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Degree Sequences of Triangulations of Convex Point Sets

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Abstract

In this thesis we consider degree sequences of triangulations of point sets in convex position in the plane. This is equivalent to degree sequences of maximal outerplanar graphs. Utilizing basic properties and transformations of triangulations we develop sufficient and necessary conditions for special cases of nonnegative integer sequences to be valid degree sequences of a triangulation of a convex point set. Furthermore, we consider more general cases including the case with more 'big nodes' (nodes with degree greater than 2) than nodes with degree 0. Also for this case we present sufficient and necessary conditions for a sequence to be a degree sequence. Additionally, we present a construction of a canonical triangulation for every discussed case.

Keywords: degree sequence, convex point set, maximal outerplanar graph, triangulation

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Contents

1 Introduction

In human history people always used graphs to present, discuss and realize their ideas and plans. Although they mostly didn't know what a graph is about. With the rise of mathematics, graph theory started to study graphs and their properties. Sometimes with real world applications in mind (Figure 1.1) but often discussing abstract buildings with no possible applications (at least at first sight).

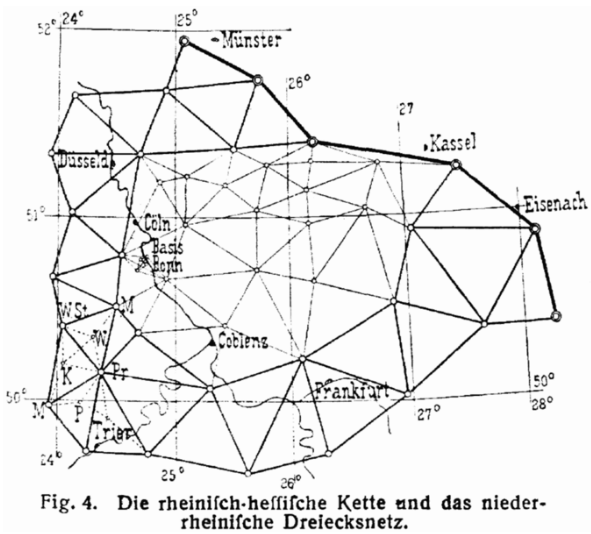


Figure 1.1: Triangulation in geodesy [12]

These studies developed subclasses of graphs like simple graphs (no loops or double edges), planar graphs (cross free embedding into the plane) or outerplanar graphs (planar with all vertices incident to the same face).

One famous subclass of planar graphs are triangulations. The idea of a triangulation is an often used concept in different fields like numerical mathematics, geodesy and even complete other areas of science like social research [14] or psychology [1] (although these triangulations have only little to do with our concept). Even the chocolate industry discovered the beauty of triangulations (see Figure 1.2) with 540 kcal per 100g.

In the already famous class of triangulations, the probably best known and most often used is the Delaunay triangulation. The Delaunay triangulation fulfills a bunch of nice properties (for an overview see [21]). For example, the criterion to be Delaunay is equivalent if tested locally or on a global scope. For an arbitrary triangulation it's easy to show that one can reach the Delaunay triangulation by flipping $O(n^2)$ edges, which gives a nice proof that two triangulations on the same point set can be transformed into each other by flipping at most $O(n^2)$ edges. It maximizes the minimal angle between two edges and the minimum spanning

Figure 1.2: Triangulation of chocolate [19]

tree is always a subgraph of the Delaunay triangulation. Even the dual structure, the Voronoi diagram is well studied [2] and very useful for many applications [5].

Looking at the degrees of nodes occurring in a graph the question about degree sets and degree sequences arise. The *degree set* of a graph is the set containing all degrees occurring in the graph. It is already known that for every set of nonnegative integers there exists a graph with this degree set. Looking at special classes of graphs like trees, planar graphs or outerplanar graphs, there are also results for the realizability of degree sets [9].

While degree sets ignore the multiplicity of a degree, degree sequences fix the number of occurrences for each degree and even the number of vertices in the realizing graph (if there is any). The question on the realizability of a given sequence of nonnegative integers as a degree sequence of a simple graph is a classical problem in graph theory and theoretical computer science. Some major results are even dating back to the work of Erdős and Gallai [6]. They discovered sufficient and necessary conditions for a sequence to be graphical (meaning realizable). Although there have been many results ([13], [15]) and different criteria ([18], [20], [7], [8]) for sequences to be graphical, they mostly consider only general simple graphs.

Focusing on the question of sequences to be realizable as a given class of graphs, like planar or outerplanar, seems to be a much harder problem. For planar sequences (graphical sequences that could be realized as a planar graph) only results for special sequences were obtained [17]. It seems that the only known cases until now are degree sequences of trees and 2-trees (which is a super class of edgemaximal outerplanar graphs) [4]. This result is pretty interesting for us, since regarding degree sequences there is no difference between triangulations of convex point sets and edgemaximal outerplanar graphs.l now are degree sequences of trees and 2-trees (which is a super class of edgemaximal outerplanar graphs) [4]. This result is pretty interesting for us, since regarding degree sequences there is no difference between triangulations of convex point sets and edgemaximal outerplanar graphs.

Relaxing from simple graphs to multi graphs opens the problem to many new applications like the problem of isomers in organic chemistry. Fortunately there exist nice results for this case [7].

In this thesis we are going to explore the properties of degree sequences of triangulations of convex point sets. The main goal is to find sufficient and necessary conditions for integer sequences to be degree sequences of triangulations of convex point sets. We also provide canonical triangulations for integer sequences that fulfill one of the conditions.

Some parts of this work were already presented in [10], but are included in this work for reasons of self containment and to give the reader a complete view on the topic.

2 Related Work

A graph is called simple if it has no multiple edges and no loops. We will now look at degrees of nodes in simple, undirected graphs.

2.1 Degree Sets

The *degree set* D_G of a graph G is the set containing all degrees of nodes in G.

Every degree set is a set of nonnegative integers. On the other hand the degree 0 means an isolated node. Therefore a set of nonnegative integers S is a degree set if and only if $S\backslash\{0\}$ is a degree set. So it is sufficient to consider sets of positive integers.

A graph G is said to *realize* the set S if $S = D_G$. Because of the properties of a set, the degree set provides no information on the multiplicity of a degree in the graph.

Regarding degree sets, there arise three important questions:

- Is every set of nonnegative integers realizable?
- What is the minimum number of nodes for such a realization?
- What about special classes of graphs like planar graphs, outerplanar graphs or trees?

We write

$$\mu(S) = inf\{|V| \mid G = (V, E) \text{ a graph with } D_G = S\}$$

for the minimal number of nodes for a graph realizing the set S.

If $S = \{d_1, \ldots, d_n\}$ with $d_1 > \ldots > d_n$ we write $\mu(d_1, \ldots, d_n)$ instead of $\mu(S)$.

2.1.1 Degree Sets for General Simple Graphs

In every simple, undirected graph with highest degree k there exists at least one node which is adjacent to k other nodes. Therefore it's easy to see that

$$\mu(d_1, \ldots, d_n) \geq d_1 + 1$$

On the other hand Kapoor et al. [9, Theorem 1 and Corollary 1a] state that this necessary condition is sufficient for a simple, undirected graph:

Theorem 2.1
For every set $S = \{d_1, \ldots, d_n\}$ of positive integers, with $d_1 > \ldots > d_n$, there exists a connected graph G such that $D_G = S$ and furthermore,

$$\mu(d_1, \ldots, d_n) = d_1 + 1$$

So there exists full information on degree sets of general simple undirected graphs. Now we look at a subclass of graphs.

2.1.2 Degree Sets for Trees

It is well known that every nontrivial tree contains at least two leaves. Every leaf is a node with degree 1. This means that for every nontrivial tree T we know that $1 \in D_T$.
This necessary condition again proves to be sufficient, shown by Kapoor et al. [9, Theorem 2]:

Theorem 2.2
For every set $S = \{d_1, \ldots, d_n\}$ of positive integers there exists a nontrivial tree T with $D_T = S$ if and only if $1 \in S$.
Moreover if $1 \in S$ then the minimum number of vertices of such a tree is $\sum_{i=1}^{n}(d_i - 1) + 2$.

This means that we again have full information about degree sets of nontrivial trees. So we look at a more general subclass of graphs.

2.1.3 Degree Sets for Planar Graphs

A graph is called *planar* if there exists a cross free embedding in the plane.
It is known that every simple connected planar graph fulfills the *Euler characteristic*:

$$\#\text{Nodes} - \#\text{Edges} + \#\text{Faces} = 2$$

A direct conclusion is that the minimal degree in every planar graph is less than 6. Again Kapoor et al. [9, Theorem 3] shows that this simple necessary condition is already sufficient:

Theorem 2.3
For every set $S = \{d_1, \ldots, d_n\}$ of positive integers with $d_1 > d_2 > \ldots > d_n$ there exists a planar graph G with $D_G = S$ if and only if $1 \leq d_n \leq 5$.

Although we have full information on the realizability of a set as a planar graph, the question regarding the minimum number of nodes for such a realization seems to be more difficult.
For $S = \{d_1, \ldots, d_n\}$ the minimum number of nodes for a planar realization is called

$$\mu_p(S) \;=\; \mu_p(d_1, \ldots, d_n)$$

In the simple case with $n = 1$ the value of $\mu_p(S)$ is well-known: $\mu_p(1) = 2$ (two connected nodes), $\mu_p(2) = 3$ (a triangle), $\mu_p(3) = 4$ (the tetrahedron), $\mu_p(4) = 6$ (the octahedron) and $\mu_p(5) = 12$ (the icosahedron).
The only further information on the value of $\mu_p(S)$ is shown by Kapoor et al. [9, Theorem 4] for the case $n = 2$:

Theorem 2.4
Let d_1 and d_2 be positive integers with $d_1 > d_2$. Then

$$\mu_p(d_1, d_2) = \begin{cases} d_1 + 1, & \text{for } 1 \leq d_2 \leq 3 \\ d_1 + 2, & \text{for } d_2 = 4 \end{cases}$$

and

$$\mu_p(d_1, d_2) \leq 2d_1 + 2 \text{ for } d_2 = 5$$

Finally we look at the degree sets of a more special subclass of planar graphs.

2.1.4 Degree Sets for Outerplanar Graphs

A graph is called *outerplanar* if there exists a cross free embedding in the plane where every node is incident to the outer face. It is easy to show that the minimal degree in an outerplanar graph is less than 3. One more time Kapoor et al. [9, Theorem 5] proves that this necessary condition is already sufficient:

Theorem 2.5
Let $S = \{d_1, \ldots, d_n\}$ be a set of positive integers with $d_1 > d_2 > \ldots > d_n$. Then there exists an outerplanar graph G with $D_G = S$ if and only if $1 \leq d_n \leq 2$.

But again the minimal number of nodes for an outerplanar realization is a more complicated question. For $S = \{d_1, \ldots, d_n\}$ the minimum number of nodes for an outerplanar realization is called $\mu_o(S) = \mu_o(d_1, \ldots, d_n)$. In the case $n = 1$ the realizing graphs are the same as in the planar case: $\mu_o(1) = 2$ and $\mu_o(2) = 3$.
For the case $n = 2$ Kapoor et al. [9, Theorem 6] provides full information:

Theorem 2.6
For $d_1 > 1$, $\mu_o(d_1, 1) = d_1 + 1$. For $d_1 > 2$,

$$\mu_o(d_1, 2) = \begin{cases} d_1 + 1, & \text{if } d_1 \text{ is even} \\ 2d_1 - 2, & \text{if } d_1 \text{ is odd} \end{cases}$$

2.2 Degree Sequences

The degree sequence of a simple, undirected graph G is the nonincreasing sequence of the degrees of its nodes.
A nonincreasing sequence of positive integers $S = (d_1, \ldots, d_n)$ is called *graphical* if there exists a simple, undirected graph G having S as its degree sequence. Since a sequence preserves the multiplicity of a degree it even provides the number of nodes for a realizing graph (if there exists one).
The handshaking lemma provides a simple necessary condition for a sequence to be graphical:

$$\sum_{i=1}^{n} d_i = 2\#\{\text{Edges}\}$$

This means that the sum of every degree sequence has to be even.
The interesting questions regarding degree sequences are:

- What kind of sequences are graphical?
- For which sequences exists a realizing graph that is planar, outerplanar or a tree?

Unfortunately these questions seem more difficult than the questions regarding degree sets.
Because of the necessary condition we will now only consider nonincreasing sequences $S = (d_1, \ldots, d_n)$ of positive integers with even sum.

2.2.1 Criteria for a Sequence to be Graphical

There exist a lot of different equivalent criteria for sequences to be graphical, see [18] for an overview and proof of the equivalence, but be aware of misprints.

The Ryser Criterion

See Ryser [16]. A sequence $(a_1, \ldots, a_p; b_1, \ldots, b_q)$ is called *bipartite-graphic* if and only if there exists a simple bipartite graph such that one component has degree sequence $(a_1, \ldots, a_p)$ and the other component has degree sequence $(b_1, \ldots, b_q)$.
Be $f = max\{i \mid d_i \geq i\}$ and $\widetilde{d_i} = d_i + 1$ if $1 \leq i \leq f$ and $\widetilde{d_i} = d_i$ otherwise.
Then the criterion of Ryser states:

$$S \text{ is graphical } \Leftrightarrow (\widetilde{d_1}, \ldots, \widetilde{d_n}; \widetilde{d_1}, \ldots, \widetilde{d_n}) \text{ is bipartite-graphic}$$

The Berge Criterion

See Berge [3]. M_S be the $(0,1)$-Matrix, containing 1s exactly in the leading d_k terms of the k-th row except for the diagonal. We define $\overline{d_i}$ as the i-th column sum of M_S and receive the sequence $(\overline{d_1}, \ldots, \overline{d_n})$.
For example, for the sequence $(3, 2, 2, 2, 1)$ we have $\overline{d_1} = 4$, $\overline{d_2} = 3$, $\overline{d_3} = 2$, $\overline{d_4} = 1$, $\overline{d_5} = 0$ and

$$M_S = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now S is graphical if and only if

$$\sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} \overline{d_i} \text{ for } 1 \leq k \leq n$$

The Erdős-Gallai Criterion

See Erdős and Gallai [6]. This criterion is probably one of the most famous criteria for sequences to be graphical. Mainly because it was stated by Erdős himself. Furthermore it doesn't need any additional definition as the criteria before. The criterion is:
S is graphical if and only if

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{n} min\{k, d_j\}$$

for $1 \leq k \leq n$.

The Havel-Hakimi Criterion

This is another well known criterion for sequences to be graphical and was developed independently by Havel [8] and Hakimi [7]. Other than the previous criterion it shows graphical equivalence between two sequences.
Let $(d_1, \ldots, d_n)$ be a nonincreasing sequence of positive integers with even sum and $n - 1 \geq d_1 \geq d_2 \geq \ldots \geq d_n$. Then the criterion states as follows:
$(d_1, \ldots, d_n)$ is graphical if and only if the sequence

$$(d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n)$$

is graphical.

2.2.2 The Kleitman-Wang Criterion

As a generalization of the Havel-Hakimi Criterion Kleitman and Wang [11] showed the following criterion:
Let $(d_1, \dots, d_n)$ be a nonincreasing sequence of positive integers with even sum and $n - 1 \geq d_1 \geq d_2 \geq \dots \geq d_n$. S^i be the sequence obtained from $S = (d_1, \dots, d_n)$ by deleting the term d_i from S and decreasing the remaining d_i largest terms by 1.

Then the criterion of Kleitman and Wang states that S is graphical if and only if S^i is graphical for $1 \leq i \leq n$.

2.2.3 Planar Graphical Sequences

A nonincreasing sequence S is called *planar* if it is realizable as a planar graph. Obviously every planar sequence has to be graphical. This means we can use the full information for sequences to be graphical as a necessary condition for sequences to be planar. A combination of the handshaking lemma and the Euler characteristic provides another necessary condition. Every planar sequence with $n \geq 3$ fulfills

$$\sum_{i=1}^{n} d_i \leq 6(n-2)$$

A sequence fulfilling this condition is called *Euler sequence*. If the inequality is fulfilled with equality, then the sequence is called *maximal Euler sequence*.
A sequence is called a k-sequence if $d_1 - d_n = k$, 0-sequences are called *regular*.
For an easier notation we write the sequence $(\underbrace{d_1, \dots, d_1}_{e_1}, \underbrace{d_2, \dots, d_2}_{e_2}, \dots, \underbrace{d_k, \dots, d_k}_{e_k})$
as $\left(d_1^{e_1}, d_2^{e_2}, \dots, d_k^{e_k}\right)$
Schmeichel and Hakimi [17] showed some information on planar graphical k-sequences. For regular sequences and 1-sequences they provide full information:

Theorem 2.7 (see [17, Theorem 1])
Every regular, graphical Euler sequence is planar except for (4^7) *and* (5^{14}).

Theorem 2.8 (see [17, Theorem 2])
Every graphical Euler 1-sequence is planar graphical except for $(5^{10}, 4^1)$, $(5^{12}, 4^1)$, $(6^1, 5^{12})$ *and* $(6^1, 5^{14})$.

For 2-sequences there are still some unresolved cases and the number of exceptions is rising.

Theorem 2.9 (see [17, Theorem 3])
Every graphical, maximal Euler 2-sequence is planar graphical except for
$(5^1, 4^4, 3^1)$, $(5^5, 4^2, 3^1)$, $(6^1, 5^{10}, 4^1)$, $(5^3, 3^3)$, $(6^1, 5^2, 4^5)$, $(7^1, 5^{13})$,
$(5^4, 4^1, 3^2)$, $(5^7, 4^1, 3^1)$, $(7^1, 6^1, 5^{13})$, $(6^1, 4^6)$, $(5^9, 3^1)$,
and possibly the following unresolved cases:

$$\begin{aligned} &(7^1, 6^2, 5^{13}), \\ &(7^k, 6^1, 5^{k+12}) \text{ for } k = 2,3,5,7, \\ &(7^k, 5^{k+12}) \text{ for } k = 3,5,7,9. \end{aligned}$$

Theorem 2.10 (see [17, Theorem 4])
Every graphical, nonmaximal Euler 2-sequence is planar graphical except for
$(4^5, 2^1)$, $(5^{11}, 3^1)$, $(6^{p-7}, 4^7)$ *for* $p > 7$, $(5^5, 3^3)$, $(7^1, 5^{15})$,
and possibly the following unresolved cases:
$(5^{13}, 3^1)$, $(7^1, 5^{17})$, $(7^3, 5^{17})$.

Unfortunately there is still no further information on general planar sequences. The only additional information is a more strict necessary condition:

Theorem 2.11 (see [17, Theorem 4])
Every graphical Euler sequence with $d_3 \leq 3$ *is planar graphical.*
Every planar graphical sequence with $d_3 > 3$ *fulfills*

$$\sum_{i=1}^{n} d_i \leq 6(n-2) - 2n_2 - 4n_1$$

where n_i *denotes the number of times* i *occurs as an element of the sequence.*

2.2.4 Degree Sequences of Trees

Let's look at a more special case of planar graphs: the trees.

It is widely known that every tree fulfills $\#Edges = \#Nodes - 1$. Therefore every degree sequence of a tree fulfills

$$\sum_{i=1}^{n} d_i = 2(n-1).$$

Bose et al. [4] improve the characterization of degree sequences of trees with the following lemma:

Lemma 2.12 ([4, Lemma 1])
A nonincreasing sequence of positive integers $S = (d_1, \ldots, d_n)$ *is the degree sequence of a tree if and only if*

$$\sum_{i=1}^{n} d_i = 2(n-1)$$

Moreover, for any $l, k \in S$ *there exists a realizing tree of* S *in which a vertex of degree* l *is adjacent to a vertex of degree* k*, unless* $n > 2$ *and* $l = k = 1$.

This means that there exists full information on degree sequences of trees. Bose et al. further looked at a generalization of trees, the k-trees.

A graph G is a k-tree if and only if one of the following conditions is true

- G is the complete graph on $k+1$ vertices
- there exists a vertex v whose neighborhood is a clique of order k and $G \backslash v$ is a k-tree.

This means that 1-trees are trees where we already have full information. Now we look at 2-trees.

Every 2-tree T with degree sequence $(d_1, \ldots, d_n)$ fulfills:

- $\sum_{i=1}^{n} d_i = 4n - 6$

- $d_n = 2$
- $d_{n-1} = 2$
- No two ears in T are adjacent unless $T = K_3$
- T has no K_4-minor
- T is 2-connected

For a proof of these properties look at [4, Lemma 2]. Furthermore Bose et al. provide full information on the realizability of sequences as 2-trees:

Theorem 2.13 ([4, Theorem 1])
Let $S = (d_1, \ldots, d_n)$ be a nonincreasing sequence of positive integers, n_2 be the multiplicity of 2 in S.
S is the degree sequence of a 2-tree if and only if the following conditions are fulfilled:

- $\sum_{i=1}^{n} d_i = 4n - 6$
- $d_1 \leq n - 1$
- $d_n = 2$ *and* $n_2 \geq 2$
- $S \notin \{(2^{n-4}, d^4) | d \geq 5\}$
- $n_2 \geq \frac{n}{3} + 1$ *if all degrees in S are even*

3 Notation and Basics

Everyone in the field of computational geometry is familiar with terms like triangulations, nodes and degrees. When it comes to degree sequences, inner triangles and 'zigzags' there exist different papers with different meanings for these terms. For this reason and because of my inner hope that not only experts (and my parents) will read this thesis, we (re)state definitions for the terms used in this thesis.

3.1 Notation of Parity Constraints

In this thesis we will often need to distinguish cases by the parity of two numbers. To avoid too much writing (and to minimize your reading time) we will use this notation:
$[x \equiv y \ mod\ 2]$ is denoted by $\left[x \overset{2}{\equiv} y\right]$ and $[x \not\equiv y \ mod\ 2]$ is denoted by $\left[x \overset{2}{\not\equiv} y\right]$.

3.2 Geometric Graph and Convex Point Set

A geometric graph $G = (V, E)$ is a pair of two sets. The first (V) represents the points in the Euclidean plane being the vertices of the graph. The second (E) contains subsets of V with two elements, each subset representing an edge in the graph by connecting the two vertices with a straight line. A geometric graph is called *planar* if no two edges intersect.

An object is called *convex* if for every two points within the object every point on the straight line between them is again within the object.

A point set is called convex if there exists a convex set such that all points from the point set are on the border of the convex set.

3.3 Triangulation

By definition a triangulation of a point set V is a maximal planar geometric graph which uses all points of V as nodes. It is easy to see that $m = 3n - n_h - 3$ where m is the number of edges, n the number of points and n_h the number of points on the convex hull.

In this thesis we will only look at triangulations on convex point sets. Setting $n_h = n$ in the equation above leads to $m = 2n - 3$. To get the number of diagonals we subtract the n edges on the convex hull:

$$\#diagonals = n - 3 \tag{3.1}$$

We refer to the degree of a node v as the number of diagonals incident to that node.

$$deg(v) = \#\{\text{to } v \text{ incident diagonals}\}$$

Geometrically we delete the edges on the convex hull and look at the degrees in the resulting graph.

Because of this we call two nodes *connected* if they are connected by an inner diagonal.

Further, we will often refer to the number of nodes with the same degree:

$$\#i = \#\{v \mid v \in V, deg(v) = i\}$$

Later on we will also need the number of nodes with a degree bigger than a given threshold. Therefore we will use this notation:

$$\#k^+ = \sum_{i \geq k} \#i$$

Definition 3.1 (inner triangle)
In a triangulation of a convex point set an *Inner Triangle* is a triangle which consists of diagonals, that is, it has no edge from the convex hull.
Geometrically all triangles that remain after deleting the edges on the convex hull are the inner triangles.

3.4 The Dual Tree

Given a triangulation of a convex point set, we receive a tree by replacing the triangles by vertices and connecting two vertices if and only if the corresponding triangles share an edge. We call this tree the *dual tree* for the given triangulation.

In a triangulation every inner triangle yields an additional ear (a vertex of degree 0) and every additional ear requires an additional inner triangle. This can be seen from the dual of the triangulation. In the tree every vertex of degree 3 leads to an additional leaf and vice versa. So we get the following equation:

$$\#\{\text{inner triangles}\} = \#0 - 2 \tag{3.2}$$

3.5 Illustrative Outlines

When drawing outlines of triangulations on a convex point set it is not important to draw the points exactly in convex position. As long as all faces (except the outer face) are triangles and all points are incident to the outer face, we are always able to move the points so that they are in convex position and the graph stays planar.

3.6 Degree Sequence

Definition 3.2 (degree sequence)
A *degree sequence* of a triangulation is a sequence of nonnegative integers, representing the degrees of the nodes as defined above. Note that, since a degree sequence doesn't have an order, we usually write it in descending order.

Definition 3.3 (degree vector)
The *degree vector* of a triangulation is a vector of nonnegative integers $(\#0, \#1, \#2, \ldots)$, representing the number of occurrences of degrees as defined above. For triangulations of finite convex point sets there always exists k so that $\forall l \geq k \;\; \#l = 0$, therefore we only write the first k elements of the degree vector.

If we want to find a triangulation according to the degree sequence we need to know the degree of each node.

Definition 3.4 (ordered degree sequence)
A degree sequence, which has got an explicit order so that there exists an according triangulation where the degrees of the nodes read clockwise, starting at the left most node, equals the degree sequence, is called an *ordered degree sequence.*

For example, be $n = 8$ and consider $\#0 = 2, \#1 = 3, \#2 = 2, \#3 = 1$. This is equivalent to the degree sequence $\{3, 2, 2, 1, 1, 1, 0, 0\}$ and the degree vector $(2, 3, 2, 1)$. This is the degree vector for the triangulations in Figure 3.2, with ordered degree sequences $[0, 2, 1, 2, 0, 1, 3, 1]$ and $[0, 3, 2, 0, 1, 2, 1, 1, 0]$.

3.6.1 From Triangulation to (Ordered) Degree Sequence and Back

For a given triangulation it is easy to see how we get an ordered degree sequence. We just start at the left-most node and write down its degree, then we walk around the point set in clockwise direction, writing down the degree of each node we pass by, till we get back to the start node.

If we have an ordered degree sequence we reverse the procedure, starting at the left-most node writing the first number from the sequence to the node. Then we walk around the point set, writing the according number from the sequence to the node. Afterwards we start 'cutting of the edges', that means we take a node with degree 0, connect its neighbors with an edge, and decrease their degrees by 1. Iterating this procedure until no positive degrees are left yields the wanted triangulation.

We can think of each step as deleting an ear (node with degree 0) from the point set. This yields another (smaller) convex point set.

The original sequence is an ordered degree sequence if and only if the changed sequence on the smaller point set is also an ordered degree sequence.

Lemma 3.5
Every triangulation on a convex point set has got at least 2 ears.

Proof. An ear in the triangulation is a leaf (node with degree 1) in the dual tree. If the dual tree is a single node this means the triangulation is a single triangle having 3 ears. Otherwise we start in a node with degree 1 (there has to be at least one because a tree is cycle free). Then we walk through the tree till we get to another node with degree 1 (we get to the second leaf because the tree is cycle free and finite). Now we have 2 nodes with degree 1 and the lemma is proven. □

This means we always get at least two choices for the iteration step.

3.6.2 Uniqueness

It is obvious that for every ordered degree sequence we can move the first number to the end, which is geometrically equivalent to rotating the point set. Similarly we can read the ordered degree sequence forward or backward, which geometrically means to reflect the according triangulation.
Once we have decided the direction and the mapping, the algorithm for drawing the triangulation is unique. So we know that every ordered degree sequence yields an unique triangulation except for rotation and reflection.

For a degree sequence (with no order) we only know, that there exists at least one triangulation with this degree sequence.

For $n = 3, 4, 5$ there exists only one triangulation except for reflection and rotation, hence there exists only one degree sequence which yields a unique triangulation. As shown in Figure 3.1, for $n = 6$ there exist 3 different triangulations with different degree sequences, that is, each of them yields an unique triangulation. For $n = 7$ there exist 4 different triangulations, each with a different degree sequence.

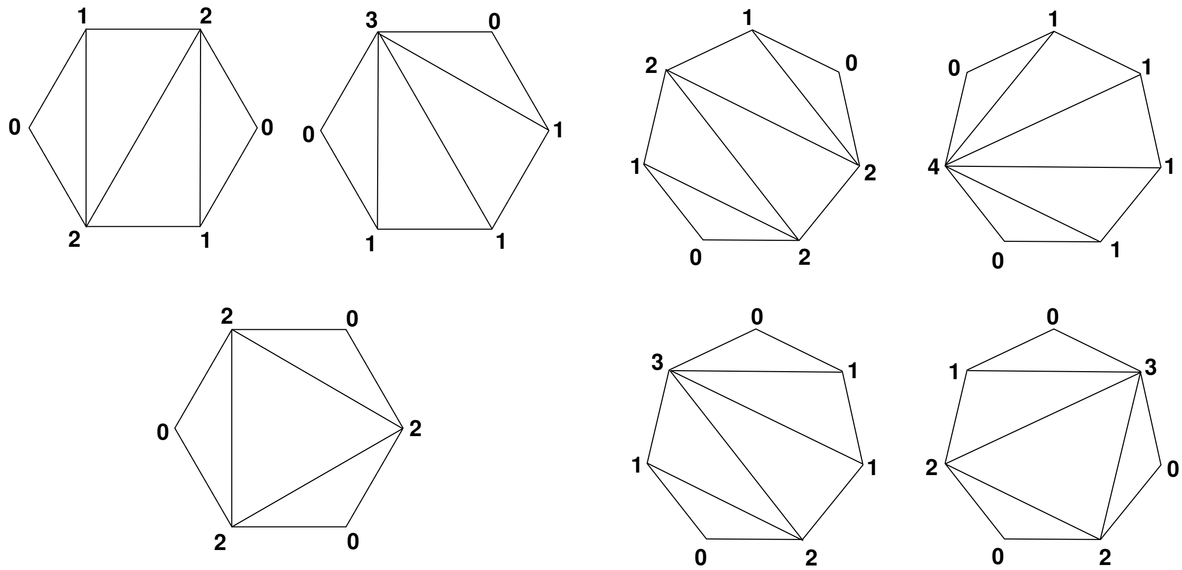


Figure 3.1: The triangulations for 6 and 7 nodes

For $n = 8$ the degree vector $(2, 3, 2, 1)$ yields two different triangulations, shown in Figure 3.2. It follows that for $n \geq 8$ a degree sequence usually does not yield a unique triangulation.

3.7 Big Nodes, Wedges and the '1's

Definition 3.6 (big node)
A node v is called a *big node* if $deg(v) > 2$. Nodes with degree 2 are called *pseudo big nodes*. A (pseudo) big node is called *isolated* if it is not incident to an inner triangle.

A big node v is incident to an inner triangle or has connected '1's. These are the nodes with only one incident diagonal. Nodes on inner triangles can also have connected '1's or multiple

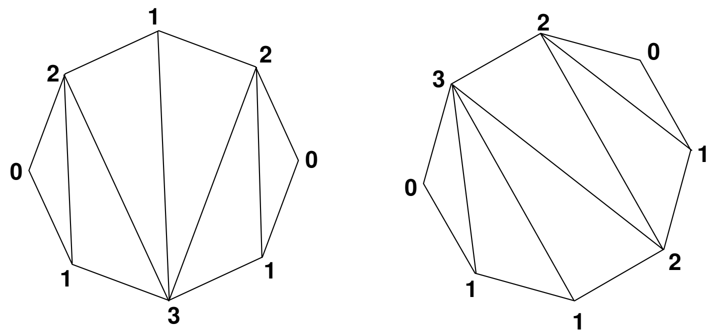


Figure 3.2: Two different triangulations with same degree sequence

inner triangles etc.
If a (pseudo) big node has got no incident inner triangles the number of connected '1's is $deg(v) - 2$. This equation holds if we define the first '1' after an ear as an *isolated '1'*. From now on, we call the '1's (except for the '1' which is next to an ear) connected to node v, the '1s of v' (consider Figure 3.3).

Definition 3.7 (wedge of a node)
Let's look at an arbitrary (pseudo) big node v ($deg(v) \geq 2$) and the nodes connected to v. After sorting these nodes by their angle between the connecting diagonal and the convex hull we get $\{u_1, \ldots, u_{deg(v)}\}$.

A *wedge of* v is a set $\{u_k\}_{i \leq k \leq j}$ such that $deg(u_k) = 1$ *for* $i \leq k \leq j$ where i is either equal to 1 or $deg(u_{i-1}) \geq 2$ and j is either equal to $deg(v)$ or $deg(u_{j+1}) \geq 2$.

If either $i = 1$ or $j = deg(v)$ we call it an *unbounded wedge*, otherwise it is called a *bounded wedge*. u_{i-1} and/or u_{j+1} are called *bounding nodes*.

For two (pseudo) big nodes u_l and u_{l+1}, where the edge $\{u_l, u_{l+1}\}$ is part of the convex hull, the empty set of nodes with degree 1 between them is called an *empty bounded wedge*. If v is incident to an ear without isolated '1', it has an *unbounded empty wedge*. See Figure 3.3 for examples.

Remark 3.8 Isolated (pseudo) big nodes have exactly one wedge.

Lemma 3.9
For every triangulation T with big nodes, there exists a triangulation T with the same degree sequence where all big nodes have got at most one non empty wedge.

Proof. Let's look at an arbitrary big node v. Assume that v got more than one non empty wedge. We can easily move '1's from one wedge to another without changing the degree sequence of the triangulation, so we move all '1's to an arbitrary chosen wedge. □

Remark 3.10 If possible we always move the '1's into a bounded wedge.

Definition 3.11 (zigzag)
In this thesis we define a *zigzag-triangulation* as a triangulation where the dual tree is a path. A zigzag in a triangulation is a partial triangulation where the dual tree is a path. This means a zigzag is a concatenation of (pseudo) big nodes and their wedges.

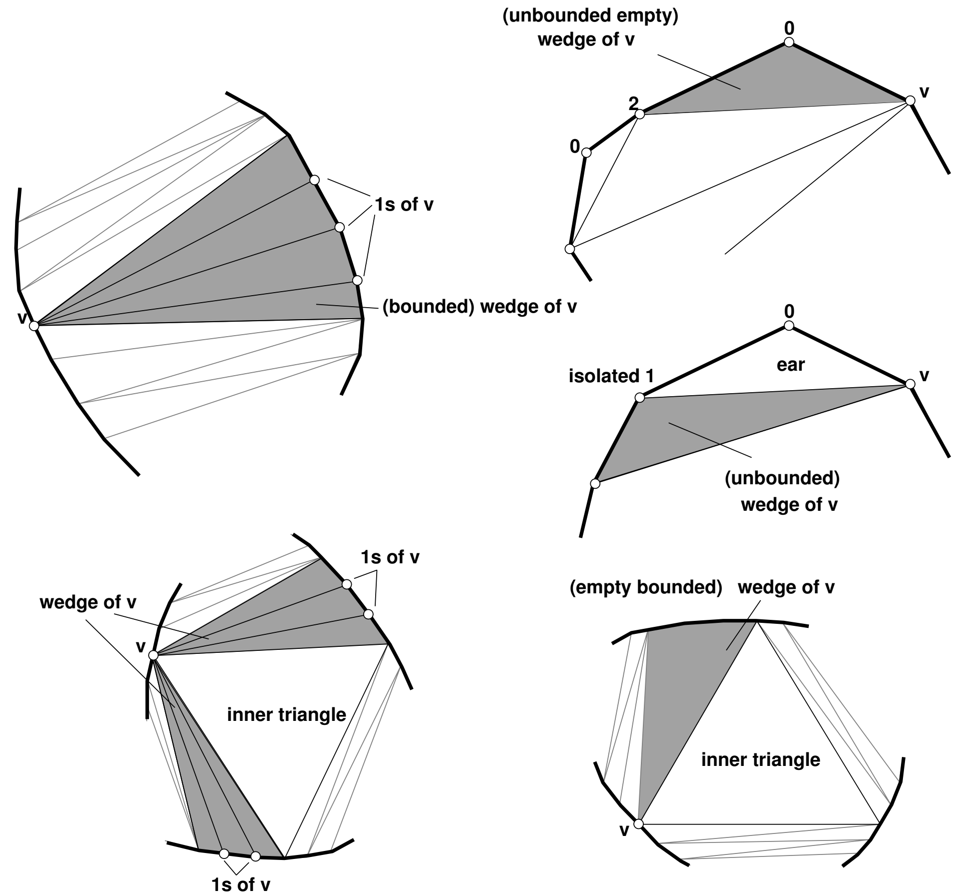


Figure 3.3: Example for wedge of v, inner '1's and isolated '1's

Remark 3.12 A triangulation with no inner triangle consists of exactly one zigzag. This definition is quite unusual, but since we don't have to distinguish between isolated pseudo big nodes and isolated big nodes it makes sense to define a general way to combine nodes not incident to inner triangles.

3.8 Common Transformations

3.8.1 Moving Isolated Big Nodes

Setting

Let T be a triangulation of a convex point set with more than one (pseudo) big node and an arbitrary isolated (pseudo) big node v. If the wedge of v is a bounded one, then v is connected to two (pseudo) big nodes u and w. Otherwise v is connected to one (pseudo) big node u and an isolated '1' w. Let $\{r, s\}$ be an edge which is not part of an inner triangle with r being a (pseudo) big node and s being either a (pseudo) big node or an isolated '1' (see Figure 3.4).

Target

We want to move v between r and s, meaning that after the transformation v is part of the path on diagonals between r and s, without changing the degree sequence.

Procedure

We detach v from u and w, and r from s. After that we can move v with the connected '1's into the free space between r and s so that, after attaching v to r, s to v and u to w, we have a triangulation. Therefor we have to horizontally mirror the partially triangulation which contains r and w and possibly the block containing v (see Figure 3.4).

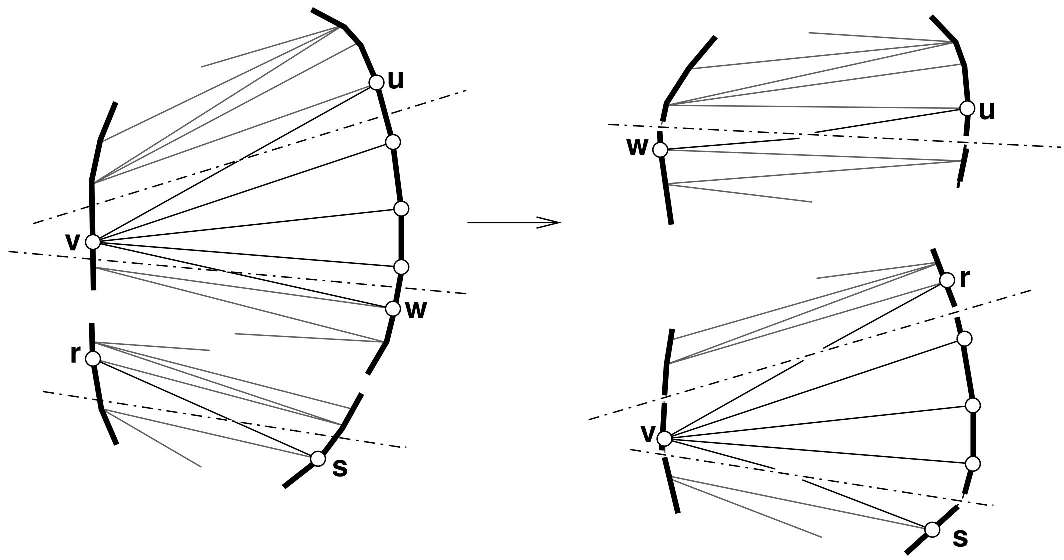


Figure 3.4: Moving isolated big nodes

3.8.2 Swapping a Partial Triangulation

Setting

Let T be a triangulation of a convex point set and nodes u, v, w, r be either (pseudo) big nodes or isolated '1's, where $\{u, v\}$ and $\{w, r\}$ are edges of the triangulation. Neither $\{u, v\}$ nor $\{w, r\}$ are part of an inner triangle. If we draw a line g through the midpoint of $\{u, v\}$ (defining that point on g as midpoint of g) so that u and v lay on different sides of g and g intersects with no edge incident to u or v except $\{u, v\}$ and the convex hull edges. Then g splits the triangulation into two partial triangulations. Let t_g be the part not containing $\{w, r\}$. Repeat this for $\{w, r\}$ with line f. t_f is the part not containing $\{u, v\}$. Obviously the intersection of t_g, t_f is empty.

Target

We want to swap t_g and t_f without changing the degree sequence of the triangulation.

Procedure

First we detach t_g and t_f from the triangulation, which is easily done by cutting of the corresponding edges on the convex hull and $\{u, v\}$ or $\{w, r\}$ respectively. Now we move t_g and t_f such that g swaps place with f and the midpoint of f matches the midpoint of g. If the edge $\{u, r\}$ lays on the convex hull, we mirror t_g at the normal on g going through the midpoint. We repeat this procedure for t_f. After that we can attach u to r and w to v. Maybe it's necessary to rearrange the points to get a convex point set again, but such a transformation does not modify the triangulation. Now we can insert the 4 missing edges on the convex hull and so we get a triangulation where t_g and t_f swapped places. Looking at the degrees we notice that the only changes were made at u, v, w, r. We detached an edge, moved the parts and attached an edge. So after the transformation all degrees stay the same.

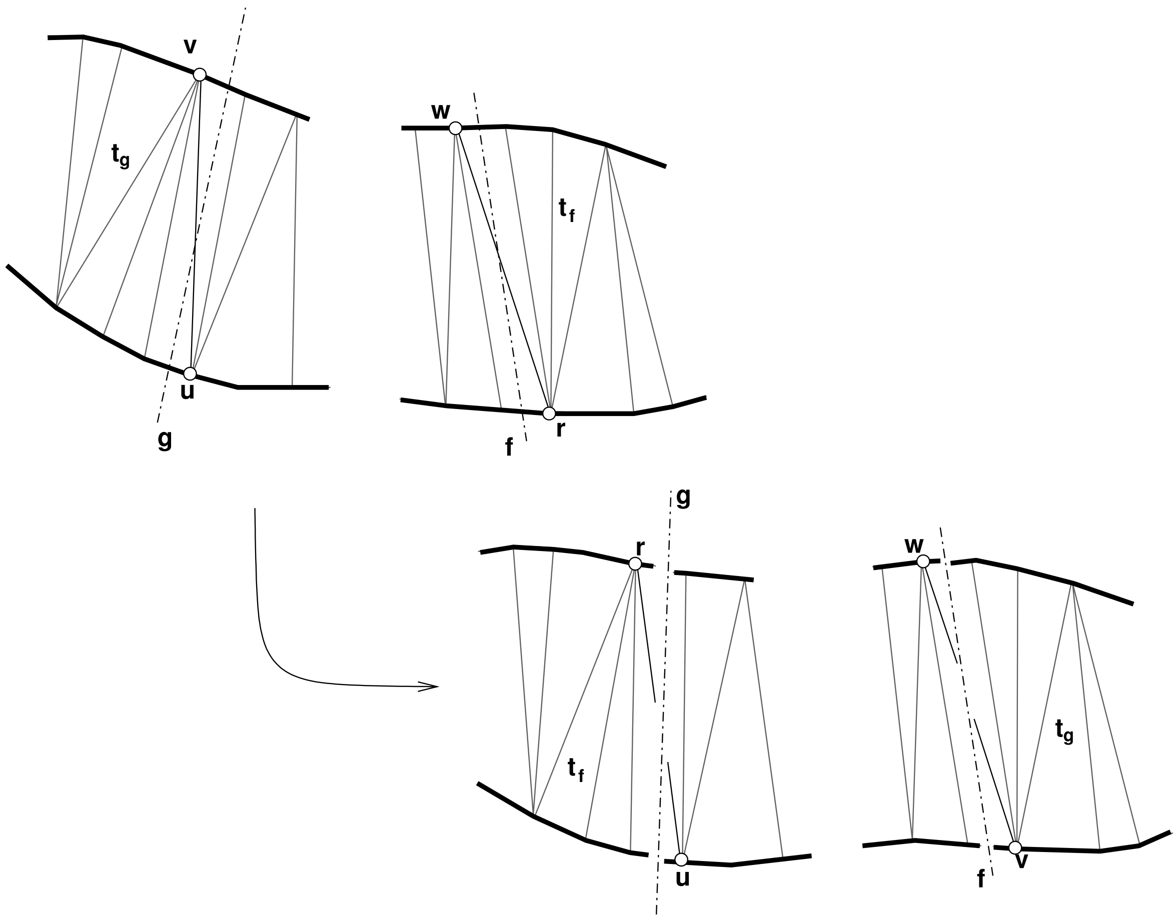


Figure 3.5: Swapping two partial triangulations

3.8.3 Eliminating a Bounded Wedge

Setting

Let T be a triangulation of a convex point set, v be a node on a block of inner triangles t with at least two bounded wedges, two of them not bounded by inner triangles incident to v.

There also has to exist at least one isolated '1' somewhere in the triangulation.

Target

We want to transform the triangulation so that one of the two bounded wedges gets unbounded and empty.

Procedure

If there is only one isolated '1' in the triangulation and one of the wedges leads to that isolated '1' (when walking in the dual tree starting at v) we choose the other wedge, otherwise it's not important which wedge we choose. We call the node which bounds the wedge we want to eliminate w. Now we can flip the partial triangulation starting at the edge $\{v, w\}$ and the isolated '1' with the incident ear as in 3.8.2. After the transformation the chosen wedge is unbounded. Move all '1's from the unbounded into the bounded wedge. Because of 3.8.2 the transformation leaves the degree sequence untouched.

3.8.4 Flipping a Partial Triangulation

Setting

Let T be a triangulation of a convex point set, v be a node on a block of inner triangles t with a bounded wedge, u being the bounding node, $\{u, v\}$ not being part of an inner triangle, and in T exists an unbounded empty wedge.

Target

We want to flip the partial triangulation starting at (and including) the bounded wedge with the unbounded empty wedge.

Procedure

Figure 3.6 shows how the transformation is done.

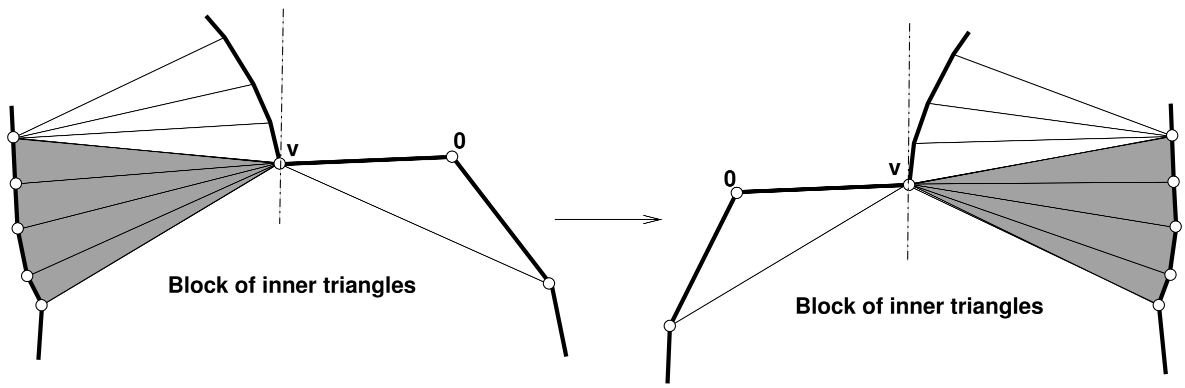


Figure 3.6: Flipping two wedges of v

3.9 Simple Constraints and Properties

3.9.1 Sums

As we have n nodes, each counting for one $\#i$, $i \geq 0$, we get

$$\#0^+ = \sum_{i \geq 0} \#i = n \tag{3.3}$$

From (3.1) we know that the number of inner diagonals is $n-3$. According to the handshaking lemma follows

$$\sum_{v \in V} deg(v) = \sum_{i \geq 0} i\#i = 2(n-3) \tag{3.4}$$

Since every triangulation fulfills these equations, we can use them to sort out sequences which cannot be degree sequences.
Further on we will often use a combination of these two:

$$\sum_{i \geq 0} (i-2)\#i = \sum_{i \geq 0} i\#i - 2\sum_{i \geq 0} \#i = 2(n-3) - 2n = -6 \tag{3.5}$$

Let k be the number of nodes incident to an inner triangle, b the number of blocks of inner triangles and w the number of wedges in the blocks.

Lemma 3.13
For a block of l inner triangles which is not a wedge-block the amount of nodes incident to the block is $l+2$.

Proof. We prove this by induction on l:
$l = 1$ means one triangle and 3 nodes incident to it.
$l \to l+1$: if we add an inner triangle to the block this means the new triangle shares one edge (and therefore 2 nodes) with the block. This means every additional inner triangle yields exactly one additional node incident to the block. □

So we know that, if there are no wedge-blocks, k equals the number of inner triangles plus two additional nodes for every block of inner triangles.

Observation 3.14 Every wedge in a block produces exactly one additional node incident to the block.

Combining this with $\#\{inner\ triangles\} = \#0 - 2$ we get:

$$k = (\#0 - 2) + 2b + w$$

or if we want to know the number of blocks:

$$b = \frac{k - \#0 - w}{2} + 1 \tag{3.6}$$

and therefore every triangulation fulfills

$$w \overset{2}{\equiv} k - \#0 \tag{3.7}$$

Remark 3.15 When considering integers modulo 2, there is no difference between plus and minus, which means that (3.7) is the same as $\left[w \overset{2}{\equiv} k + \#0\right]$.

3.9.2 Inequalities

In an ordered degree sequence with more than 3 elements a positive degree follows after every 0. This leads to:

$$\#0 \leq \left\lfloor \frac{n}{2} \right\rfloor \tag{3.8}$$

For $\#0 \geq 3$ Lemma 5.6 provides a more strict version of this inequality:

$$\#0 \leq \left\lfloor \frac{n - \#1}{2} \right\rfloor \tag{3.9}$$

Remark 3.16 This inequality is equivalent to $\#0 \leq \#2^+$

Furthermore every triangulation on a finite point set has got at least two ears.

$$\#0 \geq 2 \tag{3.10}$$

In Lemma 5.4 we will further show that every degree sequence fulfills

$$\#1 + \#2 \geq 2 \tag{3.11}$$

4 Special Cases

We begin our investigation with some special cases where everything seems to be easy. We present full information on this special cases. This provides us the freedom to skip the unwanted effects in this special cases later on, when we examine the more general sequences.

4.1 Two Ears ($\#0 = 2$)

In the case $\#0 = 2$ the according triangulation (if existing) is build in a zigzag way (see Figure 4.1). This means the dual tree is a path. We can start with one of the ears and adjust all big nodes with their wedges in a zigzag till we get to the second ear.

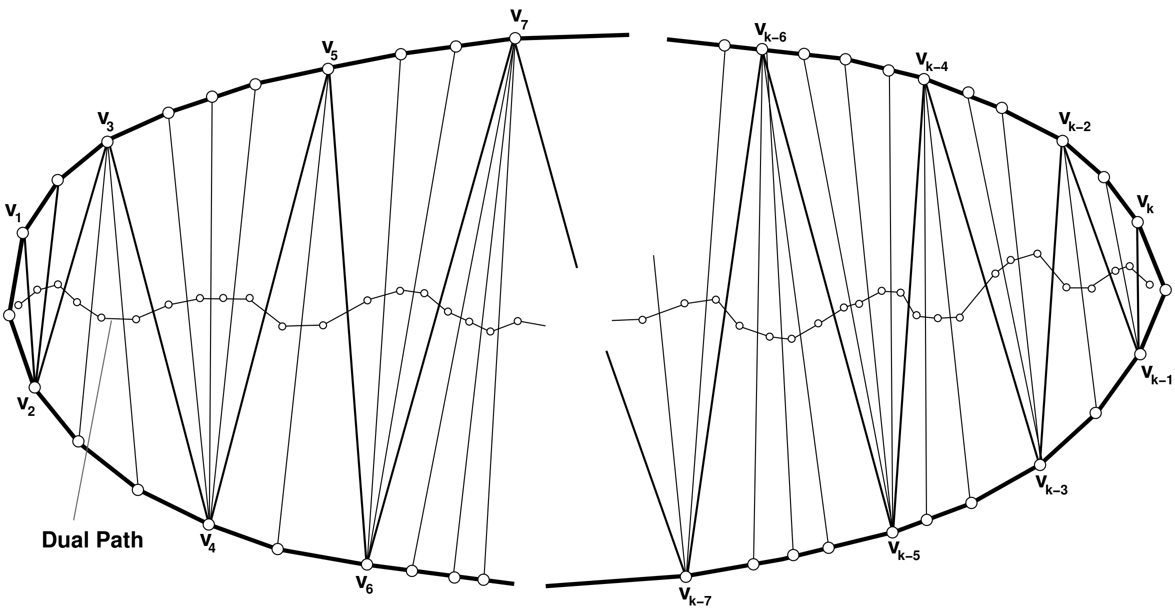


Figure 4.1: Zigzag triangulation for $\#0{=}2$

As there are no inner triangles we know that for every big node v the number of '1's in the wedge of v is $(deg(v) - 2)$ and there are two isolated '1's. Let's look at the number of '1's we need for the construction:

$$\sum_{\text{all big nodes } v} (deg(v) - 2) + 2$$

If we take (3.5) and bring $(1 - 2)\#1$ and $(0 - 2)\#0$ to the other side we get

$$\sum_{i \geq 2} (i - 2)\#i = \#1 + 2\#0 - 6 = \#1 - 2$$

For $\#2$ the terms in the sum are 0, therefore we finally get

$$\sum_{i>2}(i-2)\#i+2=\sum_{i\geq 2}(i-2)\#i+2=\#1$$

which is the exact number of '1's we need for the construction. This means that for every sequence of nonnegative integers S where $\#0=2$, which fulfills the equations (3.3) and (3.4), there exists a triangulation with degree sequence S.

On the other hand, for every triangulation the degree sequence has to fulfill the simple equations (3.3) and (3.4).

We thus conclude:

Lemma 4.1
For every sequence of nonnegative integers with $\#0=2$ which fulfills (3.3) *and* (3.4) *exists a triangulation with the same degree sequence.*

4.2 Three Ears ($\#0 = 3$)

In the case $\#0=3$ an according triangulation (if existing) consists of an inner triangle and attached zigzags. The smallest example of such a triangulation is a single triangle with the ordered degree sequence $[0,0,0]$. Now we consider sequences with $\#0^+>3$.

Lemma 4.2
Given a sequence of nonnegative integers S with $\#0=3$ and $\#0^+>3$, there exists a triangulation of a convex point set with the degree sequence S if and only if $\#2^+\geq 3$ and S fulfills (3.3), (3.4) *and one of the following conditions*

- $\#3^+=0$ *and* $\#2=3$
- $\#3^+\geq 1$

Proof.
"$\Rightarrow$"
A triangulation with 3 ears has one inner triangle so $\#2^+\geq 3$ (every node on the inner triangle having degree ≥ 2). If there are no big nodes the triangulation is the inner triangle with no further nodes and the degree vector is (3,0,3).
"$\Leftarrow$"
Let $S=\{v_1,\ldots,v_n\}$ be a sequence of nonnegative integers (in decreasing order) with $\#0=3$, $\#2^+\geq 3$. If $\#3^+\geq 1$ and $k\geq 3$ so that $v_k\geq 2$ and $v_{k+1}<2$ we build the triangulation as an inner triangle with an attached zigzag. v_1, v_2 and v_3 are the nodes on the triangle, the zigzag is attached to v_1 ($v_1\geq 3$). $v_4,\ldots,v_k$ form the zigzag. Let's look at the '1's we need for this construction: v_1 is connected to the inner triangle by two and to the zigzag by one edge, so it needs $(deg(v_1)-3)$ '1's in its wedge. v_2 and v_3 are not connected to a zigzag so they need $(deg-2)$ '1's. Every node in the zigzag also needs $(deg-2)$ '1's (without the isolated '1' at the end). If we add the isolated '1' to the '1's of v_1, every (pseudo) big node v needs $(deg(v)-2)$ '1's. From (3.5), which is a combination of (3.3) and (3.4), we get:

$$\sum_{i\geq 2}(i-2)\#i=2\#0+\#1-6\overset{\#0=3}{=}\#1$$

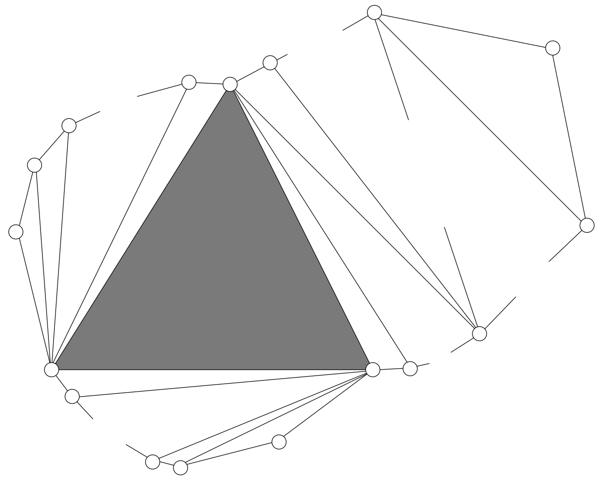


Figure 4.2: Canonical triangulation for $\#0 = 3$

This means we get exactly the number of '1's that we need (see Figure 4.2 for better understanding).
In the case $\#3^+ = 0$ follows from (3.5)

$$\#1 \overset{\#0=3}{=} \sum_{i\geq 2} (i-2)\#i \overset{\sum_{i\geq 3}\#i=0}{=} 0$$

and in combination with $\#2 = 3$ the triangulation is an inner triangle with 3 incident ears and no further nodes. $\square$

4.3 Four Ears ($\#0 = 4$)

The case $\#0 = 4$ is determined by the number of big nodes ($\#3^+$). We distinguish the sub cases $\#3^+ = 1$, $\#3^+ = 2$ and $\#3^+ \geq 3$. Every triangulation with 4 ears has 2 connected inner triangles so there has to be at least one big node. The 3 sub cases are covered by the following lemmata:

Lemma 4.3
Given a sequence of nonnegative integers S with $\#0 = 4$ and $\#3^+ = 1$, there exists a triangulation of a convex point set with the degree sequence S if and only if it fulfills (3.3), (3.4) *and one of the following conditions:*

- $\#2 = 4$ *and* $\#4 = 1$
- $\#5^+ = 1$ *and* $\#2 \geq 4$

Proof.
"$\Rightarrow$"
In every triangulation with 4 ears there are 2 inner triangles. These inner triangles have to be connected. The only connection which produces only one big node is a wedge-block. That means the two inner triangles share exactly one node v with $deg(v) \geq 4$. The other nodes on the inner triangle have got to be pseudo big nodes (which means $\#2 \geq 4$), so the only

way to connect further nodes is to connect them to v. This means that either $deg(v) \geq 5$ or $deg(v) = 4 \wedge \#2 = 4$.
"$\Leftarrow$"
Case 1 produces the ordered degree sequence $[0, 2, 0, 2, 2, 0, 2, 0, 4]$, obviously this is a legal ordered degree sequence.
In case 2 we build a triangulation consisting of a wedge-block with a zigzag connected to the big node. The zigzag only consists of pseudo big nodes with the isolated '1' at the end. So the number of '1's we need for the construction is the degree of the big node minus 4. From (3.5) we get:

$$\sum_{i\geq 3}(i-4)\#i \overset{\#3^+=1}{=} \sum_{i\geq 3}(i-2)\#i - 2 = 2\#0 + \#1 - 6 - 2 \overset{\#0=4}{=} \#1$$

This means we have exactly the number of '1's we need (see Figure 4.3 for better understanding).

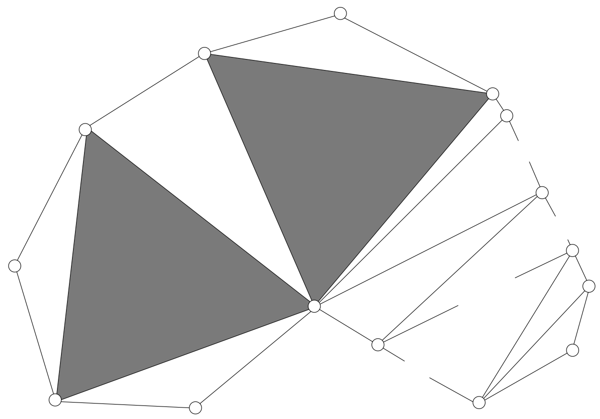


Figure 4.3: Canonical triangulation for $\#0{=}4$ with 1 big node

□

Lemma 4.4
Given a sequence of nonnegative integers S with $\#0 = 4$ and $\#3^+ = 2$ there exists a triangulation of a convex point set with the degree sequence S if and only if it fulfills (3.3), (3.4) *and one of the following conditions:*

- $\#2 \geq 4$
- $\#2 = 3$ *and* $\#4^+ \geq 1$
- $\#2 = 2$

Proof.
"$\Rightarrow$"
There are 3 types of triangulations with 4 ears that produce exactly 2 big nodes:

- 2 separated inner triangles, each inner triangle producing one big node and 2 pseudo big nodes $\Rightarrow \#2 \geq 4$
- 2 inner triangles sharing only one node (which has a degree ≥ 4). Only one of the four remaining nodes on inner triangles is big $\Rightarrow (\#2 = 3 \wedge \#4^+ \geq 1)$

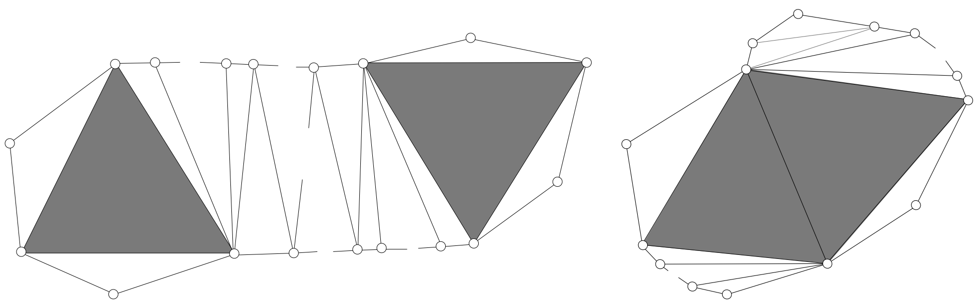
- The two inner triangles share an edge. Either $\#2 = 2$ (no zigzag attached) or $\#2 \geq 3$ and the big node the zigzag is connected to has a degree $\geq 4 \Rightarrow (\#2 \geq 3 \wedge \#4^+ \geq 1)$ or $\#2 = 2$

"$\Leftarrow$"

Given a sequence of nonnegative integers $S = \{v_1, \ldots, v_n\}$ (in decreasing order) with $\#0 = 4$ and $\#3^+ = 2$ we get three cases:

- $\#2 \geq 4$: We build the triangulation as two separated inner triangles with a zigzag of the remaining pseudo big nodes between them. Each inner triangle has one big and two pseudo big nodes. The zigzag is at each end connected to the big node of the inner triangle. The number of '1's needed is $\sum_{i \geq 3}(i-3)\#i$ (see Figure 4.4 for better understanding).

$$\sum_{i \geq 3}(i-3)\#i \overset{\#3^+=2}{=} \sum_{i \geq 3}(i-2)\#i - 2 \overset{(3.5)}{=} 2\#0 + \#1 - 6 - 2 \overset{\#0=4}{=} \#1 \tag{4.1}$$



(a) $\#2 \geq 4$ (b) $\#2 = 3$

Figure 4.4: Canonical triangulations for #0=4 with 2 big nodes

- $\#2 = 3$ and $\#4^+ \geq 1$: The triangulation consists of a block of two inner triangles with a zigzag (containing only one pseudo big node) attached. Both big nodes are incident to 3 edges from the block. One big node also has an edge leading to the zigzag which ends in an isolated '1'. So the number of needed '1's is $\sum_{i \geq 3}(i-3)\#i$. As before (3.5) provides the required number of '1's (see Figure 4.4 for better understanding).
- $\#2 = 2$: This leads to the ordered degree sequence:

$$[0, 2, \underbrace{1, \ldots, 1}_{v_1 - 3}, 0, v_1, 0, 2, \underbrace{1, \ldots, 1}_{v_2 - 3}, 0, v_2]$$

  Again the number of needed '1's is provided due to (4.1).

$\square$

Lemma 4.5

Given a sequence of nonnegative integers S with $\#0 = 4$ and $\#3^+ \geq 3$ there exists a triangulation of a convex point set with the degree sequence S if and only if it fulfills (3.3), (3.4) *and* $\#2^+ \geq 4$.

Proof.

"$\Rightarrow$"

Every triangulation fulfills (3.3) and (3.4). $\#0 = 4$ means we have two inner triangles, so there are at least 4 nodes on inner triangles and therefore $\#2^+ \geq 4$.

"$\Leftarrow$"

Given a sequence of nonnegative integers $S = \{v_1, \ldots, v_n\}$ (in decreasing order) with $\#0 = 4$ and $\#3^+ \geq 3$ we build the triangulation as a block of two inner triangles with a zigzag attached. Two big nodes (v_1 and v_2) are incident to 3 edges from the block, v_3 is incident to 2 edges from the block and 1 edge is incident to the zigzag. The other (pseudo) big nodes are either incident to 2 edges from the block or from the zigzag (treating the isolated '1' as a node of the zigzag). Now we add the isolated '1' at the end of the zigzag to the '1's of the big node connected to the zigzag and therefore get 2 nodes with $(deg - 3)$ '1's and all the other (pseudo) big nodes with $(deg - 2)$ '1's.

$$\sum_{i \geq 2} (i-2)\#i - 2 \overset{(3.5)}{=} 2\#0 + \#1 - 6 - 2 \overset{\#0=4}{=} \#1$$

This means (3.5) (which is a combination of (3.3) and (3.4)) provides exactly the number of '1's we need (see Figure 4.5 for better understanding). $\square$

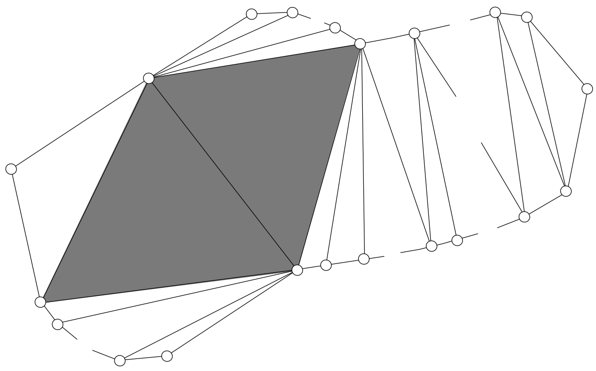


Figure 4.5: Canonical triangulation for $\#0{=}4$ with at least 3 big nodes

5 Separated Inner Triangles and Small Blocks

5.1 Only Separated Inner Triangles

We now look at the general case where $\#0 \geq 5$. Let's start simple and look at triangulations where all inner triangles are separated.

Definition 5.1 (separated inner triangle)
An inner triangle which neither has a common edge nor a common node with another inner triangle is called *separated*.

Definition 5.2 (oriented inner triangles)
A separated inner triangle t is called *oriented* if all the nonempty wedges of the nodes of t either point in clockwise or counterclockwise direction.

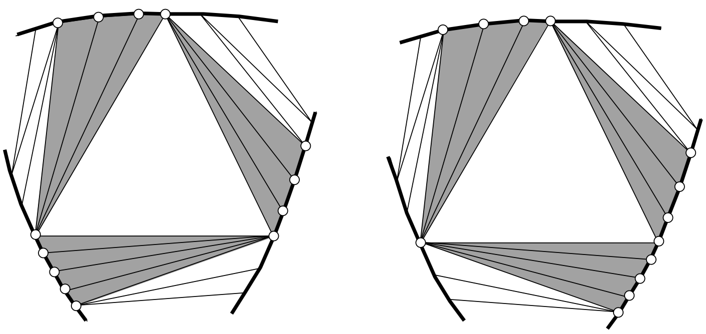


Figure 5.1: Oriented inner triangle vs. non-oriented

Lemma 5.3
Let T be a triangulation with separated inner triangles, t being a non-oriented inner triangle. If there exists at least one ear with an incident node k with $deg(k) = 1$ (an isolated '1'), then there exists a triangulation T' with the same degree sequence where t is oriented.

Proof. Let u, v, w be the nodes of the inner triangle t. t is nonoriented if a node of t has a bounded wedge on each side of the triangle or if there are two nodes with different directed wedges. If a node has a bounded and an unbounded nonempty wedge we transform the triangulation so that there is a bounded and an empty wedge (Lemma 3.9)

Case 1 Each node on the inner triangle has at most one bounded wedge. Two nodes, u and v, have different directed nonempty wedges.
This means that the edge $\{u, v\}$ has to be incident to an ear. We can easily flip the partial triangulation starting at $\{w, u\}$ with the ear at $\{u, v\}$ (see Transformation 3.8.4). Now all wedges point either in clockwise or counterclockwise direction.

Case 2 A node u has two bounded wedges. Choose the bounding node of the wedge, which destroys the orientation and call it l. Now we use Transformation 3.8.3 to eliminate the wedge bounded by l. After that the triangle is oriented.

□

Lemma 5.4
In every triangulation with more than 3 nodes the sum of isolated '1's and nodes with degree 2 which are adjacent to two '0's is always at least 2.

Proof. In the dual tree an isolated '1' is a node with degree 2 which is adjacent to a leaf.

A node in the triangulation with degree 2, which is incident to two ears, corresponds to a node in the dual tree with degree 3, which is incident to two leaves.

Case 1 There is no inner triangle, meaning $\#0 = 2$
Every ear is incident to an isolated '1' and no isolated '1' corresponds to more than one ear.

Case 2 3 ears and 6 nodes.
This means we have one inner triangle containing 3 nodes with degree 2, each of them adjacent to two '0's.

Case 3 $\#0 \geq 3$ and more than 6 nodes.
Let's look at the dual tree. We start at an arbitrary inner node. If the node has degree 2 and is adjacent to a leaf or it has degree 3 and is adjacent to two leaves, we already have the first occurrence and our first step will be away from the leaves. Otherwise we have at least 2 choices for the first step without stepping onto a leaf. Now we start walking through the tree according to the following rules:

- If we arrive at a node with degree 2 adjacent to a leaf or at a node with degree 3 adjacent to 2 leaves, we stop.
- From a node with degree 2, we step to the adjacent node where we haven't been yet.
- From a node with degree 3, we step to the adjacent node where we haven't been yet and which is not a leaf. If we have two possibilities we choose the leftmost one.

Since the dual tree is finite and has no loops we stop after a finite number of steps. If we had a choice in the beginning, we started a walk in both directions. So in the end we either started with one of the interesting nodes and stopped in one, or we walked in different directions and stopped at an occurrence of an interesting node for each direction. This means we have at least two occurrences and the sum of isolated '1's and nodes with degree 2 with two adjacent '0's is always at least 2.

□

Remark 5.5 Lemma 5.4 shows that $\#1 + \#2 \geq 2$ for every degree sequence.

Lemma 5.6
Every triangulation with $\#0 \geq 3$ *fulfills*

$$\#0 \leq \left\lfloor \frac{n - \#1}{2} \right\rfloor$$

Proof. $\#0 \geq 3$ means we have at least 1 inner triangle. We know that every node on an inner triangle must have a degree of at least 2. Furthermore we know that the number of inner triangles is $\#0 - 2$.

The number of nodes on inner triangles is minimal if we have only one big block where all inner triangles share at least one edge. In this case there are exactly $\#0 - 2 + 2 = \#0$ nodes on inner triangles. So we know:

$$\begin{aligned}
\#0 &\leq \{nodes\ on\ inner\ triangles\} \\
&\leq \#2^{+} \\
&= n - \#0 - \#1 \\
2\#0 &\leq n - \#1 \\
\#0 &\leq \left\lfloor \frac{n - \#1}{2} \right\rfloor
\end{aligned}$$

□

Lemma 5.7
Let T be a triangulation with separated inner triangles, where the nodes on the inner triangles are not the nodes with highest degrees. Then there exists a triangulation with separated inner triangles and the same degree sequence, where the nodes incident to the inner triangles do have the highest degrees.

Proof. Let u be the node with minimum degree on an inner triangle and k an isolated big node with highest degree. The remaining nodes on the inner triangle are v and w. If u is not unique, choose the node with the most incident ears. $deg(u)$ has to be less than $deg(k)$, otherwise there is nothing to show. Since u is incident to an inner triangle $deg(u)$ has to be at least 2 and therefore $deg(k) \geq 3$. This means that k has at least one '1' in its wedge.

Case 1 u has a wedge (possibly empty). Then we can easily move $deg(k) - deg(u)$ nodes from the wedge of k to the wedge of u or to the incident ear respectively. After the transformation we have $deg(u^{new}) = deg(k^{old})$ and $deg(k^{new}) = deg(u^{old})$.

Case 2 u has no wedge (not even an empty one), but $\{v, w\}$ is incident to an ear.
Now we can use Transformation 3.8.4 to flip the partial triangulation from $\{u, v\}$ and the ear incident to $\{v, w\}$ therefore u has an incident ear and we are in Case 1.

Case 3 u has no wedge and the inner triangle has no incident ear. This splits up into 2 sub cases:

- There exists an isolated '1' in the triangulation. Use Lemma 5.3 and go to the triangulation where the inner triangle is oriented. Then v and w have only one wedge with the same orientation therefore u is incident to an ear and we have Case 1 again.
- There exists no isolated '1', use Lemma 5.4 to show that there exists at least one inner triangle with 2 incident ears. This means the node between the ears has degree 2 which is less or equal than $deg(u)$ and is incident to more ears than u. This would be a contradiction to the choice of u.

Iterate this for all isolated big nodes. □

Theorem 5.8
Let T be a triangulation with separated inner triangles which fulfills $\#3^+ \geq 3(\#0-2)$. Then there exists a triangulation T' with the same degree sequence and separated inner triangles, where the nodes with highest degrees are incident to the inner triangles and the inner triangles are oriented.

Proof. From Lemma 5.7 we know that there exists a triangulation T' where the biggest nodes are on inner triangles. According to the number of big nodes this means that every node on an inner triangle has a degree greater or equal to 3. So every node on an inner triangle has a wedge. Therefore every inner triangle has to be oriented. $\square$

5.2 Explicit Construction for Separated Inner Triangles

Inspired from the previous section we now provide a construction for every sequence of non-negative integers fulfilling $5 \leq \#0 \leq \left\lfloor \frac{n-\#1}{2} \right\rfloor$, $\#3^+ \geq 3(\#0-2)$, (3.3),(3.4), (3.8) and (3.10).

Let $S = \{v_1, \ldots, v_n\}$ be a sequence of nonnegative integers in decreasing order, fulfilling these conditions, $k = \#0 - 2$ and l so that $v_l \geq 2$ and $v_{l+1} < 2$, $B = \{v_1, \ldots, v_{3k}\}$ be a sequence containing the $3k$ biggest nodes and $P = \{v_{3k+1}, \ldots, v_l\}$ be the sequence containing the (pseudo) big nodes left (the sequences contain 'nodes' represented by their degrees). Partition B into groups of 3 ($|B| = 3k$). Each group now represents an inner triangle.

Now $B = \{\{v_1^1, v_2^1, v_3^1\}, \{v_1^2, v_2^2, v_3^2\}, \ldots, \{v_1^k, v_2^k, v_3^k\}\}$ and $P = \{u_i \mid 1 \leq i \leq l-3k\}$.

Geometrically we arrange the inner triangles in a row, always connecting v_2^i and v_1^{i+1} for $1 \leq i < k$. Each node from B has its one wedge filled with the needed '1's. At the end of the row the big nodes, which are not contained in B, are positioned in a zigzag. The triangulation starts with an ear at the leftmost node.
After the first ear we get k blocks

$$v_1^i, \overbrace{1, \ldots, 1}^{deg(v_2^i)-3}, \text{ for } i = 1, \ldots, k \tag{5.1}$$

then the zigzag with the isolated big nodes appears:
If $l \overset{2}{\equiv} 3k$:

$$u_1, \overbrace{1, \ldots, 1}^{deg(u_2)-2}, u_3, \ldots, u_{l-3k-1}, \overbrace{1, \ldots, 1}^{deg(u_{l-3k})-2}, 1, 0, u_{l-3k}, \underbrace{1, \ldots, 1}_{deg(u_{l-3k-1})-2}, u_{l-3k-2}, \ldots, u_2, \underbrace{1, \ldots, 1}_{deg(u_1)-2}, \tag{5.2}$$

If $l \overset{2}{\not\equiv} 3k$:

$$u_1, \overbrace{1, \ldots, 1}^{deg(u_2)-2}, u_3, \ldots, u_{l-3k-2}, \overbrace{1, \ldots, 1}^{deg(u_{l-3k-1})-2}, u_{l-3k}, 0, 1, \underbrace{1, \ldots, 1}_{deg(u_{l-3k})-2}, u_{l-3k-1}, \ldots, u_2, \underbrace{1, \ldots, 1}_{deg(u_1)-2}, \tag{5.3}$$

Going back to the first zero underneath the row we find these blocks:

$$v_2^i, \underbrace{1, \ldots, 1}_{deg(v_3^i)-3}, 1, 0, v_3^i, \underbrace{1, \ldots, 1}_{deg(v_1^i)-3}, \text{ for } i = k \text{ down to } 1 \tag{5.4}$$

We add the last isolated '1' and are done. Finally we have an ordered degree sequence like this:

$$[0, (5.1), (5.2) \text{ or } (5.3), (5.4), 1]$$

Figure 5.2 shows a sample triangulation.

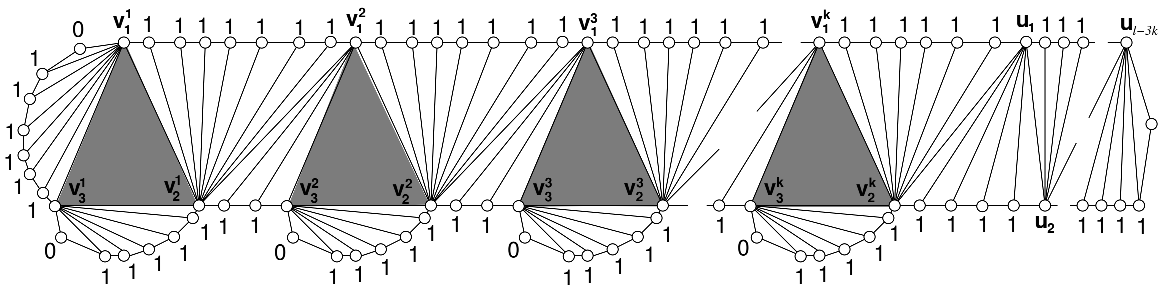


Figure 5.2: Canonical triangulation for separated inner triangles

Proposition 5.9
The described sequence is an ordered degree sequence for S.

Proof. First we show that the given sequence is a legal ordered degree sequence.
There are two big parts in the resulting triangulation: the row of inner triangles (5.1) and (5.4) and the zigzag from the nodes in S (5.2) or (5.3) respectively.

As in Figure 5.2 the inner triangles are arranged in a row where v_2^i and v_1^{i+1} are connected. The sequence starts at the leftmost '0', then every instance of (5.1) describes the wedge of the second node on the inner triangle. After that we are at the last node in the wedge of v_2^k. v_2^k is connected to u_1. Now the zigzag starts. Obviously this is a part of an ordered degree sequence. It ends with the last '1' in the wedge of u_1.

As u_1 is connected to v_2^k we start with (5.4). Every instance of (5.4) describes the part on the 'base' of the inner triangles which means v_2^i followed by the wedge of v_3^i, the isolated '1', the according '0', v_3^i itself and the wedge of v_1^i. At the end v_1^1 is missing one node, which is the isolated '1' according to the first zero.

After that all nodes according to the inner triangle i have 'filled' wedges which means, they have the expected degree. As 2 edges come from the inner triangle, one goes to the next triangle or to an isolated '1' respectively and the rest is filled with the $(deg(v_j^i) - 3)$ '1's in the wedge. All other big nodes are in the zigzag where they obviously get the right degree.

Now we show that the constructed ordered degree sequence uses exactly the numbers from the given sequence S. Let n be the length of the sequence, that is the number of nodes. From the construction we know that $\#i$ has to be the same in the constructed degree sequence and the given sequence $\forall i \geq 2$. Also $\#0$ is the same because of the construction. Let us count the '1's in the constructed ordered degree sequence:

$$\begin{aligned}\text{from (5.1) and (5.4)} &\sum_{v\in B} deg(v)-3\\ \text{from (5.2) or (5.3)} &\sum_{v\in P} deg(v)-2\\ \text{the isolated '1's from (5.4)} &\ \#0\end{aligned}$$

this sums up to

$$\begin{aligned}\sum_{v\in B}(deg(v)-3)+\sum_{v\in P}(deg(v)-2)+\#0 &= \sum_{v\in(B\cup P)}(deg(v)-2)-|B|+\#0\\ &= \sum_{v\in(B\cup P)}(deg(v)-2)-3(\#0-2)+\#0\\ &= \sum_{v\in(B\cup P)}(deg(v)-2)-2\#0+6\end{aligned}$$

so

$$\#1=\sum_{v\in(B\cup P)}(deg(v)-2)-2\#0+6$$

Since $B\cup P$ contain all (pseudo) big nodes this is exactly the constraint (3.5). $\square$

Therefore we can conclude:

Corollary 5.10
Given a sequence of nonnegative integers $S=\{v_1,\ldots,v_n\}$ with $5\leq\#0\leq\left\lfloor\frac{n-\#1}{2}\right\rfloor$ and $\#3^+\geq 3(\#0-2)$, there exists a triangulation with degree sequence S if and only if it fulfills the constraints (3.3) and (3.4). There also exists a triangulation which is build as a chain of separated inner triangles.

Remark 5.11 If we allow v_1^1,v_2^k and all v_3^i to be equal to 2 we only have to move the zigzag between the first two inner triangles, the rest of the construction works even with the lower number of big nodes.
Therefore for every sequence with $\#0\geq 5$ which fulfills $\#3^+\geq 2(\#0-3)$, $\#2^+\geq 3(\#0-2)$ and the simple constraints there exists a triangulation with that degree sequence.

5.3 Small Blocks of Inner Triangles

In this chapter we look at triangulations where inner triangles are either separated or have a common edge (which means there are no inner triangles sharing only one node). We assume that there are more big nodes than nodes on inner triangles. As all other cases are already covered in the previous sections we assume $\#3^+<3(\#0-2)$ and $\#0\geq 5$.

Definition 5.12 (block of inner triangles)
We call inner triangles joined through common edges a *block of inner triangles*.

Definition 5.13 (oriented block of inner triangles)
Analog to oriented separated inner triangles, we can define *oriented blocks of inner triangles* as blocks of inner triangles, where all the wedges of the nodes on the block either point in clockwise or counterclockwise direction. Nodes with empty, unbounded wedges are ignored.

Definition 5.14 (inner inner triangle)
If each edge of an inner triangle is shared with another inner triangle, we call it an *inner inner triangle*. In the dual tree an inner inner triangle corresponds to a node with degree 3 whose neighbours also have degree 3.

Definition 5.15 (wedge-block)
A *wedge-block* of inner triangles is similar to a normal block of inner triangles, but this time triangles can be connected through only a single node (such connections create inner wedges within the block). A wedge-block can have more than one inner wedge.

Inspired from Lemma 5.7 we state:

Lemma 5.16
Let T be a triangulation with no two inner triangles sharing only one node. Then there exists a triangulation T' with the same degree sequence as T, where the nodes with highest degrees are on inner triangles.

Proof. Let u be the node with highest degree not on an inner triangle, and v the node with lowest degree on an inner triangle (if not unique, choose the one with the greatest number of incident ears), v being on a block of inner triangles B. Suppose $deg(u)$ to be greater than $deg(v)$ (otherwise there is nothing to show). Since v is on an inner triangle $deg(v) \geq 2$ so $deg(u) \geq 3$. This means u has a nonempty wedge. Let $v = v_1, v_2, \ldots, v_k$ be the clockwise ordered nodes on B.

Case 1 v already has a wedge, even an unbounded or empty one. Then we easily move $deg(u) - deg(v)$ nodes from the wedge of u to the wedge of v. After the transformation we have have $deg(u^{new}) = deg(v^{old})$ and $deg(v^{new}) = deg(u^{old})$.

Case 2 v has no wedge, B is incident to an ear. Let $\{v_i, v_{i+1}\}$ be the edge incident to an ear with the smallest index i.

- $v_2, \ldots, v_i$ each has only one wedge. Now we can flip the parttriangulation starting at the edge $\{v_{i-1}, v_i\}$ with the ear on edge $\{v_i, v_{i+1}\}$ as in Transformation 3.8.4. This transformation leaves the degrees as they are, but moves the ear one node closer to v. Iterate this procedure till v is incident to the ear which leads to Case 1.
- $\exists v_k, 1 < k < i$ so that v_k has two bounded wedges. Till we reach v_k we use the same procedure as before. Because of Lemma 5.4 there exists either an isolated '1' or an inner triangle incident to two ears.
  If there exists an isolated '1' we use Transformation 3.8.3. This moves the ear one node closer to v.
  Otherwise there exists an inner triangle incident to two ears. On this triangle we have a node $\bar{v}$ with $deg(v) \geq deg(\bar{v}) = 2$. Since v has no incident ear, this is a contradiction to the choice of v.
  Iterate this procedure till either v has an incident ear or we reach the previous subcase.

Case 3 v has no wedge and B has no incident ear. There has to be at least one node w on B with two wedges.

From Lemma 5.4 we know that there exists either an isolated '1' or an inner triangle incident to two ears. If there exists an isolated '1' we use Transformation 3.8.3. After the transformation B is incident to an ear and we can use Case 2.

Otherwise there exists an inner triangle incident to two ears. On this triangle we have a node $\bar{v}$ with $deg(v) \geq deg(\bar{v}) = 2$. Since v has no incident ear, this would be a contradiction to the choice of v.

Iterate this procedure till T' is reached. □

Lemma 5.17
For every triangulation T, where inner triangles are either separated or share an edge where all blocks of inner triangles are filled, there exists a triangulation T' with the same degree sequence where the blocks are oriented.

Proof. Let B be a non oriented block.

Case 1 There exists a node v on B with two bounded wedges.
From Lemma 5.4 we know that there either exists an isolated '1' or an inner triangle with two incident ears. An inner triangle with two incident ears means one of the nodes on the triangle has degree 2. So in T there exists an isolated '1'. Use Transformation 3.8.3 to eliminate one of the bounded wedges. Iterate this procedure for all nodes with two bounded wedges on B (after the transformation all nodes on inner triangles again have degree ≥ 3 so there once more exists an isolated '1'). After the transformations we are either done or in Case 2.

Case 2 All nodes on B have only one bounded wedge.
Since the bounded wedges are not oriented there have to be incident ears. Choose one direction and use Transformation 3.8.4 to flip all nondirected wedges which are adjacent to an ear. Iterate this procedure till all wedges are oriented.

□

Remark 5.18 This lemma is also true for triangulations where at most one node on inner triangles has got degree 2.

Inspired by these lemmata we will now provide a triangulation for every sequence with
$2(\#0 - 2) \;<\; \#3^+ \;<\; 3(\#0 - 2)$.

5.4 Explicit Construction for Small Blocks of Inner Triangles

Given a sequence of nonnegative integers $S = \{v_1, \ldots, v_n\}$ in decreasing order,
$B = \{v_i \mid v_i \;\geq\; 3\}$, $k = |B|$, if $k \overset{2}{\not\equiv} \#0$ we take the smallest entry away from B and keep in mind that maybe $v_{k+1} \geq 3$ but $v_{k+1} \notin B$. Because of the previous sections we can assume $k < 3(\#0 - 2)$ and $\#0 > 5$.
Let us define $b = \frac{k-\#0}{2} + 1$, $d = \#0 - 2$ and m such that $deg(v_m) \geq 2$ and $deg(v_{m+1}) < 2$.

We now look at the case $d \leq 2b$:
In this case we have $l = d - b$ blocks with two inner triangles, and $2b - d$ separated inner triangles. First we partition the set of big nodes

$$B = \{\{v_1^1, v_2^1, v_3^1, v_4^1\}, \dots, \{v_1^l, v_2^l, v_3^l, v_4^l\}, \{v_1^{l+1}, v_2^{l+1}, v_3^{l+1}\}, \dots, \{v_1^b, v_2^b, v_3^b\}\}$$

Again we build the ordered sequence out of smaller blocks:
After the first ear we get for $1 \leq i \leq l$:

$$v_1^i, \overbrace{1, \dots, 1}^{deg(v_2^i)-3}, 0, v_2^i, \overbrace{1, \dots, 1}^{deg(v_3^i)-3} \tag{5.5}$$

if $m \overset{2}{\equiv} k$:

$$v_{k+1}, v_{k+3}, \dots, v_{m-2}, v_m \tag{5.6}$$

if $m \overset{2}{\not\equiv} k$:

$$v_{k+1}, v_{k+3}, \dots, v_{m-3}, v_{m-1} \tag{5.7}$$

for $l + 1 \leq i \leq b - 1$

$$v_1^i, \overbrace{1, \dots, 1}^{deg(v_2^i)-3} \tag{5.8}$$

and for the end of the "upper side":

$$v_1^b, \overbrace{1, \dots, 1}^{deg(v_2^b)-2}, 0 \tag{5.9}$$

Going back on the "lower side" we get for i from b down to $l + 1$

$$v_2^i, \underbrace{1, \dots, 1}_{deg(v_3^i)-2}, 0, v_3^i, \underbrace{1, \dots, 1}_{deg(v_1^i)-3} \tag{5.10}$$

if $m \overset{2}{\equiv} k$:

$$v_{m-1}, v_{m-3}, \dots, v_{k+2}, \underbrace{1, \dots, 1}_{deg(v_{k+1})-2} \tag{5.11}$$

if $m \overset{2}{\not\equiv} k$:

$$v_m, v_{m-2}, \dots, v_{k+2}, \underbrace{1, \dots, 1}_{deg(v_{k+1})-2} \tag{5.12}$$

for i from l down to 2:

$$v_3^i, \underbrace{1, \dots, 1}_{deg(v_4^i)-3}, 0, v_4^i, \underbrace{1, \dots, 1}_{deg(v_1^i)-3} \tag{5.13}$$

Finally:

$$v_3^1, \underbrace{1, \dots, 1}_{deg(v_4^1)-3}, 0, v_4^1, \underbrace{1, \dots, 1}_{deg(v_1^1)-2}, 0 \tag{5.14}$$

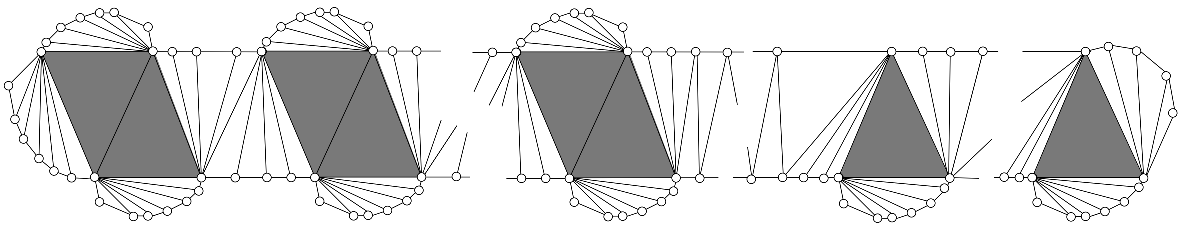


Figure 5.3: Example for the construction with 2b > d

Theorem 5.19
Given a sequence of nonnegative integers $S = \{v_1, \ldots, v_n\}$ with $5 \leq \#0 \leq \left\lfloor \frac{n-\#1}{2} \right\rfloor$ and $2(\#0-2) < \#3^+$, there exists a triangulation with degree sequence S if and only if it fulfills the constraints (3.3) *and* (3.4)*.*

Proof. The case $3(\#0-2) \leq \#3^+$ is already covered by Corollary 5.10.

Let's look at the case $2(\#0-2) < \#3^+ < 3(\#0-2)$. If there exists a triangulation it obviously fulfills the constraints.

For the other direction we show that the sequence constructed above is a legal ordered degree sequence and exactly uses the integers of S. The only constraint was $d \leq 2b$ and for the construction we need $l >= 1$. $d \leq 2b$ evaluates to $\#0 - 2 \leq k - \#0 + 2$. For an easier way to write the following, we introduce the value b and set b to 1 if $\#3^+ \overset{2}{\not\equiv} \#0$ and zero otherwise. Now we can write $k = \#3^+ - b$. So we need $2(\#0-2) \leq \#3^+ - b$ which is provided by $2(\#0-2) < \#3^+$.

Look at the other constraint ($l \geq 1$):

$$
\begin{aligned}
l &= d - b \\
&= \#0 - 2 - \left(\frac{k - \#0}{2} + 1\right) \\
&> \#0 - 2 - \frac{3(\#0-2) - \#0}{2} - 1 \\
&= \#0 - 2 - \#0 + 3 - 1 \\
&= 0
\end{aligned}
$$

This means our construction works for all sequences with $2(\#0-2) < \#3^+ < 3(\#0-2)$.

Figure 5.3 shows the constructed triangulation. We only use big nodes from the given sequence, and from the construction follows, that the number of used '0's is equal to #0. We only have to prove that we have enough '1's.

in block	amount of needed '1's	
(5.5)	$deg(v_j^i) - 3$	$j \in \{2,3\}, 1 \leq i \leq l$
(5.8)	$deg(v_2^i) - 3$	$l+1 \leq i \leq b-1$
(5.9)	$deg(v_2^b) - 2$	
(5.10)	$deg(v_3^i) - 2$	$l+1 \leq i \leq b$
(5.10)	$deg(v_1^i) - 3$	$l+1 \leq i \leq b$
(5.11) or (5.12)	$deg(v_{k+1}) - 2$	
(5.13)	$deg(v_j^i) - 3$	$j \in \{1,4\}$, $2 \leq i \leq l$
(5.14)	$deg(v_4^1) - 3$	
(5.14)	$deg(v_1^1) - 2$	

The amount sums up to:

$$\begin{aligned}
&\sum_{i=1}^{l}\sum_{j=1}^{4}(deg(v_j^i) - 3) + \sum_{i=l+1}^{b}\sum_{j=1}^{3}(deg(v_j^i) - 3) + 1 + (b-l) + 1 + deg(v_{k+1}) - 3 + 1 \\
&= \sum_{i=1}^{k+1}(deg(v_i) - 3) + b - l + 3 \\
&\overset{l=d-b}{=} \sum_{i=1}^{k+1}(deg(v_i) - 3) + 2b - d + 3 \\
&= \sum_{i=1}^{k+1}(deg(v_i) - 3) + k - \#0 + 2 - \#0 + 2 + 3 \\
&= \sum_{i=1}^{k+1}(deg(v_i) - 3) + (k+1) - 1 - 2\#0 + 7 \\
&= \sum_{i=1}^{k+1}(deg(v_i) - 2) - 2\#0 + 6 \\
&= \sum_{i \geq 2}(i-2)\#i - 2\#0 + 6
\end{aligned}$$

which is exactly the number of '1's we get from constraint (3.5). □

Remark 5.20 Again we are able to allow v_1^1 and v_4^l to be equal to 2. Therefore for every sequence with $\#0 \geq 5$ which fulfills $\#3^+ \geq 2(\#0 - 3)$, $\#2^+ \geq 2(\#0 - 2)$ and the simple constraints there exists a triangulation with this degree sequence.

6 No Twos ($\#2 = 0$)

In this chapter we will provide full information for the case $\#2 = 0$. Every sequence in this chapter is assumed to fulfill the basic constraints (3.3), (3.4), (3.8) and (3.11) (which evaluates to $\#1 \geq 2$ for $\#2 = 0$).

Chapter 4 already provides full information for $\#0 \in \{2, 3, 4\}$ therefore only sequences with $\#0 \geq 5$ are considered. Furthermore Theorem 5.19 proves that a triangulation exists for every sequence fulfilling the basic constraints and the additional constraint $\#3^+ > 2(\#0-2)$. So we only consider sequences fulfilling the basic constraints, $\#2 = 0$, $\#0 \geq 5$ and $\#3^+ \leq 2(\#0-2)$.

We will split it up into several sections, starting with the easy part and ending up with a canonical triangulation.

6.1 (u,v,0) - Only Zeros and Ones

Proposition 6.1
There exists no triangulation with $\#0 \geq 5$ and $\#2^+ = 0$

Proof. From the basic constraints we know that $-6 = \sum_{i \geq 0} (i-2)\#i = -2\#0 - \#1$ and with $\#0 \geq 5$ follows $-6 = -2\#0 - \#1 \leq -12$. Therefore there could be no such triangulation. $\square$

6.2 (u,v,0,r) - Zero, One, Three

Now we consider sequences where the highest degree is 3. From the previous section we know that $\#3 > 0$ for every valid sequence of this type.

Observation 6.2 Since the highest degree is 3, there could be no wedge-blocks in the triangulation, because a wedge-block would always yield a node with degree ≥ 4.

Lemma 6.3
Let T be a triangulation with no wedge-blocks and k be the number of nodes on inner triangles. Then $k \overset{2}{\equiv} \#0$.

Proof. We prove this by induction on $\#0$.

For $\#0 = 2$ there are no inner triangles and therefore $0 \overset{2}{\equiv} 2$.

Now we assume $\#0 \geq 2$ and $k \overset{2}{\equiv} \#0$. Since we know that the number of inner triangles equals $\#0 - 2$ every new inner triangle yields an additional node with degree 0 and every new '0' yields an additional inner triangle. Now we add an inner triangle to the triangulation and receive $\overline{T}$ with $\overline{\#0} = \#0 + 1$. When adding an inner triangle there are two possibilities:

- the new inner triangle shares one edge with an old inner triangle. This results in 1 additional node on inner triangles ($\overline{k} = k + 1$)

- the new inner triangle is separated. Then we get 3 new nodes on inner triangles ($\overline{k} = k + 3$)

Both cases result in $\overline{k} \stackrel{2}{\equiv} k + 1$ and we can conclude

$$\overline{k} \stackrel{2}{\equiv} k + 1 \stackrel{2}{\equiv} \#0 + 1 \stackrel{2}{\equiv} \overline{\#0}$$

$\square$

Observation 6.4 Since we only consider sequences with $\#4^+ = 0$ and $\#2 = 0$ all nodes on inner triangles have degree 3.

Observation 6.5 If $\#3 \stackrel{2}{\not\equiv} \#0$ there exists at least one node with degree 3 which is not on an inner triangle and therefore is connected to an additional node with degree 1 $\Rightarrow \#1 \geq 3$.

Lemma 6.6
Every triangulation with $\#2 = 0$ and highest degree 3 fulfills $\#3 \geq 2(\#0 - 2)$

Proof. We know that every such triangulation fulfills

$$n = \#0 + \#1 + \#3$$

and

$$2(n - 3) = \#1 + 3\#3$$

If we combine these two we get

$$\begin{aligned} \#1 + 3\#3 &= 2(\#0 + \#1 + \#3 - 3) \\ \#3 &= 2\#0 + \#1 - 6 \\ &= 2(\#0 - 2) + (\#1 - 2) \\ &\underset{(\#1 \geq 2)}{\geq} 2(\#0 - 2) \end{aligned}$$

$\square$

Observation 6.7 On the other hand, the proof shows that every sequence of the form $(u, v, 0, r)$ which fulfills the basic constraints, fulfills $\#3 \geq 2(\#0 - 2)$.

Lemma 6.8
Every sequence of nonnegative integers with a degree vector of the form $(u, v, 0, r)$ which fulfills the basic constraints, $\#3 = 2(\#0 - 2)$ and $\#0 \stackrel{2}{\equiv} \#3$ is a valid degree sequence.

Proof. Let's consider the case $\#3 = 2(\#0-2)$ and $\#0 \stackrel{2}{\equiv} \#3$. To prevent the need of additional '1's we build a triangulation with as few isolated big nodes as possible. This means that we put all '3's on inner triangles. Equation (3.6) shows that this is possible (consider that there are no wedge-blocks possible) and we get $\frac{\#0-2}{2}$ blocks of inner triangles (this is an integer because $\#0 \stackrel{2}{\equiv} \#3 = 2(\#0 - 2)$ which means it is even). Furthermore this equation tells us that every block consists of exactly two inner triangles. The blocks are put into a row as in Figure 6.1. Again the only thing which is left to show is that the basic constraints provide the correct $\#1$. If we look at the triangulation we see that the construction needs exactly two '1's. Looking at the basic constraints we get the following:

$$\begin{aligned}
-6 &= \sum_{i \geq 0} (i-2)\#i \\
&= -2\#0 - \#1 + \#3 \\
&= -2\#0 - \#1 + 2(\#0 - 2) \\
&= -\#1 - 4 \\
\#1 &= 2
\end{aligned}$$

□

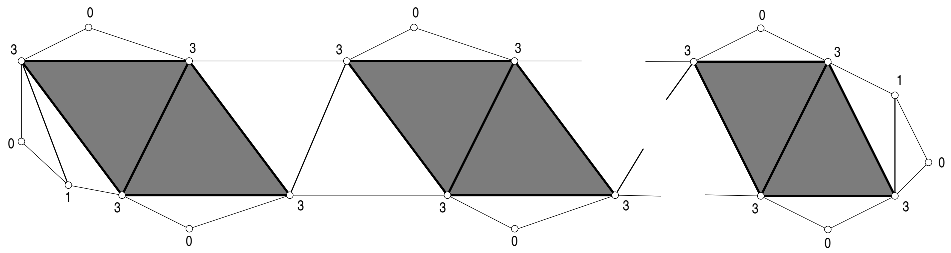


Figure 6.1: Triangulation for the case $(u, v, 0, r)$

Remark 6.9 The case $\#0 \overset{2}{\not\equiv} \#3$ and $\#3 > 2(\#0 - 2)$ is covered by Theorem 5.8 or Theorem 5.19 which proves that there always exists a triangulation for such sequences.

Lemma 6.10

There exists no triangulation with a degree vector of the form $(u, v, 0, r)$ with $\#0 \overset{2}{\not\equiv} \#3$ and $\#3 = 2(\#0 - 2)$

Proof. From Lemma 6.3 we know that there exists at least 1 node with degree 3 which is not incident to an inner triangle. Let k be the nodes incident to blocks of inner triangles, then $k \leq 2(\#0 - 2) - 1$. b being the amount of blocks of inner triangles we can conclude from (3.6):

$$\begin{aligned}
b &= \frac{k - \#0 + 2}{2} \\
&\leq \frac{2\#0 - 5 - \#0 + 2}{2} \\
&= \frac{\#0 - 3}{2}
\end{aligned}$$

Now we calculate the average amount of inner triangles per block:

$$\begin{aligned}
\frac{\textit{amount inner triangles}}{\textit{amount blocks}} &\geq \frac{2(\#0 - 2)}{\#0 - 3} \\
&> 2
\end{aligned}$$

This means that there exists at least one block with at least 3 inner triangles. But such a block can only exist if there are nodes with degree ≥ 4, a contradiction to the form $(u, v, 0, r)$ □

Lemma 6.11
Every sequence with a degree vector of the form $(u, v, 0, r)$ fulfilling the simple constraints fulfills

$$\#1 \geq 3 \Leftrightarrow \#3 > 2(\#0 - 2)$$

Proof. Combining the simple constraints (3.3) and (3.4) we get:

$$\begin{aligned} \#1 + 3\#3 &= 2(\#0 + \#1 + \#3 - 3) \\ \#3 &= 2\#0 + \#1 - 6 \\ &= 2(\#0 - 2) + (\#1 - 2) \\ \#3 - 2(\#0 - 2) &= \#1 - 2 \end{aligned}$$

Which directly leads to

$$\#1 \geq 3 \Leftrightarrow \#3 > 2(\#0 - 2)$$

□

Theorem 6.12
A sequence with a degree vector of the form $(u, v, 0, r)$ is a degree sequence if and only if it fulfills the basic constraints and $\left(\#3 \overset{2}{\not\equiv} \#0\right) \Rightarrow \#1 \geq 3$

Proof. Let S be a sequence of the given form fulfilling the basic constraints. We have 2 cases:

Case 1 $\#3 \overset{2}{\equiv} \#0$: The combination of Observation 6.7 and Lemma 6.8 proves that S is a degree sequence.

Case 2 $\#3 \overset{2}{\not\equiv} \#0$: If $\#1 \geq 3$ Lemma 6.11 provides $\#3 > 2(\#0 - 2)$ and Theorem 5.8 and Theorem 5.19 show that in this case S is a degree sequence.

Now consider a triangulation with a degree sequence $\overline{S}$ with a degree vector of the form $(u, v, 0, r)$. Since $\overline{S}$ fulfills the basic constraints, Lemma 6.6, Lemma 6.10 and Lemma 6.11 provide that $\left(\#3 \overset{2}{\not\equiv} \#0\right) \Rightarrow \#1 \geq 3$.

□

This means we have full information for sequences with degree vectors of the form $(u, v, 0, r)$.

6.3 (u,v,0,r,0,...,0,1) - The Big One

Now we go one step further and allow one node with degree ≥ 4. Since we already have full information on degree vectors of the form $(u, v, 0, r)$, we can assume that there is exactly one node with degree ≥ 4. Let's call the number of wedges in blocks of inner triangles w.

Let i be the degree of the node with degree ≥ 4.

Lemma 6.13
Let S be a sequence with a degree vector of the form $(u, v, 0, r, 0, \ldots, 0, 1)$ with i the degree of the big node. If S fulfills the basic constraints it fulfills $\#3 \geq 2(\#0 - 1) - i$

Proof. S fulfills the basic constraints which include

$$n = \#0 + \#1 + \#3 + 1$$

and

$$2(n-3) = \#1 + 3\#3 + i$$

combining these two we get:

$$\begin{aligned} 2\#0 + 2\#1 + 2\#3 + 2 - 6 &= \#1 + 3\#3 + i \\ (\#1 - 2) + 2(\#0 - 1) &= \#3 + i \\ \#3 &\geq 2(\#0 - 1) - i \end{aligned}$$

□

Remark 6.14 If $\#3 \geq 2(\#0 - 2)$ the number of big nodes is bigger than $2(\#0 - 2) + 1$ and Chapter 5 already provides full information.

Lemma 6.15
For every valid degree sequence S with a degree vector of the form $(u, v, 0, r, 0, \ldots, 0, 1)$ with $u \geq 5$ there exists a triangulation with the biggest node incident to an inner triangle.

Proof. Since S is a valid degree sequence, there exists a triangulation T with degree sequence S. Let a be the node in T with $i = deg(a) \geq 4$. If a is incident to an inner triangle we are done.

If a is a separated big node, it has at least $i - 2$ ones in his wedge. Since $\#0 \geq 5$ there are inner triangles and every block has at least one incident node who has a wedge, let's call it b. Because of the form of the sequence this node has degree 3 and we move $i - 3$ ones from the wedge of a to the wedge of b. This transformation leaves the degree sequence untouched and provides a triangulation where the biggest node is incident to an inner triangle. □

6.3.1 Without a Wedge-Block

Let's consider sequences with $n \overset{2}{\equiv} \#1$. If we look at (3.7) we see that this means that there is an even number of wedges in blocks of inner triangles.

We will now present a canonical triangulation for every sequence with a degree vector of the form $(u, v, 0, r, 0, \ldots, 0, 1)$ with $n \overset{2}{\equiv} \#1$ and $\#3 < 2(\#0 - 2)$ (because $\#3 \geq 2(\#0 - 2)$ is already handled in Chapter 5). Of course we can only provide a triangulation for sequences fulfilling the basic constraints.

Since we have so few big nodes, we put all of them on inner triangles. (3.7) tells us, that the number of wedges in blocks of inner triangles equals $\#3 + 1 + \#0 \ mod\ 2$. Wedge-blocks make things difficult and for every double-wedge we would need an additional 1. Therefore and because of $n = \#0 + \#1 + \#3 + 1$ and $n - \#1 \overset{2}{\equiv} 0$ we decide that there will be no wedge in this triangulation. Figure 6.2 shows the triangulation we are building.

(3.6) tells us, that we can now calculate the number of blocks of inner triangles:

$$b = \frac{\#3 + 1 - \#0}{2} + 1$$

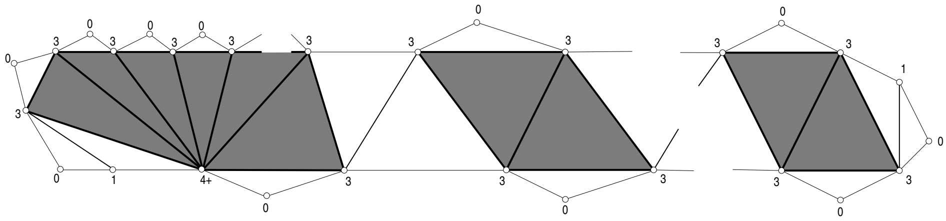


Figure 6.2: Triangulation for the case $(u, v, 0, r, 0, \ldots, 0, 1)$ without the need of a wedge

To construct a valid triangulation, we have to ensure that $b \geq 1$. This means that $\#3+1 \geq \#0$ which is true for every sequence fulfilling the basic constraints including $\#0 \leq \frac{n-\#1}{2}$.

Let's look at the number of triangles per block. Using $\#3 < 2(\#0-2)$ we can rewrite the calculation of the blocks like this:

$$\begin{aligned}\frac{\#3+1-\#0}{2}+1 &< \frac{2(\#0-2)+1-\#0+2}{2}\\ &= \frac{\#0-1}{2}\end{aligned}$$

The number of triangles is $\#0-2$ and therefore we have an average of more than two triangles per block. We will construct one big block with the big node on it and the rest as blocks of 2 inner triangles per block.

Let's see what else is guaranteed by the basic constraints. Looking at (3.5) we see that

$$\begin{aligned}-6 &= -2\#0 - \#1 + \#3 + (i-2)\\ \#3 &= 2(\#0-2) + (\#1-2) - (i-2)\end{aligned} \tag{6.1}$$

combining this again with the calculation for the amount of blocks we get

$$\begin{aligned}b &= \frac{2(\#0-2)+(\#1-2)-(i-2)+1-\#0}{2}+1\\ &= \frac{\#0+\#1-i-3}{2}+1\end{aligned}$$

According to our planned construction we have $2(b-1)$ blocks with 2 triangles. This means that the number of inner triangles on the big block can be calculated and written like this:

$$\begin{aligned}\#0-2-2(b-1) &= \#0-2-2\left(\frac{\#0+\#1-i-3}{2}\right)\\ &= i+1-\#1\\ &= 2+(i-3)-(\#1-2)\end{aligned}$$

Now we look at this formula the right way and see, that this means that for every "missing" '1' (to fill the wedge of the big node) the big block has one additional inner triangle.

Since the number of blocks is calculated from the existing number of big nodes, we ensure to have enough big nodes to "fill" the inner triangles. Now the last question is if we have enough '1's. Two are used for the beginning and the end of the row of blocks. The small blocks don't need a '1' (there are only nodes with degree 3 on these blocks) and the big block is constructed in dependency of #1 which is ensured by the basic constraints. Therefore this construction leads to a triangulation for every such sequence. #1 can't even be too big because of:

$$\begin{aligned}(\#1-2) &= \#3 + i - 2(\#0-1)\\ &< 2(\#0-2) + i - 2(\#0-1)\\ &= i-2\\ (\#1-2) &\le i-3\end{aligned}$$

Therefore we just proved:

Proposition 6.16
Every sequence with a degree vector of the form $(u, v, 0, r, 0, \ldots, 0, 1)$ with $u \geq 5$, fulfilling the basic constraints and $n \overset{2}{\equiv} \#1$ is a valid degree sequence.

6.3.2 With a Wedge-Block

Again we consider only sequences with $\#3 < 2(\#0-2)$, this time with $n \overset{2}{\not\equiv} \#1$ which is equivalent to $\#3 \overset{2}{\equiv} \#0$. Like before we construct a canonical triangulation where all big nodes are on inner triangles. As before we put all big nodes on inner triangles. Looking at (3.7) we see that this means that the number of wedges in blocks of inner triangles has to be odd and therefore at least 1. Again we try to use as few wedge-blocks as possible, which means one wedge-block in this case.

(3.6) tells us, that we have $\frac{\#3+1-\#0-1}{2} + 1 = \frac{\#3-(\#0-2)}{2}$ blocks of inner triangles. To ensure that our construction is possible the number of blocks has to be at least 1.

$$\begin{aligned}\#3 + 1 - \#0 - 1 &\geq 0\\ \#3 &\geq \#0\end{aligned}$$

Since $\#0 \overset{2}{\equiv} \#3$ it's either $\#3 \leq \#0 - 2$ or $\#3 \geq \#0$. The basic constraint $\#0 \leq \frac{n-\#1}{2}$ leads to:

$$\begin{aligned}\#0 &\leq \frac{\#0 + \#3 + 1}{2}\\ 2\#0 &\leq \#0 + \#3 + 1\\ \#0 - 1 &\leq \#3\end{aligned}$$

Therefore we always have at least 1 block in our triangulation.

Since we have only one node with degree ≥ 4 all but one block cannot have more than 2 inner triangles. Because of $\#2 = 0$ a separated inner triangle would lead to an additional '1', so we construct blocks with at least two inner triangles (meaning all but one block has exactly two inner triangles). Let's look at the inner triangles in the big block (which is a wedge-block):

$$
\begin{aligned}
\#0 - 2 - 2\left(\frac{\#3 - (\#0-2)}{2} - 1\right) &= \#0 - 2 - (\#3 - (\#0-2)) + 2 \\
&= \#0 - \#3 + \#0 - 2 \\
&= 2\#0 - \#3 - 2 \\
&= 2\#0 - (2(\#0-2) + (\#1-2) - (i-2)) - 2 \\
&= 2\#0 - 2\#0 + 4 - (\#1-2) + (i-2) - 2 \\
&= 2 + i - \#1 \qquad (6.2) \\
&\leq i
\end{aligned}
$$

On the other hand with $\#3 < 2(\#0-2)$:

$$
\begin{aligned}
2\#0 - \#3 - 2 &> 2\#0 - 2 - (2(\#0-2)) \\
&= 2
\end{aligned}
$$

So for l being the number of inner triangles in the big block we know $3 \leq l \leq i$.

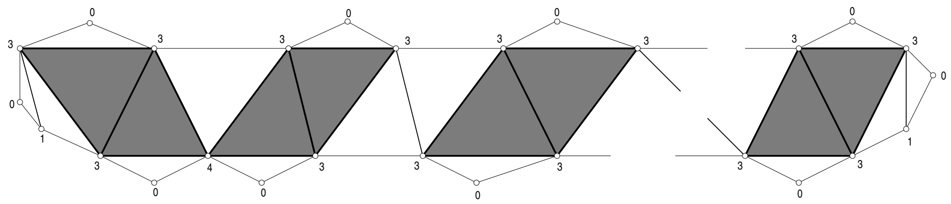


Figure 6.3: Triangulation for the case $(u, v, 0, r, 1)$

As illustrated in Figure 6.3 the big block starts with two inner triangles on the one side of the wedge. The other side of the wedge is either again an attached block with less or equal than two inner triangles (if $i - \#1 \leq 2$) or a fan of inner triangles as in Figure 6.4. In the second case we need an additional '1' ($\#1 \geq 3$) for the node next to the wedge.

In the case $i = 4$ the basic constraints guarantee that we have either 4 inner triangles on the big block (in this case we don't need an additional '1') or $\#1 \geq 3$.

For the case $i > 4$ our construction needs $\#1 \geq 3$ (either to fill the wedge of the big node or for the '3' next to the wedge).

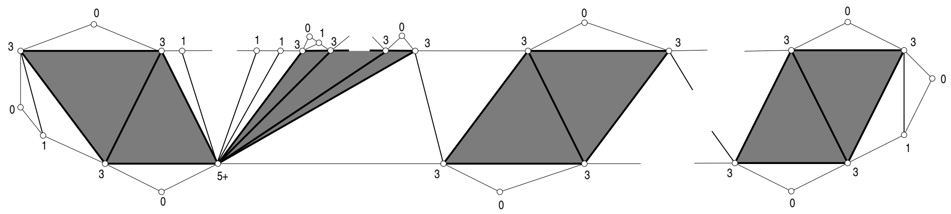


Figure 6.4: Triangulation for the case $(u, v, 0, r, 0, \ldots, 0, 1)$ with a wedge needed and biggest degree > 4

Lemma 6.17

Every sequence of this form fulfilling the basic constraints and $(i \geq 5 \Rightarrow \#1 \geq 3)$ *is a valid degree sequence.*

Proof. Again we have 2 cases:

Case 1 $i = 4$: We look at the construction above. The number of blocks is calculated from the number of existing big nodes, therefore the only thing we need to show is that we have exactly the needed number of '1's.

We need two '1's at the beginning and the end of the row of blocks. The small blocks don't need additional '1's. On the big block we need $i - 2 - (i - \#1) = \#1 - 2$ '1's to fill the degree of the big node (compare equation (6.2)).

Case 2 $i \geq 5$: (6.2) ensures that there are either enough inner triangles or '1's to fill the degree of the biggest node. The constraint $\#1 \geq 3$ is only needed to ensure the additional '1' for the '3' next to the wedge. If there are less than $i - 1$ inner triangles on the big node, this constraint is provided by the basic constraints.

□

Since we want full information, we now need to show that a sequence of this form with $\#4 = 0$ and $\#1 = 2$ is not a legal degree sequence, even if it fulfills the basic constraints.

Lemma 6.18

In every triangulation with $\#2 = 0$, $n \overset{2}{\not\equiv} \#1$ *and* $\#4 = 0$ *there exist at least 3 nodes with degree 1.*

Proof. Because of $n \overset{2}{\not\equiv} \#1$ there exists either an isolated big node (which directly leads to an additional '1') or at least one wedge-block.

Let's look at one such wedge-block. Let a be the node connecting the two parts of the wedge. Then $deg(a) \geq 4$ and because of $\#4 = 0$ it has degree ≥ 5. So there is either a diagonal from outside of the block incident to a or at least one part of the wedge-block has more than 2 inner diagonals incident to a.

If there is a diagonal from outside the block incident to a, because of $\#2 = 0$ this diagonal leads to an additional '1'.

If there are more than 2 inner diagonals from one part incident to a, we have either the node next to the wedge with an attached '1' (which is an additional '1'), or the inner triangle next to the wedge is an inner-inner-triangle and therefore leads to an additional '1'.

If there is more than one wedge incident to a, the part between two of the wedges again leads to an additional '1'.

So in every case we get $\#1 \geq 3$. □

Therefore every valid degree sequence of this form fulfills our constraints, and if a sequence fulfills the constraints we can provide a canonical triangulation. So finally we have full information on sequences with degree vectors of the form $(u, v, 0, r, 0, \ldots, 0, 1)$.

6.3.3 An Example for the Need of our Constraint

Let's consider the sequence $(10, 2, 0, 10, 0, 0, 0, 0, 1)$. $n = 23 \stackrel{2}{\not\equiv} \#1$ which means we are in the second case and our constraint claims $\#1 \geq 3$ which is not true for this sequence. Since we didn't get an additional '1' we need to put all big nodes on inner triangles (otherwise it wouldn't be a valid triangulation). Looking at the number of blocks we get $\frac{11-8-w}{2} = \frac{3-w}{2}$ where w is the number of wedge-blocks. The number of blocks is at least 1 and therefore the number of wedge-blocks is ≤ 1. We also know that $w \stackrel{2}{\equiv} (11 - 10)$ and therefore has to be 1. The big node (with degree 8) has to be incident to the wedge. Since there are no additional '1's, all slots of the biggest node need to be filled by inner triangles. At least at one side of the wedge there need to be more than 2 inner triangles. The node on that side next to the wedge has either degree 2 or we need an additional '1'. Neither is possible and therefore no triangulation is possible.

6.4 The General Case

In this section we reduce our constraints to $\#2 = 0$, but since some sub cases are already handled by the previous sections, we consider only sequences with $\#4^+ \geq 2$ and $\#3^+ \leq 2(\#0 - 2)$. Of course the basic constraints must still be fulfilled.

As before we will present canonical triangulations where all big nodes are incident to inner triangles with one big block of inner triangles and the remaining blocks with 2 inner triangles. The number of wedge-blocks is called $w \in \{0, 1\}$, b the number of blocks and $\#bB$ is the number of nodes incident to the big block.

Let's recall some formulas:

$$\begin{aligned} b &= \frac{\#3^+ - \#0 - w}{2} + 1 \\ \#bB &= \#3^+ - 4(b-1) \\ &= \#3^+ - 2\#3^+ + 2\#0 + 2w \\ &= 2\#0 - \#3^+ + 2w \\ w &\stackrel{2}{\equiv} \#3^+ - \#0 \\ &\stackrel{2}{\equiv} n - \#1 \end{aligned}$$

6.4.1 Without a Wedge-Block

Again we start with the easier part where we don't need a wedge-block to put all big nodes on inner triangles. As said above, the triangulation will consist of 1 big block and as much blocks with 2 inner triangles, as needed. The big nodes will be distributed over the blocks in descending order, where the big block is the leftmost block (see Figure 6.5).

Now we need to show that this triangulation can be build from every sequence which fulfills our constraints.

First we have to check if the number of blocks is guaranteed to be at least 1. From (3.9) we know that $\#3^+ \geq \#0$ therefore $\frac{\#3^+ - \#0}{2} + 1 \geq 1$.

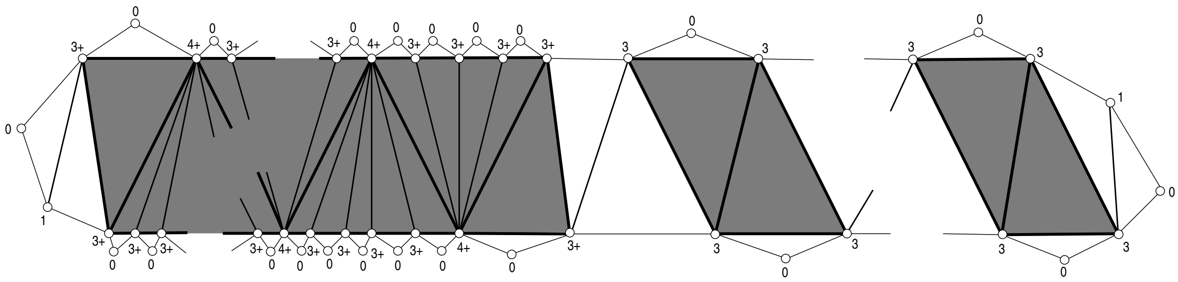


Figure 6.5: Triangulation for the case $\#2 = 0$ with no wedge needed

Now we look closer on the big block. The internal structure of the block is a zigzag between big nodes with degree ≥ 4. Let $B = \{b_1, \ldots, b_{\#bB}\}$ be the set with the $\#bB$ biggest nodes in descending order. We start the block with $b_{\#bB}$ as the ear of the block. Then we start the zigzag with $b_{\#bB-1}$ as "isolated '1'". The first big node in the internal zigzag is b_1 whose "wedge" (in the internal zigzag) is filled with the $deg(b_1) - 4$ smallest remaining nodes from B. The next big node in the zigzag (now on the lower side) is b_2 filled with the $deg(b_2) - 4$ smallest remaining nodes from B. We continue like that till we finish with an "isolated '1'" in the internal zigzag and the other ear of the inner block.

If $deg(b_{\#bB}) \geq 4$ everything is fine, otherwise we need to show that we have enough nodes with degree ≥ 4 for the zigzag. This means the number of nodes with degree 3 on the big block must be less or equal to $\sum_{i \geq 4} (i-4)\#i + 4$. If the number of nodes with degree 3 is exactly that amount, every node on the zigzag is filled and we cannot extend the zigzag. If it's smaller some of the nodes on the zigzag need additional '1's and eventually some "filling '3's" are nodes with degree greater than 3. Let's calculate the number of '3's on the big block (we still assume that all nodes with degree ≥ 4 are on the big block):

$$\begin{aligned}
\#\{3s\ on\ the\ big\ block\} &= \#3 - 4(b-1) \\
&= \#3 - 2\#3^+ + 2\#0 \\
&= 2\#0 - 2(n - \#0 - \#1) + \#3 \\
&= 2\#0 - 2n + 2\#0 + 2\#1 + \#3 \\
&= 4\#0 + 2\#1 + \#3 - 2n
\end{aligned}$$

Now we need to reformulate the upper bound for the allowed number of '3's:

$$\begin{aligned}
\sum_{i \geq 4} (i-4)\#i + 4 &= \sum_{i \geq 4} i\#i - 4(n - \#0 - \#1 - \#3) + 4 \\
&= 2(n-3) - \#1 - 3\#3 - 4n + 4\#0 + 4\#1 + 4\#3 + 4 \\
&= 4\#0 + 3\#1 + \#3 - 2n - 2
\end{aligned}$$

So our constraint is:

$$\begin{aligned}
4\#0 + 2\#1 + \#3 - 2n &\leq 4\#0 + 3\#1 + \#3 - 2n - 2 \\
0 &\leq \#1 - 2 \\
2 &\leq \#1
\end{aligned}$$

but this is already a basic constraint ($\#1 + \#2 \geq 2$) and therefore we don't need a further constraint to make our big block construction work.

At last we look at the needed amount of '1's. To know how much '1's we need, we first have to calculate how much of the degrees of the big nodes are already filled from diagonals in the blocks and between them. Every small block has 5 diagonals in the block and 1 connecting diagonal per block. Which means we have

$$(5+1)(b-1) = 3(\#3^{+} - \#0)$$

degrees from the small blocks (the isolated '1' from the end is included in this calculation).

Looking at the big block we have $2\#bB - 3$ diagonals in the big block, one connecting diagonal and the diagonal to the isolated '1'. Combined we get

$$\begin{aligned} 3(\#3^{+} - \#0) + 2\#bB - 3 + 2 &= 3\#3^{+} - 3\#0 - 1 + 2(2\#0 - \#3^{+}) \\ &= \#3^{+} + \#0 - 1 \end{aligned}$$

We know that every triangulation has $n-3$ inner diagonals. Subtracting all diagonals which connect two big nodes, we receive the needed amount of '1's. In our calculation we included the diagonals to the two isolated '1's, therefore we should receive $\#1 - 2$

$$\begin{aligned} \#1 - 2 &\stackrel{!}{=} (n-3) - (\#3^{+} + \#0 - 1) \\ &\stackrel{!}{=} \#0 + \#1 + \#3^{+} - 3 - \#3^{+} - \#0 + 1 \\ &= \#1 - 2 \end{aligned} \tag{6.3}$$

So we conclude that our construction works for every sequence with a degree vector of the form $(u, v, 0, r, \ldots)$ fulfilling the basic constraints, $\#0 \geq 5$, $n \stackrel{2}{\equiv} \#1$ and $\#3^{+} \leq 2(\#0 - 2)$.

6.4.2 In Need of a Wedge-Block

Let's look at the case where we need a wedge-block to put all big nodes on inner triangles. This means we look at sequences with $n \stackrel{2}{\not\equiv} \#1$ which is equivalent to $\#3^{+} \stackrel{2}{\not\equiv} \#0$.

Again we present a canonical triangulation with 1 big block and as much blocks with two inner triangles as needed. As in section 6.3.2 the big block is a wedge-block and has two inner triangles on the left side of the wedge and the remaining inner triangles on the right side. Starting with the wedge-node (the node connecting the two parts of the wedge-block) we again build a zigzag of nodes with degree ≥ 4.

This time we have to separate two more sub cases: $\#4 = 0$ and $\#4 > 0$. Let's recall some basic formulas (keep in mind that w the number of wedge-blocks is 1).

$$\begin{aligned} b &= \frac{\#3^{+} - \#0 - 1}{2} + 1 \\ \#bB &= 2\#0 - \#3^{+} + 2 \end{aligned}$$

The Case $\#4 > 0$

Let $B = \{b_1, \dots, b_{\#bB}\}$ be the nodes on the big block where $deg(b_{\#bB}) = 4$ and $b_1, \dots, b_{\#bB-1}$ are the biggest remaining nodes in the sequence in descending order. The big block is build as shown in Figure 6.6. Starting with 2 inner triangles on the left on which we put $b_{\#bB-1}, \dots, b_{\#bB-3}$. $b_{\#bB}$ is our wedge-node and starts the internal zigzag in the big block. The next node on the zigzag is $b_{\#bB-4}$ because we don't need a high degree here and want to claim as few high degrees as possible. After that, the zigzag is filled starting with b_1. The wedges of the zigzag-nodes are always filled with the smallest remaining nodes from B. The remaining degrees (after building the blocks of inner triangles) are filled with the remaining '1's.

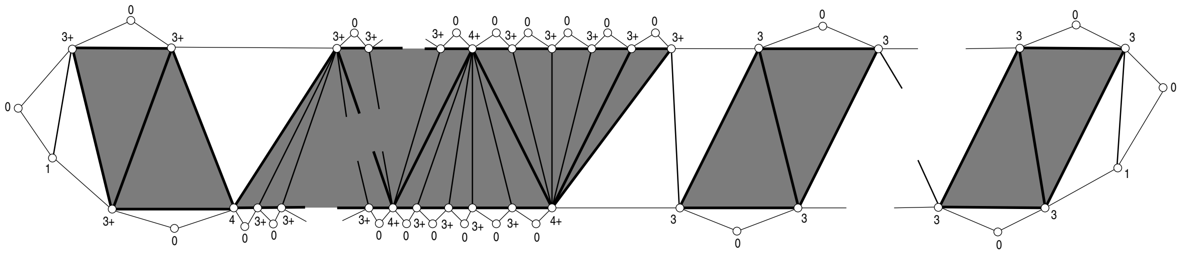


Figure 6.6: Triangulation for the case $\#2 = 0$ with wedge and $\#4 > 0$

Again we have to show that we have enough nodes and everything is possible. We start with the number of blocks. To build a valid triangulation we need at least 1 block guaranteed from the sequence. We know that $\#3^+ \geq \#0$ and in combination with $\#3^+ \overset{2}{\not\equiv} \#0$ follows $\#3^+ > \#0$

$$\begin{aligned} b &= \frac{\#3^+ - \#0 - 1}{2} + 1 \\ &> \frac{\#3^+ - \#3^+ + 1}{2} \geq 0 \end{aligned}$$

For the big block we need at least 3 inner triangles on that block. Using $\#3^+ \leq 2(\#0 - 2)$ we see:

$$\begin{aligned} b &\leq \frac{2(\#0 - 2) - \#0 + 1}{2} \\ &= \frac{\#0 - 3}{2} \\ \frac{\#0 - 2}{b} &\geq \frac{2(\#0 - 2)}{\#0 - 3} > 2 \end{aligned}$$

Since all other blocks consist of exactly 2 inner triangles, the big block is provided with at least 3 inner triangles.

What about the number of '3's? If we have more nodes with degree ≥ 4 than nodes on the big block, everything is fine. So let's assume all nodes on other blocks have degree 3. So again the number of '3's on the big block is calculated similar to (6.3)

$$\begin{aligned}\#\{3s\ on\ the\ big\ block\} &= \#3 - 4(b-1)\\ &= \#3 - 2\#3^{+} + 2\#0 + 2\\ &= 2\#0 - 2(\#3 + \#4^{+}) + \#3 + 2\\ &= 2\#0 - \#3 - 2\#4^{+} + 2\end{aligned}$$

This number has to be less than the amount of slots on the big block where we don't need a degree ≥ 4. This means the 3 nodes at the left part, the "small" node next to the wedge of the block, the '3' at the end of the block (the "ear" in the internal zigzag), the "isolated '1'" in the internal zigzag and all the nodes in wedges of the internal zigzag (for every node on the zigzag there are at most $deg(v) - 4$ slots).

$$\begin{aligned}6 + \sum_{i\geq 4}(i-4)\#i &= \sum_{i\geq 4} i\#i - 4\#4^{+} + 6\\ &= 2(n-3) - \#1 - 3\#3 - 4\#4^{+} + 6\\ &= 2(\#0 + \#1 + \#3 + \#4^{+}) - \#1 - 3\#3 - 4\#4^{+}\\ &= 2\#0 + \#1 - \#3 - 2\#4^{+}\end{aligned}$$

now we only have to check this inequality:

$$2\#0 - \#3 - 2\#4^{+} + 2 \overset{!}{\leq} 2\#0 + \#1 - \#3 - 2\#4^{+} \tag{6.4}$$

$$2 \leq \#1 \tag{6.5}$$

which is provided by the basic constraints.
The last question concerns the number of '1's. Again we calculate the remaining number of diagonals in our triangulation and see if we have enough '1's provided to fill them. As before we get $6(b-1)$ diagonals from the small blocks. The wedge block is a little more complicated: The left part is similar to a normal small block and gives 5 diagonals from the block and one to the isolated '1'. The right part of the block contains $2\#\{nodes\ on\ the\ part\} - 3$ inner diagonals and one connecting diagonal. So the number of diagonals connecting two big nodes or an isolated '1' with a big node can be calculated like this:

$$\begin{aligned}6(b-1) + 6 + 2(\#bB - 3) - 3 + 1 &= 6(\frac{\#3^{+} - \#0 - 1}{2} + 1) + 2(2\#0 - \#3^{+} - 1) - 2\\ &= 3\#3^{+} - 3\#0 - 3 + 6 + 4\#0 - 2\#3^{+} - 2 - 2\\ &= \#3^{+} + \#0 - 1\end{aligned} \tag{6.6}$$

The number of remaining diagonals should now be equal to $\#1 - 2$

$$\begin{aligned}\#1 - 2 &\overset{!}{=} (n-3) - (\#3^{+} + \#0 - 1)\\ &\overset{!}{=} \#0 + \#1 + \#3^{+} - 3 - \#3^{+} - \#0 + 1\\ &= \#1 - 2\end{aligned} \tag{6.7}$$

So we can conclude that we have enough blocks of inner triangles, enough nodes with degree ≥ 4 for the zigzag in the big block and enough '1's to fill the remaining degrees. Since the construction doesn't need further constraints, we now know that there exists a triangulation for every sequence fulfilling the basic constraints, $\#0 \geq 5$, $\#2 = 0$, $n \overset{2}{\not\equiv} \#1$ and $\#4 > 0$.

The Case $\#4 = 0$

As we saw in section 6.3.2 we need the additional constraint $\#1 \geq 3$. The canonical triangulation is similar to the case $\#4 > 0$. The only difference is that this time the wedge-node is the biggest node and the internal zigzag starts directly from there (see Figure 6.7 for clarification). The calculation of the number of blocks and nodes on the big block is exactly the same as in the case $\#4 > 0$ so we directly look at the number of '3's on the big block. Again we assume that all nodes with degree ≥ 4 are on the big block, because otherwise everything is fine. The number of '3's on the big block is the same as before but what happens to the upper bound for the slots where these nodes can be put in?

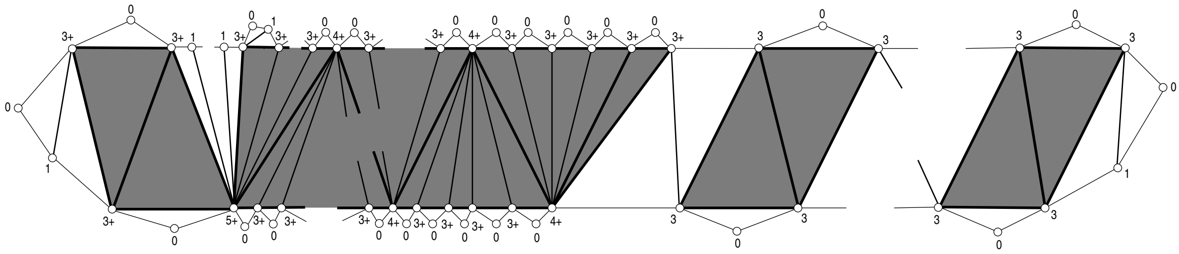


Figure 6.7: Triangulation for the case $\#2 = 0$ with a wedge needed and $\#4 = 0$

The left part of the wedge-block again provides 3 slots. The right part provides the end of the zigzag, the "isolated '1'" in the internal zigzag and $deg(v) - 4$ slots for every node on the internal zigzag (meaning every node with degree ≥ 4).

$$\begin{aligned}
5 + \sum_{i \geq 4} (i-4)\#i &= \sum_{i \geq 4} i\#i - 4\#4^+ + 5 \\
&= 2(n-3) - \#1 - 3\#3 - 4\#4^+ + 5 \\
&= 2(\#0 + \#1 + \#3 + \#4^+) - \#1 - 3\#3 - 4\#4^+ - 1 \\
&= 2\#0 + \#1 - \#3 - 2\#4^+ - 1
\end{aligned}$$

and the inequality this time evaluates to

$$\begin{aligned}
2\#0 - \#3 - 2\#4^+ + 2 &\overset{!}{\leq} 2\#0 + \#1 - \#3 - 2\#4^+ - 1 \\
3 &\leq \#1
\end{aligned}$$

which is our additional constraint.

Again there is only the number of '1's left. This calculation is exactly the same as in (6.6) and (6.7) and therefore the construction works for every sequence with $\#2 = 0$, $\#0 \geq 5$, $\#3^+ \leq 2(\#0 - 2)$, $n \overset{2}{\not\equiv} \#1$, $\#4 = 0$ and $\#1 \geq 3$.

Lemma 6.18 already showed that the constraint $\#1 \geq 3$ is always fulfilled for such triangulations.

Therefore we can conclude the following theorem which provides full information for the case $\#2 = 0$.

Theorem 6.19
A sequence with $\#0 \geq 5$ and $\#2 = 0$ is a degree sequence if and only if it fulfills the basic constraints and $\left[\left(\left(\#3^+ \stackrel{2}{\not\equiv} \#1\right) \wedge [\#4 = 0]\right) \Rightarrow (\#1 \geq 3)\right]$

As an example for a sequence fulfilling the basic constraints but not being a degree sequence, the reader may try to find a triangulation for $(9, 2, 0, 8, 0, 2)$. Consider that this sequence does not fit into the constraint of the theorem and therefore is no degree sequence.

6.5 Explicit Construction of the Ordered Degree Sequence

In this section we will present in an algorithmic way how to construct an ordered degree sequence for a sequence of nonnegative integers that fulfill our constraints. The way from the ordered degree sequence to the triangulation is easy and described generally in section 3.6.1.

Again we stick to the notation b being the number of blocks, k the number of nodes on inner triangles, $\#bB$ the number of nodes on the big block and w the number of wedge-blocks in the triangulation.

6.5.1 Without a Wedge-Block

Let's start with the easy part again. Let S be a sequence of non-negative integers fulfilling the basic constraints, $\#0 \geq 5$, $\#2 = 0$ and $\#3^+ \stackrel{2}{\equiv} \#0$. As shown in the previous sections there exists a triangulation with the degree sequence S if and only if S fulfills the basic constraints. At first we calculate the number of blocks of inner triangles and the number of nodes on the big block. Keep in mind that because of $\#3^+ \stackrel{2}{\equiv} \#0$ we know that $w = 0$.

$$\begin{aligned}
k &= \#3^+ \\
b &= \frac{\#3^+ - \#0}{2} + 1 \\
\#bB &= \#3^+ - 4(b-1) \\
&= \#3^+ - 2\#3^+ + 2\#0 \\
&= 2\#0 - \#3^+
\end{aligned}$$

Let $B = \{b_1, \ldots, b_{\#3^+}\}$ be the multiset with the degrees of the big nodes in decreasing order, bB the subset of B containing the $\#bB$ first elements from B (meaning the nodes for the big block) and $sB = \{u_1, \ldots, u_{4(b-1)}\}$ the multiset containing the remaining big nodes. We partition sB into subsets of cardinality 4: $sB = \{\{u_1^1, u_2^1, u_3^1, u_4^1\}, \ldots, \{u_1^{(b-1)}, u_2^{(b-1)}, u_3^{(b-1)}, u_4^{(b-1)}\}\}$ each subset representing a small block of inner triangles.

For a more convenient way to build the ordered degree sequence, we will always start at the leftmost node and build the sequence parallel on the upper and lower side. This means we build it simultaneous from the beginning and the end. Every time we add a big node from bB or sB to the ordered degree sequence we remove it from the multiset.

The ordered degree sequence starts with a zero. Then we have $(b_{\#bB} - 3)$ '1's on the lower side and $b_{\#bB}$ on the upper side. This is the ear of the big block. Next we have the isolated '1' from the big block: Again on the upper side we add $(b_{\#bB-1} - 3)$ '1's, the '0' and $b_{\#bB-1}$ itself.

Now the beginning is made and we build the inner zigzag from the big block until only 1 node remains in bB:

Let l be the highest remaining degree in bB. We add l and a '0' to the lower side, then we add the $(l-4)$ smallest remaining nodes in bB to the upper side including their wedge with the necessary '1's and a '0' for each one. If we don't have enough nodes left, we fill the remaining slots with '1's at the lower side. Then the upper and lower side switch places (that is the biggest remaining node is now placed on the opposite side of the previous). We iterate this till only 1 node is left in bB.

The last node in bB is always added to the lower side of the triangulation because we need to ensure the orientation of the wedges. Let l be the degree of the last node. We add l to the lower side and $(l-3)$ '1's to the upper side.

So the big block is finished and we build the small blocks as already shown before:

For every $1 \le i \le b-1$ we add

$$u_1^k, \underbrace{1, \ldots, 1}_{u_2^k - 3}, 0, u_2^k, \underbrace{1, \ldots, 1}_{u_3^k - 3}$$

to the upper side and

$$\underbrace{1, \ldots, 1}_{u_1^k - 3}, u_4^k, 0, \underbrace{1, \ldots, 1}_{u_4^k - 3}, u_3^k$$

to the lower side.

Since we don't have any further nodes left, we finish with a '1' and a '0' on the upper side.

Now we don't have any nodes left and therefore we are done.

6.5.2 In Need of a Wedge-Block

Again S is a sequence of nonnegative integers fulfilling the basic constraints, this time with $\#0 \ge 5$, $\#2 = 0$ and $\#3^+ \overset{2}{\not\equiv} \#0$. Theorem 6.19 shows that for every degree sequence of this form we either have $\#4 > 0$ or $\#1 \ge 3$.

Again we calculate the number of blocks of inner triangles and the number of nodes on the big block. This time $w = 1$ because $\#3^+ \overset{2}{\not\equiv} \#0$ forces us to add a wedge-block.

$$\begin{aligned}
k &= \#3^+ \\
b &= \frac{\#3^+ - \#0 - 1}{2} + 1 \\
\#bB &= \#3^+ - 4(b-1) \\
&= \#3^+ - 2\#3^+ + 2\#0 + 2 \\
&= 2\#0 - \#3^+ + 2
\end{aligned}$$

As before $B = \{b_1, \ldots, b_{\#3^+}\}$ being the multiset of the degrees of the big nodes in decreasing order. The subset bB needs to be adapted to the additional constraint. We again have the subset $bB = \{v_1, \ldots, v_{\#bB}\}$ but this time it's not always filled with the biggest degrees. If $\#4 > 0$ we set $v_1 = 4$, remove one '4' from B and set $v_i = b_{i-1}$ for $2 \leq i \leq \#bB$. If $\#4 = 0$ we simply set $v_i = b_i$ for $1 \leq i \leq \#bB$. In both cases the remaining big nodes are stored into the subset sB which we again partition into subsets of 4 nodes: $sB = \{\{u_1^1, u_2^1, u_3^1, u_4^1\}, \ldots, \{u_1^{(b-1)}, u_2^{(b-1)}, u_3^{(b-1)}, u_4^{(b-1)}\}\}$. Again, every time we add a node to the ordered degree sequence, we remove it from bB.

The ordered degree sequence again starts with a '0' on the upper side and a '1' on the lower side. Then we build the first part of the wedge-block consisting of 2 inner triangles by adding

$$v_{\#bB}, \underbrace{1, \ldots, 1}_{v_{(\#bB-1)}-3}, 0, v_{(\#bB-1)}$$

to the upper side and

$$\underbrace{1, \ldots, 1}_{v_{\#bB}-3}, v_{(\#bB-2)}, 0, \underbrace{1, \ldots, 1}_{v_{(\#bB-2)}-3}, v_1$$

to the lower side.

If $\#bB = 6$, meaning we have exactly 3 inner triangles on the big block, we add

$$\underbrace{1, \ldots, 1}_{v_1-4}, v_2$$

to the upper side and

$$\underbrace{1, \ldots, 1}_{v_3-2}, 0, v_3, \underbrace{1, \ldots, 1}_{v_2-3}$$

to the lower side. In this case the wedge-block is done.

Otherwise set $l = max(v_1 - (\#bB - 2), 0)$ and add l '1's to the upper side (these are the '1's within the wedge of the wedge-block). If $v_1 > 4$ we add

$$v_{(\#bB-3)}, 0, \underbrace{1, \ldots, 1}_{v_{(\#bB-3)}-2}, v_{(\#bB-4)}, 0, \underbrace{1, \ldots, 1}_{v_{(\#bB-4)}-3}, \ldots, v_{(\#bB-4)}, 0, \underbrace{1, \ldots, 1}_{v_{(\#bB-v_1+l+2)}-3}$$

to the upper side and then add the internal zigzag of the right part of the block as before in the no-wedge case, but this time we start at the upper side.

Until only 1 node remains in bB we do the following:

Let m be the highest remaining degree in bB. We add m and a '0' to the upper side, then we add the $(m - 4)$ smallest remaining nodes in bB to the lower side including their wedge with the necessary '1's and a '0' for each one. If we don't have enough nodes left, we fill the remaining slots by adding '1's to the lower side. Then the upper and lower side switch places (that is the biggest remaining node is now placed on the opposite side of the previous). We iterate this till only 1 node is left in bB.

Now we only need to add the last node to the big block. Because of the orientation in the wedge-case the last node is always added to the upper side. Let l be the degree of the last remaining node. We add l to the upper side and $(l - 3)$ '1's to the lower side.

So the big block is finished and we build the small blocks as already shown before:

For every $1 \leq i \leq b - 1$ we add

$$u_1^k, \underbrace{1, \ldots, 1}_{u_2^k-3}, 0, u_2^k, \underbrace{1, \ldots, 1}_{u_3^k-3}$$

to the upper side and

$$\underbrace{1,\dots,1}_{u_1^k-3}, u_4^k, 0, \underbrace{1,\dots,1}_{u_4^k-3}, u_3^k$$

to the lower side.

Since we don't have any further nodes left, we finish with a '1' and a '0' on the upper side. Now we don't have any nodes left and therefore we are done.

7 At Least as Many Big Nodes as Zeros

Now we want to extend the result from Theorem 6.19 to a more general case.

If we look at an arbitrary triangulation with isolated '1's it's easy to see that we can add an arbitrary amount of '2's to the triangulation by inserting them at one of the isolated '1's. The other way around we can easily remove such isolated '2's from a triangulation. If we get no '1's things may get more difficult.

Looking at the basic constraints we see that only the constraints $\#1+\#2 \geq 2$ and $\#0 \leq \frac{n-\#1}{2}$ may become invalid when we remove a '2'. To ensure the validity of $\#0 \leq \frac{n-\#1}{2}$ even after removing up to all '2's, we claim $\#3^+ \geq \#0$. This constraint should now apply to all sequences in this chapter.

7.1 At Least one '1' ($\#1 > 0$)

In the case $\#2 = 0$ the only additional constraint was in the case $\#3^+ \overset{2}{\not\equiv} \#0$. Since the case $\#2 = 0$ is complete, we will now consider only sequences with $\#2 > 0$. If $\#3^+ \overset{2}{\not\equiv} \#0$ we simply "convert" a '2' to a '3' by adding an additional '1' to the sequence. More formally:

Let S be a sequence of nonnegative integers with $\#3^+ \overset{2}{\not\equiv} \#0$ and $\#2 > 0$. We construct $\overline{S}$ as follows:

$$\begin{aligned}
\overline{\#i} &= \#i \quad \forall i \notin \{1,2,3\} \\
\overline{\#1} &= \#1 + 1 \\
\overline{\#2} &= \#2 - 1 \\
\overline{\#3} &= \#3 + 1
\end{aligned}$$

So $\overline{\#3^+} = \#3^+ + 1 \overset{2}{\equiv} \#0 = \overline{\#0}$. If $\#1 \geq 1$ we can always transform a triangulation for $\overline{S}$ to a triangulation of S by removing the added '1' from the additional '3'.

Obviously $\overline{S}$ fulfills the basic constraints as soon as S fulfills the basic constraints. The following lemma uses this idea to provide full information on the case $\#1 \geq 1$.

Lemma 7.1
Every sequence with $\#2 > 1$ fulfilling the basic constraints, $\#3^+ \geq \#0$ and $\#1 \geq 1$ is a valid degree sequence.

Proof. We prove this by using the information from Theorem 6.19. Therefor we need to convert the sequence to fit into the constraints of the theorem and then convert the resulting triangulation.
Let S be our sequence.

If $\#1 = 1$ we convert S to $\overline{S}$ by adding an additional '1' and transforming a '2' to a '3':

$$\begin{aligned}\overline{\#i} &= \#i \quad \forall i \notin \{1,2,3\}\\ \overline{\#1} &= \#1 + 1\\ \overline{\#2} &= \#2 - 1\\ \overline{\#3} &= \#3 + 1\end{aligned}$$

Claim: $\overline{S}$ fulfills the basic constraints as soon as S fulfills the basic constraints.

Proof of the Claim. We first look at the constraint for the inner diagonals:

$$\begin{aligned}\sum_{i\geq 0} i\overline{\#i} &= 2(\overline{n} - 3)\\ \sum_{i\geq 0} i\#i + 1 - 2 + 3 &= 2((n+1) - 3)\\ \sum_{i\geq 0} i\#i + 2 &= 2(n-3) + 2\end{aligned}$$

Now we check $\overline{\#0} \leq \frac{\overline{n}-\overline{\#1}}{2}$:

$$\overline{\#0} = \#0 \leq \frac{n - \#1}{2} = \frac{\overline{n} - 1 - (\overline{\#1} - 1)}{2} = \frac{\overline{n} - \overline{\#1}}{2}$$

$\overline{\#1} + \overline{\#2} \geq 2$ is obviously fulfilled. □

If$\#1 > 1$ we set $\overline{S} = S$. Therefore we know that $\overline{\#1} \geq 2$ and $\overline{\#2} \geq 1$.

If $\overline{\#3^+} \overset{2}{\not\equiv} \overline{\#0}$ we need to transform $\overline{S}$ to $\widetilde{S}$ by converting another '2':

$$\begin{aligned}\widetilde{\#i} &= \overline{\#i} \quad \forall i \notin \{1,2,3\}\\ \widetilde{\#1} &= \overline{\#1} + 1\\ \widetilde{\#2} &= \overline{\#2} - 1\\ \widetilde{\#3} &= \overline{\#3} + 1\end{aligned}$$

As before $\widetilde{S}$ fulfills the basic constraints as soon as $\overline{S}$ fulfills them (and therefore as soon as S fulfills them). As the transformation leaves the '0's untouched but adds a big node, we now have $\widetilde{\#3^+} \overset{2}{\equiv} \widetilde{\#0}$. If $\overline{\#3^+} \overset{2}{\equiv} \overline{\#0}$ we set $\widetilde{S} = \overline{S}$.

So now we have a sequence $\widetilde{S}$ fulfilling the basic constraints, $\widetilde{\#1} \geq 2$ and $\widetilde{\#3^+} \overset{2}{\equiv} \widetilde{\#0}$. Additionally we know that S fulfilled $\#3^+ \geq \#0$ and as the transformations only add big nodes but don't change $\#0$ we still have $\widetilde{\#3^+} \geq \widetilde{\#0}$. This means we can remove all remaining '2's and receive a sequence $\widehat{S}$, which fulfills the basic constraints and $\widehat{\#2} = 0$.

Finally we got $\widehat{S}$ fulfilling the basic constraints, $\widehat{\#2} = 0$ and $\widehat{\#3^+} \overset{2}{\equiv} \widehat{\#0}$. Therefore Theorem 6.19 shows that $\widehat{S}$ is a valid degree sequence.

If $\widehat{\#1} \leq \#1 + 1$ we append the $\widetilde{\#2}$ '2's at the leftmost isolated '1'. Otherwise we increased the number of big nodes (which is the number of nodes on inner triangles) by 2. If we look at (3.6) we see:

$$b = \frac{\#3^{+} + 2 - \#0}{2} + 1 \underset{(\#3^{+} \geq \#0)}{\geq} 2$$

Therefore we can always insert the additional '2's between the block of inner triangles. This was the transformation back to $\widetilde{S}$.

Now we remove $\widetilde{\#1} - \#1$ isolated '1's which are connected to ears of blocks of inner triangles. If the ear on the big block was a node with degree greater than 3, we move the excessive '1's to the wedge of an other '3' somewhere in the triangulation (such a '3' exists because of the construction). Therefore we just transformed the triangulation so that it has the degree sequence S. $\square$

The case $\#2 = 1$ is a bit more complicated. If $\#1 \geq 2$, or $\#1 = 1$ and $\#2^{+} \overset{2}{\equiv} \#0$ the transformation of the previous proof still works. But if $\#1 = 1$ and $\#2^{+} \overset{2}{\not\equiv} \#0$ we would need another '2' to prevent the additional constraint from Theorem 6.19. Since we won't be able to totally avoid this constraint, we provide a more general version:

Lemma 7.2
There exists no triangulation with $\#2^{+} \overset{2}{\not\equiv} \#0$, $\#4 = 0$ and $\#1 + \#2 = 2$.

Proof. Let's try to build such a triangulation.

We know that every triangulation has at least 2 nodes which are either isolated '1's or has degree 2 with two adjacent '0's. Because of $\#1 + \#2 = 2$ this means that all '2's (if there are any) have to be on inner triangles.

Every separated big node would lead to an additional '1' (in the wedge of the big node), but all '1's (if there are any) are already used for the isolated '1's. Therefore all big nodes have to be incident to an inner triangle.

Because of $\#2^{+} \overset{2}{\not\equiv} \#0$, (3.7) tells us that there has to be an odd number of wedge-blocks. Similar to the proof of Lemma 6.18 we now show that the combination of $\#4 = 0$ and a wedge-block leads to an additional node with degree 1 or 2:

First assume that there is one wedge in the block. The big node incident to the wedge, call it v, has degree > 4. If there are less than $deg(v)$ diagonals from inside the block incident to v, v either has connected '1's (which would be an additional '1' and therefore $\#1 + \#2 > 2$) or a diagonal incident to v connects the wedge-block with another block (as shown in Figure 7.1). The wedge-block already has two ears which are different to v, so the connection to another block again leads to an additional '1' or '2'.

Now assume that there are $deg(v)$ diagonals from inside the block incident to v. This means that on at least one part of the wedge-block there have to be more than 2 diagonals be incident to v. On this part, the node next to the wedge has either degree 2 (an additional '2') or leads to another block of inner triangles (additional '1' or '2' as before) or directly to an additional '1' (as shown in Figure 7.2).

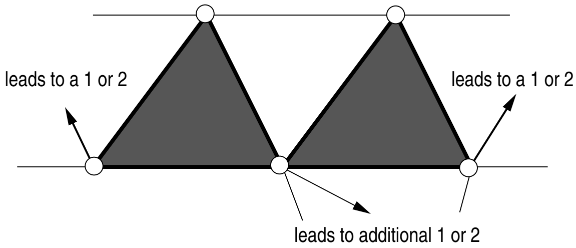


Figure 7.1: A wedge-block with $\#4 = 0$ and 4 diagonals from inside the block leads to an additional '1' or '2'

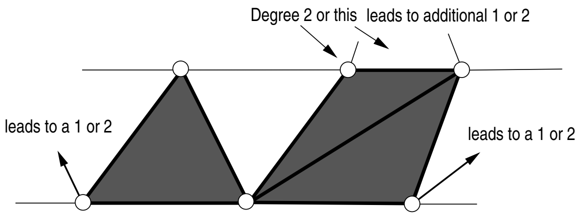


Figure 7.2: A wedge-block with $\#4 = 0$ and more than 4 diagonals from inside the block leads to an additional '1' or '2'

Now consider more than one wedge (in a block of inner triangles) incident to v. The part between two of the wedges directly leads to an additional '1' or '2' (as shown in Figure 7.3).

Therefore we can conclude that every triangulation with $\#3^+ \stackrel{2}{\not\equiv} \#0$ and $\#4 = 0$ fulfills $\#1 + \#2 > 2$ $\square$

Coming back to our situation before, we now know that $\#1 = 1$, $\#2 = 1$ and $\#2^+ \stackrel{2}{\not\equiv} \#0$ means that there have to be $\#4 > 0$ (otherwise there exists no such triangulation). Therefore we can convert the '2' into a '3' as above and the new sequence fits into the constraint of Theorem 6.19. After building the triangulation for the modified sequence, we simply delete the isolated '1' at the end with degree '3' and add the remaining '2's at the other isolated '1'.

7.2 No '1's ($\#1 = 0$)

Finally we look at the case where we have no '1' in the triangulation. Again we try to use the results and triangulations from the previous chapter, but unfortunately it won't work in every case.

Let's start easy:

Lemma 7.3
Every nonnegative integer sequence fulfilling the basic constraints with $\#3^+ \geq \#0$, $\#3^+ \stackrel{2}{\equiv} \#0$ and $\#1 = 0$ is a degree sequence.

Proof. Let S be such a sequence. As in Lemma 7.1 we simply convert two of the '2's (there are at least two because of (3.11)) to '3's and delete the remaining '2's. Let $\overline{S}$ be the sequence

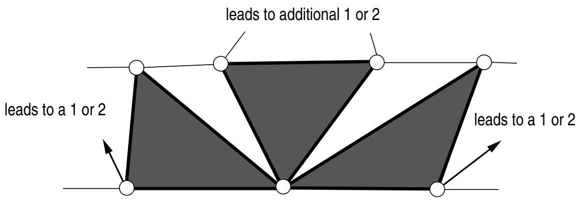


Figure 7.3: A wedge-block with $\#4 = 0$ and more than one wedge leads to an additional '1' or '2'

after the transformation. $\overline{S}$ still fulfills $\overline{\#3^+} \overset{2}{\equiv} \overline{\#0}$ and Theorem 6.19 provides a triangulation for this sequence.

Now we need to transform the triangulation so that it fits the original sequence. The two '3's are easily transformed back by removing the two isolated '1's from the triangulation as before. Now we need at least two blocks guaranteed, so we are able to add the remaining '2's between these blocks.

The number of blocks is calculated as follows.

$$\begin{aligned} \frac{\overline{\#3^+} - \overline{\#0}}{2} + 1 &= \frac{\#3^+ + 2 - \#0}{2} + 1 \\ &\geq \frac{\#3^+ - \#0}{2} + 2 \\ &\underset{(\#3^+ \geq \#0)}{\geq} 2 \end{aligned}$$

This means we always have at least two blocks and are able to add the additional '2's. □

But what happens if the parity constraint ($\#3^+ \overset{2}{\equiv} \#0$) is not fulfilled? Let's look at an easy subcase where we have at least one '4'.

Lemma 7.4

Every sequence of nonnegative integers fulfilling the basic constraints with $\#3^+ \geq \#0$, $\#3^+ \overset{2}{\not\equiv} \#0$, $\#1 = 0$ and $\#4 > 0$ is a degree sequence.

Proof. Let S be such a sequence. As before, we convert the guaranteed two '2's into '3's and remove the remaining '2's receiving $\overline{S}$. Because of $\#4 > 0$ $\overline{S}$ fits into the constraints of Theorem 6.19 which provides a triangulation for $\overline{S}$. Again $\#3^+ \geq \#0$ guarantees at least two blocks and we are able to transform the triangulation to fit S. □

Now let's get to the more difficult part, the case $\#3^+ \overset{2}{\not\equiv} \#0$ and $\#4 = 0$. Lemma 7.2 provides at least three '2's for this case, but if we convert only two into '3's and delete the remaining '2's, Theorem 6.19 claims three '1's (when we have only two). If we convert three '2's into '3's we would fit into the constraints of the theorem, but because of $\overline{\#3^+} \overset{2}{\equiv} \overline{\#0}$ we would not be able to convert the triangulation back to fit the original sequence. Therefore we need to present a new triangulation for this case.

Lemma 7.5
Every sequence of nonnegative integers fulfilling the basic constraints, $\#3^+ \geq \#0$, $\#3^+ \stackrel{2}{\not\equiv} \#0$, $\#1 = 0$ and $\#2 \geq 3$ is a degree sequence.

Proof. Let S be such a sequence. The triangulation we are building is similar to the case $(\#2 = 0) \wedge \left(\#3^+ \stackrel{2}{\equiv} \#0\right)$. The only difference will be that the rightmost block is a separated inner triangle with two incident '2's and the big block starts with a '2'. Figure 7.4 shows the triangulation we want to build, but let's look at it in detail.

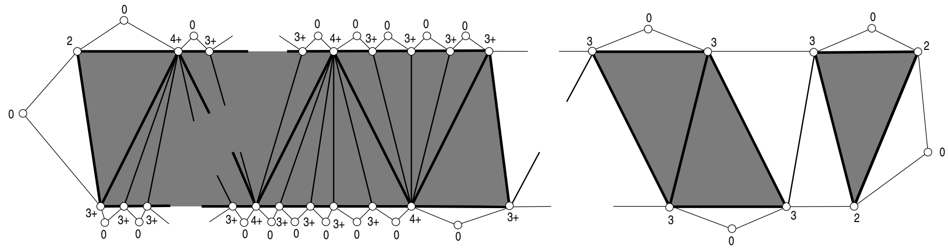


Figure 7.4: Triangulation for the case $2(\#0-2) \geq \#3^+ \geq \#0$, $\#3^+ \stackrel{2}{\not\equiv} \#0$ and $\#1 = 0$

As in the mentioned case we put all big nodes on inner triangles. Additionally we put the three guaranteed '2's on inner triangles. So the number of nodes on inner triangles equals $\#3^+ + 3$. Because of $\#3^+ + 3 \stackrel{2}{\equiv} \#0$ we don't need a wedge-block and therefore don't use it. Again b is the number of blocks and $\#bB$ is the number of nodes incident to the big block. $\#3 \geq \#0$ in combination with $\#3 \stackrel{2}{\not\equiv} \#0$ guarantees $\#3^+ \geq \#0 + 1$.
Let's recall some formulas:

$$\begin{aligned} b &= \frac{\#3^+ + 3 - \#0}{2} + 1 \\ &= \frac{\#3^+ - (\#0+1)}{2} + 3 \\ &\geq 3 \\ \#bB &= \#3^+ + 3 - 4(b-2) - 3 \\ &= \#3^+ + 3 - 2\#3^+ - 2 + 2\#0 - 3 \\ &= 2\#0 - \#3^+ - 2 \end{aligned}$$

So we know we have at least 3 blocks. This is good because later it will be easy to add the remaining '2's between them. Because of $\#3^+ \leq 2(\#0-2)$ (otherwise this case is already covered in Theorem 5.19) the number of nodes on the big block is at least 2.
Now we look closer at the big block. The internal structure of the block is a zigzag between big nodes with degree ≥ 4. Let $bB = \{b_1, \ldots, b_{\#bB}\}$ be the set with the nodes for the big block, $b_1 = 2$ and $b_2, \ldots, b_{\#bB}$ the biggest nodes in descending order. The big block starts with b_1 as ear of the block. Then the internal zigzag starts with $b_{\#bB}$ as "isolated '1'". The first big node is b_2 whose "wedge" (in the internal zigzag) is filled with the $deg(b_2) - 4$ smallest remaining nodes from bB. The next in the zigzag (now on the lower side) is b_3 filled with the $deg(b_3) - 4$ smallest remaining nodes from bB. We iterate this till we finish with an "isolated '1'" in the internal zigzag.

We now need to show that we have the right amount of nodes with degree ≥ 4 for the zigzag. This means the number of nodes with degree 3 on the big block must be equal $\sum_{i\geq 4}(i-4)\#i+3$ (don't forget the '2' at the beginning). If the number of nodes with degree 3 is exactly that amount every node on the zigzag is filled and we don't need any '1's. If it would be smaller, some of the nodes on the zigzag would need additional '1's and maybe some "filling '3's" are nodes with degree greater than 3 which again needs additional '1's which we don't have. Let's calculate the number of '3's on the big block (nodes with degree ≥ 4 are on the big block). Keep in mind that the last triangle contains only one '3' and $\#1 = 0$.

$$\begin{aligned}
\#\{\text{'3's on the big block}\} &= \#3 - 4(b-2) - 1 \\
&= \#3 - 2\#3^+ - 2 + 2\#0 - 1 \\
&= 2\#0 - 2(n - \#0 - \#2) + \#3 - 3 \\
&= 2\#0 - 2n + 2\#0 + 2\#2 + \#3 - 3 \\
&= 4\#0 + 2\#2 + \#3 - 2n - 3
\end{aligned}$$

Now we reformulate the upper bound for the allowed number of '3's:

$$\begin{aligned}
\sum_{i\geq 4}(i-4)\#i + 3 &= \sum_{i\geq 4} i\#i - 4(n - \#0 - \#2 - \#3) + 3 \\
&= 2(n-3) - 2\#2 - 3\#3 - 4n + 4\#0 + 4\#2 + 4\#3 + 3 \\
&= 4\#0 + 2\#2 + \#3 - 2n - 3
\end{aligned}$$

So our constraint is:

$$\begin{aligned}
4\#0 + 2\#2 + \#3 - 2n - 3 &\stackrel{?}{=} 4\#0 + 2\#2 + \#3 - 2n - 3 \\
0 &= 0
\end{aligned}$$

This means that the amount of '3's is always exactly the amount needed to fill the wedges in the internal zigzag of the big block. Because of $\#1 = 0$ we don't have '1's to fill any remaining spot and this is the only way the triangulation would work.

So we conclude that our construction works for every sequence with $\#3^+ \geq \#0$, $\#3^+ \stackrel{2}{\not\equiv} \#0$ and $\#1 = 0$. $\square$

Theorem 7.6
A sequence S with $\#3^+ \geq \#0$ is a degree sequence if and only if it fulfills the basic constraints and

$$\left[\left(\left(\#2^+ \stackrel{2}{\not\equiv} \#0\right) \wedge (\#4 = 0)\right) \Rightarrow (\#1 + \#2 \geq 3)\right]$$

Proof. For any given triangulation Lemma 7.2 shows that the additional constraint is always fulfilled.
If $\#3^+ > 2(\#0-2)$ Theorem 5.19 shows that there always exist a triangulation. The remaining part splits up into several cases:

Case 1 $\#2 = 0$: Theorem 6.19 already proves that S is a degree sequence if and only if it fulfills the basic constraints and $\left[\left(\left(\#3^+ \stackrel{2}{\not\equiv} \#0\right) \wedge (\#4 = 0)\right) \Rightarrow (\#1 \geq 3)\right]$

Case 2 $\#1 = 0$: Lemma 7.3, Lemma 7.4 and Lemma 7.5 prove that there always exists a triangulation with such a degree sequence.

Case 3 $(\#1 \geq 1) \wedge (\#2 > 1)$: In this case Lemma 7.1 shows that S is a degree sequence if and only if it fulfills the basic constraints.

Case 4 $(\#1 \geq 1) \wedge (\#2 = 1)$: Lemma 7.2 shows that S can't be a degree sequence if $\#2^+ \overset{2}{\not\equiv} \#0$, $\#4 = 0$ and $\#1 + \#2 = 2$.

If $\#1 \geq 2$ we can easily convert the '2' into a '3' by adding an additional '1'. Now we are in the case $(\#2 = 0) \wedge (\#1 \geq 3)$ which fits into the constraints of Theorem 6.19. S is a degree sequence as soon as it fulfills the basic constraints.

If $\#1 = 1$ and $\#2^+ \overset{2}{\equiv} \#0$ we again convert the '2' to a '3'. And again $\overline{S}$ fits into Theorem 6.19 and is a degree sequence. By removing an isolated '1' which is incident to a '3', we receive a triangulation with degree sequence S.

In the case $\#2^+ \overset{2}{\not\equiv} \#0$ we assume $\#4 > 0$ (because otherwise we know that there can't be a triangulation). Once again we convert the '2' to a '3' and build the triangulation from Theorem 6.19 (which is possible because of $\#4 > 0$). Then we convert back by removing an isolated '1' and receive a triangulation with degree sequence S.

□

7.3 Construction of the Ordered Degree Sequence

The proof of Lemma 7.1 motivates an easy way to build the ordered degree sequence. First we build the ordered degree sequence for the transformed sequence as described for the case $\#2 = 0$.

If we have to transform a '2' into a '3', the last big node in the transformed sequence, and therefore the node on the rightmost ear, has degree 3 and a connected '1' (which is an isolated '1'). We remove this '1' and get back to the original '2'. If we have to transform two '2's, we just have to ensure, that the second '3' is on the other ear (the leftmost big node) which again can easily be transformed back.

The last question is how to deal with possible additional '2's. Because of $\#3^+ \geq \#0$ we either have an isolated '1' (which would be the leftmost node in our construction) or we have more than one block after the transformation.

In the first case we easily add the additional '2's at the isolated '1'. More precisely, l being the amount of additional '2's, we insert $\lfloor \frac{l}{2} \rfloor$ on the upper side directly after the first '0', and $\lceil \frac{l}{2} \rceil$ on the lower side directly after the first '1'. If $l \overset{2}{\equiv} 1$ we then switch the first '0' on the upper side with the first '1' on the lower side.

If $\#1 = 0$ we insert this zigzag of '2's before the last block of inner triangles.

8 Conclusion

8.1 Summary

In this thesis we presented full information on sequences with "enough" big nodes, that is, for all sequences fulfilling $\#3^+ \geq \#0$.

We developed some basic constraints which have to be fulfilled by every degree sequence:

- $2 \leq \#0 \leq \left\lfloor \frac{\#0^+}{2} \right\rfloor$
- $\#0 \leq \#2^+$ (for $\#0 \geq 3$)
- $\#1 + \#2 \geq 2$
- $\sum_{i \geq 0} i\#i = 2(\#0^+ - 3)$

For the special cases with $\#0 \in \{2, 3, 4\}$ we showed necessary and sufficient criteria for a sequence to be a degree sequence. These criteria are summarized in Table 8.1.

The general case with $\#0 \geq 5$ was divided into sub cases distinguished by the relation between $\#3^+$ and $\#0$:

- $3(\#0 - 2) \leq \#3^+$
  The easy part, where all inner triangles could be separated and aligned in a row. Theorem 5.8 proves that in this case every sequence fulfilling the basic constraints is a degree sequence.
- $2(\#0 - 2) < \#3^+ < 3(\#0 - 2)$
  Things started to be tricky. There are not enough big nodes for separated triangles but we managed to present a triangulation with small blocks of inner triangles aligned in a row. Theorem 5.19 proves that in this case again every sequence fulfilling the basic constraints is a degree sequence.
- $\#0 \leq \#3^+ \leq 2(\#0 - 2)$
  The tough part. In this case we needed to put all big nodes on inner triangles and deal with the need of wedge-blocks. By examining the special case $\#2 = 0$ we slowly got a feeling how to solve this case. After developing the canonical triangulation for $\#2 = 0$ it was a short step to the generalization and finally Theorem 7.6 provides full information.

The essential (and only) additional constraint for the general case is, that every degree sequence has to fulfill

$$\left[\left(\left(\#2^+ \overset{2}{\not\equiv} \#0\right) \wedge [\#4 = 0]\right) \Rightarrow (\#1 + \#2 \geq 3)\right]$$

The best part of this constraint is that it has to be fulfilled for every degree sequence (by Lemma 7.2).
In the case $\#3^+ > 2(\#0 - 2)$ this constraint is already guaranteed by the basic constraints (more specific by (3.4)).

$\#0$	$\#3^+$	additional condition(s) to the basic constraints	Section
$= 2$		no further condition	4.1
$= 3$		no further condition	4.2
$= 4$	$= 1$	$\#2 = 4$ and $\#4 = 1$	4.3
		$\#2 \geq 4$ and $\#5^+ = 1$	
	$= 2$	$\#2 = 3 \Rightarrow \#4^+ \geq 1$	
	≥ 3	no further condition	
≥ 5	$> 2(\#0 - 2)$	no further condition	5
	$2(\#0 - 2) \geq \#3^+ \geq \#0$	$\left[\left(\left(\#2^+ \overset{2}{\not\equiv} \#0\right) \wedge [\#4 = 0]\right) \Rightarrow (\#1 + \#2 \geq 3)\right]$	7

Table 8.1: Known cases covered by this thesis. The columns are connected through logical 'and'

8.2 Outlook

From the basic constraint we know $\#0 \leq \#2^+$. So the only open problem regarding degree sequences of triangulations of convex point sets is the case $\#2^+ \geq \#0 > \#3^+$. In this case we are forced to put $(\#0 - \#3^+)$ '2's onto inner triangles even if we got more than two '1's. But every additional '2' on an inner triangle leads to an additional ear in the corresponding block of inner triangles. With more than two ears on one block we get inner inner triangles ("inner triangles" within a block of inner triangles) and the nice approach of the big block with an internal zigzag doesn't work anymore.

Even the approach to split some separated inner triangles from the block to put a '2' on each one doesn't work in general because it may use too many pseudo big nodes.

In [4] it is shown that every degree sequence, where all degrees are even, have to fulfill $\#0 \geq \frac{n}{3} + 1$. This means that in the case $\#1 = 0$ we may get an additional constraint. But in fact we don't know if this constraint has any real effect.

Therefore this part is left for the reader to solve. If you do so, don't hesitate to contact me.

List of Figures

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Glossary

$\#k^+$	the number of nodes with degree of at least k	12
$\overset{2}{\equiv}$	short form for congruence modulo 2	11
$\overset{2}{\not\equiv}$	short form for incongruence modulo 2	11
big node	a node with degree of at least 3	14
degree sequence	sequence of nonnegative integers representing the degrees of nodes in a graph	12
degree vector	vector of nonnegative integers representing the occurrences of degrees in the according degree sequence	13
dual tree	the tree with nodes for every triangle in the triangulation. Two nodes in the dual tree are connected iff the corresponding triangles share one edge	12
inner inner triangle	an inner triangle where all edges leads to another inner triangle	35
inner triangle	a triangle within the triangulation where all edges are inner diagonals	12
isolated '1'	a node with degree 1 which is adjacent to a node with degree 0	15
pseudo big node	a node with degree 2	14
wedge of a node	maximal adjacent list of '1's connected to the node	15
wedge-block	a block of inner triangles which can be separated into at least two parts connected only through one node	35
zigzag	a (part) triangulation where the dual tree is a path	15