathrm{m}^2$
J_{F}	$333.3 \cdot 10^{-3}$	$\mathrm{kg} \cdot \mathrm{m}^2$
J_{V}	135.7	$\mathrm{kg} \cdot \mathrm{m}^2$
d	20	$\mathrm{Nm} \cdot \mathrm{s}$
d_{J}	10^{-4}	$\mathrm{Nm} \cdot \mathrm{s}$
k	$4 \cdot 10^3$	$\mathrm{N} \cdot \mathrm{m}$
r	0.317	m

Table 6.5: System parameters

symbol	value
i_{F}	4.616
$i_{\mathrm{P}}^{\mathrm{R1}}$	0.260
$i_{\mathrm{P}}^{\mathrm{R2}}$	0.480
$i_{\mathrm{P3}}^{\mathrm{R3}}$	−0.676
$i_{\mathrm{S1}}^{\mathrm{R1}}$	−0.317
$i_{\mathrm{S1}}^{\mathrm{R2}}$	1.633
$i_{\mathrm{S2}}^{\mathrm{R1}}$	0.745
$i_{\mathrm{S2}}^{\mathrm{R2}}$	−0.491

Table 6.6: Gear ratios

Control system

symbol	value	unit	explanation
$\Delta\bar{T}_{\mathrm{gs}}$	0.5	s	planned gears shift duration
ΔT_{cl}	0.5	s	clutch torque ramp duration
$\Delta\tau_{\mathrm{c,max}}$	200	Nm	maximum clutch torque
τ_{F}	0.05	s	low-pass filter time constant
R_{E}	10		LQR input weighting combustion engine
R_{M}	10		LQR input weighting electric motor
Q_3	1		LQR state weighting x_3
Q_4	1		LQR state weighting x_4
Q_5	10^5		LQR state weighting x_5
Q_6	1		LQR state weighting x_6

Table 6.7: Control parameters

$$\mathbf{Q} = \begin{bmatrix} Q_3 & 0 & 0 & 0 \\ 0 & Q_4 & 0 & 0 \\ 0 & 0 & Q_5 & 0 \\ 0 & 0 & 0 & Q_6 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} R_\mathrm{E} & 0 \\ 0 & R_\mathrm{M} \end{bmatrix} \tag{6.72}$$

6.5.2 Stability check

According to chapter 3.3 stability of the controlled subsystems has to be evaluated. The evaluation is done for the gear shift: $\mathrm{G}_a \rightarrow \mathrm{G}_{\mathrm{i,a}\leftrightarrow\mathrm{c}} \rightarrow \mathrm{G}_c$. Since the real parts of the eigenvalues are true negative (see table 6.8) all three controlled subsystems are asymptotically stable.

gear	eigenvalues		
G_i	$-3.5 \pm j \cdot 18.87$	-8.17	-5.37
G_1	$-2.78 \pm j \cdot 18.30$	-5.77	
G_2	$-3.76 \pm j \cdot 18.26$	-7.11	

Table 6.8: Eigenvalues: Σ_{Gi}, Σ_{G1}, Σ_{G2}

6.5.3 Qualitative evaluation of optimality

Chapter 4.3.2 shows how to reduce a linear state feedback in order to control subsystems with respect to additional linear constraints. The advantage of this approach is that there is no need for redesigning the controller. Although certain conditions can guarantee stability of the reduced controller (see chapter 3.3), its optimality in general gets lost (see chapter 4.3). A possible approach to quantify the effects of the loss of optimality in a specific application is to compare them to the effects of parameter uncertainties. This comparison is done in the frequency domain considering the controlled system's overall transfer function. Therefore vehicle mass is varied up to plus 10%, modeling for example additional vehicle load, and damping constant ±20%, since its exact determination is quite hard in practice. Figure 6.9 shows on the one hand uncertain but optimal controlled system's overall transfer function $\mathbf{T}_{\mathrm{opt,d}}$ and on the other hand suboptimal controlled (reduced controller) $\mathbf{T}_{\mathrm{subopt}}$ in gears $\mathrm{G}_1 = \mathrm{G}_a$ and $\mathrm{G}_2 = \mathrm{G}_c$.

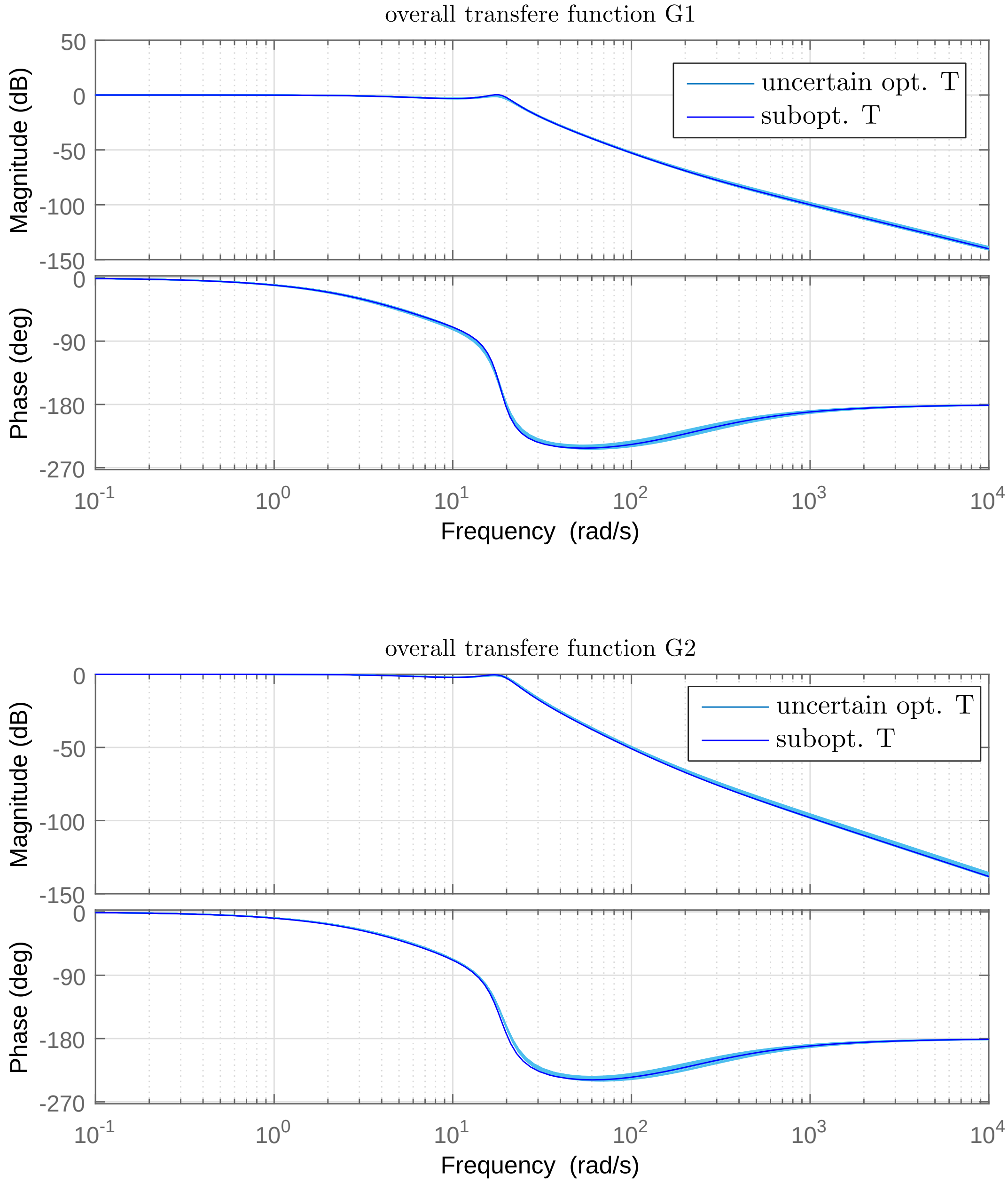

Figure 6.9: Evaluation of the performance of the suboptimal controllers

Although this approach is only a graphical analysis, it shows that the loss of optimality of the reduced controllers in this application is negligible.

6.5.4 Simulation

This section finally provides results of four specific gear shift simulations (Simulation 1-4). Table 6.9 is an additional legend for simulation results (see figures 6.10 to 6.13) with respect to the line styles. For explanation of the used variables see figure 6.8 and chapter 6.4.

linestyle	states		actuation	control error	clutch state	vehicle dynamic
——— (solid)	$\mathbf{x}$	$\mathbf{z}$	$\begin{bmatrix}\mathbf{u} & \mathbf{u}_{\mathrm{cl}}\end{bmatrix}$	$\Delta\mathbf{x}$	C_0,C_1,C_2,C_3	v, a, j
— — —	$\mathbf{x}_{\mathrm{r}}$	–	$\mathbf{u}_{\mathrm{ff}}$	–	–	v_{r}
......	$\tilde{\mathbf{x}}_{\mathrm{r}}$	$\tilde{\mathbf{z}}_{\mathrm{r}}$	–	–	–	–

Table 6.9: Additional line style legend

Simulation 1

The first experiment performs a gear shift while tracking a constant vehicle speed. The necessary propulsion torque is stationary provided by the combustion engine, while the electric motor just supports the required gear shift. Further in this first simulation the LQR feedback control is disabled in order to see performance of the flatness-based linear feedforward propulsion torque control. Table 6.10 summarizes the simulation setup and figure 6.10 shows the results of simulation 1.

symbol	value	unit	explanation
$v_{\mathrm{r,start}}$	10	m/s	starting vehicle velocity (reference)
$v_{\mathrm{r,final}}$	10	m/s	final vehicle velocity (reference)
ΔT_{acc}	–	s	acceleration time
$T_{\mathrm{gs,init}}$	2	s	gear shift initiation time
G_1	G_a		current gear (see table 6.2)
G_2	G_c		required gear (see table 6.2)
ff	1		enable feedforward control
fb	0		enable feedback control
k_{E}	1		torque split combustion engine

Table 6.10: Parameters simulation 1

Discussion Due to the fact that the drivetrain is affected by air drag and rolling resistance, which are not part of the mathematical model, the control errors increase with increasing simulation time, i. e. angular velocities in the drivetrain as well as the vehicle speed diverge from their required values. According to the clutch state required gear shift took place from simulation time 2.5 to 3s (at the beginning and the end of the simulation each one additional simulation second is inserted). This time approximately equals the planned transition time (see $\Delta\bar{T}_{\mathrm{gs}}$ in table 6.7). Considering the vehicle jerk one can notice that vehicle dynamic is hardly influenced by the performed gear shift. Due to subsequent calculation of the vehicle jerk by differentiation of the vehicle acceleration there is a peak in vehicle jerk at the beginning of the simulation. Internal damping in the drivetrain is small (see damping constant d_{J} in table 6.5) and therefore there is only a need of slight stationary propulsion torques for keeping speeds. Actually step like changes in the actuation torques are not feasible in reality, since they involve unbounded energy. Therefore chapter 7.2 presents an approach to include rate saturations for actuators into the flatness-based feedforward concept.

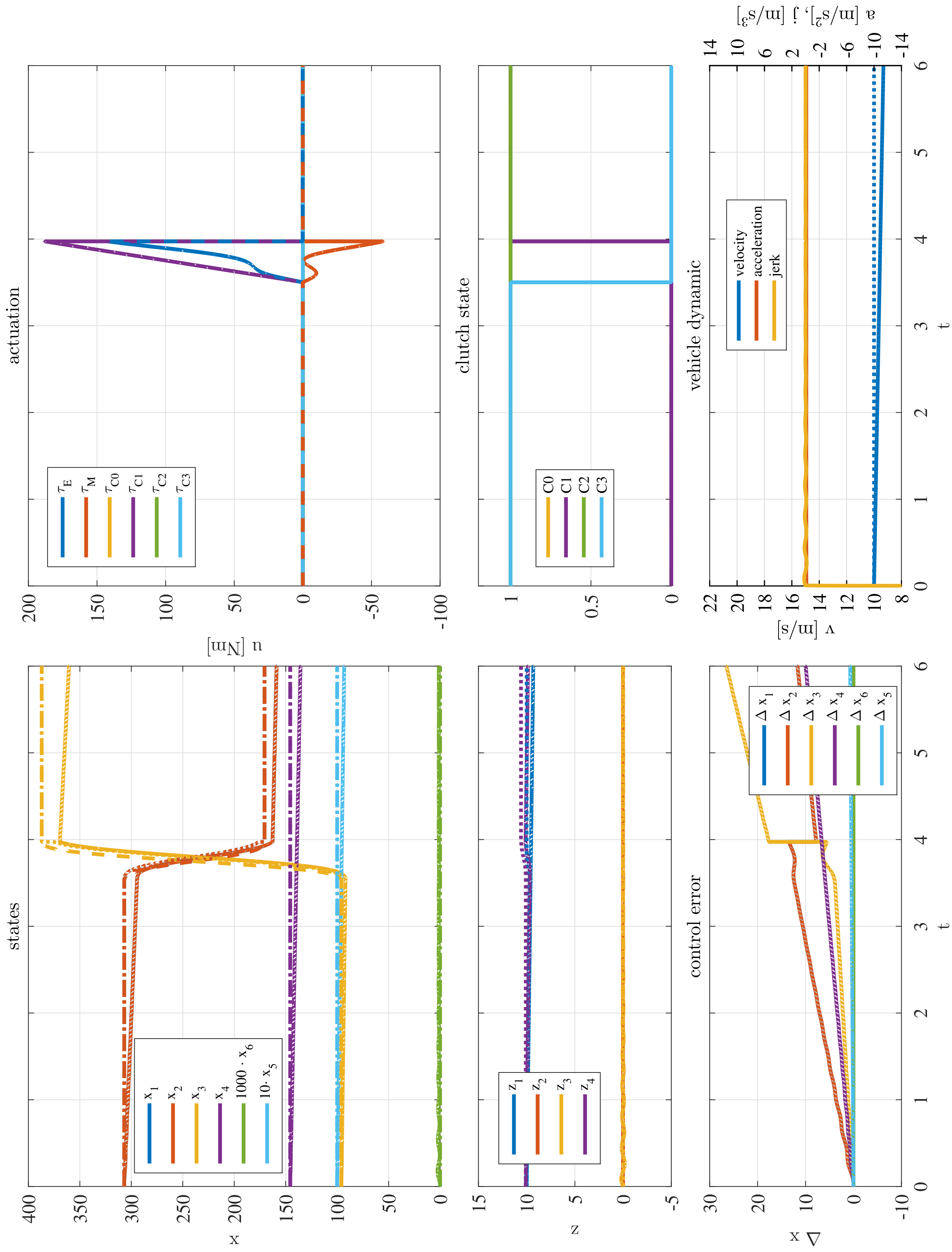

Figure 6.10: Simulation results 1

Simulation 2

The second simulation experiment repeats simulation 1 with enabled feedback loop, in order to show the performance of combined feedforward and feedback control. Therefore the remaining simulation setup (see table 6.11) equals the setup used in simulation 1 (see table 6.10). Figure 6.11 shows the results of simulation 2.

symbol	value	unit	explanation
$v_{\mathrm{r,start}}$	10	m/s	starting vehicle velocity (reference)
$v_{\mathrm{r,final}}$	10	m/s	final vehicle velocity (reference)
ΔT_{acc}	–	s	acceleration time
$T_{\mathrm{gs,init}}$	2	s	gear shift initiation time
G_1	G_a		current gear (see table 6.2)
G_2	G_c		required gear (see table 6.2)
ff	1		enable feedforward control
fb	1		enable feedback control
k_{E}	1		torque split combustion engine

Table 6.11: Parameters simulation 2

Discussion In contrast to simulation 1 in this simulation control errors do not increase with increasing simulation time, but are constant. Increasing of the control errors is prevented by the feedback controller. Consequently there is hardly no variance between actual and required vehicle speed. Due to the use of a linear state feedback and the presence of constant disturbances there exist steady control errors. Their further decreasing could be realized by introducing an additional integral part to the feedback control law. The influence of the performed gear shift on the vehicle dynamic is again negligible.

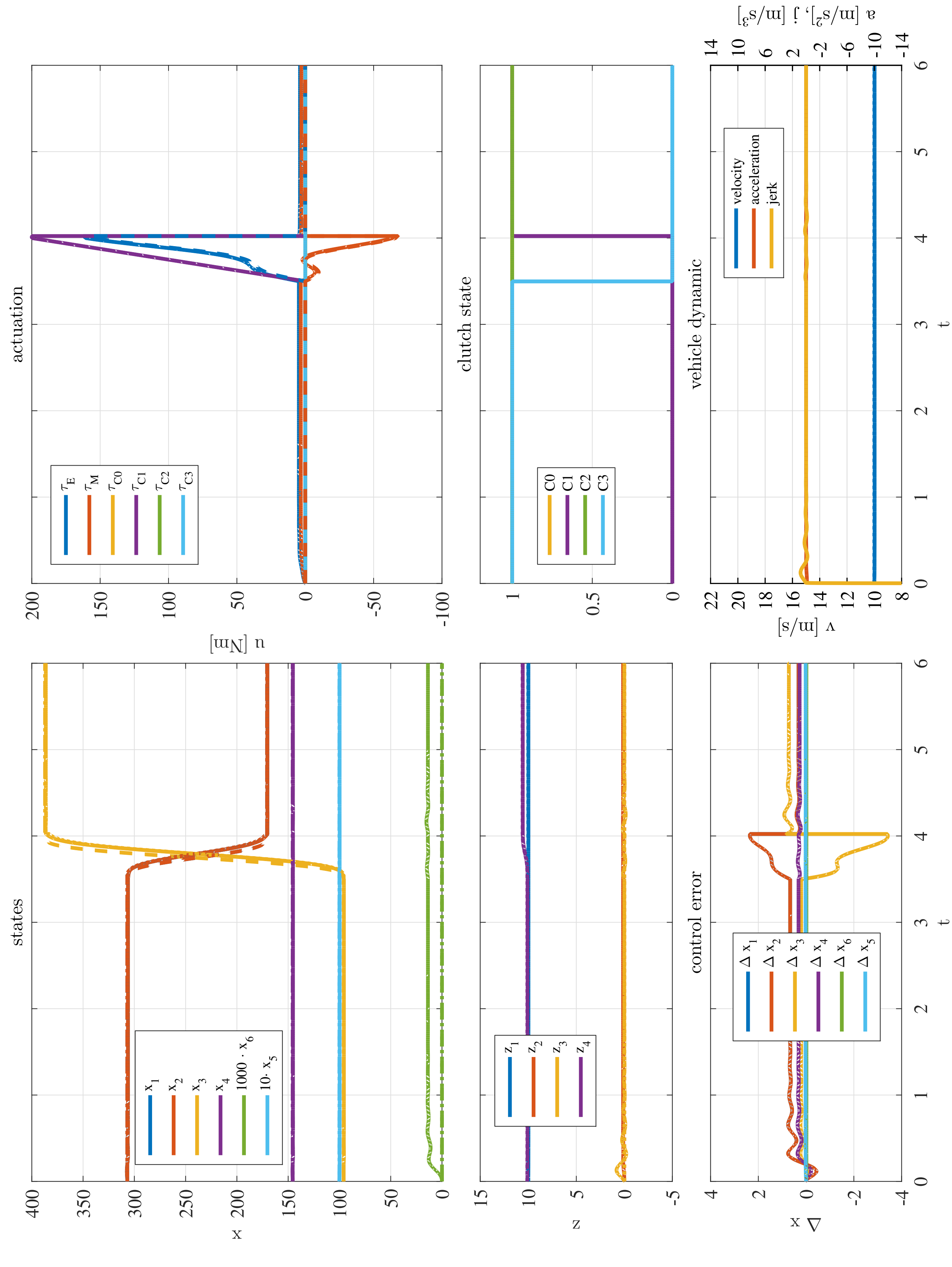

Figure 6.11: Simulation results 2

Simulation 3

While simulations 1 and 2 deal with tracking a constant vehicle speed, simulation 3 deals with tracking a vehicle acceleration from vehicle speed 10m/s to 20m/s within four seconds (ramp from simulation time 1s to 5s), while performing the required gear shift. The other simulation settings, with respect to gear shift and torque split (see table 6.12) are unmodified. Figure 6.12 shows the results of simulation 3.

symbol	value	unit	explanation
$v_{\mathrm{r,start}}$	10	m/s	starting vehicle velocity (reference)
$v_{\mathrm{r,final}}$	20	m/s	final vehicle velocity (reference)
ΔT_{acc}	4	s	acceleration time
$T_{\mathrm{gs,init}}$	2	s	gear shift initiation time
G_1	G_a		current gear (see table 6.2)
G_2	G_c		required gear (see table 6.2)
ff	1		enable feedforward control
fb	1		enable feedback control
k_{E}	1		torque split combustion engine

Table 6.12: Parameters simulation 3

Discussion The required gear shift is indeed performed, but the actual gear shift duration ΔT_{gs} (approximately 0.3 s) is shorter than the planned gear shift duration $\Delta \bar{T}_{\mathrm{gs}}$. This fact implies that the transition to gear G_2 can not be that smooth as it was the case in simulations 1 and 2. Ripple in vehicle jerk confirms this assumption, although its magnitude is still acceptable. Salient points of the required vehicle velocity profile, at the beginning and the end of the ramp, result in peaks in vehicle jerk. These peaks can be reduced by applying a smooth velocity profile. Considering the used actuations, the requirement of stationary propulsion by combustion engine is satisfied. Actually feedback controller employs both the combustion engine and the electric motor to compensate the acting disturbances (air drag, rolling resistance), therefore propulsion torque of the electric motor stationary is not equal to zero. Due to the higher transmission ratio in gear G_2 combustion engine stationary has to apply a higher propulsion torque than in gear G_1. For explanation of the acting clutch torques $\tau_{\mathrm{C},1}$ and $\tau_{\mathrm{C},3}$ in this simulation see figure 6.2. Note that signs of the torques consider signs of the according differential angular velocities.

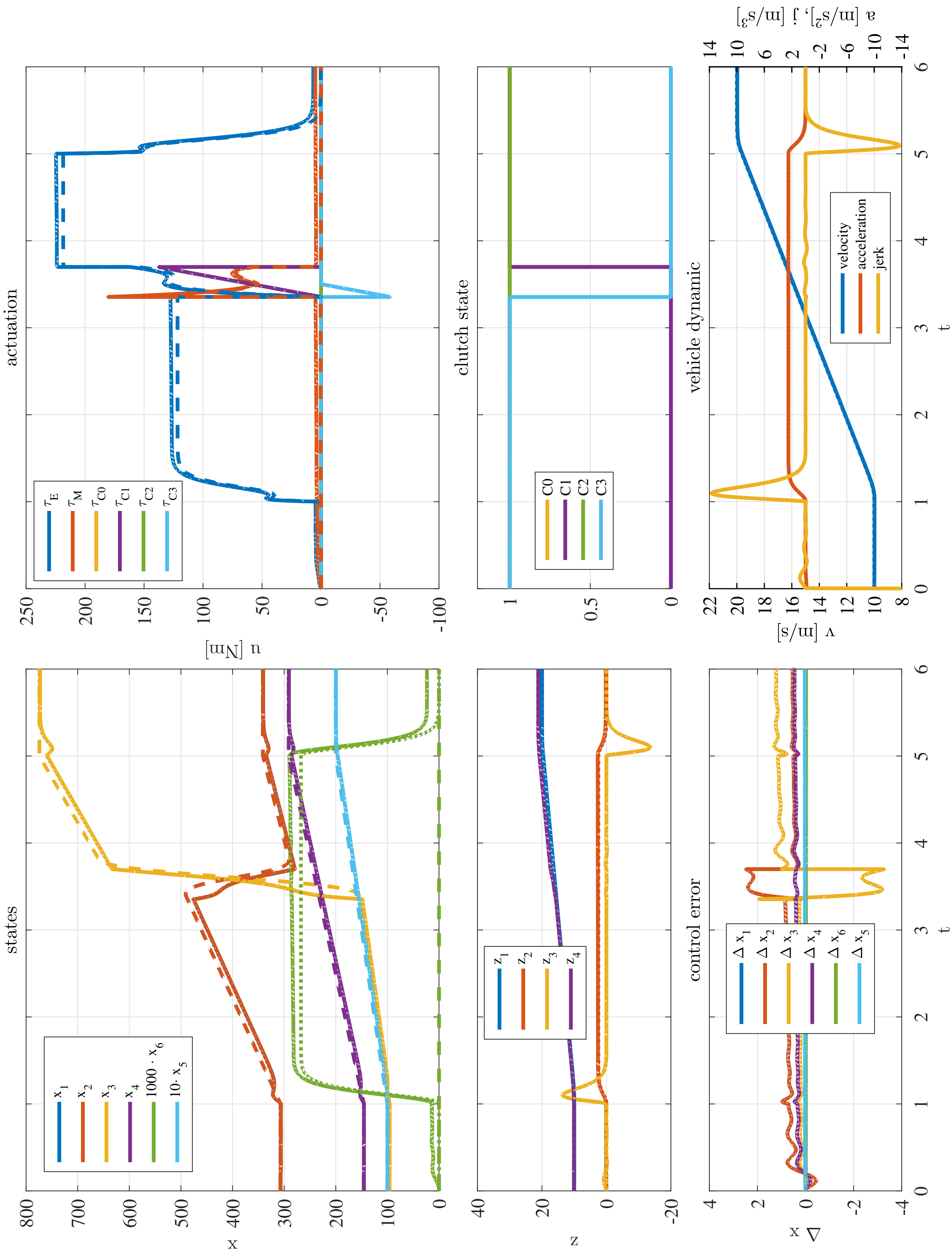

Figure 6.12: Simulation results 3

Simulation 4

The forth simulation experiment repeats simulation 3 with a varied stationary torque split. According to simulation setup (see table 6.13) 70% of the required propulsion torque and consequently power (see chapter 6.4.3) are stationary provided by the combustion engine. The remaining 30% are provided by the electric motor.

symbol	value	unit	explanation
$v_{\mathrm{r,start}}$	10	m/s	starting vehicle velocity (reference)
$v_{\mathrm{r,final}}$	20	m/s	final vehicle velocity (reference)
ΔT_{acc}	4	s	acceleration time
$T_{\mathrm{gs,init}}$	2	s	gear shift initiation time
G_1	G_a		current gear (see table 6.2)
G_2	G_c		required gear (see table 6.2)
ff	1		enable feedforward control
fb	1		enable feedback control
k_{E}	0.7		torque split combustion engine

Table 6.13: Parameters simulation 4

Discussion The requirement on propulsion torque split results in specific torques τ_{E} and τ_{M}, due to transmission ratios. Apart from this the simulation results are similar to simulation 3. Comparing vehicle dynamic and coordinates $\mathbf{z}$ illustrates the approximation $z_1(t) \approx v(t)$ (see remark 24).

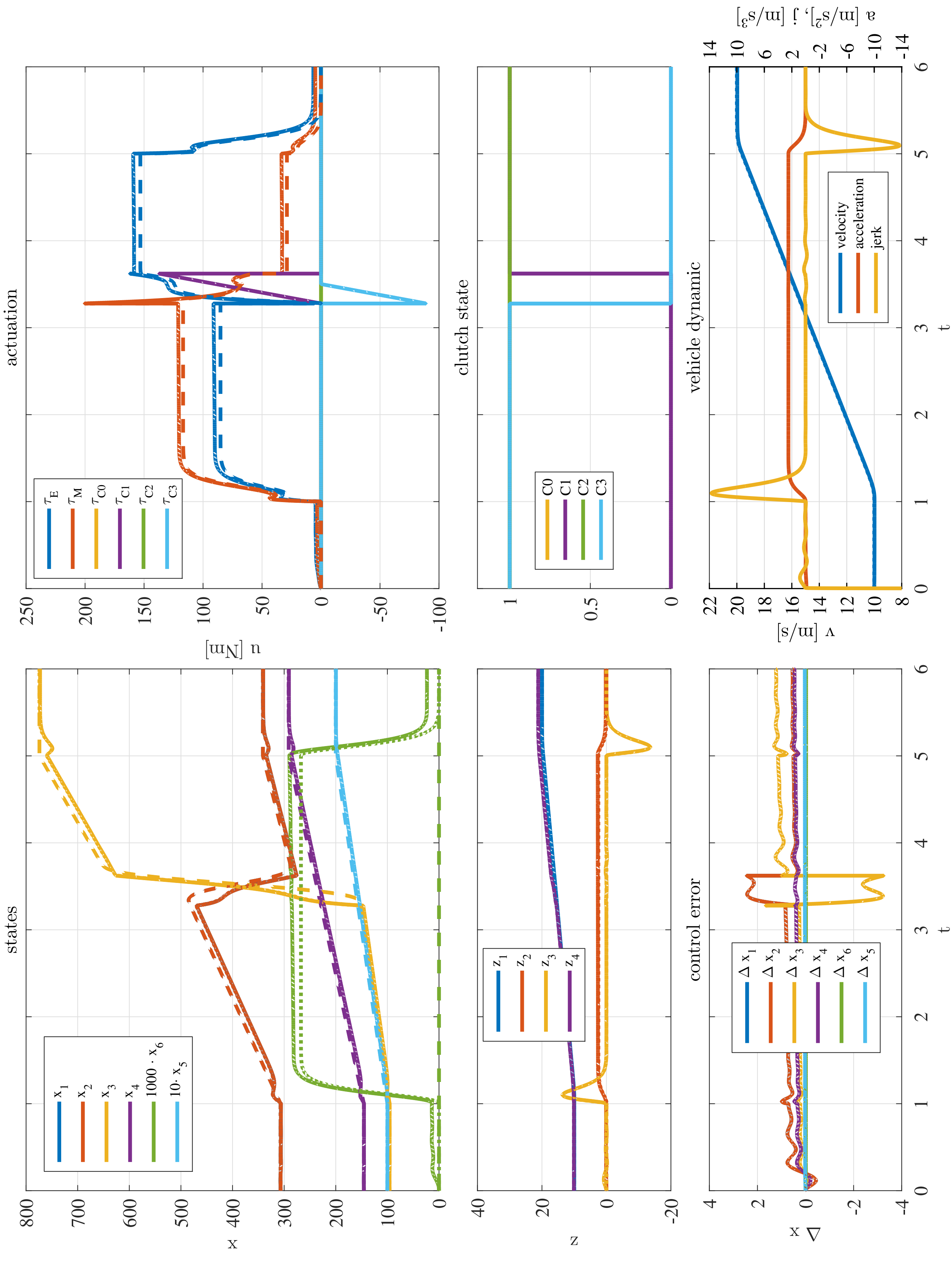

Figure 6.13: Simulation results 4

7

Conclusion and perspectives

Since chapters 2 to 5 provide the mechanical and mathematical knowledge used for the design of the control system in chapter 6, the focus of this chapter is on conclusions of and perspectives to the applied control strategy.

7.1 Conclusion

Chapter 6 provides a control strategy for the specific drivetrain topology that uses both the advantages of the modeling approach presented in general in chapter 2 and more specific in chapter 5 and the mechanical peculiarities of the considered drivetrain. The results of specific gear shift experiments presented in the last section show quite good performance with respect to the control task stated in section 6.3.

Nevertheless the control system so far includes several points for improvement:

- Actuators are considered to be unconstrained in magnitude and rate. Although such constraints can be included into the feedforward strategy (see chapter 7.2), a deterioration of the gear shift performance with respect to vehicle jerk has to be expected.
- As already mentioned in the discussions of the single simulation results in section 6.5.4 planned gear shift duration $\Delta\bar{T}_{\mathrm{gs}}$ has to estimate actual gear shift duration ΔT_{gs}. Discrepancies between these two values can have considerable impact on the gear shift performance. Actual gear shift duration is further heavily influenced by the open loop design parameters *clutch ramp time* ΔT_{cl} and *maximal clutch torque* $\tau_{\mathrm{c,max}}$. Choice of the parameters $\Delta\bar{T}_{\mathrm{gs}}$, ΔT_{cl} and $\tau_{\mathrm{c,max}}$ depending on the current angular velocities, torques and required gear shift could further improve shift quality in the general case (see section 7.2).

7.2 Perspectives

At the end of this chapter and this work this section provides an outlook for further improvement of the control system as suggested in section 7.1 as well as possible additions and generalizations of the used concept.

Determination of Lagrangian multipliers

As already mentioned in section 7.1, choice of the parameters $\Delta\bar{T}_{\mathrm{gs}}$, ΔT_{cl} and $\tau_{\mathrm{c,max}}$ is challenging in order to achieve a good gear shift performance. Transmitted torque on a locked clutch is defined by the corresponding Lagrangian multiplier (see remark 20) and can be calculated in dependency of the current driving situation in advance. This calculation offers an estimation of the minimal necessary clutch torque τ_{C}, and consequently minimal necessary pressure on clutch plates, to keep it locked. Starting ramp down from a torque $\tau_{\mathrm{C,max}}$ more closely to the locking torque, clutch ramp time ΔT_{cl} would be more directly related to the actual gear shift duration ΔT_{gs}. Consequently its estimation ($\Delta\bar{T}_{\mathrm{gs}}$) could be done in a more consistent way.

Constrained actuators

In practice every actuator is constrained. Constraints can be characterized by maximum and minimum limits as well as maximum increasing and decreasing rates. Flatness-based linear feedforward control enables a smart possibility to include actuator constraints into the design as suggested in [19]. The idea is to enforce the components $\tilde{\mathbf{z}}_i^{(\mu_i)}$ for $i = 1, \ldots, m$ to satisfy the constraints and afterwards calculate corresponding coordinates $\tilde{\mathbf{z}}$ by applying integration. The approach is similar to anti-windup measures like back calculation proposed by Hanus in [20]. To ensure intelligibility this idea is considered in detail in SISO case.

In the first step the suggested filter structure has to be adapted (see figure 7.1).

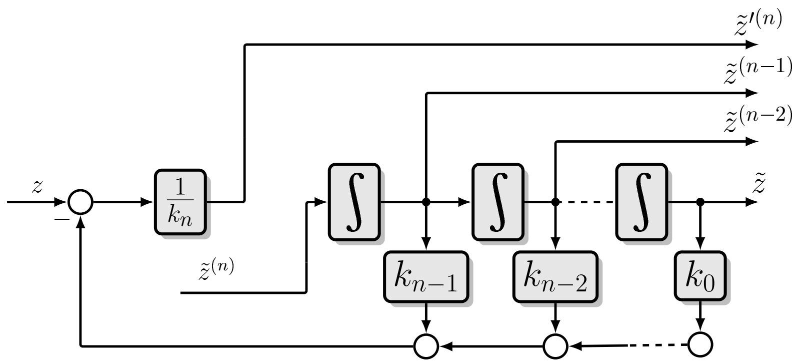

Figure 7.1: Adapted low-pass filter n-th order

Figure 7.2 shows the structure of the feedforward control considering actuator saturation in SISO case. Usage of $\tilde{\mathbf{z}}$ and unsaturated component $z'^{(n)}$ enables calculation of the necessary actuation u'. Applying the system model, saturated actuation u_{sat} and state vector $\tilde{\mathbf{z}}$ define the component $\tilde{z}^{(n)}$. This component is channeled into the filter in an feedback loop.

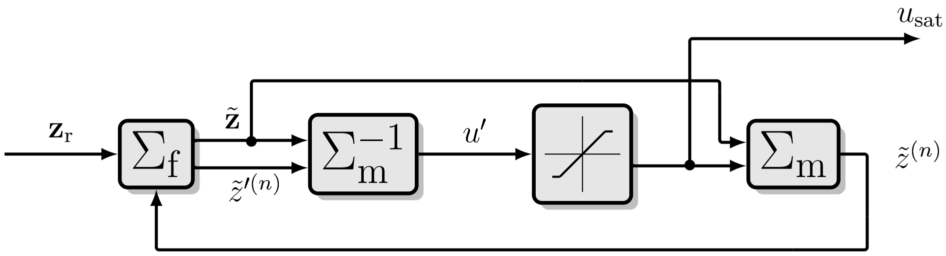

Figure 7.2: Feedforward control structure including actuator saturation: Σ_f filter (see figure 7.1), Σ_m plant model, Σ_m^{-1} inverse plant model

This approach can be used to avoid non feasible step like changes in the actuation signals, in order to generate more realistic gear shifts.

Generalization of the control strategy

The used modeling with respect to the additional mechanical constraints, and the special structure of the drivetrain topology with respect to mechanics, enabled a smart approach in control. Consequently it is an interesting question, if this control strategy can be generalized to general (non-conventional) gear shifts or more general on other drivetrain topologies respectively also on similar switching linear multibody systems.

Clutch actuation

Although clutch torque actuation is part of the discussed control strategy, clutch torques are not yet considered in a model-based way with respect to control. Intelligent actuation of the slipping clutches' torques could probably significantly improve the gear shift performance. Since clutch torques are dissipative actuators, i. e. they can not impress energy into the system, and further disappear in case of locked clutches, such an extension of the control strategy will be quite challenging, but definitely desirable.

State observation

Since there exists a strong duality between controlling and observing a system, the design of a state observer, based on modeling and considerations with respect to control discussed in this work, immediately suggests itself. Chapter 4.2 already offers preparatory work to that from the mathematical point of view.

List of Figures

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