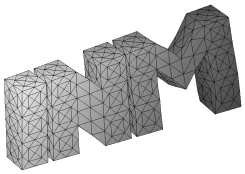

Master Thesis

A symmetric boundary element domain decomposition method for a free space transmission problem

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Preface

Originally, the implementation for numerical methods to solve various boundary value and free space transmission problems started in spring 2008 in course of a master project with my studying colleague Martin Puster. At this time, the goal was to calculate boundary element methods for the Laplace problem with the help of a software program. Therefore, the program, developed in the language C, started from a blank sheet of paper. After a year, the project was finished and I was ready to discuss and implement further techniques such as domain decomposition methods. The final goal of my master thesis was to deal with interface problems. All in all, the greatest effort concerning this work was to build the computational program.

During that whole period of my studies, Dr. Of and mainly Prof. Steinbach took a lot of time for explanations and discussions about the background theory and options for different implementation procedures. Both were reachable and available for a meeting at all times. At the beginning of my master thesis, Prof. Steinbach patiently introduced me to the subject.

So this is why I would like to thank Dr. Of for his time and openness.

Above all, I would like to express my appreciation to my advisor Prof. Steinbach for his abundant help, understanding and helpful suggestions.

My gratitude also goes to my study colleagues for countless mathematical discussions and particularly for a enjoyable period of life in Graz.

Finally, the most special thanks go to my girlfriend Barbara and to my parents and brothers, for their unconditional and lovely support during my years of study and final work.

Contents

Introduction

Many physical phenomena, procedures and problems of different natures can be described by partial differential equations and formulated as boundary value problems. Unfortunately, there often is no possibility to calculate the solutions in an explicit way. Therefore, numerical methods have been developed to obtain approximate solutions. The most common ones are the Finite Element Methods (FEM), see for example [2] and [16], and the Boundary Element Methods (BEM), see [32] and [27]. The latter has been developed since the 1950s, and has become more popular with the invention of electronic computers. The full emergence of boundary element methods was then in the late 1970s [4]. They play an important role to solve problems in such diverse topics as solid and fluid mechanics, electromagnetics, fluid dynamics, heat transfer and elastostatics, to name just a few. In this work we concentrate on the Laplace and Poisson equation as a model problem, where applications can be found in subjects like perfect fluids, electromagnetism and gravitation theory.
In FEM, the entire domain under consideration has to be discretised. The main advantage of BEM is that only a discretisation of the surface of the domain is needed, instead of a volume discretisation, so the element mesh is simpler. Furthermore, we do not necessarily have to reorder the mesh for moving structures. Other advantages are in dealing with exterior boundary value problems in unbounded domains, which is an important topic in this work, and there are also advantages in the high accuracy of the approximate solutions. Generally, once the problem has been solved with BEM, we are able to compute the solution at any point of the considered domain. Last but not least, with BEM we calculate the complete Cauchy data on the boundary explicitly. On the other way round, there are some disadvantages. Since we have to know the fundamental solution of the partial differential equation for Green's third formula, the utilisation of BEM is mainly restricted to partial differential equations with piecewise constant coefficients. Moreover, as far as boundary integral operators are concerned, we have to deal with singular integrals. One of the greatest drawbacks compared to FEM are the fully populated matrices, because of which the memory requirements and computational costs grow at least quadratically in the number of boundary elements when implementing BEM. This is why fast BEM have been developed. Most of these techniques deal with a more efficient evaluation of the kernels of the appearing potential operators. To name some of possible faster procedures, there are multipole techniques [23], and there is panel clustering [12] as well as the Adaptive Cross Approximation [26].
In what follows, we want to give a short overview about the theory of BEM. It is known for a long time that boundary value problems for elliptic partial differential equations can be formulated in terms of boundary integral equations. After discretising the derived vari-

ational formulations, the Cauchy data can be evaluated numerically. Afterwards, with the representation formula basically derived from Green's first and second formula, the solution of the original problem can be determined. This procedure is called direct approach, see for example [5]. On the contrary, there exist indirect approaches. Since the single or double layer potential operator solve the partial differential equations under consideration, they can be used as a solution too. Since the calculated density functions have, in general, no physical meaning, the approach is called indirect.
Undoubtedly, a proper stability and error analysis is important. On the one hand there are the Galerkin formulations, on the other hand we may use collocation methods. Concerning the former technique, the theoretical study is well established. Otherwise, the engineering community often apply collocation procedures because of an easier and more practical implementation. But, the theory about stability and error estimates is only known for two-dimensional models [26]. Besides, the rate of convergence for Galerkin schemes is often better. For some general informations about BEM and FEM see for example [32], [34] and [20].
Another important topic, which can be formulated with BEM, are domain decomposition methods (DDM), for general concepts see for example [25], as well as [3], [15], [18], [17], [13] and [19]. They were originally developed to solve boundary value problems in complicated domains. Here we mention the famous alternating Schwarz method [28] as the first work on DDM. Thereafter, additive and multiplicative Schwarz methods have been developed. Some information about these techniques can be taken from [10] and [9]. Since the idea is to decompose a domain into substructures, DDM are well suited for the coupling of different discretisation schemes as FEM and BEM, see for example [7], [18] and [29]. Thus, there is a possibility to use the former method for an interior problem, and the latter technique for an exterior, unbounded domain. Moreover, considering software implementations of numerical procedures, DDM are a relatively simple possibility for parallel solutions, for instance each CPU solves a partial differential equation in one subdomain. What is more, we can compute different, non-conformal meshes, or use different trial spaces, for each subdomain. Besides, DDM is also suitable for shapes which are combined with different materials, so each subdomain covers different material parameters. Furthermore, the decomposition of the domain is mostly due to the geometrical form of the domain under consideration, or due to properties of the partial differential operator. Generally spoken, the main idea of DDM is to reduce a complicated, global domain to simpler subdomains. In this work we concentrate on geometry-based DDM, where a global boundary value problem is reduced to local subproblems, linked via transmission conditions on the interface. Since the complete Cauchy data have to be found on the skeleton, we formulate variational methods of these coupling conditions. Therefore we use local Dirichlet-to-Neumann maps, derived from the boundary integral equations. Applying standard discretisation techniques such as the collocation scheme and the Galerkin method, we obtain local approximate solutions and at the end the final global solution of the original problem.
The following chapters are organised as follows:
In Chapter 1, at first we state some preliminary examples for boundary value problems for the two-dimensional case. After giving a motivation for different types of boundary value

problems, we have to introduce suitable function spaces. Moreover, we provide an overview of general variational formulations, which we apply to BEM afterwards. We introduce the fundamental solution for exterior and interior domains and deal with diverse appearing boundary integral operators. At the end we state the boundary integral equations of the exterior and interior boundary value problems. Chapter 2 describes the different boundary value problems treated in the course of this work, as well as an error and stability analysis for each of them. Firstly, there are interior and exterior boundary value problems, and therefore various solving techniques. Secondly, we discuss an interface problem, where we introduce the transmission conditions which we need for domain decomposition methods too. Chapter 3 is concerned with the concept of BEM. At the beginning, we formulate discretisation schemes of variational methods, for which we need a suitable boundary discretisation and particularly two different trial spaces. Afterwards we solve the problems of Chapter 2 with BEM. At last, in Chapter 4 we show a range of numerical examples for the solution of boundary value and free space transmission problems as well as for domain decomposition methods via BEM, developed with the C programming language. This work ends with a short outlook about continuative themes, mainly concerning faster solution methods.

1 Boundary Integral Equations

At the beginning, in Section 1.1 we want to introduce different examples of partial differential equations in general. Therefore, we also need some definitions, which are important for the formulation of the boundary value and free space problems, as well as for domain decomposition methods. Besides, we note that this work describes problems and their solution procedures for the two-dimensional case only. However, this approach can also be applied to related boundary value problems in three space dimensions.

1.1 Motivation

We state a self-adjoint linear partial differential operator of second order in two space dimensions as

$$(Lu)(x) := -\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial}{\partial x_j}\Big[a_{ji}(x)\frac{\partial}{\partial x_i}u(x)\Big] \quad \text{for } x \in \Omega \subset \mathbb{R}^2, \tag{1.1}$$

with a scalar, real function $u(x)$ mapping from $\mathbb{R}^2$ to $\mathbb{R}$, and coefficients $a_{ji}(x) = a_{ij}(x)$, which define a symmetric matrix

$$A(x) = (a_{ij}(x))_{i,j=1}^{2}. \tag{1.2}$$

The differential operator L defined in (1.1) is called *uniformly elliptic* in the domain Ω, if the two eigenvalues $\lambda_k(x)$ of the matrix $A(x)$ in (1.2) are greater than a lower positive limit, i.e.

$$\lambda_k(x) \geq \lambda_0 > 0 \quad \text{for } k = 1, 2 \text{ and for all } x \in \Omega.$$

In this work we mainly consider the *Laplace equation*

$$-\Delta u(x) = -\sum_{i=1}^{2}\frac{\partial^2}{\partial x_i^2}u(x) = 0 \quad \text{for } x \in \Omega \subset \mathbb{R}^2, \tag{1.3}$$

with the uniformly elliptic Laplace operator $-\Delta$, and a bounded, multiply or simply connected domain Ω with Lipschitz boundary $\Gamma = \partial\Omega$. Another important differential equation is the *Poisson equation*

$$-\Delta u(x) = f(x) \quad \text{for } x \in \Omega \subset \mathbb{R}^2, \tag{1.4}$$

with a function $f(x) \not\equiv 0$ on the right hand side. To handle given data on the boundary, we need to introduce some trace operators concerning an interior domain Ω and an exterior domain $\Omega^c = \mathbb{R}^2 \setminus \overline{\Omega}$.

Definition 1.1 (Trace operator)**.** *Let Ω be a bounded domain with the boundary $\Gamma = \partial\Omega$. Then the interior and exterior trace operator are defined as*

$$\gamma_0^{int} u(x) = \lim_{\tilde{x} \ni \Omega \to x \in \Gamma} u(\tilde{x}) \quad \text{for } x \in \Gamma \tag{1.5}$$

and

$$\gamma_0^{ext} u(x) = \lim_{\tilde{x} \ni \Omega^c \to x \in \Gamma} u(\tilde{x}) \quad \text{for } x \in \Gamma, \tag{1.6}$$

respectively.

Definition 1.2 (Conormal derivative)**.** *Let Ω be a bounded domain with the boundary $\Gamma = \partial\Omega$. Then the interior and exterior conormal derivative are defined as*

$$\gamma_1^{int} u(x) = \lim_{\tilde{x} \ni \Omega \to x \in \Gamma} (n_x, \nabla_{\tilde{x}} u(\tilde{x})) \quad \text{for almost all } x \in \Gamma \tag{1.7}$$

and

$$\gamma_1^{ext} u(x) = \lim_{\tilde{x} \ni \Omega^c \to x \in \Gamma} (n_x, \nabla_{\tilde{x}} u(\tilde{x})) \quad \text{for almost all } x \in \Gamma, \tag{1.8}$$

respectively, with n_x as the outer normal vector.

Remark 1.1. *Per convention, the normal vector n_x is always directed out of the domain Ω. Hence, the normal vectors of the exterior and interior conormal derivative have the same direction.*

Using the above introduced definitions, we can finally present the *Dirichlet boundary value problem*

$$\begin{aligned} -\Delta u(x) &= f(x) \quad \text{for } x \in \Omega \subset \mathbb{R}^2, \\ \gamma_0^{int} u(x) &= g(x) \quad \text{for } x \in \Gamma. \end{aligned} \tag{1.9}$$

Besides the problems on interior domains, there are also boundary value problems on the exterior domain $\Omega^c = \mathbb{R}^2 \setminus \overline{\Omega}$, e.g.

$$\begin{aligned} -\Delta u(x) &= 0 \quad\;\; \text{for } x \in \Omega^c, \\ \gamma_0^{ext} u(x) &= g(x) \quad \text{for } x \in \Gamma = \partial\Omega, \end{aligned} \tag{1.10}$$

where u has to satisfy a radiation condition at infinity,

$$\exists\, a \in \mathbb{R} : u(x) = a \log|x| + o(1) \quad \text{for } |x| \to \infty.$$

Finally, we introduce a combination of an interior and an exterior boundary value problem, the so-called *interface problem*

$$\begin{aligned} -\Delta u_1(x) &= f(x) \quad \text{for } x \in \Omega, \\ -\Delta u_2(x) &= 0 \quad\;\; \text{for } x \in \Omega^c. \end{aligned}$$

To solve an interface problem, the first step is to formulate suitable transmission conditions, which will be done in Section 2.3.
Last but not least, as far as *domain decomposition methods* [13] are concerned, there is the possibility to partition a given domain Ω into subdomains Ω_i,

$$\overline{\Omega} = \bigcup_{i=1}^{p} \overline{\Omega}_i.$$

As a consequence, the Ω_i have local subdomain boundaries Γ_i. We deal with boundary value problems on each subdomain. Since we obtain local coupling boundaries

$$\Gamma_{ij} = \Gamma_i \cap \Gamma_j$$

on which we can formulate associated transmission conditions, the local problems are linked and will be solved via these conditions. At the end we get a solution for the original problem on the global domain Ω.

1.2 Function Spaces

In theanalytical and numerical theory about variational formulations, integral operators and boundary element methods, we need appropriate function spaces and dedicated norms, which we introduce in this section.
First of all, we start with some partial derivatives and their abbreviations. For a multi index $\alpha = (\alpha_1, \alpha_2)$, $\alpha_i \in \mathbb{N}_0$, we define

$$D^\alpha u(x) := \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} u(x).$$

What is more,

$$C^k(\Omega) := \{u, D^\alpha u \text{ exists and is continuous on } \Omega \text{ for } |\alpha| = \alpha_1 + \alpha_2 \leq k\}$$

for $k \in \mathbb{N}_0$. If $u \in C^k(\Omega)$ for all $k = 0, 1, \ldots$, we have $u \in C^\infty(\Omega)$, where we will need the closure of this function space afterwards for a definition more often used in this work. Moreover,

$$C_0^\infty(\Omega) := \{u \in C^\infty(\Omega) : \text{ supp } u = \overline{\{x \in \Omega : \ u(x) \neq 0\}} \subset \Omega\}$$

defines the space of infinite times continuously differentiable functions with compact support.

Definition 1.3. *The definition of the function space $L_2(\Omega)$ of square integrable functions is*

$$L_2(\Omega) := \{u : \int_\Omega |u(x)|^2 dx < \infty\}.$$

With the inner product

$$\langle u, v\rangle_{L_2(\Omega)} := \int_\Omega u(x)v(x)dx,$$

and therefore equipped with the L_2-norm

$$\|u\|_{L_2(\Omega)} := \left(\int_\Omega |u(x)|^2 dx\right)^{1/2},$$

$L_2(\Omega)$ is a Hilbert space.

Definition 1.4. *We define the norm*

$$\|u\|_{W_2^k(\Omega)} := \left(\sum_{|\alpha|\le k}\int_\Omega |D^\alpha u(x)|^2 dx\right)^{1/2} \tag{1.11}$$

$$= \left(\sum_{|\alpha|\le k}\|D^\alpha u\|^2_{L_2(\Omega)}\right)^{1/2} \quad \text{for } k\in\mathbb{N}_0. \tag{1.12}$$

For $0 < s \in \mathbb{R}\setminus\mathbb{N}_0$, we first separate the index s as $s = k+\kappa$, with $\kappa\in(0,1)$ and again $k\in\mathbb{N}_0$. Then the Sobolev-Slobodeckii norm is defined as

$$\|u\|_{W_2^s(\Omega)} := \left(\|u\|^2_{W_2^k(\Omega)} + |u|^2_{W_2^k(\Omega)}\right)^{1/2},$$

with

$$|u|^2_{W_2^k(\Omega)} = \sum_{|\alpha|=k}\int_\Omega\int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{2+2\kappa}}dxdy.$$

The corresponding Sobolev space $W_2^k(\Omega)$ is defined as follows:

Definition 1.5.

$$W_2^k(\Omega) := \{u\in L_2(\Omega) : D^\alpha u \in L_2(\Omega), |\alpha|\le k\}.$$

To go into more detail, for $0\le s$ the Sobolev space $W_2^s(\Omega)$ is the closure of $C^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W_2^s}$, so we have

$$W_2^s(\Omega) = \overline{C^\infty(\Omega)}^{\|\cdot\|_{W_2^s(\Omega)}}.$$

In other words

$$\forall u\in W_2^s(\Omega)\ \exists\{\varphi_j\}_{j\in\mathbb{N}}\subset C^\infty(\Omega):\ \lim_{j\to\infty}\|u-\varphi_j\|_{W_2^s(\Omega)} = 0.$$

To treat the missing case $s<0$ for the norm in $W_2^s(\Omega)$, we first need another definition:

Definition 1.6. *With respect to the $\|\cdot\|_{W_2^s(\Omega)}$-norm, the completion of $C_0^\infty(\Omega)$ implies the Sobolev space*

$$\overset{\circ}{W}{}_2^s(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W_2^s(\Omega)}} \quad \text{for } s > 0.$$

In what follows, we can define the Sobolev space $W_2^s(\Omega)$ for indices $s < 0$:

Definition 1.7. *For $s < 0$, the Sobolev space $W_2^s(\Omega)$ is the dual space of $\overset{\circ}{W}{}_2^{-s}(\Omega)$ with the norm*

$$\|f\|_{W_2^s(\Omega)} := \sup_{0 \neq v \in \overset{\circ}{W}{}_2^{-s}(\Omega)} \frac{|\langle f, v\rangle_\Omega|}{\|v\|_{W_2^{-s}(\Omega)}},$$

with respect to the duality pairing

$$\langle f, v\rangle_\Omega := \int_\Omega f(x)v(x)dx.$$

Correspondingly,

$$\overset{\circ}{W}{}_2^s(\Omega) = \big[W_2^{-s}(\Omega)\big]'.$$

Adopting the approach for example of [35], we define Sobolev spaces by distributions, which leads to the function space

$$H^s(\mathbb{R}^2) = W_2^s(\mathbb{R}^2) \quad \text{for all } s \in \mathbb{R}.$$

Definition 1.8. *For a bounded domain $\Omega \subset \mathbb{R}^2$ we have*

$$H^s(\Omega) := \Big\{ v = \tilde{v}_{|\Omega} : \ \tilde{v} \in H^s(\mathbb{R}^2) \Big\},$$

equipped with the norm

$$\|v\|_{H^s(\Omega)} := \inf_{\tilde{v} \in H^s(\mathbb{R}^2), \tilde{v}_{|\Omega} = v} \|\tilde{v}\|_{H^s(\mathbb{R}^2)}.$$

Theorem 1.9 ([35])**.** *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain and $s \geq 0$. Then*

$$W_2^s(\Omega) = H^s(\Omega).$$

More Sobolev spaces are defined by duality:

$$\widetilde{H}^s(\Omega) = \big[H^{-s}(\Omega)\big]', \qquad H^s(\Omega) = \big[\widetilde{H}^{-s}(\Omega)\big]', \quad \text{for all } s \in \mathbb{R}.$$

The next point we want to discuss is the restriction of the function space $H^s(\Omega)$ onto the boundary $\Gamma = \partial\Omega$.

Definition 1.10. *For $s = 0$ we define*

$$\|v\|_{L_2(\Gamma)} := \left(\int_\Gamma |v(x)|^2 ds_x \right)^{1/2}.$$

For $s \in (0,1)$ the Sobolev-Slobodeckii norm is defined as

$$\|v\|_{H^s(\Gamma)} := \left(\|v\|^2_{L_2(\Gamma)} + \int_\Gamma \int_\Gamma \frac{|v(x) - v(y)|^2}{|x-y|^{1+2s}} ds_x ds_y \right)^{1/2}.$$

As for the Sobolev spaces in the domain, the H^s-spaces on the boundary for negative indices are defined via duality:

Definition 1.11. *For $s < 0$ we define*

$$H^s(\Gamma) := \left[H^{-s}(\Gamma) \right]',$$

with the corresponding H^s-norm

$$\|v\|_{H^s(\Gamma)} := \sup_{0 \neq \tilde{v} \in H^{-s}(\Gamma)} \frac{\langle v, \tilde{v} \rangle_\Gamma}{\|\tilde{v}\|_{H^{-s}(\Gamma)}},$$

and with the duality pairing

$$\langle v, \tilde{v} \rangle_\Gamma = \int_\Gamma v(x) \tilde{v}(x) ds_x.$$

What is more, we define a Sobolev space for some open boundary part $\Gamma_0 \subset \Gamma = \partial\Omega$,

$$H^s(\Gamma_0) := \left\{ v = \tilde{v}_{|\Gamma_0} : \tilde{v} \in H^s(\Gamma) \right\},$$

for $s \geq 0$, equipped with the norm

$$\|v\|_{H^s(\Gamma_0)} = \inf_{\tilde{v} \in H^s(\Gamma) : \tilde{v}_{|\Gamma_0} = v} \|\tilde{v}\|_{H^s.(\Gamma)}$$

Eventually, there is one more Sobolev space which we will need later on for discretisations.

Definition 1.12. *For a piecewise smooth boundary*

$$\Gamma = \bigcup_{i=1}^N \overline{\tau}_i, \quad \tau_i \cap \tau_j = \emptyset \quad \text{for } i \neq j,$$

the function space

$$H^s_{pw}(\Gamma) := \left\{ v \in L_2(\Gamma) : \ v_{|\tau_i} \in H^s(\tau_i), i = 1, \ldots, N \right\}$$

defines a Sobolev space for $s \geq 0$, with the norm

$$\|v\|_{H^s_{pw}}(\Gamma) := \left(\sum_{i=1}^N \|v_{|\tau_i}\|^2_{H^s(\tau_i)} \right)^{1/2}.$$

So these are the main function spaces we will need among others to formulate variational formulations, for which we describe the general concept in the next section. Further theories and applications of Sobolev spaces can be found in [20] and [32].

1.3 Variational Methods

In this section we introduce the concept of variational methods in general, in order to describe the variational formulations for the Laplace and Poisson equations. For this purpose we need some assumptions of functional analysis:

X ... Hilbert space

Inner product $\langle \cdot, \cdot \rangle_X$

Induced norm $\| \cdot \|_X = \sqrt{\langle \cdot, \cdot \rangle_X}$

X' ... dual space of X

Duality pairing $\langle \cdot, \cdot \rangle : X' \times X \to \mathbb{R}$

It holds the representation

$$\|f\|_{X'} = \sup_{0 \neq v \in X} \frac{|\langle f, v \rangle|}{\|v\|_X} \quad \text{for all } f \in X'. \tag{1.13}$$

We also consider a bounded linear operator $A : X \to X'$, mapping from the introduced Hilbert space X to the dual space X', with

$$\|Av\|_{X'} \leq c_2^A \|v\|_X \quad \text{for all } v \in X.$$

Additionally, A is assumed to be self-adjoint, i.e.

$$\langle Au, v \rangle = \langle u, Av \rangle \quad \text{for all } u, v \in X.$$

Now let $u \in X$ be a solution of the *operator equation*

$$Au = f, \tag{1.14}$$

for a given $f \in X'$. Then the following statement is true:
The *variational formulation* to find $u \in X$, such that

$$\langle Au, v \rangle = \langle f, v \rangle \quad \text{for all } v \in X, \tag{1.15}$$

is equivalent to the operator equation (1.14). This can be proven as follows: On the one hand, for a solution $u \in X$ of the operator equation (1.14), the variational formulation (1.15) is obviously fulfilled. On the other hand, we now assume that $u \in X$ is a solution of the variational formulation (1.15). Inserting the difference into Definition (1.13), we have

$$\|Au - f\|_{X'} = \sup_{0 \neq v \in X} \frac{|\langle Au - f, v \rangle|}{\|v\|_X} = 0,$$

and this means $0 = Au - f \in X'$, so $u \in X$ is a solution of the operator equation (1.14). Moreover, the operator $A : X \to X'$ induces a bilinear form

$$\begin{aligned} a(\cdot,\cdot) : X \times X &\to \mathbb{R}, \\ a(u,v) &:= \langle Au, v\rangle \quad \text{for all } u, v \in X. \end{aligned} \tag{1.16}$$

On the other way round, for each bilinear form (1.16) there is an operator $A : X \to X'$ [32, Lemma 3.1]. Besides the boundedness of the linear operator A, another assumption is necessary to prove the unique solvability of the operator equation with the important lemma of Lax-Milgram, see below.

Definition 1.13. *An operator $A : X \to X'$ is called X-elliptic, if*

$$\langle Au, u\rangle \geq c_1^A \|u\|_X^2 \quad \textit{for all } u \in X,$$

with a positive constant c_1^A.

The unique solvability of the operator equation (1.14) and the variational formulation (1.15) can now be given by the following well-known theorem:

Theorem 1.14 (Lemma of Lax-Milgram)**.** *Let $A : X \to X'$ be a bounded and X-elliptic operator. Then there is a unique solution $u \in X$ of the operator equation (1.14) for any $f \in X'$, and it holds*

$$\|u\|_X \leq \frac{1}{c_1^A} \|f\|_{X'}.$$

For the proof, a Riesz operator is introduced, and the operator equation (1.14) is written as an equivalent fix point equation, see for example [32].
As we have seen in the motivation in Section 1.1, the operator equation $Au = f$ has to be solved with an additional condition $Bu = g$ in general. For that reason we need again some functional analysis concepts:
Let Π be a Banach space and $B : X \to \Pi'$ a bounded, linear operator. For the operator equation

$$Bu = g \tag{1.17}$$

we have to find a solution $u \in X$ for a given $g \in \Pi'$. A necessary additional assumption is the *solvability condition*

$$g \in \mathrm{Im}_X B := \{Bv \in \Pi', \quad \text{for all } v \in X\}.$$

Now let

$$V_g := \{v \in X : Bv = g\}$$

be the solution manifold in X for a given $g \in \Pi'$. In particular,

$$V_0 = \ker B = \{v \in X : Bv = 0\}.$$

If $f \in X'$ fulfills the solvability condition

$$f \in \mathrm{Im}_{V_g} A := \{Au \in X', \quad \text{for all } u \in V_g\},$$

we can reformulate the variational formulation (1.15) in the following way:
Find $u \in V_g$, such that

$$\langle Au, v\rangle = \langle f, v\rangle \quad \text{for all } v \in V_0. \tag{1.18}$$

The final theorem shows the unique solvability of the variational formulation (1.18):

Theorem 1.15. *Let $B : X \to \Pi'$ be a bounded operator, $A : X \to X'$ be bounded and V_0-elliptic, i.e.*

$$\langle Av, v\rangle \geq c_1^A \|v\|_X^2 \quad \textit{for all } v \in V_0.$$

Then there exists a unique solution $u \in X$ for the operator equation with an additional condition,

$$\begin{aligned} Au &= f, \\ Bu &= g, \end{aligned}$$

for $f \in Im_{V_q} A$ and $g \in Im_X B$.

For deeper discussions and more general concepts on variational methods, see for example [1].

1.4 Representation Formulae

The aim of this section is to formulate at first the fundamental solution for the Poisson equation. For this purpose, we have to introduce Green's first and second formulae. With the help of these equations, we can explain the representation formula for the solution of an interior boundary value problem. Additionally, we also describe the representation formula for the Laplace equation for an exterior domain.

1.4.1 Interior Boundary Value Problems

We multiply the Poisson equation (1.4) with a test function v, integrate over the domain Ω and apply integration by parts. This procedure leads to *Green's first formula*:

$$\int_\Omega (-\Delta u(y))v(y)dy = a(u, v) - \int_\Gamma \gamma_1^{int} u(y)\gamma_0^{int} v(y)ds_y, \tag{1.19}$$

with the symmetric bilinear form

$$a(u,v) = \int_\Omega \nabla u(y) \nabla v(y) dy.$$

The symmetry $a(u,v) = a(v,u)$ leads to *Green's second formula*:

$$\begin{aligned} \int_\Omega (-\Delta v(y)) u(y) dy + \int_\Gamma \gamma_1^{int} v(y) \gamma_0^{int} u(y) ds_y \\ = \int_\Omega (-\Delta u(y)) v(y) dy + \int_\Gamma \gamma_1^{int} u(y) \gamma_0^{int} v(y) ds_y. \end{aligned} \tag{1.20}$$

For the test function v in Green's second formula (1.20) we choose the *fundamental solution*

$$\begin{aligned} U^*(\cdot,\cdot) : \mathbb{R}^2 \times \mathbb{R}^2 &\to \mathbb{R}, \\ U^*(x,y) &= -\frac{1}{2\pi} \log |x-y|, \end{aligned}$$

satisfying

$$\int_\Omega (-\Delta_y U^*(x,y)) u(y) dy = u(x) \quad \text{for } x \in \Omega,$$

see for example [20]. After rewriting, Green's second formula (1.20) results in the *representation formula for the Poisson equation*:

$$u(x) = \int_\Gamma U^*(x,y) \gamma_1^{int} u(y) ds_y - \int_\Gamma \gamma_{1,y}^{int} U^*(x,y) g(y) ds_y + \int_\Omega U^*(x,y) f(y) dy \tag{1.21}$$

for $x \in \Omega$. Looking at the Dirichlet boundary value problem (1.9), the challenge is to determine the unknown Neumann data $\gamma_1^{int} u$ on the boundary Γ to gain the solution u on the whole domain Ω.

1.4.2 Exterior Boundary Value Problems

In order to derive the representation formula for the exterior domain, we choose a point $x \in \Omega^c$, a ball $B_r(x)$ and the radius r in such a way that the ball completely contains the interior domain Ω, i.e. $\overline{\Omega} \subset B_r(x)$. The result is a domain $\Omega_r = B_r(x) \backslash \overline{\Omega}$, for which we can use the representation formula (1.21) for the Laplace equation. By minding the algebraic signs of the normal vectors, we have

$$\begin{aligned} u(x) &= -\int_\Gamma U^*(x,y) \gamma_1^{ext} u(y) ds_y + \int_\Gamma \gamma_{1,y}^{ext} U^*(x,y) \gamma_0^{ext} u(y) ds_y \\ &\quad + \int_{\partial B_r(x)} U^*(x,y) \gamma_1^{int} u(y) ds_y - \int_{\partial B_r(x)} \gamma_{1,y}^{int} U^*(x,y) \gamma_0^{int} u(y) ds_y. \end{aligned} \tag{1.22}$$

Our goal is to calculate the limit $r \to \infty$, so we require a suitable radiation condition to deal with the behavior of the solution u at infinity. As in McLean's book [20] and originally in the paper of Costabel and Dauge [6], there is a special approach in $\mathbb{R}^2$, namely the *radiation condition*

$$\exists\, a \in \mathbb{R} : u(x) = a \log|x| + o(1) = a \log|x| + \mathcal{O}\left(\frac{1}{|x|}\right) \quad \text{as } |x| \to \infty. \tag{1.23}$$

Theorem 8.9 in [20] proves that if and only if the solution u satisfies the decay condition (1.23), the last two integrals in equation (1.22) tend to zero as $r \to \infty$. Putting the above together, we get the *representation formula for the solution of the exterior boundary value problem*,

$$u(x) = -\int_\Gamma U^*(x,y)\gamma_1^{ext}u(y)ds_y + \int_\Gamma \gamma_{1,y}^{ext}U^*(x,y)\gamma_0^{ext}u(y)ds_y \quad \text{for } x \in \Omega^c. \tag{1.24}$$

1.5 Boundary Integral Operators

In this section we introduce boundary integral operators, which we will need for the boundary integral equations of the exterior and interior boundary value problems in the next two sections. We describe the boundary and volume potentials of the introduced representation formulae, as well as their interior and exterior traces and conormal derivatives, respectively.

1.5.1 Newton Potential

The first potential we are going to consider is a volume potential, namely the *Newton potential*

$$(\widetilde{N}_0 f)(x) := \int_\Omega U^*(x,y)f(y)dy = -\frac{1}{2\pi}\int_\Omega \log|x-y| f(y)dy \quad \text{for } x \in \Omega. \tag{1.25}$$

It is a continuous mapping

$$\widetilde{N}_0 : \widetilde{H}^{-1}(\Omega) \to H^1(\Omega).$$

For the mapping properties of the Newton potentials, see for example [32].

Lemma 1.16. *Since the interior trace operator γ_0^{int} maps from $H^1(\Omega)$ to $H^{1/2}(\Gamma)$, we obtain the linear and bounded operator*

$$N_0 := \gamma_0^{int}\widetilde{N}_0 : \widetilde{H}^{-1}(\Omega) \to H^{1/2}(\Gamma),$$

with

$$\|N_0 f\|_{H^{1/2}(\Gamma)} \le c_2^{N_0} \|f\|_{\widetilde{H}^{-1}(\Omega)} \quad \textit{for all } f \in \widetilde{H}^{-1}(\Omega).$$

For $f \in L_\infty(\Omega)$ and $x \in \Gamma$ the representation

$$(N_0 f)(x) = \gamma_0^{int}(\widetilde{N}_0 f)(x) = \int_\Omega U^*(x,y) f(y) dy = -\frac{1}{2\pi} \int_\Omega \log|x-y| f(y) dy$$

exists as a weakly singular domain integral.

Moreover, by applying the interior conormal derivative to the Newton potential (1.25) defined above, there exists the continuous mapping

$$N_1 := \gamma_1^{int} \widetilde{N}_0 : \widetilde{H}^{-1}(\Omega) \to H^{-1/2}(\Gamma),$$

with

$$(N_1 f)(x) = \gamma_1^{int}(\widetilde{N}_0 f)(x) \quad \text{for } x \in \Gamma,$$

and

$$\|N_1 f\|_{H^{-1/2}(\Gamma)} \leq c_2^{N_1} \|f\|_{\widetilde{H}^{-1}(\Omega)} \quad \text{for all } f \in \widetilde{H}^{-1}(\Omega).$$

In what follows, we are going to describe boundary potentials in the next Subsections.

1.5.2 Single Layer Potential

The next potential we are discussing is the *single layer potential*

$$(\widetilde{V} w)(x) := \int_\Gamma U^*(x,y) w(y) ds_y = -\frac{1}{2\pi} \int_\Gamma \log|x-y| w(y) ds_y \quad \text{for } x \in \Omega, \tag{1.26}$$

for a given density function $w \in H^{-1/2}(\Gamma)$. The potential (1.26) maps the density function w on the boundary Γ to a function $\widetilde{V} w$ in the domain Ω. Particularly, it is a continuous mapping

$$\widetilde{V} : H^{-1/2}(\Gamma) \to H^1(\Omega).$$

An important condition of the single layer potential is the weak solvability of the Laplace equation (1.3), i.e.

$$-\Delta(\widetilde{V} w)(x) = 0 \quad \text{for } x \in \Omega$$

for any function $w \in H^{-1/2}(\Gamma)$, which means that $\widetilde{V} w$ is a harmonic function. The boundary integral operator

$$\begin{aligned} (Vw)(x) := \gamma_0^{int}(\widetilde{V} w)(x) &= \int_\Gamma U^*(x,y) w(y) ds_y \\ &= -\frac{1}{2\pi} \int_\Gamma \log|x-y| w(y) ds_y \quad \text{for } x \in \Gamma \end{aligned}$$

exists as a weakly singular surface integral, with the mapping properties

$$V = \gamma_0^{int}\widetilde{V} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma),$$

with

$$\|Vw\|_{H^{1/2}(\Gamma)} \le c_2^V \|w\|_{H^{-1/2}(\Gamma)} \quad \text{for all } w \in H^{-1/2}(\Gamma).$$

Remark 1.2. *The representation of the operator V on the boundary Γ is still the same when applying the exterior trace operator (1.6),*

$$(Vw)(x) = \gamma_0^{ext}(\widetilde{V}w)(x) \quad \textit{for } x \in \Gamma.$$

Regarding the solvability of variational formulations with the observed boundary integral operator, and therefore in view of the lemma of Lax-Milgram 1.14, the following lemma is important:

Lemma 1.17 ([32])**.** *With the additional restriction to the domain Ω, $diam(\Omega) < 1$, the bounded single layer potential operator V is $H^{-1/2}(\Gamma)$-elliptic, i.e.*

$$\langle Vw, w\rangle \ge c_1^V \|w\|^2_{H^{-1/2}(\Gamma)} \quad \textit{for all } w \in H^{-1/2}(\Gamma).$$

1.5.3 Double Layer Potential

The next potential we are looking at is the *double layer potential*

$$\begin{aligned}(Wv)(x) &:= \int_\Gamma \gamma_{1,y}^{int} U^*(x,y)v(y)ds_y \\ &= -\frac{1}{2\pi}\int_\Gamma \gamma_{1,y}^{int}\log|x-y|v(y)ds_y \\ &= -\frac{1}{2\pi}\int_\Gamma \frac{(y-x,n_y)}{|x-y|^2}v(y)ds_y \quad \text{for } x \in \Omega. \end{aligned} \tag{1.27}$$

It is again a continuous map

$$W : H^{1/2}(\Gamma) \to H^1(\Omega),$$

and the function $Wv \in H^1(\Omega)$ is a weak solution of the Laplace equation (1.3), so

$$-\Delta(Wv)(x) = 0 \quad \text{for } x \in \Omega.$$

Applying the interior trace operator (1.5), we obtain the linear and bounded *double layer potential operator*

$$\gamma_0^{int}W : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma),$$

with

$$\|\gamma_0^{int}Wv\|_{H^{1/2}(\Gamma)} \le c_2^{\gamma_0^{int}W} \|v\|_{H^{1/2}(\Gamma)} \quad \text{for all } v \in H^{1/2}(\Gamma).$$

Lemma 1.18 ([34]). *For $v \in H^{1/2}(\Gamma)$ there holds*

$$\gamma_0^{int}(Wv)(x) = \Big(-1 + \sigma(x)\Big)v(x) + (Kv)(x) \quad \textit{for } x \in \Gamma,$$

with

$$\sigma(x) := \lim_{\varepsilon \to 0} \frac{1}{2\pi} \frac{1}{\varepsilon} \int_{y \in \Omega : |y-x| = \varepsilon} ds_y \quad \textit{for } x \in \Gamma, \tag{1.28}$$

and

$$\begin{aligned} (Kv)(x) &:= \lim_{\varepsilon \to 0} \int_{y \in \Gamma : |y-x| \geq \varepsilon} \gamma_{1,y}^{int} U^*(x,y) v(y) ds_y \\ &= -\frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{y \in \Gamma : |y-x| \geq \varepsilon} \frac{(y-x, n_y)}{|x-y|^2} ds_y \quad \textit{for } x \in \Gamma. \end{aligned}$$

Analogously, there holds the following representation of the double layer potential operator when we apply the exterior trace operator (1.6):

$$\gamma_0^{ext}(Wv)(x) = \sigma(x)v(x) + (Kv)(x) \quad \text{for } x \in \Gamma.$$

The next Remark 1.3 is about an easier representation of the function $\sigma(x)$.

Remark 1.3. *If the boundary Γ is smooth, i.e. at least differentiable, in the neighbourhood $U(x) \cap \Gamma$ of the observed point $x \in \Gamma$, the description (1.28) of $\sigma(x)$ is*

$$\sigma(x) = \frac{1}{2} \quad \textit{for almost all } x \in \Gamma.$$

Since we have introduced the first three boundary integral operators up to now, the interior representation formula (1.21) can be written in a more compact way,

$$u(\tilde{x}) = (\widetilde{V}\gamma_1^{int} u)(\tilde{x}) - (W\gamma_0^{int})u(\tilde{x}) + (\tilde{N}_0 f)(\tilde{x}) \quad \text{for } \tilde{x} \in \Omega. \tag{1.29}$$

Calculating the limit $\Omega \ni \tilde{x} \to x \in \Gamma$, or rather applying the interior trace operator γ_0^{int}, we gain the *first boundary integral equation*

$$\gamma_0^{int} u(x) = (V\gamma_1^{int} u)(x) + \frac{1}{2}\gamma_0^{int} u(x) - (K\gamma_0^{int})u(x) + (N_0 f)(x) \quad \text{for } x \in \Gamma. \tag{1.30}$$

Analogously we have a compact formula for the exterior representation formula (1.24) for the Laplace problem:

$$u(\tilde{x}) = -(\widetilde{V}\gamma_1^{ext} u)(\tilde{x}) + (W\gamma_0^{ext})u(\tilde{x}) \quad \text{for } \tilde{x} \in \Omega^c. \tag{1.31}$$

Again, the application of the exterior trace operator γ_0^{ext} leads to the first integral equation for an exterior problem,

$$\gamma_0^{ext} u(x) = -(V\gamma_1^{ext} u)(x) + \frac{1}{2}\gamma_0^{ext} u(x) + (K\gamma_0^{ext})u(x) \quad \text{for } x \in \Gamma. \tag{1.32}$$

Remark 1.4. *For a constant function $v_0 \equiv 1$ it holds*

$$(\frac{1}{2}I + K)v_0(x) = 0 \quad \textit{for } x \in \Gamma.$$

Proof. Let us consider the introduced first boundary integral equation (1.30) for the Laplace equation with $f \equiv 0$. For $v_0 \equiv 1$ it holds

$$1 = (\frac{1}{2}I - K)1 + V\gamma_1^{int}1 = (\frac{1}{2}I - K)1.$$

Now the statement follows immediately. □

The equation in Remark 1.4 is very useful. Thinking about software implementations of discretized operators, the above given relation is an easy possibility to check if the double layer potential is calculated in a right way. Looking at Table 4.2 in Subsection 4.2.1, we see that this kernel property is still valid for the discretised scheme, which will be introduced in Chapter 3.

1.5.4 Adjoint Double Layer Potential

For the corresponding boundary integral operators of the single and double layer potential, we applied the interior trace operator γ_0^{int} so far. The interior conormal derivative (1.7) with the mapping properties for the space $H^1(\Omega, \Delta) = \left\{v \in H^1(\Omega) : \ \Delta v \in \widetilde{H}^{-1}(\Omega)\right\}$,

$$\gamma_1^{int} : H^1(\Omega, \Delta) \to H^{-1/2}(\Gamma)$$

yields another opportunity to define further boundary integral operators, which we describe in the following and next section.

Lemma 1.19 ([32])**.** *The interior conormal derivative of the single layer potential defines a linear boundary integral operator*

$$\gamma_1^{int}\widetilde{V} : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma),$$

which is bounded with

$$\|\gamma_1^{int}\widetilde{V}w\|_{H^{-1/2}(\Gamma)} \leq c_2^{\gamma_1^{int}\widetilde{V}}\|w\|_{H^{-1/2}(\Gamma)} \quad \textit{for all } w \in H^{-1/2}(\Gamma).$$

For $w \in H^{-1/2}(\Gamma)$ it holds

$$\gamma_1^{int}(\widetilde{V}w)(x) = \sigma(x)w(x) + (K'w)(x) \quad \textit{for } x \in \Gamma$$

in the sense of $H^{-1/2}(\Gamma)$, with

$$\begin{aligned}
(K'w)(x) &:= \lim_{\varepsilon\to 0}\int_{y\in\Gamma:|y-x|\geq\varepsilon} \gamma_{1,x}^{int}U^*(x,y)w(y)ds_y \\
&= -\frac{1}{2\pi}\lim_{\varepsilon\to 0}\int_{y\in\Gamma:|y-x|\geq\varepsilon} \gamma_{1,x}^{int}\log|x-y|w(y)ds_y \\
&= -\frac{1}{2\pi}\lim_{\varepsilon\to 0}\int_{y\in\Gamma:|y-x|\geq\varepsilon} \frac{(x-y,n_x)}{|x-y|^2}w(y)ds_y \quad \textit{for } x \in \Gamma.
\end{aligned}$$

Again, there is a similar representation when applying the exterior conormal derivative (1.8):

$$\gamma_1^{ext}(\widetilde{V}w)(x) = \Big(\sigma(x) - 1\Big)w(x) + (K'w)(x) \quad \text{for } x \in \Gamma$$

in the sense of $H^{-1/2}(\Gamma)$.

1.5.5 Hypersingular Integral Operator

The interior conormal derivative of the double layer potential (1.27) defines the last operator we want to introduce, the bounded *hypersingular integral operator*

$$D := -\gamma_1^{int} W : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma), \tag{1.33}$$

with

$$\|Dv\|_{H^{-1/2}(\Gamma)} \leq c_2^D \, \|v\|_{H^{1/2}(\Gamma)} \quad \text{for all } v \in H^{1/2}(\Gamma).$$

Basically, it holds

$$\begin{aligned}(Dv)(x) &= -\gamma_1^{int}(Wv)(x) = -\lim_{\tilde{x} \ni \Omega \to x \in \Gamma} (n_x, \nabla_{\tilde{x}}(Wv)(\tilde{x})) \\ &= -\frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{y \in \Gamma : |y-x| \geq \varepsilon} \left[\frac{(n_x, n_y)}{|x-y|^2} - 2\frac{(x-y, n_x)(x-y, n_y)}{|x-y|^4} \right] v(y) ds_y\end{aligned}$$

for $x \in \Gamma$. Unfortunately, this integral does not exist as a Cauchy singular surface integral. The outcome of this is the name as a hypersingular integral operator. For a useful explicit representation of it, we need to do some regularisations.

Remark 1.5. *For a constant function $u_0 \equiv 1$, the hypersingular integral operator is zero, i.e.*

$$(Du_0)(x) = 0 \quad \textit{for } x \in \Gamma.$$

Proof. Let us consider the representation formula (1.21) for the Laplace equation. Since $u_0 \equiv 1$ is a solution of $-\Delta u_0(x) = 0$, we have

$$1 = -\int_\Gamma \gamma_{1,y}^{int} U^*(\tilde{x}, y) ds_y \quad \text{for } \tilde{x} \in \Omega.$$

As the derivative of constant functions is zero, after applying the gradient with respect to $\tilde{x} \in \Omega$ we receive

$$\nabla_{\tilde{x}}(Wu_0)(\tilde{x}) = 0 \quad \text{for } \tilde{x} \in \Omega. \tag{1.34}$$

Multiplying the equation with the outer normal vector of the domain Ω and calculating the limit $\tilde{x} \ni \Omega \to x \in \Gamma$, relation (1.34) results in

$$(Du_0)(x) = 0 \quad \text{for } x \in \Gamma.$$

□

Besides, as for the double layer potential, this relation is still valid for the discretised scheme, see for instance the results in Table 4.2 in Subsection 4.2.1. Remark 1.5 can now be used in the following way:

$$\begin{aligned}(Dv)(x) &= -\lim_{\tilde{x}\ni\Omega\to x\in\Gamma}(n_x, \nabla_{\tilde{x}}(Wv)(\tilde{x})) \\ &= -\lim_{\tilde{x}\ni\Omega\to x\in\Gamma}(n_x, \nabla_{\tilde{x}}\int_\Gamma \gamma_{1,y}^{int}U^*(\tilde{x},y)v(y)ds_y) \\ &= -\lim_{\tilde{x}\ni\Omega\to x\in\Gamma}(n_x, \nabla_{\tilde{x}}\int_\Gamma \gamma_{1,y}^{int}U^*(\tilde{x},y)\big(v(y)-v(x)\big)ds_y). \qquad (1.35)\end{aligned}$$

The integral (1.35) now exists as a Cauchy singular surface integral, and for a continuous density function $v \in H^{1/2}(\Gamma)$ there are several possibilities of representations, one of them given below:

$$(Dv)(x) = -\gamma_{1,x}^{int}\int_\Gamma \gamma_{1,y}^{int}U^*(x,y)u(y)ds_y ds_x \quad \text{for } x\in\Gamma.$$

Moreover, we note that due to the induced bilinear form of the hypersingular integral operator and integration by parts, an alternative representation can be reached, see [32].

Remark 1.6. *Once more, we can apply the exterior conormal derivate to the double layer potential W. This procedure makes no difference in the representation of the hypersingular operator, hence it still holds*

$$(Dv)(x) = -(\gamma_1^{ext}Wv)(x) \quad \textit{for } x\in\Gamma.$$

As far as the ellipticity of the hypersingular integral operator is concerned, there is a problem with the kernel condition $(Du_0)(x) = 0$ with the eigensolution $u_0 \equiv 1$ for $x \in \Gamma$ as noted in Remark 1.5. Hence there is no ellipticity of D on the whole space $H^{1/2}(\Gamma)$. For the following lemma showing the special ellipticity properties of D, we define the function space

$$H_{**}^{1/2}(\Gamma) := \Big\{v\in H^{1/2}(\Gamma) :\ \langle v,1\rangle_\Gamma = 0\Big\}.$$

Lemma 1.20 ([32])**.** *The hypersingular integral operator D defined as in (1.33) is $H^{1/2}(\Gamma)$-semi-elliptic, that is why it holds*

$$\langle Dv, v\rangle_\Gamma \geq |v|^2_{H^{1/2}(\Gamma)} \quad \textit{for all } v\in H^{1/2}(\Gamma).$$

For functions $v \in H_{**}^{1/2}(\Gamma)$, D is $H_{**}^{1/2}(\Gamma)$-elliptic, i.e.

$$\langle Dv, v\rangle_\Gamma \geq \|v\|^2_{H^{1/2}(\Gamma)} \quad \textit{for all } v\in H_{**}^{1/2}(\Gamma).$$

Since we can now apply the interior conormal derivative to the introduced operators too, the interior representation formula (1.21) yields a *second boundary integral equation*

$$\gamma_1^{int}u(x) = \frac{1}{2}\gamma_1^{int}u(x) + (K'\gamma_1^{int}u)(x) + (D\gamma_0^{int}u)(x) + (N_1 f)(x) \tag{1.36}$$

for $x \in \Gamma$. Again, the application of the exterior conormal derivative yields a second boundary integral equation for the exterior problem,

$$\gamma_1^{ext}u(x) = \frac{1}{2}\gamma_1^{ext}u(x) - (K'\gamma_1^{ext}u)(x) - (D\gamma_0^{ext}u)(x) \quad \text{for } x \in \Gamma. \tag{1.37}$$

1.5.6 Mapping Properties

Finally, we want to summarise the mapping properties of the boundary integral operators introduced in the sections above.

Lemma 1.21 ([5]). *For the boundary Γ of the Lipschitz domain Ω, the boundary integral operators*

$$\begin{array}{rcccl} V & : & H^{-1/2+s}(\Gamma) & \to & H^{1/2+s}(\Gamma), \\ D & : & H^{1/2+s}(\Gamma) & \to & H^{-1/2+s}(\Gamma), \\ K & : & H^{1/2+s}(\Gamma) & \to & H^{1/2+s}(\Gamma), \\ K' & : & H^{-1/2+s}(\Gamma) & \to & H^{-1/2+s}(\Gamma) \end{array}$$

are bounded for all $s \in [-\frac{1}{2}, \frac{1}{2}]$.

1.5.7 Steklov-Poincaré Operators

Looking ahead to domain decomposition methods, the operator we deal with in this subsection is very important. For this purpose, we consider the interior first boundary integral equation (1.30) for the Laplace equation, i.e. $f \equiv 0$. The single layer potential V is invertible when assuming $\operatorname{diam}(\Omega) < 1$, so we can solve the equation for the interior conormal derivative

$$\gamma_1^{int}u(x) = V^{-1}(\frac{1}{2}I + K)\gamma_0^{int}u(x) \quad \text{for } x \in \Gamma.$$

Thus, the bounded interior *Steklov-Poincaré operator*

$$S^{int} := V^{-1}(\frac{1}{2}I + K) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

defines a *Dirichlet to Neumann map* for the interior Laplace equation. When inserting $\gamma_1^{int}u(x) = (S^{int}\gamma_0^{int})u(x)$ into the second boundary integral equation (1.36), we obtain an equivalent symmetric representation

$$\begin{aligned} \gamma_1^{int}u(x) &= (D\gamma_0^{int}u)(x) + (\frac{1}{2}I + K')\gamma_1^{int}u(x) \\ &= \Big(D + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)\Big)\gamma_0^{int}u(x), \end{aligned}$$

with

$$S^{int} := D + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma).$$

Remark 1.7. *For a constant function $v_0 \equiv 1$ it holds*

$$(S^{int}v_0)(x) = 0 \quad \textit{for } x \in \Gamma.$$

Proof. This relation follows from the kernel properties of the hypersingular integral operator and of $(\frac{1}{2}I + K)$, see Remarks 1.5 and 1.4, respectively. □

For ellipticity discussions, we know that the inverse single layer potential is $H^{1/2}(\Gamma)$-elliptic, see [32, Lemma 3.3] for a general discussion. Hence,

$$\begin{aligned}
\langle S^{int}v, v\rangle_\Gamma &= \langle Dv, v\rangle_\Gamma + \langle V^{-1}(\frac{1}{2}I + K)v, (\frac{1}{2}I + K)v\rangle_\Gamma \\
&\geq \langle Dv, v\rangle_\Gamma + \frac{1}{c_2^V}\|(\frac{1}{2}I + K)v\|^2_{H^{1/2}(\Gamma)} \\
&\geq \langle Dv, v\rangle_\Gamma \quad \text{for all } v \in H^{1/2}(\Gamma).
\end{aligned}$$

Thus, S is spectrally equivalent to D, meaning that the Steklov-Poincaré operator has the same semi-ellipticity conditions as the hypersingular integral operator, particularly

$$\langle S^{int}v, v\rangle_\Gamma \geq c_1^D\,|v|^2_{H^{1/2}(\Gamma)} \quad \text{for all } v \in H^{1/2}(\Gamma).$$

Regarding the symmetric representation, we finally note that for software implementations S is only given implicitly. Using boundary element methods, we describe an alternative formulation of the Steklov-Poincaré operators in Section 2.3. Doing so, we then are able to calculate these operators explicitly.
In the end, we also want to define the *exterior Steklov-Poincaré operator*. For this purpose, we obtain from the first exterior boundary integral equation (1.32)

$$\gamma_1^{ext}u(x) = -V^{-1}(\frac{1}{2}I - K)\gamma_0^{ext}u(x) \quad \text{for } x \in \Gamma, \tag{1.38}$$

which leads us to the definition

$$S^{ext} := V^{-1}(\frac{1}{2}I - K) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma).$$

The second boundary integral equation (1.37) for the exterior problem is

$$\gamma_1^{ext}u(x) = -(D\gamma_0^{ext}u)(x) + (\frac{1}{2}I - K')\gamma_1^{ext}u(x).$$

Inserting now the exterior conormal derivative $\gamma_1^{ext}u(x)$ from (1.38), we get

$$\gamma_1^{ext}u(x) = -\Big(D + (\frac{1}{2}I - K')V^{-1}(\frac{1}{2}I - K)\Big)\gamma_0^{ext}u(x).$$

Thus, the symmetric representation is

$$S^{ext} = D + (\frac{1}{2}I - K')V^{-1}(\frac{1}{2}I - K) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma).$$

The exterior Steklov-Poincaré operator is $H^{1/2}(\Gamma)$-elliptic. For the proof, we shall adopt the approach of [32], where

$$\|v\|_{V^{-1}} := \sqrt{\langle V^{-1}v, v\rangle_\Gamma} \quad \text{for all } v \in H^{1/2}(\Gamma)$$

defines a norm on $H^{1/2}(\Gamma)$ equivalent to $\|\cdot\|_{H^{1/2}(\Gamma)}$. So let us start with

$$\begin{aligned}
\langle S^{ext}v, v\rangle_\Gamma &= \langle Dv, v\rangle_\Gamma + \langle(\frac{1}{2}I - K')V^{-1}(\frac{1}{2}I - K)v, v\rangle_\Gamma \\
&\geq \langle V^{-1}(\frac{1}{2}I - K)v, (\frac{1}{2}I - K)v\rangle_\Gamma \\
&= \|(\frac{1}{2}I - K)v\|^2_{V^{-1}}.
\end{aligned}$$

For the next step, we need

$$\begin{aligned}
\|v\|_{V^{-1}} &= \|(\frac{1}{2}I - K)v + (\frac{1}{2}I + K)v\|_{V^{-1}} \\
&\leq \|(\frac{1}{2}I - K)v\|_{V^{-1}} + \|(\frac{1}{2}I + K)v\|_{V^{-1}} \\
&\leq \|(\frac{1}{2}I - K)v\|_{V^{-1}} + c_K\|v\|_{V^{-1}},
\end{aligned}$$

with a contraction constant $c_K < 1$ depending on the ellipticity constants of the single layer operator and the hypersingular operator, see [32, Folgerung 6.6]. Hence, it follows

$$(1 - c_K)\|v\|_{V^{-1}} \leq \|(\frac{1}{2}I - K)v\|_{V^{-1}},$$

and since $\|\cdot\|_{V^{-1}}$ defines an equivalent norm, we finally get

$$\begin{aligned}
\langle S^{ext}v, v\rangle_\Gamma &= (1 - c_K)^2\|v\|^2_{V^{-1}} \\
&\geq c_1^{S^{ext}}\|v\|^2_{H^{1/2}(\Gamma)},
\end{aligned}$$

which completes the proof for the $H^{1/2}(\Gamma)$-ellipticity of the exterior Steklov-Poincaré operator S^{ext}.

1.6 Interior Boundary Integral Equations

As already mentioned in Section 1.4, the goal is to find the unknown Neumann data $\gamma_1^{int}u(x)$ for $x \in \Gamma$. Then the solution of either the Poisson or the Laplace problem can be described

by the representation formula (1.21). This can be done by the use of boundary integral equations.
Let us start with the interior representation formula (1.29),

$$u(\tilde{x}) = (\widetilde{V}\gamma_1^{int}u)(\tilde{x}) - (W\gamma_0^{int}u)(\tilde{x}) + (\tilde{N}_0 f)(\tilde{x}) \quad \text{for} \quad \tilde{x} \in \Omega.$$

The application of the interior trace operator γ_0^{int} and the interior conormal derivative γ_1^{int} yields the *first boundary integral equation*

$$\gamma_0^{int}u(x) = (V\gamma_1^{int}u)(x) + \frac{1}{2}\gamma_0^{int}u(x) - (K\gamma_0^{int})u(x) + (N_0 f)(x) \quad \text{for } x \in \Gamma, \tag{1.39}$$

and the *second boundary integral equation*

$$\gamma_1^{int}u(x) = \frac{1}{2}\gamma_1^{int}u(x) + (K'\gamma_1^{int}u)(x) + (D\gamma_0^{int}u)(x) + (N_1 f)(x) \quad \text{for } x \in \Gamma,$$

respectively. With the help of the *interior Calderon projector*

$$\mathcal{C}^{int} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix},$$

these two relations can be written in a *system of boundary integral equations*

$$\begin{pmatrix} \gamma_0^{int}u \\ \gamma_1^{int}u \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix} \begin{pmatrix} \gamma_0^{int}u \\ \gamma_1^{int}u \end{pmatrix} + \begin{pmatrix} N_0 f \\ N_1 f \end{pmatrix} \tag{1.40}$$

on the boundary Γ. Since the Operator $\mathcal{C}^{int}$ is a projection operator, which means that $\mathcal{C}^{int} = (\mathcal{C}^{int})^2$, we get the following relations:

Remark 1.8 ([32])**.**

$$\begin{aligned} VD &= (\frac{1}{2}I + K)(\frac{1}{2}I - K), \\ DV &= (\frac{1}{2}I + K')(\frac{1}{2}I - K'), \\ VK' &= KV, \\ K'D &= DK. \end{aligned}$$

Moreover, the conormal derivative of the Newton potential can be expressed by the Newton potential itself:

Lemma 1.22. *For $x \in \Gamma$, the volume potential $(N_1 f)(x)$ fulfills the equation*

$$(N_1 f)(x) = (\frac{1}{2}I + K')V^{-1}(N_0 f)(x).$$

Proof. The relation follows by inverting the single layer potential V in the first equation of the system of boundary integral equations (1.40), and applying the result into the second boundary integral equation. □

Lemma 1.22 is very useful for software implementations of boundary element methods, because we do not have to deal with the calculation of the operator $N_1 f$ explicitly, but it can be described by the other integral operators involved.

1.7 Exterior Boundary Integral Equations

We can do the same procedure as in Section 1.6 above for an exterior boundary value problem in Ω^c. So we start with the representation formula (1.31) for the exterior problem,

$$u(\tilde{x}) = -(\widetilde{V}\gamma_1^{ext}u)(\tilde{x}) + (W\gamma_0^{ext})u(\tilde{x}) \quad \text{for} \quad \tilde{x} \in \Omega^c.$$

With the exterior operators γ_0^{ext} and γ_1^{ext} it follow the *first exterior boundary integral equation*

$$\gamma_0^{ext}u(x) = -(V\gamma_1^{ext}u)(x) + \frac{1}{2}\gamma_0^{ext}u(x) + (K\gamma_0^{ext})u(x) \quad \text{for } x \in \Gamma,$$

and the *second exterior boundary integral equation*

$$\gamma_1^{ext}u(x) = \frac{1}{2}\gamma_1^{ext}u(x) - (K'\gamma_1^{ext}u)(x) - (D\gamma_0^{ext}u)(x) \quad \text{for } x \in \Gamma,$$

respectively. With the *exterior Calderon projector*

$$\mathcal{C}^{ext} = \begin{pmatrix} \frac{1}{2}I + K & -V \\ -D & \frac{1}{2}I - K' \end{pmatrix} \tag{1.41}$$

we obtain an alternative system of boundary integral equations for the exterior Laplace problem, i.e.

$$\begin{pmatrix} \gamma_0^{ext}u \\ \gamma_1^{ext}u \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I + K & -V \\ -D & \frac{1}{2}I - K' \end{pmatrix} \begin{pmatrix} \gamma_0^{ext}u \\ \gamma_1^{ext}u \end{pmatrix} \tag{1.42}$$

on Γ. Besides, the exterior Calderon projector $\mathcal{C}^{ext}$ is a projection operator too, so it holds $\mathcal{C}^{ext} = (\mathcal{C}^{ext})^2$.

2 Boundary Value and Free Space Transmission Problems

In the following sections we state various models of partial differential equations for different problems in physics. We start with the Laplace and Poisson equation and their different solving approaches. Since boundary element methods are especially suitable for free space problems, we handle an exterior Dirichlet boundary value problem. As a combination of the previous problems we could see the interface problem, which considers two domains. In the interior domain Ω_1 we have given a Poisson equation, and for the exterior domain Ω_2 we use a Laplace problem. The transmission conditions on the boundary of Ω_1, which is equal to the boundary of Ω_2 too, connect our two problems. Hence, there is no boundary data given explicitly. The second main topic in this work are domain decomposition methods. The idea is to divide a given domain into many subdomains. Hence, we obtain a skeleton of subdomain boundaries, as well as local boundary value problems, linked on the occurring coupled boundaries with Dirichlet and Neumann transmission conditions. Using the Steklov-Poincaré operator S, we are able to describe variational formulations on the skeleton to determine the missing data on the subdomain boundaries. Finally, we are able to calculate the solution for the original, global problem in the usual way.

2.1 Interior Dirichlet Boundary Value Problems

We consider two common partial differential equations, namely the Laplace and Poisson equation. As already mentioned in the introduction of this work, there exist two different types of boundary integral approaches to solve the corresponding boundary value problems, which will be shown for the Laplace problem. There is the indirect approach, where we use the solvability properties of the single layer potential operator for the Laplace equation. Concerning the direct approach, we apply the boundary integral equations of Section 1.6 to determine the solution with the help of the representation formula (1.21). At the end, we state the direct approach for the Poisson problem too, and deal with the evaluation of the Newton potential operator.

2.1.1 Approaches for the Laplace Problem

For a better overview, we state again the interior Dirichlet boundary value problem

$$\begin{aligned} -\Delta u(x) &= 0 \qquad \text{for } x \in \Omega \subset \mathbb{R}^2, \\ \gamma_0^{int} u(x) &= g(x) \quad \text{for } x \in \Gamma. \end{aligned} \tag{2.1}$$

Indirect Approach

As mentioned in Subsection 1.5.2,

$$u(x) = (\widetilde{V}w)(x) \quad \text{for } x \in \Omega$$

is harmonic for $w \in H^{-1/2}(\Gamma)$. Using the interior trace operator, we obtain

$$(Vw)(x) = \int_\Gamma U^*(x,y)w(y)ds_y = \gamma_0^{int}u(x) = g(x) \quad \text{for } x \in \Gamma, \tag{2.2}$$

with the given Dirichlet datum g on the boundary. The goal is to determine the unknown density function w. In this case, we know that the boundary integral equation (2.2) is equivalent to the following variational formulation: Find $w \in H^{-1/2}(\Gamma)$, such that

$$\langle Vw, z\rangle_\Gamma = \langle g, z\rangle_\Gamma \quad \text{for all } z \in H^{-1/2}(\Gamma), \tag{2.3}$$

or rather stated in integral notation

$$-\frac{1}{2\pi}\int_\Gamma z(x)\int_\Gamma \log|x-y|w(y)ds_y ds_x = \int_\Gamma g(x)z(x)ds_x.$$

Theorem 2.1. *For a given boundary datum $g \in H^{1/2}(\Gamma)$, and assuming diam$(\Omega) < 1$, there exists a unique solution $w \in H^{-1/2}(\Gamma)$ of the variational formulation (2.3) with*

$$\|w\|_{H^{-1/2}(\Gamma)} \leq \frac{1}{c_1^V}\|g\|_{H^{1/2}(\Gamma)}.$$

Proof. Since the single layer potential V is bounded and elliptic for diam$(\Omega) < 1$ according to Lemma 1.17, the discussed variational formulation is uniquely solvable due to the lemma of Lax-Milgram (Theorem 1.14), which yields the described inequality too. Therefore, the main steps are

$$c_1^V \|w\|^2_{H^{-1/2}(\Gamma)} \leq \langle Vw, w\rangle_\Gamma \overset{(2.3)}{=} \langle g, w\rangle_\Gamma.$$

Using the Cauchy-Schwarz inequality, we obtain

$$c_1^V \|w\|^2_{H^{-1/2}(\Gamma)} \leq \|g\|_{H^{1/2}(\Gamma)}\|w\|_{H^{-1/2}(\Gamma)},$$

and finally

$$\|w\|_{H^{-1/2}(\Gamma)} \leq \frac{1}{c_1^V}\|g\|_{H^{1/2}(\Gamma)}.$$

□

The approach for solving the Dirichlet boundary value problem by the single layer potential alone is called *indirect*. Apart from this, the density function w we are looking for has no physical meaning in general.

Direct Approach

Other than for the indirect approach, the solution of the boundary value problem (2.1) can also be calculated on the whole domain Ω with the representation formula for the Laplace equation given in Section 1.4,

$$u(x) = \int_\Gamma U^*(x,y)\gamma_1^{int}u(y)ds_y - \int_\Gamma \gamma_{1,y}^{int}U^*(x,y)g(y)ds_y \quad \text{for } x \in \Omega. \tag{2.4}$$

Since the Dirichlet datum $\gamma_0^{int}u = g$ is given on the boundary Γ, the intention is to determine the unknown Neumann datum $\gamma_1^{int}u \in H^{-1/2}(\Gamma)$. The retyped first boundary integral equation in system (1.40) gives

$$(V\gamma_1^{int}u)(x) = (\frac{1}{2}I + K)g(x) \quad \text{for } x \in \Gamma,$$

and accordingly

$$-\frac{1}{2\pi}\int_\Gamma \log|x-y|\gamma_1^{int}u(x)ds_y = \frac{1}{2}g(x) - \frac{1}{2\pi}\int_\Gamma \frac{(y-x,n_y)}{|x-y|^2}g(y)ds_y.$$

The equivalent variational problem reads as follows: Find $\gamma_1^{int}u \in H^{-1/2}(\Gamma)$, such that

$$\langle V\gamma_1^{int}u, z\rangle_\Gamma = \langle(\frac{1}{2}I + K)g, z\rangle_\Gamma \tag{2.5}$$

is satisfied for all $z \in H^{-1/2}(\Gamma)$, and alternatively

$$\begin{aligned}-\frac{1}{2\pi}\int_\Gamma z(x)\int_\Gamma \log|x-y|t(y)ds_y ds_x \\ = \frac{1}{2}\int_\Gamma g(x)z(x)ds_x - \frac{1}{2\pi}\int_\Gamma z(x)\int_\Gamma \frac{(y-x,n_y)}{|x-y|^2}g(y)ds_y ds_x.\end{aligned}$$

Theorem 2.2. *Consider the given Dirichlet datum $g \in H^{1/2}(\Gamma)$. When we assume $diam(\Omega) < 1$, then there exists a uniquely determined Neumann datum $\gamma_1^{int}u \in H^{-1/2}(\Gamma)$ as a solution of the variational formulation (2.5) of the direct approach, and it holds*

$$\|\gamma_1^{int}u\|_{H^{-1/2}(\Gamma)} \le \frac{1}{c_1^V}\left(1 + c_2^{\gamma_0^{int}W}\right)\|g\|_{H^{1/2}(\Gamma)}.$$

Proof. Once again, the lemma of Lax-Milgram (Theorem 1.14) yields the unique solvability, since the difference to the indirect approach is just another definition of the right hand side. As far as the estimate is concerned, we also have the inequality given by the Lax-Milgram lemma for the first step:

$$\|\gamma_1^{int}u\|_{H^{-1/2}(\Gamma)} \le \frac{1}{c_1^V}\|(\frac{1}{2}I + K)g\|_{H^{1/2}(\Gamma)}.$$

Secondly, as shown in Subsection 1.5.3, the double layer potential is bounded. With the triangle inequality the asserted estimate can finally be proven:

$$\|\gamma_1^{int}u\|_{H^{-1/2}(\Gamma)} \le \frac{1}{c_1^V}\|g + \gamma_0^{int}Wg\|_{H^{1/2}(\Gamma)} \le \frac{1}{c_1^V}\left(1 + c_2^{\gamma_0^{int}W}\right)\|g\|_{H^{1/2}(\Gamma)}.$$

□

2.1.2 Direct Approach for the Poisson Problem

At the beginning, let us state the interior Dirichlet boundary value problem for the Poisson equation,

$$\begin{aligned} -\Delta u(x) &= f(x) \quad \text{for } x \in \Omega \subset \mathbb{R}^2, \\ \gamma_0^{int} u(x) &= g(x) \quad \text{for } x \in \Gamma. \end{aligned} \tag{2.6}$$

The difference to the Laplace problem (2.1) is in a given function f on the right hand side. The solution u in the domain Ω is then given by the representation formula (1.21),

$$u(x) = \int_\Gamma U^*(x,y)\gamma_1^{int}u(y)ds_y - \int_\Gamma \gamma_{1,y}^{int}U^*(x,y)g(y)ds_y + \int_\Omega U^*(x,y)f(y)dy.$$

To obtain the unknown Neumann datum $\gamma_1^{int}u$ on the boundary Γ, the application of the interior trace operator leads to the rewritten first boundary integral equation already given in Section 1.6,

$$(V\gamma_1^{int}u)(x) = (\frac{1}{2}I + K)\gamma_0^{int}u(x) - (N_0 f)(x) \quad \text{for } x \in \Gamma. \tag{2.7}$$

Theorem 2.3. *Let $g \in H^{1/2}(\Gamma)$ be the given Dirichlet datum, and $f \in \widetilde{H}^{-1}(\Omega)$ a given function on the right hand side of the Poisson problem (2.6). Assuming diam$(\Omega) < 1$, there exists a unique solution $\gamma_1^{int}u \in H^{-1/2}(\Gamma)$ of the variational formulation*

$$\langle V\gamma_1^{int}u, z\rangle_\Gamma = \langle(\frac{1}{2}I + K)g, z\rangle_\Gamma + \langle N_0 f, z\rangle_\Gamma \quad \textit{for all } z \in H^{-1/2}(\Gamma),$$

with

$$\|\gamma_1^{int}u\|_{H^{-1/2}(\Gamma)} \leq \frac{1}{c_1^V}\Big(1 + c_2^{\gamma_0^{int}W}\Big)\|g\|_{H^{1/2}(\Gamma)} + \frac{c_2^{N_0}}{c_1^V}\|f\|_{\widetilde{H}^{-1}(\Omega)}.$$

Proof. The unique solvability follows immediately as in the Theorems 2.1 and 2.2 for the indirect and the direct approach for the Laplace problem respectively from the lemma of Lax-Milgram (Theorem 1.14). As far as the stability analysis is concerned, similarly to the estimate in Theorem 2.2 we have

$$\|\gamma_1^{int}u\|_{H^{-1/2}(\Gamma)} \leq \frac{1}{c_1^V}\Big(1 + c_2^{\gamma_0^{int}W}\Big)\|g\|_{H^{1/2}(\Gamma)} + \frac{1}{c_1^V}\|N_0 f\|_{H^1(\Omega)}.$$

The boundedness of the Newton potential shown in Lemma 1.16 yields

$$\|\gamma_1^{int}u\|_{H^{-1/2}(\Gamma)} \leq \frac{1}{c_1^V}\Big(1 + c_2^{\gamma_0^{int}W}\Big)\|g\|_{H^{1/2}(\Gamma)} + \frac{c_2^{N_0}}{c_1^V}\|f\|_{\widetilde{H}^{-1}(\Omega)},$$

and the proof is complete. □

Remembering the direct approach for the Laplace equation in the Subsection 2.1.1 above, we additionally have to deal with the Newton potential. There are several methods to do so, intuitively one approach is a direct calculation. Nonetheless, we realise $N_0 f$ by the aid of a particular solution of the Poisson equation.

Calculation of the Newton Potential

We start with the Poisson equation (2.6) for a particular solution u_p,

$$-\Delta u_p(x) \;=\; f(x) \quad \text{for } x \in \Omega \subset \mathbb{R}^2,$$

with the same f on the right hand side as in the Poisson problem (2.6). Again, the rewritten first boundary integral equation gives

$$(V\gamma_1^{int}u_p)(x) = (\frac{1}{2}I + K)\gamma_0^{int}u_p(x) - (N_0 f)(x) \quad \text{for } x \in \Gamma,$$

and therefore

$$(N_0 f)(x) = (\frac{1}{2}I + K)\gamma_0^{int}u_p(x) - (V\gamma_1^{int}u_p)(x) \quad \text{for } x \in \Gamma. \tag{2.8}$$

This means that we formulate the Newton potential by the surface potentials, namely the single and double layer potential operator. We mind that we have to provide a particular solution u_p. Finally, we have to solve the following equation for the Neumann datum $\gamma_1^{int}u$ on the boundary Γ:

$$(V\gamma_1^{int}u)(x) = (\frac{1}{2}I + K)\gamma_0^{int}u(x) - (\frac{1}{2}I + K)\gamma_0^{int}u_p(x) + (V\gamma_1^{int}u_p)(x) \quad \text{for } x \in \Gamma.$$

Once the solution $\gamma_1^{int}u$ is determined, it will be inserted into the representation formula. For this purpose, let us consider the formula for the particular solution u_p of the Poisson problem,

$$u_p(x) = \int_\Gamma U^*(x,y)\gamma_1^{int}u_p(y)ds_y - \int_\Gamma \gamma_{1,y}^{int}U^*(x,y)\gamma_0^{int}u_p(y)ds_y + \int_\Omega U^*(x,y)f(y)dy$$

for $x \in \Omega$. In the end, by subtraction of the representation formula (1.21) of the original problem, we get the solution u of the Poisson problem (2.6) for $x \in \Omega$:

$$u(x) = u_p(x) + \int_\Gamma U^*(x,y)\gamma_1^{int}\big(u(y) - u_p(y)\big)ds_y - \int_\Gamma \gamma_{1,y}^{int}U^*(x,y)\big(g(y) - \gamma_0^{int}u_p(y)\big)ds_y,$$

or written in a more compact way,

$$u(x) = u_p(x) + (\widetilde{V}\gamma_1^{int}u)(x) - (\widetilde{V}\gamma_1^{int}u_p)(x) - (Wg)(x) + (W\gamma_0^{int}u_p)(x).$$

2.2 Exterior Dirichlet Boundary Value Problem

Boundary integral equation methods and boundary element methods are particularly suitable for boundary value problems on the exterior domain Ω^c. Once a computational program for an interior boundary value problem is implemented, there is no big additional

afford necessary to solve a similar exterior problem too. As a model problem, let us consider the exterior Dirichlet boundary value problem (1.10),

$$\begin{aligned} -\Delta u(x) &= 0 \qquad \text{for } x \in \Omega^c, \\ \gamma_0^{ext} u(x) &= g(x) \quad \text{for } x \in \Gamma = \partial\Omega, \end{aligned}$$

with the radiation condition (1.23) for the behavior of u at infinity. The solution of this problem is given by the representation formula (1.24),

$$u(x) = -\int_\Gamma U^*(x,y)\gamma_1^{ext} u(y) ds_y + \int_\Gamma \gamma_{1,y}^{ext} U^*(x,y)\gamma_0^{ext} u(y) ds_y \quad \text{for } x \in \Omega^c.$$

Calculating the limit $x \ni \Omega^c \to x \in \Gamma$, from the exterior Calderon projector (1.41) we obtain the first boundary integral equation to evaluate the missing exterior Neumann data on the boundary for the representation formula,

$$(V\gamma_1^{ext} u)(x) = (-\frac{1}{2}I + K)g(x) \quad \text{for } x \in \Gamma.$$

Multiplying with a test function z and integrating over the boundary Γ, we formulate the corresponding variational formulation to find $\gamma_1^{ext} u \in H^{-1/2}(\Gamma)$, such that

$$\langle V\gamma_1^{ext} u, z\rangle_\Gamma = \langle(-\frac{1}{2}I + K)g, z\rangle_\Gamma \quad \text{for all } z \in H^{-1/2}(\Gamma).$$

Since there is no essential difference to the variational formulation of the interior Dirichlet boundary value problem (2.5) as far as the unique solvability and the stability is concerned, the hypothesis of Theorem 2.2 can be applied analogously to the exterior problem.

2.3 Interface Problem

When talking about an *interface problem*, we have given as example a Poisson problem in the interior domain Ω and a Laplace problem in the exterior domain Ω^c,

$$\begin{aligned} -\alpha_1 \Delta u_1(x) &= f(x) \quad \text{for } x \in \Omega_1 = \Omega, \\ -\alpha_2 \Delta u_2(x) &= 0 \qquad \text{for } x \in \Omega_2 = \Omega^c. \end{aligned}$$

The function u_2 has to satisfy the decay condition (1.23) for $a = 0$, so

$$u(x) = \mathcal{O}\left(\frac{1}{|x|}\right) \quad \text{as } |x| \to \infty. \tag{2.9}$$

We further note that there is no data explicitly given at the boundary. Moreover, for the description of the continuity of the potential and the flux, we need some *transmission conditions* on the interface boundary $\Gamma = \partial\Omega_1 = \partial\Omega_2$,

$$\gamma_0^{int} u_1(x) = \gamma_0^{ext} u_2(x), \qquad \alpha_1 \gamma_1^{int} u_1(x) = \alpha_2 \gamma_1^{ext} u_2(x) \quad \text{for } x \in \Gamma. \tag{2.10}$$

For the description of the exterior trace operator and the conormal derivative, see Definition (1.6) and (1.8) respectively. As a side note, transmission conditions play an important role in domain decomposition methods, which will be introduced in Section 2.4.
On the one hand, we already know that the solution u_1 for the interior domain is given by the representation formula (1.21),

$$u_1(x) = \int_\Gamma U^*(x,y)\gamma_1^{int}u_1(y)ds_y - \int_\Gamma \gamma_{1,y}^{int}U^*(x,y)\gamma_0^{int}u_1(y)ds_y + \frac{1}{\alpha_1}\int_{\Omega_1} U^*(x,y)f(y)dy$$

for $x \in \Omega_1$. On the other hand, there is the representation formula (1.24) for the exterior domain:

$$u_2(x) = -\int_\Gamma U^*(x,y)\gamma_1^{ext}u_2(y)ds_y + \int_\Gamma \gamma_{1,y}^{ext}U^*(x,y)\gamma_0^{ext}u_2(y)ds_y \quad \text{for } x \in \Omega_2.$$

There is now one important remark to be made as far as the given function f on the right hand side of the Poisson problem is concerned. First of all, the far field condition (2.9) has to be fulfilled. Therefore, the conormal derivative has to satisfy the scaling condition

$$\alpha_2 \int_\Gamma \gamma_1^{ext}u_2(x)ds_x = 0, \tag{2.11}$$

see for example [34, Lemma 6.21]. Taking Green's first formula (1.19) into consideration, choosing as a test function $v = 1$, and combining the result with the transmission conditions (2.10), we obtain

$$-\alpha_2 \int_\Gamma \gamma_1^{ext}u_2(x)ds_x = \int_{\Omega_1} f(x)dx.$$

So we see, if we want to ensure the scaling condition (2.11), the *solvability condition*

$$\int_{\Omega_1} f(x)dx = 0$$

has to be satisfied. Thus, adopting the approach of [29], we get the scaling condition

$$\langle u_1, w_{eq}\rangle_\Gamma = 0$$

for a natural density $w_{eq} = V^{-1}1$. Therefore, we may introduce the space

$$H_*^{1/2} := \big\{v \in H^{1/2}(\Gamma) : \langle v, w_{eq}\rangle_\Gamma = 0\big\}.$$

Now let us turn to find a possibility to solve the interface problem. At first, we write down the first kind boundary integral equation (2.7) for the Poisson problem of the interior domain,

$$(V\gamma_1^{int}u_1)(x) = (\frac{1}{2}I + K)\gamma_0^{int}u_1(x) - \frac{1}{\alpha_1}(N_0 f)(x) \quad \text{for } x \in \Gamma.$$

Due to the invertibility of the single layer potential operator V, we obtain

$$\gamma_1^{int} u_1(x) = V^{-1}(\frac{1}{2}I + K)\gamma_0^{int} u_1(x) - \frac{1}{\alpha_1} V^{-1}(N_0 f)(x) \quad \text{for } x \in \Gamma. \tag{2.12}$$

As shown in Section 2.1.2, we can calculate the Newton potential operator $N_0 f$ on the boundary with the aid of a particular solution u_p of the Poisson equation $-\Delta u_p(x) = f(x)$ for $x \in \Omega_1$,

$$(N_0 f)(x) = (\frac{1}{2}I + K)\gamma_0^{int} u_p(x) - (V\gamma_1^{int} u_p)(x) \quad \text{for } x \in \Gamma.$$

Inserting this representation into equation (2.12), we obtain

$$\gamma_1^{int} u_1(x) = V^{-1}(\frac{1}{2}I + K)\gamma_0^{int} u_1(x) - \frac{1}{\alpha_1} V^{-1}(\frac{1}{2}I + K)\gamma_0^{int} u_p(x) + \frac{1}{\alpha_1}\gamma_1^{int} u_p(x)$$

for $x \in \Gamma$. Using the interior Steklov-Poincaré operator S^{int} introduced in Subsection 1.5.7, we get

$$\gamma_1^{int} u_1(x) = (S^{int}\gamma_0^{int} u_1)(x) - \frac{1}{\alpha_1}(S^{int}\gamma_0^{int} u_p)(x) + \frac{1}{\alpha_1}\gamma_1^{int} u_p(x) \quad \text{for } x \in \Gamma.$$

As far as the exterior domain Ω_2 is concerned, we have the rewritten boundary integral equation (1.32) for the exterior Laplace problem,

$$(V\gamma_1^{ext} u_2)(x) = (-\frac{1}{2}I + K)\gamma_0^{ext} u_2(x) \quad \text{for } x \in \Gamma,$$

and correspondingly

$$\gamma_1^{ext} u_2(x) = -V^{-1}(\frac{1}{2}I - K)\gamma_0^{ext} u_2(x) \quad \text{for } x \in \Gamma.$$

By the use of the exterior Steklov-Poincaré operator S^{ext} of Subsection 1.5.7, we obtain

$$\gamma_1^{ext} u_2(x) = -(S^{ext}\gamma_0^{ext} u_2)(x) \quad \text{for } x \in \Gamma.$$

To combine the exterior and interior problem, we rewrite the first transmission condition (2.10) as

$$u = \gamma_0^{int} u_1(x) = \gamma_0^{ext} u_2(x) \quad \text{for } x \in \Gamma.$$

With the second transmission condition, and putting all known data to the right hand side, we formulate our final interface problem:
Find $u \in H_*^{1/2}(\Gamma)$, such that

$$\alpha_1(S^{int}u)(x) + \alpha_2(S^{ext}u)(x) = (S^{int}\gamma_0^{int} u_p)(x) - \gamma_1^{int} u_p(x) \quad \text{for all } x \in \Gamma.$$

In the usual way, by multiplying with a test function $v \in H^{1/2}(\Gamma)$ and integrating over the boundary Γ, it results in the equivalent variational formulation to find $u \in H_*^{1/2}(\Gamma)$, satisfying

$$\left\langle (\alpha_1 S^{int} + \alpha_2 S^{ext})u, v \right\rangle_\Gamma = \left\langle S^{int}\gamma_0^{int} u_p - \gamma_1^{int} u_p, v \right\rangle_\Gamma \quad \text{for all } v \in H^{1/2}(\Gamma). \tag{2.13}$$

Theorem 2.4. *Let $u_p \in H^1(\Gamma)$ be given. Then there exists a unique solution $u \in H_*^{1/2}(\Gamma)$ of the variational formulation (2.13) with*

$$\|u\|_{H^{1/2}(\Gamma)} \leq c \, \|u_p\|_{H^1(\Omega)}.$$

Proof. Talking about unique solvability of the variational problem, firstly the interior Dirichlet to Neumann map S^{int} fulfills the semi-ellipticity condition, and the exterior Steklov-Poincaré operator is $H^{1/2}(\Gamma)$-elliptic. Secondly, both operators are bounded. For that reason, with the lemma of Lax-Milgram there exists a unique solution $u \in H_*^{1/2}(\Gamma)$. As far as the error estimate is concerned, we get

$$\|u\|_{H^{1/2}(\Gamma)} \leq \frac{1}{\min\{\alpha_1, \alpha_2\}} \frac{1}{\min\{c_1^D, c_1^{S^{ext}}\}} \Big(c_2^S \, \|\gamma_0^{int} u_p\|_{H^{1/2}(\Gamma)} + \|\gamma_1^{int} u_p\|_{H^{-1/2}(\Gamma)} \Big)$$

due to the constant c_1^D of the ellipticity estimate of S^{int} and the constant $c_1^{S^{ext}}$ of the ellipticity estimate of the exterior Steklov-Poincaré operator. For the next step, we take the square of the inequality above, and obtain for an alternative constant $\tilde{c}$

$$\begin{aligned} \|u\|^2_{H^{1/2}(\Gamma)} &\leq \tilde{c} \Big(\|\gamma_0^{int} u_p\|_{H^{1/2}(\Gamma)} + \|\gamma_1^{int} u_p\|_{H^{-1/2}(\Gamma)} \Big)^2 \\ &\leq 2\tilde{c} \Big(\|\gamma_0^{int} u_p\|^2_{H^{1/2}(\Gamma)} + \|\gamma_1^{int} u_p\|^2_{H^{-1/2}(\Gamma)} \Big) \end{aligned}$$

With a trace theorem [32, Theorem 2.9] and an inverse trace theorem [32, Theorem 2.10], we get

$$\|u\|^2_{H^{1/2}(\Gamma)} \leq \hat{c} \Big(\|u_p\|^2_{H^1(\Omega)} + |u_p|^2_{H^1(\Omega)} \Big).$$

Applying definition (1.11) concerning norms in $H^1(\Omega)$, and taking the square root of the finally obtained inequality, the theorem is proven. □

2.4 Domain Decomposition Methods

In this work, for domain decomposition methods we consider the Dirichlet boundary value problem

$$\begin{aligned} -\Delta u(x) &= 0 \qquad \text{for } x \in \Omega, \\ \gamma_0^{int} u(x) &= g(x) \quad \text{for } x \in \Gamma. \end{aligned} \tag{2.14}$$

Now let us start with a subdivision of the bounded Lipschitz domain Ω into p non-overlapping subdomains Ω_i, i.e.

$$\overline{\Omega} = \bigcup_{i=1}^{p} \overline{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset \quad \text{for } i \neq j.$$

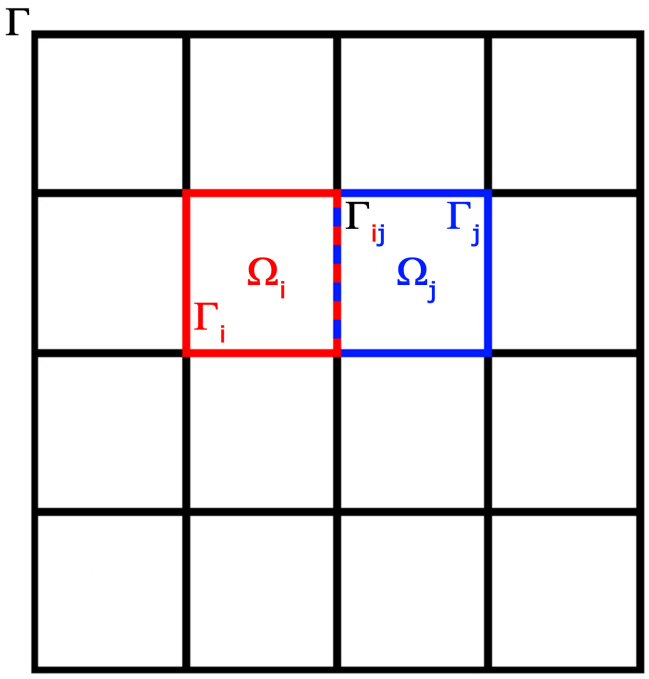

Figure 2.1: Coupling boundaries

Firstly, each subdomain has a corresponding subdomain boundary $\Gamma_i = \partial\Omega_i$, assuming they are Lipschitz again. Secondly, we get local interfaces between two neighboured subdomains,

$$\Gamma_{ij} = \Gamma_i \cap \Gamma_j \quad \text{for } i, j = 1, \ldots, p,$$

with $\Gamma_i \cap \Gamma_j = \emptyset$ for all non-overlapping subdomain boundaries. The global interface is then defined by

$$\Gamma_I = \bigcup_{i,j} \Gamma_{ij}.$$

Thirdly, the union of all local subdomain boundaries, or the global boundary and the global interface, define the *skeleton*

$$\Gamma_s = \bigcup_{i=1}^{p} \Gamma_i = \Gamma \cup \Gamma_I.$$

For the resulting decomposition of the boundary see Figure 2.1.
If we want to formulate the Laplace equation on each subdomain, we have to define local solutions u_i, i.e. $u_i(x) = u(x)$ for $x \in \Omega_i$, and we obtain

$$\begin{aligned} -\Delta u_i(x) &= 0 && \text{for } x \in \Omega_i, \\ \gamma_0^{int} u_i(x) &= g(x) && \text{for } x \in \Gamma \cap \Gamma_i. \end{aligned}$$

These local problems are connected via *transmission conditions* on the interface:

$$\gamma_{0,i}^{int} u_i(x) = \gamma_{0,j}^{int} u_j(x) \quad \text{and} \quad \gamma_{1,i}^{int} u_i(x) = -\gamma_{1,j}^{int} u_j(x) \quad \text{for } x \in \Gamma_{ij}, \tag{2.15}$$

where the second equation is due to the continuity of the flux. The next important step is to replace the conormal derivative by local Dirichlet to Neumann maps

$$\gamma_{1,i}^{int} u_i(x) = (S_i \gamma_{0,i}^{int} u_i)(x) = \Big(D_i + (\frac{1}{2} I + K_i')V_i^{-1}(\frac{1}{2} I + K_i) \Big) \gamma_{0,i}^{int} u_i(x).$$

Henceforward, for domain decomposition methods we denote the used operators restricted to subdomains Ω_i and subdomain boundaries Γ_i with some indices i. Moreover, let $u \in H^{1/2}(\Gamma_S)$ and $u_i(x) = u(x)$ for $x \in \Gamma_i$. That means the Dirichlet transmission condition is automatically fulfilled point-wise. To sum up, for $i = 1, \ldots, p$ we want so solve the local transmission problems

$$\begin{aligned} \gamma_{0,i}^{int} u_i(x) &= g(x) \quad \text{for } x \in \Gamma \cap \Gamma_i, \\ (S_i \gamma_{0,i}^{int} u_i)(x) + (S_j \gamma_{0,j}^{int} u_j)(x) &= 0 \quad \text{for } x \in \Gamma_{ij}. \end{aligned}$$

In what follows, we multiply the second transmission condition with a suitable test function and integrate over the local interface Γ_{ij}, which means that the Neumann coupling condition will be formulated in a weak sense:

$$\int_{\Gamma_{ij}} \Big((S_i \gamma_{0,i}^{int} u_i)(x) + (S_j \gamma_{0,j}^{int} u_j)(x) \Big) v(x) ds_x = 0 \quad \text{for all } v \in H^{1/2}(\Gamma_{ij}).$$

We define the function space

$$H^{1/2}(\Gamma_S, \Gamma) := \Big\{ v \in H^{1/2}(\Gamma_S) : \ v(x) = 0 \quad \text{for } x \in \Gamma \Big\},$$

and after summering up over all coupling boundaries Γ_{ij}, we obtain the variational formulation to find $u \in H^{1/2}(\Gamma_S)$ with $u(x) = g(x)$ for $x \in \Gamma$ and

$$\sum_{i=1}^{p} \int_{\Gamma_i} (S_i u_{|\Gamma_i})(x) v_{|\Gamma_i}(x) ds_x = 0 \quad \text{for all } v \in H^{1/2}(\Gamma_S, \Gamma).$$

Moreover, we define an arbitrary but fixed extension $\widetilde{g} \in H^{1/2}(\Gamma_S)$ of the given Dirichlet datum satisfying $\widetilde{g} = g$ on the global boundary Γ, and accordingly $u = \widetilde{u} + \widetilde{g} \in H^{1/2}(\Gamma_S)$. Then the final variational problem reads as follows:
Find $\widetilde{u} \in H^{1/2}(\Gamma_S, \Gamma)$, such that

$$\sum_{i=1}^{p} \int_{\Gamma_i} (S_i \widetilde{u}_{|\Gamma_i})(x) v_{|\Gamma_i}(x) ds_x = -\sum_{i=1}^{p} \int_{\Gamma_i} (S_i \widetilde{g}_{|\Gamma_i})(x) v_{|\Gamma_i}(x) ds_x \tag{2.16}$$

for all $v \in H^{1/2}(\Gamma_S, \Gamma)$.

Theorem 2.5. *([15]) The left hand side in (2.16) defines a global boundary integral bilinear form*

$$a(u,v) = \sum_{i=1}^{p} \int_{\Gamma_i} (S_i u_{|\Gamma_i})(x) v_{|\Gamma_i}(x) ds_x,$$

which is bounded and $H^{1/2}(\Gamma_S, \Gamma)$*- elliptic.*

Due to the lemma of Lax-Milgram (Theorem 1.14), the finally considered variational formulation (2.16) is uniquely solvable. On a second thought, we can not realise the Steklov-Poincaré operator in a direct way, because we would have to know the inverse of the single layer potential operator explicitly. That is why we have to deal with approximate operators, which will be done in the course of discretisations.

3 Boundary Element Methods

If we want to use boundary element methods to solve the problems described in Chapter 2, we have to deal with numerical methods for variational problems in general at first. After introducing convenient trial spaces, we write down discrete variational formulations and their equivalent linear systems of equations. As we know from Section 2.1, there exist indirect and direct approaches, which can be solved either by a collocation or a Galerkin method. For given functions on the boundary, there are various approximation techniques, mainly the piecewise constant and piecewise linear interpolation, as well as a L_2 projection, which all lead to different orders of convergence for the final error estimates.

3.1 Numerical Methods for Variational Problems

In this section we want to formulate general numerical procedures to solve variational formulations as introduced in Section 1.3. For this purpose we introduce some trial spaces, and we have to understand the relation between variational methods and resulting linear systems of equations. We start with a sequence

$$X_M := \{\varphi_i\}_{i=1}^M \subset X$$

of conformal trial spaces, with which a function u_M can be represented as

$$u_M := \sum_{j=1}^{M} u_j \varphi_j \in X_M.$$

Consequently, we rewrite the general variational formulation (1.15) to find $u \in X$, such that

$$\langle Au, v\rangle = \langle f, v\rangle \quad \text{for all } v \in X,$$

to find an approximate solution $u_M \in X_M$ of the *Galerkin-Bubnov variational formulation*

$$\langle Au_M, v_M\rangle = \langle f, v_M\rangle \quad \text{for all } v_M \in X_M. \tag{3.1}$$

If we choose the test function $v_M = \varphi_i$, the Galerkin problem (3.1) reads as

$$\sum_{j=1}^{M} u_j \langle A\varphi_j, \varphi_i\rangle = \langle f, \varphi_i\rangle \quad \text{for} \quad i = 1, \ldots, M.$$

More specially, with the entries

$$A_M[i,j] = \langle A\varphi_j, \varphi_i \rangle, \quad f_i = \langle f, \varphi_i \rangle$$

for $i,j = 1,\dots,M$ we obtain an equivalent linear system of equations

$$A_M \underline{u} = \underline{f}$$

for a vector $\underline{u} \in \mathbb{R}^M$. Furthermore, there exists a unique relation between an arbitrary function $v_M = \sum_{j=1}^{M} v_j \varphi_j \in X_M \subset X$ and a vector $\underline{v} \in \mathbb{R}^M$. Since

$$\begin{aligned} (A_M \underline{u}, \underline{v}) &= \sum_{i=1}^{M}\sum_{j=1}^{M} A[i,j] u_i v_j = \sum_{i=1}^{M}\sum_{j=1}^{M} \langle A\varphi_j, \varphi_i \rangle u_i v_j \\ &= \langle A \sum_{i=1}^{M} \varphi_i u_i, \sum_{j=1}^{M} \varphi_j v_j \rangle = \langle A u_M, v_M \rangle, \end{aligned}$$

we get two important properties of the matrix $A_M \in \mathbb{R}^{M\times M}$ from the operator $A : X \to X'$. Firstly, the positive definiteness of A_M follows from the X-ellipticity of A,

$$(A_M \underline{u}, \underline{u}) = \langle A u_M, u_M \rangle \geq c_1^A \, \|u_M\|_X^2.$$

Secondly, we gain the symmetry of the matrix with the help of the self-adjointness of the operator A,

$$(A_M \underline{u}, \underline{v}) = \langle A u_M, v_M \rangle = \langle u_M, A v_M \rangle = (\underline{u}, A_M \underline{v}).$$

Theorem 3.1 (Cea's lemma)**.** *Let $X_M \subset X$ and $A : X \to X'$ be a bounded operator with*

$$\|Av\|_{X'} \leq c_2^A \|v\|_X \quad \textit{for all } v \in X.$$

Furthermore, we assume the X-ellipticity of A. Then there exists a unique solution $u_M \in X_M$ of the discrete variational formulation (3.1), and it holds the stability estimate

$$\|u_M\|_X \leq \frac{1}{c_1^A} \|f\|_{X'}$$

as well as the error estimate

$$\|u - u_M\|_X \leq \frac{c_2^A}{c_1^A} \inf_{v_M \in X_M} \|u - v_M\|_X.$$

Proof. As mentioned above, the X-ellipticity of A implies the positive definiteness of the matrix A_M. Hence it follows the unique solvability of the equivalent system of linear equations. For the stability estimate we use the Cauchy-Schwarz inequality and the X-ellipticity of A, whereas for the error estimate we need a so-called Galerkin orthogonality. For an explicit proof see for example [32]. □

Looking at the given error estimate in Cea's lemma, for a convergence of the approximate solution u_M to $u \in X$, the trial space X_M has to satisfy the *approximation property*

$$\lim_{M\to\infty} \sup_{v\in X} \inf_{v_M \in X_M} \|v - v_M\|_X = 0.$$

As far as the boundary elements are concerned, there exist approximation properties for the corresponding local polynomial trial spaces, which will be shown in Section 3.2.

Approximation of the Right Hand Side

In this work, we nearly always use approximations of the linear form f of the right hand side in equation (3.1). This is why we introduce an operator $B : Y \to X'$ with $f = Bg$ for a given function $g \in Y$, which is bounded with

$$\|Bg\|_{X'} \le c_2^B \|g\|_Y \quad \text{for all } g \in Y.$$

Inserting the representation $f = Bg$ into the Galerkin-Bubnov variational formulation, we have to evaluate the vector values

$$f_i = \langle Bg, \varphi_i \rangle \quad \text{for all } i = 1, \dots, M.$$

Now let

$$g_N := \sum_{l=1}^{N} g_l \psi_l \in Y_N \subset Y$$

be an approximation of g. Then we have to solve the perturbed variational formulation to find an approximate solution $\widetilde{u}_M \in X_M$ in

$$\langle A\widetilde{u}_M, v_M \rangle = \langle Bg_N, v_M \rangle \quad \text{for all } v_M \in X_M, \tag{3.2}$$

which is equivalent to the linear system

$$A_M \underline{\widetilde{u}} = B_N \underline{g} =: \underline{\widetilde{f}}. \tag{3.3}$$

The matrix A_M is the same as for the non-perturbed problem, and for B_N we have

$$B_N[i,l] = \langle B\psi_l, \varphi_i \rangle,$$

resulting from

$$\widetilde{f}_i = \langle Bg_N, \varphi_i \rangle = \sum_{l=1}^{N} g_l \langle B\psi_l, \varphi_i \rangle.$$

Theorem 3.2 (Strang-Lemma). *Let $A : X \to X'$ be a bounded and X-elliptic operator. Moreover, let $u \in X$ be the unique solution of the continuous variational formulation (1.15) and $u_M \in X_M \subset X$ be the unique solution of the Galerkin-Bubnov problem (3.3). Then $\widetilde{u}_M \in X_M$ is the unique solution of the perturbed variational formulation (3.2), and it holds*

$$\|u - \widetilde{u}_M\|_X \leq \frac{1}{c_1^A}\Big(c_2^A \inf_{v_M \in X_M} \|u - v_M\|_X + c_2^B \|g - g_N\|_Y\Big).$$

Proof. For the unique solvability we have the same arguments as for Cea's lemma, mainly that the matrix A_M is positive definite. For the proof of the error estimate, the first step is the triangle inequality,

$$\|u - \widetilde{u}_M\|_X \leq \|u - u_M\|_X + \|u_M - \widetilde{u}_M\|_X.$$

The first term on the right hand side can be estimated with Cea's lemma. For the resulting error of the approximation of the linear form, see for example [32]. □

Approximate Operators

As we will see later on, we need some approximations of integral operators too, because it may be too much effort to calculate the original operator directly. Instead of the usual variational formulation (3.1) we consider the perturbed problem to find $\hat{u}_M \in X_M$, satisfying

$$\langle \widetilde{A}\hat{u}_M, v_M\rangle = \langle f, v_M\rangle \quad \text{for all } v_M \in X_M, \tag{3.4}$$

with the corresponding bounded approximate operator $\widetilde{A} : X \to X'$. Another variant of the well known Strang lemma yields the unique solvability as well as a first error estimate.

Theorem 3.3 (Strang-Lemma). *Let $\widetilde{A}$ be a bounded operator with*

$$\|\widetilde{A}v\|_{X'} \leq c_2^{\tilde{A}} \|v\|_X \quad \textit{for all } v \in X.$$

Moreover, we assume that $\widetilde{A}$ is X_M-elliptic, i.e.

$$\langle \widetilde{A}v_M, v_M\rangle \geq c_1^{\tilde{A}} \|v_M\|_X^2 \quad \textit{for all } v_M \in X_M.$$

Then the perturbed variational formulation (3.4) is uniquely solvable, and it holds

$$\|u - \hat{u}_M\|_X \leq c_1 \inf_{v_M \in X_M} \|u - v_M\|_X + c_2 \|(A - \widetilde{A})u\|_{X'}.$$

Proof. Since it is assumed that the approximate operator is bounded and X_M-elliptic, Cea's lemma yields the unique solvability of (3.4). Furthermore, with the triangle inequality we get

$$\|u - \hat{u}_M\|_X \leq \|u - u_M\|_X + \|u_M - \hat{u}_M\|_X.$$

For the first difference we have the error estimate given by Cea's lemma. For the proof of the second term in the estimate, see for example [32]. □

3.2 Boundary Discretisation and Trial Spaces

Let Γ be a piecewise polygonal boundary, and let

$$\Gamma = \bigcup_{i=1}^{M} \overline{\tau}_i \tag{3.5}$$

be a discretisation of the boundary with straight lines as boundary elements τ_i. The local mesh size is

$$h_i = \int_{\tau_i} ds_x.$$

With respect to the decomposition of Γ we introduce trial spaces of local polynomial functions, mainly the space $S_h^0(\Gamma)$ of piecewise constant functions, and $S_h^1(\Gamma)$ as the space of piecewise linear, globally continuous functions, respectively.
So let

$$S_h^0(\Gamma) := \quad \{\varphi_j^0\}_{j=1}^M \subset H^{-1/2}(\Gamma)$$

be the space of piecewise constant basis functions

$$\varphi_j^0(x) = \begin{cases} 1 & \text{for } x \in \tau_j, \\ 0 & \text{elsewhere.} \end{cases}$$

For this reason, an approximate solution $z_h(x) \in S_h^0(\Gamma)$ can be written as

$$z_h(x) = \sum_{j=1}^{M} z_j \varphi_j^0(x).$$

As fas as the approximation property of $S_h^0(\Gamma)$ is concerned, we introduce the L_2 projection $Q_h w \in S_h^0(\Gamma)$ of $w \in L_2(\Gamma)$ as the unique solution of the variational problem to find $Q_h w$, with

$$\langle Q_h w, z_h \rangle_\Gamma = \langle w, z_h \rangle_\Gamma \quad \text{for all } z_h \in S_h^0(\Gamma). \tag{3.6}$$

To have a closer look at the variational formulation for the L_2 projection, equation (3.6) is equivalent to

$$\sum_{j=1}^{M} w_j \langle \varphi_j^0, \varphi_i^0 \rangle_\Gamma = \langle w, \varphi_i^0 \rangle_\Gamma \quad \text{for } i = 1, \ldots, M.$$

In this case, since

$$\langle \varphi_j^0, \varphi_i^0 \rangle_\Gamma = \int_{\tau_j} \varphi_j^0(x) \varphi_i^0(x) ds_x = \begin{cases} h_j & \text{for } j = i, \\ 0 & \text{for } j \neq i, \end{cases}$$

the components of the resulting coefficient vector are

$$w_j = \frac{1}{h_j} \int_{\tau_j} w(x) ds_x \quad \text{for } j = 1, \ldots, M.$$

We note that for the L_2 projection $Q_h w$ the function w has to be given explicitly. For convergence discussions, there is the important *approximation property of the trial space* $S_h^0(\Gamma)$, see for example [26]:

Theorem 3.4. *Let $\sigma \in [-1, 0]$ and $s \in [0, 1]$. For $w \in H^s_{pw}(\Gamma)$ there holds*

$$\inf_{z_h \in S_h^0(\Gamma)} \|w - z_h\|_{H^\sigma(\Gamma)} \leq c\, h^{s-\sigma}\, |w|_{H^s_{pw}(\Gamma)}.$$

As already mentioned at the beginning of this section, we also introduce the trial space

$$S_h^1(\Gamma) := \{\varphi_l^1\}_{l=1}^N \subset H^{1/2}(\Gamma).$$

Its basis functions are described in the nodes x_l of the decomposition of the boundary (3.5),

$$\varphi_l^1(x) = \begin{cases} 1 & \text{for } x = x_l, \\ 0 & \text{for } x = x_k \neq x_l, \\ \text{piecewise linear} & \text{elsewhere.} \end{cases}$$

Hence, a function $v_h \in S_h^1(\Gamma)$ can be represented by

$$v_h(x) = \sum_{l=1}^{N} v_l \varphi_l^1(x).$$

Again, to formulate the approximation property for $S_h^1(\Gamma)$, we explain the L_2 projection $Q_h u \in S_h^1(\Gamma)$ of $u \in L_2(\Gamma)$ as the unique solution of the variational formulation

$$\langle Q_h u, v_h \rangle_\Gamma = \langle u, v_h \rangle_\Gamma \quad \text{for all } v_h \in S_h^1(\Gamma).$$

At the end of this section we state the *approximation property for piecewise linear, globally continuous trial space* $S_h^1(\Gamma)$, for example given in [32]:

Theorem 3.5. *Let $\sigma \in [0, 1]$ and $s \in [\sigma, 2]$. For a sufficiently smooth boundary Γ, there holds*

$$\inf_{v_h \in S_h^1(\Gamma)} \|u - v_h\|_{H^\sigma(\Gamma)} \leq c\, h^{s-\sigma}\, |u|_{H^s(\Gamma)}$$

for $u \in H^s(\Gamma)$.

3.3 Interior Dirichlet Problem

In Section 2.1 we introduced two main approaches to solve an interior Dirichlet boundary value problem. We begin with the indirect approach for the interior Laplace equation and the different possibilities to solve the resulting problems, namely the collocation and Galerkin method. Afterwards we will deal with the direct approach. All together, there are several ways to calculate the given Dirichlet datum on the boundary. Apparently, if a function is explicitly given on the boundary, we can use the exact evaluation in every point on Γ. Nevertheless, there are approximations of the Dirichlet datum like a linear interpolation and a L_2 projection, which yield different convergence rates of the caused error.

3.3.1 Solving with the Indirect Approach

Let us begin with the interior Laplace problem

$$\begin{aligned} -\Delta u(x) &= 0 \qquad \text{for } x \in \Omega, \\ \gamma_0^{int} u(x) &= g(x) \quad \text{for } x \in \Gamma. \end{aligned}$$

Since the single layer potential is harmonic, we want to solve the integral equation

$$(Vw)(x) = -\frac{1}{2\pi} \int_\Gamma \log|x-y| w(y) ds_y = g(x) \quad \text{for } x \in \Gamma. \tag{3.7}$$

By using the trial space $S_h^0(\Gamma)$ introduced in Section 3.2, we replace the density function w with the ansatz $w_h(x) = \sum_{j=1}^{M} w_j \varphi_j^0(x)$, which results in

$$-\frac{1}{2\pi} \int_\Gamma \log|x-y| w_h(y) ds_y \approx g(x) \quad \text{for } x \in \Gamma. \tag{3.8}$$

Therefore, an approximate solution of the Laplace problem is given by

$$\tilde{u}(x) = -\frac{1}{2\pi} \int_\Gamma \log|\tilde{x}-y| w_h(y) ds_y \quad \text{for } x \in \Omega.$$

Hence, the solution is just an approximation, so we have to treat the resulting error estimates afterwards. In this work, we are basically interested in the point-wise error $|u(x) - \widetilde{u}(x)|$ for x in the domain Ω. We see that for deeper discussions it depends on how the approximation $\widetilde{u}$ looks like.

Collocation Method

As far as the collocation method is concerned, equation (3.8) is considered just in the centers of the boundary elements τ_i, namely in the collocation points x_i^*. Accordingly, we

have to find the solution $w_h \in S_h^0(\Gamma)$ of

$$(Vw_h)(x_i^*) = \sum_{j=1}^{M} w_j \cdot \Big(-\frac{1}{2\pi} \int_{\tau_j} \log|x_i^* - y| \, ds_y \Big) = g(x_i^*) \tag{3.9}$$

for $i = 1, \ldots, M$. This is equivalent to the linear system of equations

$$V_h^C \underline{w} = \underline{g}$$

for vectors $\underline{w}, \underline{g} \in \mathbb{R}^M$ and matrix entries

$$V_h^C[i,j] = -\frac{1}{2\pi} \int_{\tau_j} \log|x_i^* - y| ds_y \quad \text{for } i,j = 1, \ldots, M.$$

Talking about an according error estimate for the collocation technique, by subtracting the analytic representation formula from the approximated one, at first we have

$$|u(x) - \tilde{u}(x)| \leq \|U^*(x,\cdot)\|_{H^{-\sigma}(\Gamma)} \|w - w_h\|_{H^{\sigma}(\Gamma)}.$$

Since the evaluation of the representation formula is done in the domain Ω, there is no singularity in the fundamental solution. Thus, $U^*(x,y)$ is infinitely differentiable, and as a consequence $U^*(x,\cdot) \in H^{-\sigma}(\Gamma)$ for any $\sigma \in \mathbb{R}$. That is why we consider $\|U^*(x,\cdot)\|_{H^{-\sigma}(\Gamma)}$ as a constant term. Besides, this assumption is not true for the limiting case $x \in \Gamma$. All in all, the main focus is on estimating $\|w - w_h\|_{H^{\sigma}(\Gamma)}$. Assuming the stability of the collocation scheme (3.9), the quasi optimal error estimate in Cea's lemma, Theorem 3.1, yields

$$\|w - w_h\|_{H^{-1/2}(\Gamma)} \leq c \inf_{v_h \in S_h^0(\Gamma)} \|w - v_h\|_{H^{-1/2}(\Gamma)}.$$

In the next step we can use the approximation property (3.4) for the trial space of piecewise constant functions for $\sigma = -1/2$,

$$\|w - w_h\|_{H^{-1/2}(\Gamma)} \leq c\, h^{s+1/2} \, |w|_{H^s_{pw}(\Gamma)},$$

with $w \in H^s_{pw}(\Gamma)$ for some $s \in [0,1]$. Applying the Aubin-Nitsche Trick [14], we obtain

$$\|w - w_h\|_{H^{\sigma}(\Gamma)} \leq c\, h^{s-\sigma} \, |w|_{H^s_{pw}(\Gamma)},$$

when we still assume $w \in H^s_{pw}(\Gamma)$ with $s \in [0,1]$, but $\sigma \in [-1, -1/2]$. By using the collocation method, a lower value of σ is not possible [34]. All in all, that means we can choose the lowest value $\sigma = -1$, and we have

$$|u(x) - \tilde{u}(x)| \leq \|U^*(x,\cdot)\|_{H^{-\sigma}(\Gamma)} \|w - w_h\|_{H^{\sigma}(\Gamma)} \leq c\, h^{s+1} \, |w|_{H^s_{pw}(\Gamma)}.$$

If $w \in H^1_{pw}$, e.g. the density function is sufficiently smooth, we get the final h^2-convergence of the error estimate by

$$|u(x) - \tilde{u}(x)| \leq c\, h^2 \, |w|_{H^1_{pw}(\Gamma)}. \tag{3.10}$$

Galerkin Method

Using the Galerkin Bubnov method, we have to multiply the single layer equation (3.7) with a test function φ_i^0, integrate over the boundary Γ and insert the ansatz $w_h \in S_h^0(\Gamma)$:

$$\sum_{j=1}^{M} w_j \cdot \Big(-\frac{1}{2\pi} \int_{\tau_i} \int_{\tau_j} \log|x-y| ds_y ds_x \Big) \;=\; \int_{\tau_i} g(x) ds_x \quad \text{for } i = 1, \ldots, M,$$

which is equivalent to the linear system

$$V_h^G \underline{w} \;=\; \underline{g}, \tag{3.11}$$

with the matrix entries

$$V_h^G[i,j] = \langle V\varphi_j^0, \varphi_i^0 \rangle_\Gamma = -\frac{1}{2\pi} \int_{\tau_i} \int_{\tau_j} \log|x-y| \, ds_y \, ds_x \tag{3.12}$$

for $i, j = 1, \ldots, M$. The matrix $V_h^G \in \mathbb{R}^{M \times M}$ is symmetric and positive definite, so we use the conjugate gradient scheme to calculate the uniquely defined solution vector $\underline{w}$. As far as the error analysis is concerned, we use the Aubin-Nitsche trick, where in its proof for the Galerkin method the so-called Galerkin orthogonality can be used and we obtain

$$\|w - w_h\|_{H^\sigma(\Gamma)} \le c \; h^{s-\sigma} |w|_{H^s_{pw}(\Gamma)}, \tag{3.13}$$

if we assume $w \in H^s_{pw}(\Gamma)$ for some $s \in [0,1]$, but now $\sigma \in [-2, -1/2]$. For the lowest value $\sigma = -2$, we then get

$$|u(x) - \tilde{u}(x)| \le \|U^*(x,\cdot)\|_{H^{-\sigma}(\Gamma)} \|w - w_h\|_{H^\sigma(\Gamma)} \le c \, h^{s+2} \, |w|_{H^s_{pw}(\Gamma)}, \tag{3.14}$$

and finally for $w \in H^1_{pw}(\Gamma)$

$$|u(x) - \tilde{u}(x)| \le c \, h^3 \, |w|_{H^1_{pw}(\Gamma)}.$$

Up to now, we have always assumed a given function g on the boundary. Essentially, more often an approximation of the Dirichlet datum is applied. Let us begin by using a linear interpolation $g_h \in S_h^1(\Gamma)$ of g, i.e.

$$g_h(x) = \sum_{l=1}^{N} g(x_l) \varphi_l^1(x) \quad \text{for } x \in \Gamma.$$

This results in a changed right hand side in system (3.11),

$$\int_{\tau_i} g_h(x) \, ds_x = \sum_{l=1}^{N} g(x_l) \int_{\tau_i} \varphi_l^1(x) ds_x \quad \text{for } i = 1, \ldots, M.$$

We write the resulting perturbed system with the new solution vector $\widetilde{w}$ as

$$V_h^G \underline{\widetilde{w}} \;=\; M_h \underline{g},$$

with the mass matrix $M_h \in \mathbb{R}^{M\times N}$,

$$M_h[i,l] = \int_{\tau_i} \varphi_l^1(x) ds_x \tag{3.15}$$

for $i = 1,\ldots,M$ and $l = 1,\ldots,N$.
The Strang lemma, Theorem 3.2, yields a corresponding estimate, where the caused error of the approximated function g on the boundary is involved,

$$\|w - \widetilde{w}_h\|_{H^\sigma(\Gamma)} \le c_1 \|w - w_h\|_{H^\sigma(\Gamma)} + c_2 \|g - g_h\|_{H^{\sigma+1}(\Gamma)},$$

for $\sigma \in [-1,0]$. Combining the estimates for the non-perturbed formulation (3.13) and for the interpolation in general, we get

$$\|w - \widetilde{w}_h\|_{H^\sigma(\Gamma)} \le c\, h^{s-\sigma} \Big(|w|_{H^s_{pw}(\Gamma)} + |g|_{H^{s+1}(\Gamma)} \Big),$$

for $s \in [0,1]$ and assuming $w \in H^s_{pw}(\Gamma)$ and $g \in H^{s+1}(\Gamma)$, respectively. Hence, an optimal error estimate can be reached by choosing $s = 1$, therefore $w \in H^1_{pw}(\Gamma)$ and $g \in H^2(\Gamma)$, and $\sigma = -1$. Moreover, for the approximated solution of the Laplace problem

$$\hat{u}(x) = -\frac{1}{2\pi} \int_\Gamma \log|\tilde{x} - y| \widetilde{w}_h(y) ds_y \quad \text{for } x \in \Omega,$$

this results in a quadratic convergence of the point-wise error estimate

$$\begin{aligned} |u(x) - \hat{u}(x)| \;&\le\; \|U^*(x,\cdot)\|_{H^1(\Gamma)} \|w - \widetilde{w}_h\|_{H^{-1}(\Gamma)} \\ &\le\; c\, h^2 \Big(|w|_{H^1_{pw}(\Gamma)} + |g|_{H^2(\Gamma)} \Big). \end{aligned}$$

Finally, another possibility for an approximation of the Dirichlet datum is a L_2 projection $g_h = \sum_{l=1}^{N} g_l \varphi_l^1$, satisfying

$$\sum_{l=1}^{N} g_l \langle \varphi_l^1, \varphi_k^1 \rangle_\Gamma = \langle g, \varphi_k^1 \rangle_\Gamma \quad \text{for } k = 1,\ldots,N.$$

Therefore, we can choose $\sigma = -2$ for a best convergence rate in the Strang lemma, Theorem 3.2, yielding

$$|u(x) - \hat{u}(x)| \;\le\; c\, h^3 \Big(|w|_{H^1_{pw}(\Gamma)} + |g|_{H^2(\Gamma)} \Big),$$

if we assume $w \in H^1_{pw}(\Gamma)$ and $g \in H^2(\Gamma)$.

3.3.2 Solving with the Direct Approach

For the direct approach, as a model problem we consider the Poisson equation

$$\begin{aligned} -\Delta u(x) &= f(x) \quad \text{for } x \in \Omega, \\ \gamma_0^{int} u(x) &= g(x) \quad \text{for } x \in \Gamma. \end{aligned}$$

The solution is given by the representation formula (1.21),

$$u(x) = \int_\Gamma U^*(x,y)\gamma_1^{int}u(y)ds_y - \int_\Gamma \gamma_{1,y}^{int}U^*(x,y)g(y)ds_y + \int_\Omega U^*(x,y)f(y)dy$$

for $x \in \Omega$. To determine the unknown Neumann data, we solve

$$(V\gamma_1^{int}u)(x) = (\frac{1}{2}I + K)g(x) - (N_0 f)(x) \quad \text{for } x \in \Gamma.$$

Inserting the approximation $t_h = \sum_{j=1}^M t_j \varphi_j^0 \in S_h^0(\Gamma)$ for the conormal derivative $\gamma_1^{int}u$, we obtain

$$\begin{aligned} -\frac{1}{2\pi}\int_\Gamma \log|x-y| t_h(y)ds_y & \\ \approx \frac{1}{2}g(x) - \frac{1}{2\pi}\int_\Gamma \frac{(y-x,n_y)}{|x-y|^2}g(y)ds_y + \frac{1}{2\pi}\int_\Gamma \log|x-y| f(y)ds_y & \end{aligned}$$

for $x \in \Gamma$. Compared to the indirect approach, the main difference is in the right hand side of the system to be solved. As shown for the Poisson equation in Subsection 2.1.2, we calculate the Newton potential $N_0 f$ via the approach with a particular solution $g_p = \gamma_0^{int}u_p$ and its corresponding conormal derivative $t_p = \gamma_1^{int}u_p$ on the boundary, respectively. In this work, we always use a L_2 projection of both, i.e.

$$g_{p,h}(x) = \sum_{l=1}^N g_{p,l}\varphi_l^1(x) \in S_h^1(\Gamma) \quad \text{and} \quad t_{p,h}(x) = \sum_{j=1}^M t_{p,j}\varphi_j^0(x) \in S_h^0(\Gamma),$$

satisfying

$$\langle g_{p,h}, \varphi_k^1\rangle_\Gamma = \langle g_p, \varphi_k^1\rangle_\Gamma \quad \text{for } k = 1,\dots,N,$$

and

$$\langle t_{p,h}, \varphi_i^0\rangle_\Gamma = \langle t_p, \varphi_i^1\rangle_\Gamma \quad \text{for } i = 1,\dots,M,$$

respectively. To sum up, the corresponding approximate boundary integral equation reads as

$$\begin{aligned} -\frac{1}{2\pi}\int_\Gamma \log|x-y| t_h(y)ds_y \approx \frac{1}{2}g(x) - \frac{1}{2\pi}\int_\Gamma \frac{(y-x,n_y)}{|x-y|^2}g(y)ds_y & \\ -\left(\frac{1}{2}g_{p,h}(x) - \frac{1}{2\pi}\int_\Gamma \frac{(y-x,n_y)}{|x-y|^2}g_{p,h}(y)ds_y + \frac{1}{2\pi}\int_\Gamma \log|x-y| t_{p,h}(y)ds_y\right). & \end{aligned}$$

Moreover, we additionally describe the entries for the double layer matrix. Apart from this, the procedures for building the different systems of equations and error estimates for the collocation and Galerkin method are the same in principal.

Collocation Method

The boundary integral equation is given in the collocation nodes,

$$(Vt_h)(x_i^*) = \frac{1}{2}g(x_i^*) + (Kg)(x_i^*) - \Big(\frac{1}{2}g_{p,h}(x_i^*) + (Kg_{p,h})(x_i^*) - (Vt_{p,h})(x_i^*)\Big),$$

or rather

$$\begin{aligned}-\frac{1}{2\pi}\int_\Gamma \log|x_i^* - y| t_h(y) ds_y &= \frac{1}{2}g(x_i^*) - \frac{1}{2\pi}\int_\Gamma \frac{(y - x_i^*, n_y)}{|x_i^* - y|^2} g(y) ds_y \\ &\quad - \left(\frac{1}{2}g_{p,h}(x_i^*) - \frac{1}{2\pi}\int_\Gamma \frac{(y - x_i^*, n_y)}{|x_i^* - y|^2} g_{p,h}(y) ds_y + \frac{1}{2\pi}\int_\Gamma \log|x_i^* - y| t_{p,h}(y) ds_y\right)\end{aligned}$$

for $i = 1, \dots, M$. The matrix on the left hand side is the same as for the indirect method, so the solution is uniquely determined. The difference compared to the indirect approach is in another right hand side, firstly because of the double layer potential for the collocation method,

$$(Kg)(x_i^*) = -\frac{1}{2\pi}\int_\Gamma \frac{(y - x_i^*, n_y)}{|x_i^* - y|^2} g(y) ds_y.$$

Secondly we calculate the Newton potential by approximations too. The equivalent system reads as

$$V_h^C \underline{t} = \frac{1}{2}\underline{g} + K_h^C \underline{g} - \Big(\frac{1}{2}\underline{g}_p + K_h^C \underline{g}_p - V_h^C \underline{t}_p\Big),$$

with the double layer matrix entries for a piecewise linear ansatz

$$K_h^C[i,l] = -\frac{1}{2\pi}\int_\Gamma \frac{(y - x_i^*, n_y)}{|x_i^* - y|^2} \varphi_l^1(y) ds_y$$

for $i = 1, \dots, M$ and $l = 1, \dots, N$. The approximate representation formula is

$$\widetilde{u}(x) = u_p(x) + (\widetilde{V}t_h)(x) - (\widetilde{V}t_{p,h})(x) - (Wg)(x) + (Wg_{p,h})(x) \quad \text{for } x \in \Omega.$$

All in all, since we use L_2 projections of the particular solution and its conormal derivative, we end up in quadratic convergence as the best approximation for the point-wise error estimate,

$$|u(x) - \tilde{u}(x)| \leq c(x, t, t_p, g_p)\, h^2,$$

assuming $t, t_p \in H^1_{pw}(\Gamma)$ and $g_p \in H^2(\Gamma)$. For a greater overview, we only write a constant and its dependencies on the involved functions. We further note that we may also use a linear interpolation or a L_2 projection instead of the exact evaluation of the Dirichlet datum g, but there is no change in the h^2 convergence for the best point-wise error estimate.

Galerkin Method

The Galerkin-Bubnov formulation for the direct approach is to find the unique solution $t_h \in S_h^0(\Gamma)$, such that

$$\langle Vt_h, \varphi_i^0\rangle_\Gamma = \langle (\frac{1}{2}I + K)g, \varphi_i^0\rangle_\Gamma - \langle (\frac{1}{2}I + K)g_{p,h}, \varphi_i^0\rangle_\Gamma + \langle Vt_{p,h}, \varphi_i^0\rangle_\Gamma \tag{3.16}$$

for $i = 1, \ldots, M$. The first possibility is to use a piecewise linear and continuous function $g_h(x) = \sum_{l=1}^{N} g(x_l)\varphi_l^1(x) \in S_h^1(\Gamma)$ for the Dirichlet datum, which leads to the system of linear equations

$$V_h^G \underline{t} = \underline{f}.$$

The single layer matrix V_h^G (3.12) is the same as for the indirect Galerkin approach. The right hand side vector $\underline{f} \in \mathbb{R}^M$ is given by

$$\begin{aligned} f[i] = {} & \frac{1}{2}\sum_{l=1}^{N} g(x_l) \int_{\tau_i} \varphi_l^1(x) ds_x - \frac{1}{2\pi}\sum_{l=1}^{N} g(x_l) \int_{\tau_i}\int_{\Gamma} \frac{(y-x, n_y)}{|x-y|^2}\varphi_l^1(y) ds_y ds_x \\ & -\frac{1}{2}\sum_{l=1}^{N} g_p(x_l) \int_{\tau_i} \varphi_l^1(x) ds_x + \frac{1}{2\pi}\sum_{l=1}^{N} g_p(x_l) \int_{\tau_i}\int_{\Gamma} \frac{(y-x, n_y)}{|x-y|^2}\varphi_l^1(y) ds_y ds_x \\ & -\frac{1}{2\pi}\sum_{j=1}^{M} t_p(x_j) \int_{\tau_i}\int_{\tau_j} \log|x-y| ds_y ds_x \end{aligned}$$

for $i = 1, \ldots, M$. Using matrix notation, this becomes

$$\underline{f} = (\frac{1}{2}M_h + K_h)\underline{g} - (\frac{1}{2}M_h + K_h)\underline{g}_p + V_h^G \underline{t}_p,$$

with the mass matrix M_h (3.15) already known form the indirect approach, and the double layer matrix

$$K_h[i,l] = -\frac{1}{2\pi}\int_{\tau_i}\int_{\Gamma} \frac{(y-x, n_y)}{|x-y|^2}\varphi_l^1(y) ds_y ds_x$$

for $i = 1, \ldots, M$ and $l = 1, \ldots, N$, $K_h \in \mathbb{R}^{M\times N}$. Although the error estimates for the L_2 projected particular solution would yield cubic convergence in the best case, we only obtain h^2 convergence of the point-wise error of the approximated solution $\widetilde{u}$ because of the influence of the linear interpolation of the given boundary datum. Eventually, the second possibility for an approximation of the Dirichlet datum is a L_2 projection $g_h = \sum_{l=1}^{N} g_l\varphi_l^1$, satisfying

$$\sum_{l=1}^{N} g_l \langle \varphi_l^1, \varphi_k^1\rangle_\Gamma = \langle g, \varphi_k^1\rangle_\Gamma \quad \text{for } k = 1, \ldots, N.$$

The advantage is in a faster convergence of the error $\|g - g_h\|_{H^{\sigma+1}(\Gamma)}$ in the Strang lemma 3.2. Thus, if $t, t_p \in H^1_{pw}(\Gamma)$ and $g, g_p \in H^2(\Gamma)$, in the end we have

$$|u(x) - \hat{u}(x)| \leq c(x, t, g, t_p, g_p)\, h^3. \tag{3.17}$$

3.4 Exterior Dirichlet Problem

The exterior Dirichlet boundary value problem is

$$\begin{aligned} -\Delta u(x) &= 0 \qquad \text{for } x \in \Omega^c, \\ \gamma_0^{ext} u(x) &= g(x) \quad \text{for } x \in \Gamma, \end{aligned}$$

with the radiation condition

$$u(x) = a \log|x| + \mathcal{O}\left(\frac{1}{|x|}\right) \quad \text{as } |x| \to \infty$$

for some $a \in \mathbb{R}$. The solution is given by the exterior representation formula

$$u(x) = -\int_\Gamma U^*(x,y)\gamma_1^{ext} u(y) ds_y + \int_\Gamma \gamma_{1,y}^{ext} U^*(x,y) g(y) ds_y \quad \text{for } x \in \Omega^c,$$

where the missing Neumann datum $\gamma_1^{ext} u \in H^{-1/2}(\Gamma)$ can be determined by the first exterior boundary integral equation

$$(V\gamma_1^{ext} u)(x) = (-\frac{1}{2} I + K) g(x) \quad \text{for } x \in \Gamma.$$

We replace the Neumann datum on the boundary with the piecewise constant approximation $t_h(x) = \sum_{j=1}^{M} t(x_j)\varphi_j^0(x) \in S_h^0(\Gamma)$, and therefore we have to determine the uniquely defined solution t_h from the approximate boundary integral equation

$$-\frac{1}{2\pi}\int_\Gamma \log|x-y| t_h(y) ds_y \approx -\frac{1}{2} g(x) - \frac{1}{2\pi}\int_\Gamma \frac{(y-x, n_y)}{|x-y|^2} g(y) ds_y.$$

The only difference to the interior problems is the slightly changed right hand side. Consequently, we can use the already described direct and indirect approaches, with the exact, linear approximated or L_2 projected boundary datum, and we obtain the same corresponding systems of equations and error estimates.

3.5 Interface Problem

For the interface problem, we only use a direct approach and a corresponding Galerkin method. The problem is generally given by

$$\begin{aligned} -\alpha_1 \Delta u_1(x) &= f(x) \quad \text{for } x \in \Omega_1 = \Omega, \\ -\alpha_2 \Delta u_2(x) &= 0 \qquad \text{for } x \in \Omega_2 = \Omega^c. \end{aligned}$$

The solution u_2 has to fulfill the radiation condition (2.9). Moreover, the two separated domains and therefore the functions u_1 and u_2 are linked on the boundary via the transmission conditions

$$\begin{aligned} \gamma_0^{int} u_1(x) &= \gamma_0^{ext} u_2(x) \quad \text{for } x \in \Gamma, \\ \alpha_1 \gamma_1^{int} u_1(x) &= \alpha_2 \gamma_1^{ext} u_2(x) \quad \text{for } x \in \Gamma. \end{aligned}$$

The first equation yields $u(x) = \gamma_0^{int} u_1(x) = \gamma_0^{ext} u_2(x)$ for $x \in \Gamma$. For the Galerkin variational formulation we introduce the associated approximate function $u_h \in S_h^1(\Gamma)$ as well as L_2 projections of the particular solution $u_{p,h} \in S_h^1(\Gamma)$ and its conormal derivative $t_{p,h} \in S_h^0(\Gamma)$, respectively:

$$\left\langle (\alpha_1 S^{int} + \alpha_2 S^{ext}) u_h, \varphi_k^1 \right\rangle_\Gamma = \left\langle S^{int} u_{p,h} - t_{p,h}, \varphi_k^1 \right\rangle_\Gamma \quad \text{for } k = 1, \ldots, N.$$

This leads to the equivalent system of linear equations

$$S_h \underline{u} = \underline{f}.$$

Generally, the matrix on the left hand side is

$$S_h[k,l] = \left\langle (\alpha_1 S^{int} + \alpha_2 S^{ext}) \varphi_l^1, \varphi_k^1 \right\rangle_\Gamma \quad \text{for } k, l = 1, \ldots, N,$$

and we have $\underline{f}$ in components as

$$f[k] = \sum_{l=1}^{N} u_{p,l} \langle S^{int} \varphi_l^1, \varphi_k^1 \rangle_\Gamma - \sum_{j=1}^{M} t_{p,j} \langle \varphi_j^0, \varphi_k^1 \rangle_\Gamma \quad \text{for } k = 1, \ldots, N.$$

On the contrary, looking at the symmetric representation of the interior and exterior Steklov-Poincaré operator, they can not be evaluated in a direct way because we do not calculate the inverse of the single layer potential operator explicitely. Therefore, S^{int} and S^{ext} have to be evaluated much more over solving local variational formulations. So let us start with

$$\begin{aligned} (S^{int} u)(x) &= \Big(D + (\frac{1}{2} I + K') V^{-1} (\frac{1}{2} I + K) \Big) u(x) \\ &= (Du)(x) + (\frac{1}{2} I + K') t_1(x) \quad \text{for } x \in \Gamma, \end{aligned}$$

and

$$\begin{aligned}(S^{ext}u)(x) &= \Big(D + (\frac{1}{2}I - K')V^{-1}(\frac{1}{2}I - K)\Big)u(x)\\ &= (Du)(x) + (\frac{1}{2}I - K')t_2(x) \quad \text{for } x \in \Gamma,\end{aligned}$$

where the Neumann data $t_1 \in H^{-1/2}(\Gamma)$ is the unique solution of the interior variational problem

$$\langle Vt_1, z\rangle_\Gamma = \langle(\frac{1}{2}I + K)u, z\rangle_\Gamma \quad \text{for all } z \in H^{-1/2}(\Gamma),$$

and $t_2 \in H^{-1/2}(\Gamma)$ the unique solution of the exterior variational problem

$$\langle Vt_2, z\rangle_\Gamma = \langle(\frac{1}{2}I - K)u, z\rangle_\Gamma \quad \text{for all } z \in H^{-1/2}(\Gamma),$$

respectively. Using the above introduced discretisation techniques, we obtain the Galerkin equation to determine $t_{1,h} \in S_h^0(\Gamma)$ in

$$\langle Vt_{1,h}, z_h\rangle_\Gamma = \langle(\frac{1}{2}I + K)u_h, z_h\rangle_\Gamma \quad \text{for all } z_h \in S_h^0(\Gamma),$$

and $t_{2,h} \in S_h^0(\Gamma)$ via

$$\langle Vt_{2,h}, z_h\rangle_\Gamma = \langle(\frac{1}{2}I - K)u_h, z_h\rangle_\Gamma \quad \text{for all } z_h \in S_h^0(\Gamma), \tag{3.18}$$

respectively. Accordingly, we get changed approximations of the Steklov-Poincaré operators defined by

$$(\widetilde{S}^{int}u_h)(x) = (Du_h)(x) + (\frac{1}{2}I + K')t_{1,h}(x) \quad \text{for } x \in \Gamma,$$

and

$$(\widetilde{S}^{ext}u_h)(x) = (Du_h)(x) + (\frac{1}{2}I - K')t_{2,h}(x) \quad \text{for } x \in \Gamma.$$

For these approximate operators there holds the following theorem:

Theorem 3.6. *The interior and exterior approximate operator $\widetilde{S}^{int}$ and $\widetilde{S}^{ext}$, respectively, are bounded. Furthermore, $\widetilde{S}^{int}$ is elliptic on $H^{1/2}_{**}(\Gamma) := \big\{v \in H^{1/2}(\Gamma) : \langle v, 1\rangle_\Gamma = 0\big\}$, whereas $\widetilde{S}^{ext}$ is $H^{1/2}(\Gamma)$-ellpitic, so it holds*

$$\begin{aligned}\langle\widetilde{S}^{int}v, v\rangle_\Gamma &\geq c_1^{\widetilde{S}^{int}}\|v\|^2_{H^{1/2}(\Gamma)} \quad \textit{for all } v \in H^{1/2}_{**}(\Gamma),\\ \langle\widetilde{S}^{ext}v, v\rangle_\Gamma &\geq c_1^{\widetilde{S}^{ext}}\|v\|^2_{H^{1/2}(\Gamma)} \quad \textit{for all } v \in H^{1/2}(\Gamma).\end{aligned}$$

Moreover, both approximate operators satisfy the quasi-optimal error estimates

$$\begin{aligned}\|(S^{int} - \widetilde{S}^{int})v\|_{H^{-1/2}(\Gamma)} &\leq c \inf_{z_h \in S_h^0(\Gamma)}\|S^{int}v - z_h\|_{H^{-1/2}(\Gamma)},\\ \|(S^{ext} - \widetilde{S}^{ext})v\|_{H^{-1/2}(\Gamma)} &\leq c \inf_{z_h \in S_h^0(\Gamma)}\|S^{ext}v - z_h\|_{H^{-1/2}(\Gamma)}.\end{aligned}$$

Proof. Let us give a sketch of the proof. We begin with the boundedness of $\widetilde{S}^{ext}$ with the help of the variational formulation (3.18) and the ellipticity of the single layer potential operator V:

$$c_1^V \, \|t_{2,h}\|^2_{H^{-1/2}(\Gamma)} \;\leq\; \langle V t_{2,h}, t_{2,h}\rangle_\Gamma \overset{(3.18)}{=} \langle (\frac{1}{2} I - K) u_h, t_{2,h}\rangle_\Gamma.$$

Since the double layer potential is bounded, with Cauchy-Schwarz inequality we obtain

$$c_1^V \, \|t_{2,h}\|^2_{H^{-1/2}(\Gamma)} \leq \langle (\frac{1}{2} I - K) u_h, t_{2,h}\rangle_\Gamma \leq \Big(1 + c_2^{\gamma_0^{ext} W}\Big) \|u_h\|_{H^{1/2}(\Gamma)} \|t_{2,h}\|_{H^{-1/2}(\Gamma)},$$

and finally arrive at

$$\|t_{2,h}\|_{H^{-1/2}(\Gamma)} \leq \frac{1}{c_1^V}\Big(1 + c_2^{\gamma_0^{ext} W}\Big) \|u_h\|_{H^{1/2}(\Gamma)}. \tag{3.19}$$

With the representation of $\widetilde{S}^{ext}$, the boundedness of the hypersingular operator as well as with triangle inequality we conclude the corresponding stability estimate

$$\begin{aligned}
\|\widetilde{S}^{ext} u_h\|_{H^{-1/2}(\Gamma)} &\leq \|D u_h\|_{H^{-1/2}(\Gamma)} + \|(\frac{1}{2} I - K') t_{2,h}\|_{H^{-1/2}(\Gamma)} \\
&\leq c_2^D \, \|u_h\|_{H^{1/2}(\Gamma)} + c_2^{\gamma_1^{ext} \tilde{V}} \|t_{2,h}\|_{H^{-1/2}(\Gamma)} \\
&\overset{(3.19)}{\leq} c_2^D \, \|u_h\|_{H^{1/2}(\Gamma)} + c_2^{\gamma_1^{ext} \tilde{V}} \frac{1}{c_1^V}\Big(1 + c_2^{\gamma_0^{ext} W}\Big) \|u_h\|_{H^{1/2}(\Gamma)} \\
&= c \, \|u_h\|_{H^{1/2}(\Gamma)}.
\end{aligned}$$

As far as the boundedness of the interior approximate Steklov-Poincaré operator is concerned, the proof works analogously. For proving the ellipticity of $\widetilde{S}^{int}$, we use again the ellipticity property of V:

$$\begin{aligned}
\langle \widetilde{S}^{int} v, v\rangle_\Gamma &= \langle D v, v\rangle_\Gamma + \langle (\frac{1}{2} I + K') t_{1,h}, v\rangle_\Gamma \\
&= \langle D v, v\rangle_\Gamma + \langle t_{1,h}, (\frac{1}{2} I + K) v\rangle_\Gamma \\
&\overset{(3.18)}{=} \langle D v, v\rangle_\Gamma + \langle V t_{1,h}, t_{1,h}\rangle_\Gamma \\
&\geq \langle D v, v\rangle_\Gamma.
\end{aligned}$$

Using the mapping properties of the assembled local hypersingular integral operator, we conclude the ellipticity property for $\widetilde{S}^{int}$. For the $H^{1/2}(\Gamma)$-ellipticity of the exterior Steklov-Poincaré operator we refer to [24]. Eventually, see [33] for the quasi-optimal error estimates. □

The corresponding unique solution $\widetilde{u}_h \in S_h^1(\Gamma)$ may be evaluated by solving the perturbed formulation

$$\big\langle (\alpha_1 \widetilde{S}^{int} + \alpha_2 \widetilde{S}^{ext}) \widetilde{u}_h, \varphi_k^1 \big\rangle_\Gamma = \big\langle \widetilde{S}^{int} u_{p,h} - t_{p,h}, \varphi_k^1 \big\rangle_\Gamma \quad \text{for } k = 1, \ldots, N.$$

The equivalent linear system of equations reads as

$$\begin{aligned}\alpha_1\Big(D_h\widetilde{\underline{u}} + (\frac{1}{2}M_h^\top + K_h^\top)\underline{t}_1\Big) + \alpha_2\Big(D_h\widetilde{\underline{u}} + (\frac{1}{2}M_h^\top - K_h^\top)\underline{t}_2\Big)\\ = D_h\underline{u}_p + (\frac{1}{2}M_h^\top + K_h^\top)\underline{t}^* - M_h^\top\underline{t}_p,\end{aligned} \tag{3.20}$$

satisfying the local systems

$$\begin{aligned} V_h^G\underline{t}_1 &= (\frac{1}{2}M_h + K_h)\widetilde{\underline{u}},\\ V_h^G\underline{t}_2 &= (\frac{1}{2}M_h - K_h)\widetilde{\underline{u}},\\ V_h^G\underline{t}^* &= (\frac{1}{2}M_h + K_h)\underline{u}_p.\end{aligned} \tag{3.21}$$

The matrices $V_h^G \in \mathbb{R}^{M\times M}$, $M_h \in \mathbb{R}^{M\times N}$ and $K_h \in \mathbb{R}^{M\times N}$ are all well known from the previous sections. The only new one is the hypersingular matrix $D_h \in \mathbb{R}^{N\times N}$ with the entries

$$D_h[k,l] = \langle D\varphi_l^1, \varphi_k^1\rangle_\Gamma \quad \text{for } k,l = 1,\dots,N.$$

Nevertheless, with integration by parts, we can formulate D_h via the single layer matrix V_h (for further discussions see [7] and [31]):

$$\begin{aligned} D_h[k,l] &= \frac{1}{h_{l-1}h_{k-1}}\langle V\varphi_{l-1}^0, \varphi_{k-1}^0\rangle_\Gamma - \frac{1}{h_{l-1}h_k}\langle V\varphi_{l-1}^0, \varphi_k^0\rangle_\Gamma\\ &\quad -\frac{1}{h_l h_{k-1}}\langle V\varphi_l^0, \varphi_{k-1}^0\rangle_\Gamma + \frac{1}{h_l h_k}\langle V\varphi_l^0, \varphi_k^0\rangle_\Gamma\end{aligned}$$

for $k,l = 1,\dots,N$ and elements lengths h_k. System (3.20), derived from the second transmission condition, can now be computed in the following way: For each step in a conjugate gradient scheme which solves the global system (3.20), the solutions of the local systems (3.21) will be calculated by a Cholesky decomposition, since V_h^G is symmetric and positive definite.

As far as the point-wise error estimate is concerned, we have to deal with the solution of a perturbed system at first. Triangle inequality and mainly the Strang lemma for approximate operators, Theorem 3.3, yield

$$\begin{aligned}\|u - \widetilde{u}_h\|_{H^{1/2}(\Gamma)} &\le \|u - u_h\|_{H^{1/2}(\Gamma)} + \|u_h - \widetilde{u}_h\|_{H^{1/2}(\Gamma)}\\ &\le c_1 \inf_{v_h\in S_h^1(\Gamma)} \|u - v_h\|_{H^{1/2}(\Gamma)} + c_2\,\|(S^{int} - \widetilde{S}^{int})u\|_{H^{-1/2}(\Gamma)}\\ &\quad + c_3\,\|(S^{ext} - \widetilde{S}^{ext})u\|_{H^{-1/2}(\Gamma)}.\end{aligned}$$

Using the quasi optimal error estimates for the approximation of S^{int} and S^{ext} of Theorem 3.6, we obtain

$$\begin{aligned}\|u-\widetilde{u}_h\|_{H^{1/2}(\Gamma)} \;\leq\; & c\Big(\inf_{v_h\in S_h^1(\Gamma)}\|u-v_h\|_{H^{1/2}(\Gamma)} \\ & + \inf_{w_h\in S_h^0(\Gamma)}\|S^{int}u-w_h\|_{H^{-1/2}(\Gamma)} \\ & + \inf_{w_h\in S_h^0(\Gamma)}\|S^{ext}u-w_h\|_{H^{-1/2}(\Gamma)}\Big).\end{aligned}$$

As we know from previous sections, the best approximation can be reached by assuming $u\in H^2(\Gamma)$ and $S^{int}u, S^{ext}u\in H^1_{pw}(\Gamma)$. The final estimate is due to the Aubin-Nitsche trick:

$$\|u-\widetilde{u}_h\|_{L_2(\Gamma)}\leq c(u)\,h^2.$$

Once the solution $\widetilde{u}_h$ is found, we can calculate the missing Neumann data t_1 and t_2 with the help of the Poisson problem in the interior domain, and the Laplace equation in the exterior domain, respectively. Hence, the discussed techniques and error estimates for these final problems are still valid. We end this section by noting that with the here described methods there is no possibility to gain a better error analysis than a h^2 estimate, compared to the possible cubic convergence for the separated Poisson and Laplace problems. The reason is due to the approximation property for $S_h^0(\Gamma)$, where the best order of convergence for the approximation of the particular Neumann datum is quadratic.

3.6 Domain Decomposition Methods

In the analytical part of this work, we considered the local Dirichlet boundary value problems

$$\begin{aligned}-\Delta u_i(x) &= 0 \qquad \text{for } x\in\Omega_i,\\ \gamma_0^{int}u_i(x) &= g(x) \quad \text{for } x\in\Gamma\cap\Gamma_i,\end{aligned}$$

involving appropriate coupling conditions

$$\gamma_{0,i}^{int}u_i(x)=\gamma_{0,j}^{int}u_j(x)\quad\text{and}\quad\gamma_{1,i}^{int}u_i(x)=-\gamma_{1,j}^{int}u_j(x)\quad\text{for } x\in\Gamma_{ij}.$$

With the aid of the local Dirichlet to Neumann maps S_i we obtained the final Galerkin variational formulation (2.16) to find $\widetilde{u}\in H^{1/2}(\Gamma_S,\Gamma)$, such that

$$\sum_{i=1}^{p}\int_{\Gamma_i}(S_i\widetilde{u}_{|\Gamma_i})(x)v_{|\Gamma_i}(x)ds_x=-\sum_{i=1}^{p}\int_{\Gamma_i}(S_i\widetilde{g}_{|\Gamma_i})(x)v_{|\Gamma_i}(x)ds_x \tag{3.22}$$

for all $v\in H^{1/2}(\Gamma_S,\Gamma)$. For our usual discretisation techniques, we need a trial space of piecewise linear and continuous functions defined on the whole skeleton,

$$S_h^1(\Gamma_S)=\;\{\varphi_l^1\}_{l=1}^N\subset H^{1/2}(\Gamma_S,\Gamma),$$

as well as restrictions on the local subdomain boundaries,

$$\begin{aligned} S_h^0(\Gamma_i) = &\ \{\varphi_{i,j}^0\}_{j=1}^{M_i} \subset H^{-1/2}(\Gamma_i), \\ S_h^1(\Gamma_i) = &\ \{\varphi_{i,l}^1\}_{l=1}^{N_i} \subset H^{1/2}(\Gamma_i, \Gamma). \end{aligned}$$

Furthermore, $S_h^1(\Gamma)$ denotes the restriction of the defined trial space $S_h^1(\Gamma_S)$ onto the boundary Γ of the domain Ω. With these preliminaries, we can formulate the Galerkin boundary element discretisation of the variational problem (3.22):
Find $\widetilde{u}_h \in S_h^1(\Gamma_S)$, such that

$$\sum_{i=1}^{p} \int_{\Gamma_i} (S_i \widetilde{u}_{h|\Gamma_i})(x) v_{h|\Gamma_i}(x) ds_x = -\sum_{i=1}^{p} \int_{\Gamma_i} (S_i \widetilde{g}_{|\Gamma_i})(x) v_{h|\Gamma_i}(x) ds_x$$

is satisfied for all functions $v_h \in S_h^1(\Gamma_S)$. The equivalent linear system of equations is

$$S_h \underline{\widetilde{u}}_h = \underline{f},$$

with the global matrix on the left hand side,

$$S_h[k,l] = \sum_{i=1}^{p} \int_{\Gamma_i} (S_i \varphi_l^1)(x) \varphi_k^1(x) ds_x.$$

As we know, this matrix can not be calculated explicitly, because of the representation of the Steklov-Poincaré operator. Since we nearly always use the symmetric formulation, at first we have to solve the local variational problems

$$\langle V_i t_{i,h}, z_h \rangle_\Gamma = \langle (\frac{1}{2} I + K_i) u_{h|\Gamma_i}, z_h \rangle_\Gamma \quad \text{for all } z_h \in S_h^0(\Gamma_i). \tag{3.23}$$

Secondly, as in the case of an interface problem, we get approximate operators defined by

$$(\widetilde{S}_i u_{h|\Gamma_i})(x) = (D_i u_{h|\Gamma_i})(x) + (\frac{1}{2} I + K_i') t_{i,h}(x) \quad \text{for } x \in \Gamma_i.$$

In view of this, we now want to find a boundary element approximation $\hat{u}_h \in S_h^1(\Gamma_S)$ of

$$\sum_{i=1}^{p} \int_{\Gamma_i} (\widetilde{S}_i \hat{u}_{h|\Gamma_i})(x) v_{h|\Gamma_i}(x) ds_x = -\sum_{i=1}^{p} \int_{\Gamma_i} (\widetilde{S}_i \widetilde{g}_{|\Gamma_i})(x) v_{h|\Gamma_i}(x) ds_x \tag{3.24}$$

for all $v_h \in S_h^1(\Gamma_S)$.

Theorem 3.7. *Let*

$$\widetilde{a}(u_h, v_h) = \sum_{i=1}^{p} \int_{\Gamma_i} (\widetilde{S}_i u_{h|\Gamma_i})(x) v_{h|\Gamma_i}(x) ds_x$$

be the bilinear form of the corresponding variational problem (3.24). Then $\widetilde{a}(\cdot,\cdot)$ is bounded in $H^{1/2}(\Gamma_S)$ and $S_h^1(\Gamma_S)$-elliptic. Hence there exists a unique solution $\hat{u}_h \in S_h^1(\Gamma_S)$, satisfying

$$\|\widetilde{u} - \hat{u}_h\|_{H^{1/2}(\Gamma_S)} \leq c \left(\inf_{v_h \in S_h^1(\Gamma_S)} \|\widetilde{u} - v_h\|_{H^{1/2}(\Gamma_S)} + \sum_{i=1}^{p} \inf_{w_h \in S_h^0(\Gamma_i)} \|S_i\widetilde{u}_i - w_{i,h}\|_{H^{-1/2}(\Gamma_i)} \right).$$

Proof. Principally, the hypotheses of this theorem are due to the Strang lemma 3.3 for some perturbation of an elliptic bilinear form. With Theorem 3.6, the mapping properties of the local approximate Steklov-Poincaré operators as well as a summation over the subdomain boundaries yield the boundedness of $\widetilde{a}(\cdot,\cdot)$. Secondly, as far as the ellipticity is concerned, we have

$$\begin{aligned} \langle \widetilde{S}_i v, v\rangle_{\Gamma_i} &= \langle D_i v, v\rangle_{\Gamma_i} + \langle \big(\frac{1}{2} I + K_i'\big) t_{i,h}, v\rangle_{\Gamma_i} \\ &= \langle D_i v, v\rangle_{\Gamma_i} + \langle V_i t_{i,h}, t_{i,h}\rangle_{\Gamma_i} \geq \langle D_i v, v\rangle_{\Gamma_i} \end{aligned}$$

for all $v \in H^{1/2}(\Gamma_S, \Gamma)$. Thus, after summarising over all subdomain boundaries from 1 to p, and with the ellipticity properties of the local hypersingular integral operators we obtain the ellipticity of the bilinear form. Thirdly, for the error analysis we start with the estimate of the corresponding Strang lemma in combination with the triangle inequality in the second term for the local subdomain boundaries Γ_i:

$$\|\widetilde{u} - \hat{u}_h\|_{H^{1/2}(\Gamma_S)} \leq c \left(\inf_{v_h \in S_h^1(\Gamma_S)} \|\widetilde{u} - v_h\|_{H^{1/2}(\Gamma_S)} + \sum_{i=1}^{p} \|(S_i - \widetilde{S}_i)\widetilde{u}_i\|_{H^{-1/2}(\Gamma_i)} \right).$$

With the quasi-optimal error estimate of Theorem 3.6 for the local operators $\widetilde{S}_i$ we get the desired inequality. □

Hence, this theorem yields an optimal error estimate in the following way: assuming $S_i\widetilde{u}_i \in H^1_{pw}(\Gamma_i)$ and $\widetilde{u} \in H^2(\Gamma_S)$, and applying the approximation property (3.4) for the trial space of piecewise constant functions, we obtain the best convergence rate by

$$\|\widetilde{u} - \hat{u}_h\|_{H^{1/2}(\Gamma_S)} \leq c(\widetilde{u})\, h^{3/2}.$$

All in all, at the end we are interested in the final solution, namely the missing Neumann data on the subdomain boundaries Γ_i and accordingly on the boundary Γ. But as for the interface problem, once the solution u_h is found, we have to solve the local boundary value problems (3.23) and obtain the corresponding Neumann data. Again, the discussed solving techniques and error estimates for the local problems are still valid, so our best order of convergence is cubic.

The final point is to write down the according system to solve and its matrices. Firstly, we know that an approximation $v_h \in S_h^1(\Gamma_S)$ results in a coefficient vector $\underline{v} \in \mathbb{R}^N$. With the

help of the connectivity matrix A_i, the local restriction $v_{i,h} \in S_h^1(\Gamma_i)$ can be represented by some transformations of a global vector $\underline{v}$ to its local one,

$$A_i : \mathbb{R}^N \to \mathbb{R}^{N_i}, \tag{3.25}$$

$$\underline{v}_i = A_i \underline{v}. \tag{3.26}$$

Secondly, the Galerkin problem (3.24) is equivalent to the system of linear equations

$$\widetilde{S}_h \hat{\underline{u}} = \sum_{i=1}^{p} A_i^\top \widetilde{S}_{i,h} A_i \hat{\underline{u}} = -\sum_{i=1}^{p} A_i^\top \widetilde{S}_{i,h} A_i \underline{g} = -\widetilde{S}_h \underline{g}, \tag{3.27}$$

with the connectivity matrices A_i for $A_i \hat{\underline{u}} = \hat{\underline{u}}_i$. The local representations of $\widetilde{S}_{i,h}$ are

$$\begin{aligned} \widetilde{S}_{i,h} \hat{\underline{u}}_i &= D_{i,h} \hat{\underline{u}}_i + (\frac{1}{2} M_{i,h}^\top + K_{i,h}^\top) \underline{t}_i, \\ \widetilde{S}_{i,h} \underline{g}_i &= D_{i,h} \underline{g}_i + (\frac{1}{2} M_{i,h}^\top + K_{i,h}^\top) \underline{t}_i^*, \end{aligned}$$

respectively, where the local systems

$$\begin{aligned} V_{i,h}^G \underline{t}_i &= (\frac{1}{2} M_{i,h} + K_{i,h}) \hat{\underline{u}}_i, \\ V_{i,h}^G \underline{t}_i^* &= (\frac{1}{2} M_{i,h} + K_{i,h}) \underline{g}_i, \end{aligned}$$

have to be solved. For a better overview, we write down the already known local stiffness matrices

$$\begin{aligned} V_{i,h}^G[s,t] &= \langle V\varphi_t^0, \varphi_s^0 \rangle_\Gamma, & M_{i,h}[s,l] &= \langle \varphi_l^1, \varphi_s^0 \rangle_\Gamma, \\ D_{i,h}[k,l] &= \langle D\varphi_l^1, \varphi_k^1 \rangle_\Gamma, & K_{i,h}[s,l] &= \langle K\varphi_l^1, \varphi_s^0 \rangle_\Gamma, \end{aligned}$$

for $s,t = 1,\ldots,M_i$ and $k,l = 1,\ldots,N_i$. The system matrix $\widetilde{S}_h$ in (3.27) is positive definite and symmetric, hence we use again a conjugate gradient scheme as a solution method.

3.6.1 Condition Numbers and Preconditioning

Generally, the condition number of a symmetric and positive definite matrix A_M is the division of the largest over the smallest eigenvalue, i.e.

$$\kappa_2(A_M) = \frac{\lambda_{\max}(A_M)}{\lambda_{\min}(A_M)} \geq 1.$$

On the whole, condition numbers of matrices for boundary element methods depend on mesh parameters. To determine $\kappa_2(V_h)$, the condition number of the single layer matrix, we first need spectral equivalent equations.

Lemma 3.8 ([32], Lemma 12.2)**.** *The spectral equivalent equations for the single layer matrix are given by*

$$c_1 h^2 \|\underline{w}\|_2^2 \leq (V_h \underline{w}, \underline{w}) \leq c_2 h \|\underline{w}\|_2^2$$

for all $\underline{w} \in \mathbb{R}^M \leftrightarrow w_h \in S_h^0(\Gamma)$ *with a global uniform boundary discretisation.*

Using the definition of the Raleigh quotient, we obtain

$$\lambda_{\max}(V_h) = \max_{\underline{x} \neq \underline{0}} \frac{(V_h \underline{x}, \underline{x})}{(\underline{x}, \underline{x})} \leq c_2 h$$

and

$$\lambda_{\min}(V_h) = \min_{\underline{x} \neq \underline{0}} \frac{(V_h \underline{x}, \underline{x})}{(\underline{x}, \underline{x})} \geq c_1 h^2,$$

respectively, and therefore an upper bound of the condition number of V_h by

$$\kappa_2(V_h) \leq \frac{c_2\, h}{c_1\, h^2} = c\, \frac{1}{h}.$$

Moreover, we get

$$\kappa_2(V_{h/2}) \approx 2\, \kappa_2(V_h)$$

for one step of refinement. For further discussions about condition numbers for boundary element methods, see for example [8].
Since the system matrices of the derived linear systems of equations in the previous sections are symmetric and positive definite, we nearly always use a conjugate gradient scheme as a solving method. Therefore we are able to describe the behavior of the increasing iteration number, but at first we have to know the error estimate of the conjugate gradient scheme.

Theorem 3.9 ([30])**.** *For a symmetric and positive definite matrix* A_M *and the corresponding uniquely determined solution* $\underline{u}$ *of the system* $A_M \underline{u} = \underline{f}$ *via the conjugate gradient scheme, there holds*

$$\|\underline{u}^k - \underline{u}\|_{A_M} \leq \frac{2q^k}{1 + q^{2k}} \|\underline{u}^0 - \underline{u}\|_{A_M}$$

for every initial solution $\underline{u}^0$*, and with*

$$q = \frac{\sqrt{\kappa_2(A_M)} + 1}{\sqrt{\kappa_2(A_M)} - 1} > 1.$$

The norm $\|\cdot\|_{A_M} := \sqrt{(A_M \cdot, \cdot)}$ *is equivalent to the Euclidean norm in* $\mathbb{R}^M$*.*

Since it holds

$$\|\underline{u}^k - \underline{u}\|_{A_M} \leq \frac{2q^k}{1+q^{2k}}\|\underline{u}^0 - \underline{u}\|_{A_M} \leq \frac{2}{q^k}\|\underline{u}^0 - \underline{u}\|_{A_M},$$

the order of convergence for the conjugate gradient scheme is proportional to $\sqrt{\kappa_2(A_M)}$. To sum up, we know that for boundary elements it holds $\sqrt{\kappa_2(V_h)} = \mathcal{O}\Big(\sqrt{\frac{1}{h}}\Big)$ and consequently $\sqrt{\kappa_2(V_{h/2})} = \sqrt{2}\,\sqrt{\kappa_2(V_h)}$. This means that for a relative accuracy of $\varepsilon = 10^{-10}$ for the conjugate gradient scheme, with one step of refinement the number of iterations increases with the factor of $\sqrt{2}$.

Notes on Preconditioning

To gain constant iterations independent from the refinement level for the conjugate gradient scheme, we may use a preconditioning of the system (3.27) for domain decomposition methods. Hence we have to know about a preconditioning matrix for the Steklov-Poincaré operator. In this work, we only give a short summary of possibilities. For deeper discussions on preconditioning techniques for domain decomposition boundary element methods, see for example [22] and [3].
Generally spoken, as shown in Remark 1.5, the hypersingular integral operator D has a non-trivial kernel, so we may use a stabilisation $\widetilde{D}$, see for example [32]. For that stabilised operator, the following lemma shows the spectral equivalence for the inverse of the single layer potential $V : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ and the modified hypersingular integral operator $\widetilde{D} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$.

Lemma 3.10 ([22])**.** *For the single layer potential and the modified hypersingular integral operator there hold the spectral equivalent inequalities*

$$c_1\langle V^{-1}v, v\rangle_\Gamma \leq \langle \widetilde{D}v, v\rangle_\Gamma \leq c_2\langle V^{-1}v, v\rangle_\Gamma$$

for all functions $v \in H^{1/2}(\Gamma)$ and special positive constants c_1 and c_2, depending on the ellipticity constants of V and D, as well as on the modification parameter of $\widetilde{D}$.

Moreover, there is a relation between the inverse single layer potential and the Steklov-Poincaré operator too.

Lemma 3.11 ([22])**.** *For the single layer potential and the modified Steklov-Poincaré operator $S = \widetilde{D} + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)$, there hold the spectral equivalent inequalities*

$$\widetilde{c}_1\langle V^{-1}v, v\rangle_\Gamma \leq \langle \widetilde{S}v, v\rangle_\Gamma \leq \widetilde{c}_2\langle V^{-1}v, v\rangle_\Gamma$$

for all functions $v \in H^{1/2}(\Gamma)$ and special constants $\widetilde{c}_1$ and $\widetilde{c}_2$, depending on the ellipticity constants of V and $\widetilde{D}$, as well as on the modification parameter of $\widetilde{D}$.

These two lemma show that the modified hypersingular integral operator is a suitable preconditioner for the Steklov-Poincaré operator in general. Now as far as domain decomposition methods are concerned, the Galerkin variational formulations are evaluated on the coupling boundaries Γ_{ij}. If we calculate $\widetilde{S}$ only on Γ_{ij}, no modification of the hypersingular integral operator is necessary. Hence, $D_{i,h}$ restricted on the coupling boundary is a possible preconditioner for the resulted linear system of equations (3.27), which can be solved efficiently with a preconditioned conjugate gradient scheme. See for example [21] for further discussions about boundary element preconditioners for hypersingular integral equations.

4 Numerical Implementation

Here the focus is on the results of the calculations with the developed computational program. At first we deal with the explicit formulation of the potential operators for the corresponding stiffness matrices. We know that the adjoint double layer potential operator for the discrete scheme is the transposed matrix of K_h. Additionally, since the hypersingular matrix entries will be described via the single layer matrix V_h, we only have to evaluate the matrix entries of V_h and K_h. Afterwards we show a range of numerical examples for different boundary value and free space transmission problems as well as for domain decomposition methods introduced in Chapter 2, computed with the C-programming language. Moreover, we will see that the theoretical orders of convergence derived in Chapter 3 are reached for the different problems.

4.1 Calculation of the Integral Operators

This section is about the explicit calculation of the single and double layer matrix for the collocation and Galerkin method. For the latter one, we need the Gaussian quadrature, which we will shortly introduce here.
The general idea of numerical integration is to approximate for a given function $f(x)$ the definite integral

$$I = \int_a^b f(x)dx$$

by the approach

$$I_n = \sum_{i=0}^{n} \omega_i f(x_i),$$

with weights ω_i and nodes x_i. In this work we use the Gauss-Legendre integration rule given by

$$\int_{-1}^{1} f(x)dx = \sum_{i=0}^{n} \omega_i f(x_i) + \frac{f^{(2n+2)}(y)}{(2n+2)!} \int_{-1}^{1} \left(\prod_{i=0}^{n} (x - x_i) \right)^2 dx \tag{4.1}$$

with weights

$$\omega_i = \int_{-1}^{1} \prod_{j \neq i} \frac{x - x_j}{x_i - x_j} dx \quad \text{for } i = 1, \ldots, n.$$

The nodes x_i are the roots of the Legendre polynomials

$$P_{n+1}(x) = \frac{1}{2^{n+1}(n+1)!}\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^{n+1}.$$

If we cut off the last term in (4.1), we obtain the desired approximation I_n. As far as boundary element methods are concerned, we need an approximation for an integration over a boundary element τ_l with the initial node x_{l_1} and the end node x_{l_2}, i.e $\tau_l \in [x_{l_1}, x_{l_2}]$. Therefore, we perform the transformation $x = x_{l_1} + \xi(x_{l_2} - x_{l_1})$ with $\xi \in [0,1]$. For a given function $f(x)$ on the boundary element we get

$$f[l] = \int_{\tau_l} f(x)ds_x = \int_0^1 f(\xi)|\tau_l|d\xi \approx |\tau_l| \sum_{j=1}^{p} \omega_j f(x_{l_1} + x_j(x_{l_2} - x_{l_1})),$$

with p Gauss points x_j on the interval $[0,1]$. For a deeper discussion of the Gaussian quadrature and other numerical integration techniques, see for example [30].

4.1.1 Parametrisation

For calculations of the integral operators, we introduce the following parametrisation:

$$\Gamma_k = \quad \{y = x_k + t \cdot r_k, t \in (0, h_k), h_k = |x_{k+1} - x_k|, r_k = \frac{1}{h_k}(x_{k+1} - x_k)\}, \qquad (4.2)$$

with the normalized direction vector

$$r_k = \begin{pmatrix} r_{k,1} \\ r_{k,2} \end{pmatrix},$$

and its normal

$$n_k = \begin{pmatrix} r_{k,2} \\ -r_{k,1} \end{pmatrix},$$

respectively. Furthermore, x_k is the initial node of the element τ_k. For the collocation node there holds

$$x_l^* = x_k + t_l \cdot r_k + s_l \cdot n_k.$$

Since r_k and n_k are normal to each other, we calculate t_l and s_l by

$$t_l = \langle x_l^* - x_k, r_k \rangle,$$
$$s_l = \langle x_l^* - x_k, n_k \rangle,$$

respectively. Figure 4.1 shows the vectors of the local coordinate system.

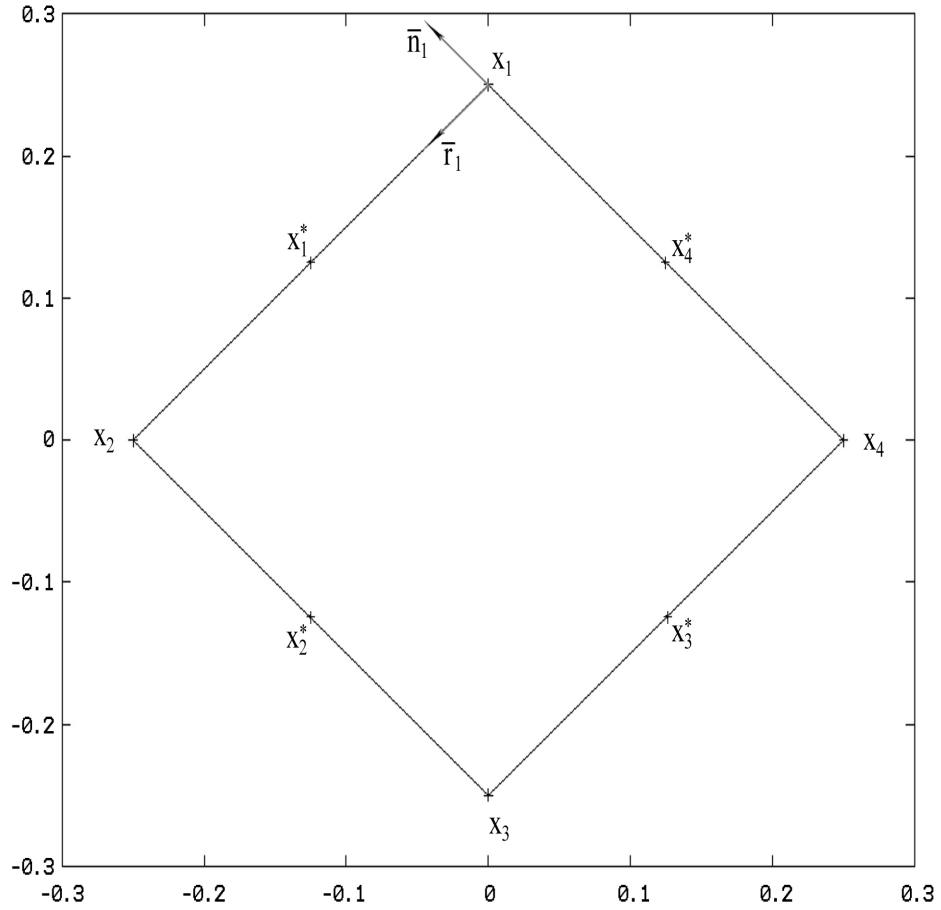

Figure 4.1: Local coordinate system for a first level circle with 4 elements

4.1.2 Single Layer Matrix

We need to calculate the single layer matrix entries for the collocation method,

$$V_h^C[l,k] = -\frac{1}{2\pi}\int_{\tau_l} \log|x_k^* - y| ds_y \quad \text{for } l,k = 1,\dots,M,$$

where x_k^* denotes the collocation point on the boundary element τ_k. Therefore we need a formulation of the argument of the logarithm, represented by the local coordinate system:

$$\begin{aligned} y - x_l^* &= 0 + (t-t_l)\cdot r_k - s_l \cdot n_k, \\ |y - x_l^*|^2 &= (t-t_l)^2 + s_l^2. \end{aligned}$$

Now the relation $\log|.|^2 = 2\log|.|$ yields

$$V_h^C[l,k] = -\frac{1}{2\pi}\frac{1}{2}\int_0^{h_k} \log\left((t-t_l)^2 + s_l^2\right) dt = -\frac{1}{4\pi}\big(F(h_k) - F(0)\big),$$

with the primitive

$$F(t) = \int \log\left((t-t_l)^2 + s_l^2\right) dt.$$

Due to the logarithm, $F(t)$ has singularities for $t = t_l$ and simultaneously $s_l = 0$. Integration by parts yields

$$F(t) = (t-t_l)\log\left((t-t_l)^2 + s_l^2\right) - \int \frac{(t-t_l)2(t-t_l)}{(t-t_l)^2 + s_l^2}\, ds.$$

With partial fraction expansion we obtain

$$\begin{aligned} F(t) &= (t-t_l)\log\left((t-t_l)^2 + s_l^2\right) - 2\int \frac{(t-t_l)^2 + s_l^2 - s_l^2}{(t-t_l)^2 + s_l^2}\, dt \\ &= (t-t_l)\log\left((t-t_l)^2 + s_l^2\right) - 2t + 2s_l \arctan\frac{t-t_l}{s_l}. \end{aligned}$$

To sum up, this is the representation for the entries of the primitive to calculate the stiffness matrix V_h^C. Moreover, there are no singularities any more, because if $t = t_l$, we have $(t-t_l)\log\left((t-t_l)^2 + s_l^2\right) = 0$, and if $s_l = 0$, we obtain $2s_l \arctan\dfrac{t-t_l}{s_l} = 0$. As far as the Galerkin method is concerned, the matrix entries are

$$V_h^G[l,k] = \langle V\varphi_k^0, \varphi_l^0\rangle_\Gamma = -\frac{1}{2\pi}\int_{\tau_l}\int_{\tau_k} \log|x-y|\, ds_y ds_x$$

for $l,k = 1,\dots,M$. We calculate the outer integral with Gaussian quadrature, and the inner one exactly as for the collocation method. Instead of the collocation nodes x_k^*, we

evaluate the inner integral in the Gauss points. Hence, we obtain the representation

$$\begin{aligned} V_h^G[l,k] &= -\frac{1}{2\pi}\int_{\tau_l}\int_{\tau_k}\log|x-y|\,ds_y ds_x \\ &\approx h_l\sum_{j=1}^{p} w_j\Big(-\frac{1}{2\pi}\int_{\tau_k}\log|x_l^j-y|ds_y\Big), \end{aligned}$$

with p Gauss points x_l^j on the element τ_l, Gauss weights w_j and lengths h_l.

4.1.3 Double Layer Matrix

Let us consider the double layer matrix entries for piecewise linear and continuous functions for the collocation method,

$$K_h^C[i,l] = -\frac{1}{2\pi}\int_\Gamma \frac{(y-x_i^*,n_y)}{|x_i^*-y|^2}\varphi_l^1(y)ds_y.$$

Here we shall adopt the approach of [31], and we just write down the resulting operations. Since we use piecewise linear test functions, there are two main calculations for one matrix entry, coming from the increasing and decreasing part of the test functions, denoting one component by '1+' and the other one by '1−', respectively. Using the local parametrisation (4.2), for $s_i = 0$ we obtain

$$K_h^{C,1+}[i,l] = K_h^{C,1-}[i,l] = 0,$$

due to the kernel representation of double layer potential. Furthermore, for $s_i \neq 0$, we get

$$\begin{aligned} K_h^{C,1+}[i,l] &= \frac{1}{2\pi}\int_{t_l}^{t_{l+1}} \frac{s_i}{s_i^2+t^2}\frac{t-t_l}{h_l}dt \\ &= \frac{1}{2\pi h_l}\Big(F(t_{l+1};t_l)-F(t_l;t_l)\Big), \\ K_h^{C,1-}[i,l] &= \frac{1}{2\pi}\int_{t_l}^{t_{l+1}} \frac{s_i}{s_i^2+t^2}\frac{t_{l+1}-t}{h_l}dt \\ &= \frac{1}{2\pi h_l}\Big(F(t_l;t_{l+1})-F(t_{l+1};t_{l+1})\Big), \end{aligned}$$

with element lengths h_l and the primitive

$$F(t;t_0) = \int \frac{s_i}{s_i^2+t^2}(t-t_0)dt = \frac{1}{2}s_i\log(s_i^2+t^2) - t_0\arctan\frac{t}{s_i}.$$

Concerning the Galerkin method, we know the double layer matrix entries are

$$K_h[i,l] = \langle K\varphi_l^1,\varphi_i^0\rangle_\Gamma = -\frac{1}{2\pi}\int_{\tau_i}\int_\Gamma \frac{(y-x,n_y)}{|x-y|^2}\varphi_l^1(y)ds_y ds_x$$

for $i = 1, \ldots, M$ and $l = 1, \ldots, N$. Analogously to the single layer matrix, we evaluate the outer integral with Gaussian integration. The inner integral is calculated as for case of collocation, but using Gauss points instead of collocation nodes x_i^*. Thus, for p Gauss points x_i^j on the element τ_i, Gauss weights w_j and lengths h_i we get

$$\begin{aligned} K_h[i,l] &= -\frac{1}{2\pi}\int_{\tau_i}\int_{\Gamma}\frac{(y-x,n_y)}{|x-y|^2}\varphi_l^1(y)ds_y ds_x \\ &\approx h_i\sum_{j=1}^{p} w_j\Big(-\frac{1}{2\pi}\int_{\Gamma}\frac{(y-x_i^j,n_y)}{|x_i^j-y|^2}\varphi_l^1(y)ds_y\Big). \end{aligned}$$

4.2 Numerical Examples

Since we have presented the theory for interior and exterior boundary value problems, interface problems as well as for domain decomposition methods in this work, we now want to provide numerical results of a computational program for each of these topics. We solve these partial differential equations for some simple domains, use particular solutions for the Poisson problems and different boundary functions as Dirichlet data. At the beginning, we state some additional information which is necessary for the calculations:

In domain decomposition methods, we solve the local boundary value problems with a Cholesky decomposition.

All other problems will be solved with a conjugate gradient scheme with an epsilon $\varepsilon = 10^{-8}$ as relative accuracy.

As far as the Gaussian quadrature is concerned, we always use 5 Gauss points.

Most of the time we are finally interested in the rate of convergence of the approximate solution. Therefore we mostly evaluate the absolute value of the difference of the exact and the approximate representation formula in a specific point in the domain. To compare this point-wise error for finer meshes, we provide the experimental order of convergence (eoc), which meshes the numerical order of convergence by comparing computations on two sequenced meshes. It is defined as

$$\text{eoc} = \frac{\ln\left(E_i/E_{2i}\right)}{\ln\left(2\right)},$$

with $E_i = |u(x) - \tilde{u}(x)|$ for $x \in \Omega$, where i corresponds to the degrees of freedom. In this work, each subsequent refinement level has twice as much boundary elements as the level before, dividing one element into two equal parts.

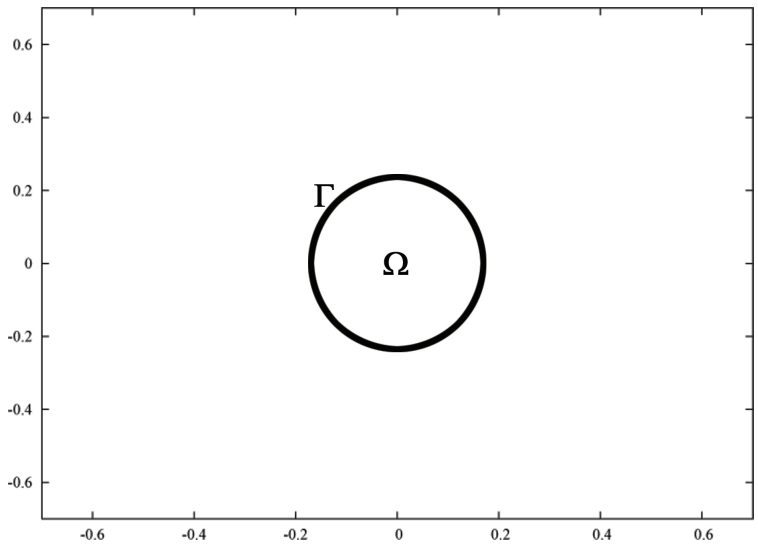

Figure 4.2: Domain Ω for the Poisson problem: Circle with radius $r = 0.25$

4.2.1 Interior Poisson Problem

The Poisson problem is given by

$$\begin{aligned} -\Delta u(x) &= |x| - \frac{r}{2} \quad \text{for } x \in \Omega, \\ \gamma_0^{int} u(x) &= \gamma_0^{int} u_p(x) + \gamma_0^{int} u^h(x) \quad \text{for } x \in \Gamma, \end{aligned} \tag{4.3}$$

with

$$\begin{aligned} \gamma_0^{int} u_p(x) &= \frac{1}{9}|x|^3 - \frac{r}{8}|x|^2 \quad \text{for } x \in \Gamma, \\ \gamma_0^{int} u^h(x) &= \log|x_0 - x| \quad \text{for } x \in \Gamma \text{ and } x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \end{aligned}$$

so the Dirichlet datum is a combination of a homogeneous and a particular solution of the Poisson problem. Additionally, we have the corresponding conormal derivative of the particular solution

$$\gamma_1^{int} u_p(x) = (\frac{1}{3}|x| - \frac{r}{4}) n_x \cdot x \quad \text{for } x \in \Gamma.$$

The domain Ω is a circle, centered at the origin with radius $r = 0.25$ (see Fig. 4.2) to fulfill the condition $\text{diam}(\Omega) < 1$ for the ellipticity of the single layer potential operator.
To determine the unknown Neumann datum $t_h \in S_h^0(\Gamma)$ for the representation formula, we use the direct approach as well as the ansatz of a particular solution to handle the Newton potential, see Subsection 2.1.2. Hence we have to solve the Galerkin formulation (3.16)

$$\langle V t_h, \varphi_i^0 \rangle_\Gamma = \langle (\frac{1}{2} I + K) g, \varphi_i^0 \rangle_\Gamma - \langle (\frac{1}{2} I + K) g_{p,h}, \varphi_i^0 \rangle_\Gamma + \langle V t_{p,h}, \varphi_i^0 \rangle_\Gamma$$

	Interpolation		L_2 projection	
Elements	$Error_1$	eoc	$Error_1$	eoc
32	1,87E-05		1,78E-05	
64	4,58E-06	2,03	2,01E-06	3,14
128	1,13E-06	2,01	2,36E-07	3,09
256	2,82E-07	2,01	2,95E-08	3,00
512	7,04E-08	2,00	3,60E-09	3,04
1024	1,76E-08	2,00	1,92E-10	4,23
Theory:		2		3

Table 4.1: Accuracy for the interior Poisson problem using piecewise linear approximation and L_2 projection for the given Dirichlet data

for $i = 1, \ldots, M$. This leads to the equivalent system

$$V_h^G \underline{t} = (\frac{1}{2} M_h + K_h)\underline{g} - (\frac{1}{2} M_h + K_h)\underline{g}_p + V_h^G \underline{t}_p,$$

using either a linear approximation of the boundary data g, or a L_2 projection.
The resulting error for this boundary value problem is given in Table 4.1, using at first a piecewise linear and continuous approximation of the given function (4.3) on the boundary, which yields a quadratic convergence rate in theory.
The second column in Table 4.1 shows the point-wise error in a prescribed inner point,

$$Error_1 = |u(x) - \tilde{u}(x)|, \quad x = (0.1, 0.05)^\top,$$

where $\tilde{u}(x)$ is the approximate representation formula

$$\tilde{u}(x) = u_p(x) + (\widetilde{V} t_h)(x) - (\widetilde{V} t_{p,h})(x) - (W g_h)(x) + (W g_{p,h})(x) \quad \text{for } x \in \Omega.$$

If we use L_2 projections of the particular solution, its conormal derivative and the homogenuous part of the Dirichlet function, we obtain cubic convergence, cf. (3.17), which will be shown in the last two columns in Table 4.1 for a different homogeneous boundary function

$$u^h(x) \;=\; \log|x_0 - x| \quad \text{with } x_0 = \begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix}.$$

To sum up, Table 4.1 verifies the theoretical results concerning the error estimates. At the end of this subsection we want to show that the kernel properties of the double layer and the hypersingular integral operators are fulfilled in the discrete scheme too, see therefore Remarks 1.4 and 1.5, respectively. Representative for all calculations of the different boundary value problems, we show these properties in Table 4.2 for this interior Poisson problem.

Elements	$(\frac{1}{2}M_h + K_h) \cdot \underline{1}$	$D_h \cdot \underline{1}$
16	5,6E-16	3,8E-16
32	9,4E-16	7,4E-16
64	1,8E-15	1,1E-15
128	2,4E-15	2,1E-15
256	2,8E-15	4,5E-15
512	4,6E-15	9,3E-15
1024	7,1E-15	1,9E-14
Theory:	0.0	0.0

Table 4.2: Kernel properties

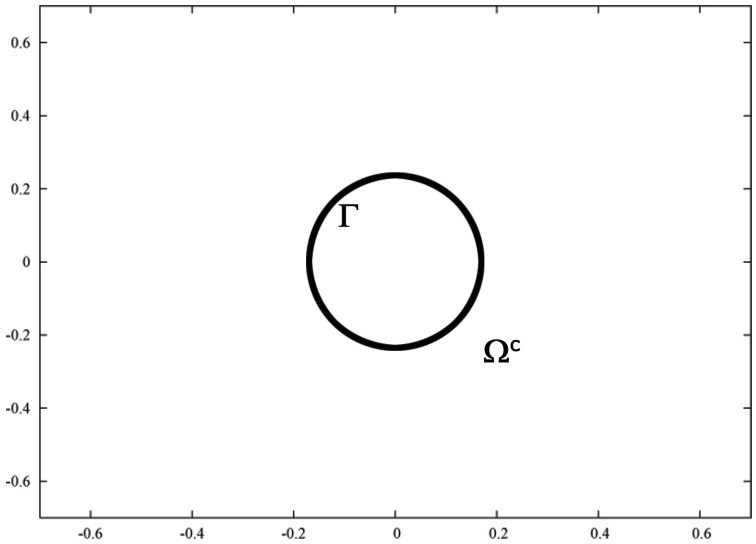

Figure 4.3: Domain Ω^c for the exterior Laplace problem: Circle with radius $r = 0.25$

4.2.2 Exterior Laplace Problem

The purpose of this subsection is to verify the theoretical results for the exterior Laplace problem

$$\begin{aligned} -\Delta u(x) &= 0 && \text{for } x \in \Omega^c, \\ \gamma_0^{ext} u(x) &= \log|x_0 - x| && \text{for } x \in \Gamma \text{ and } x_0 = \begin{pmatrix} 0.01 \\ 0.01 \end{pmatrix}. \end{aligned}$$

The domain Ω is a circle, centered at the origin with radius $r = 0.25$ (see Fig. 4.3), so the condition $\text{diam}(\Omega) < 1$ is fulfilled.
Choosing $a = 1$ in

$$u(x) = a \log|x| + \mathcal{O}\left(\frac{1}{|x|}\right) \quad \text{as } |x| \to \infty,$$

	Interpolation		L_2 projection	
Elements	$Error_1$	eoc	$Error_1$	eoc
32	3,48E-03		4,00E-05	
64	8,75E-04	1,99	5,01E-06	3,00
128	2,19E-04	2,00	6,26E-07	3,00
256	5,49E-05	2,00	7,83E-08	3,00
512	1,37E-05	2,00	9,78E-09	3,00
1024	3,44E-06	2,00	1,22E-09	3,00
Theory:		2		3

Table 4.3: Accuracy for the exterior Laplace problem using piecewise linear approximation and L_2 projection for the given Dirichlet data

we see that the solution u satisfies the decay condition (1.23). For the Galerkin formulation we have to find the uniquely determined function $t_h \in S_h^0(\Gamma)$ in

$$\langle V t_h, \varphi_i^0 \rangle_\Gamma = \langle (-\frac{1}{2} I + K) g, \varphi_i^0 \rangle_\Gamma$$

for $i = 1, \ldots, M$. The equivalent system to solve is

$$V_h^G \underline{t} = (-\frac{1}{2} M_h + K_h) \underline{g}.$$

The second and the fourth column in Table 4.3 show the absolute error

$$Error_1 = |u(x) - \tilde{u}(x)|, \quad x = (3, 3.5)^\top,$$

using linear interpolation of the Dirichlet datum, and in the second case we use a L_2 projection, yielding quadratic and cubic convergence, respectively.

4.2.3 Interface Problem

The purpose of this subsection is to give three examples of interface problems.

Example with a Circle

We use a circle centered at the origin with radius $r = 0.25$ as a domain Ω for the interface problem

$$\begin{aligned} -\Delta u_1(x) &= |x| - \frac{r}{2} \quad \text{for } x \in \Omega_1 = \Omega, \\ -\Delta u_2(x) &= 0 \qquad\quad \text{for } x \in \Omega_2 = \Omega^c, \end{aligned} \tag{4.4}$$

with one boundary $\Gamma_C = \partial\Omega_1 = \partial\Omega_2$, see Fig. 4.4.

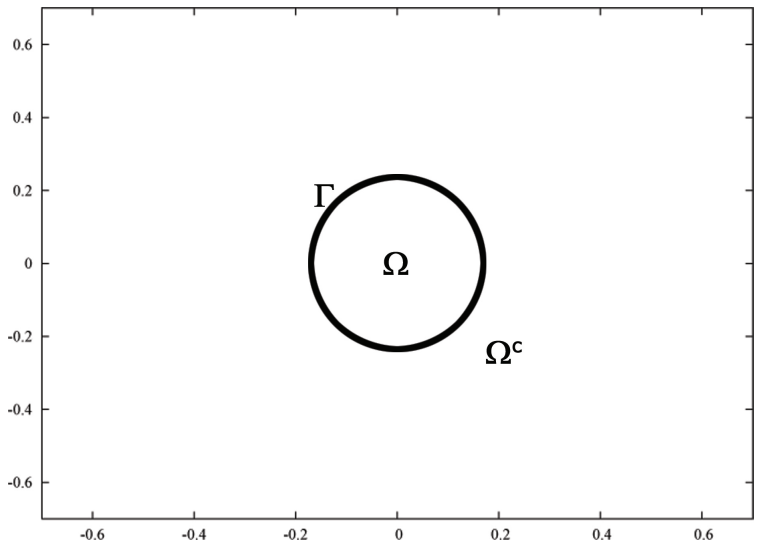

Figure 4.4: Domains for the first interface problem

The exterior solution u_2 fulfills the radiation condition (2.9). Moreover, the condition $\mathrm{diam}(\Omega_1) < 1$ for the ellipticity of the single layer potential operator is fulfilled. From Section 3.5 we know that the problems can be solved via the transmission conditions

$$\begin{aligned} u(x) = \gamma_0^{int} u_1(x) &= \gamma_0^{ext} u_2(x) \quad \text{for } x \in \Gamma_C, \\ \gamma_1^{int} u_1(x) &= \gamma_1^{ext} u_2(x) \quad \text{for } x \in \Gamma_C. \end{aligned}$$

The unique solution $\widetilde{u}_h \in S_h^1(\Gamma)$ will be evaluated by solving the perturbed formulation

$$\left\langle (\alpha_1 \widetilde{S}^{int} + \alpha_2 \widetilde{S}^{ext}) \widetilde{u}_h, \varphi_k^1 \right\rangle_{\Gamma_C} = \left\langle \widetilde{S}^{int} u_{p,h} - t_{p,h}, \varphi_k^1 \right\rangle_{\Gamma_C} \tag{4.5}$$

for $k = 1, \ldots, N$. The equivalent linear system of equations (3.20) reads as

$$\begin{aligned} &\alpha_1 \Big(D_h \underline{\widetilde{u}} + (\frac{1}{2} M_h^\top + K_h^\top) \underline{t}_1 \Big) + \alpha_2 \Big(D_h \underline{\widetilde{u}} + (\frac{1}{2} M_h^\top - K_h^\top) \underline{t}_2 \Big) \\ &\qquad = D_h \underline{u}_p + (\frac{1}{2} M_h^\top + K_h^\top) \underline{t}^* - M_h^\top \underline{t}_p, \end{aligned} \tag{4.6}$$

where the temporary vectors $\underline{t}_1, \underline{t}_2$ and $\underline{t}^*$ will be calculated via the local systems (3.21),

$$\begin{aligned} V_h^G \underline{t}_1 &= (\frac{1}{2} M_h + K_h) \underline{\widetilde{u}}, \\ V_h^G \underline{t}_2 &= (\frac{1}{2} M_h - K_h) \underline{\widetilde{u}}, \\ V_h^G \underline{t}^* &= (\frac{1}{2} M_h + K_h) \underline{u}_p, \end{aligned} \tag{4.7}$$

		Interior domain			Exterior domain		
Level	Elements	$\hat{u}(x)$	$Error_1$	eoc	$\hat{u}(x)$	$Error_2$	eoc
3	32	-0,00035569	5,75E-06		0,00023340	2,32E-06	
4	64	-0,00036000	1,45E-06	1,99	0,00023514	5,81E-07	2,00
5	128	-0,00036108	3,61E-07	2,00	0,00023558	1,45E-07	2,00
6	256	-0,00036135	8,92E-08	2,02	0,00023568	3,58E-08	2,02
7	512	-0,00036142	2,12E-08	2,07	0,00023571	8,52E-09	2,07
8	1024	-0,00036144	4,25E-09	2,32	0,00023572	1,70E-09	2,32
9	2048	-0,00036144			0,00023572		
Theory:				2			2

Table 4.4: Accuracy for the first interface problem with one circle boundary

on Γ_C. After solving (4.6) with a modified conjugate gradient scheme, the missing Neumann datum will be calculated with the first and the second line of the local systems (4.7). Moreover, since there is no data given on the boundary, there is no analytic solution to compare with. Thus, the approximate representation formula at level i will be compared with the evaluated solution of the highest level 9, so

$$Error_1 = |u_9(x) - \hat{u}_i(x)|, \quad x = (0.1, 0.15)^\top$$

for the interior domain, and

$$Error_2 = |u_9(x) - \hat{u}_i(x)|, \quad x = (3, 3.5)^\top$$

for the exterior domain, respectively. The results of the computations, especially the expected quadratic convergence of the point-wise error, are reported in Table 4.4.

Example with a Square

Additionally, we state the interface problem for a square (see Fig. 4.5):

$$\begin{aligned} -\Delta u_1(x) &= -6x_1 - 2c \quad \text{for } x \in \Omega_1 = \Omega, c \in \mathbb{R}, \\ -\Delta u_2(x) &= 0 \qquad\quad\ \ \text{for } x \in \Omega_2 = \Omega^c, \end{aligned} \tag{4.8}$$

where u_2 fulfills the radiation condition (2.9).
The vertices of the domain are

$$(0,0), (0.5,0), (0.5,0.5), (0,0.5),$$

so Ω satisfies the condition $\text{diam}(\Omega_1) < 1$. The particular solution is

$$u_p(x) = x_1^3 + cx_1^2, \tag{4.9}$$

for any constant $c \in \mathbb{R}$. As shown in Section 2.3, the function on the right hand side of the interior Poisson equation has to fulfill the solvability condition

$$\int_{\Omega_1} f(x)dx = 0.$$

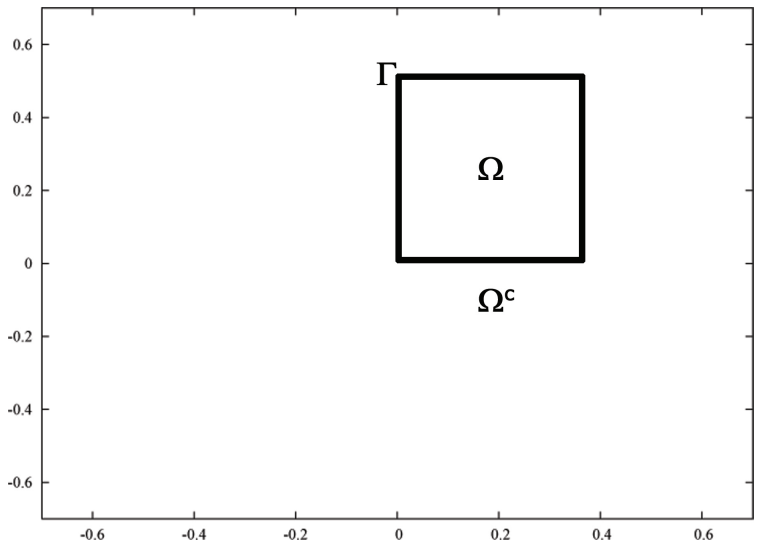

Figure 4.5: Domains for the second interface problem

		Interior domain			Exterior domain		
Level	Elements	$\hat{u}(x)$	$Error_1$	eoc	$\hat{u}(x)$	$Error_2$	eoc
3	32	0,01441931	5,61E-07		-0,00075675	1,67E-06	
4	64	0,01441886	1,11E-07	2,34	-0,00075544	3,62E-07	2,20
5	128	0,01441877	1,56E-08	2,83	-0,00075515	7,36E-08	2,30
6	256	0,01441876	2,10E-09	2,90	-0,00075509	1,40E-08	2,40
7	512	0,01441875	4,14E-10	2,34	-0,00075508	2,20E-09	2,66
8	1024	0,01441875			-0,00075508		
Theory:				2			2

Table 4.5: Accuracy for the second interface problem with one square boundary

Hence, we have to determine the constant c in (4.9) in such a way, that the solvability condition is fulfilled. Using the square given above as the domain Ω_1, we get $c = -\frac{3}{4}$. The resulting systems of linear equations are the same as for the previous interface problem. The solutions and errors are given in Table 4.5.

Example with a Tire

Now we calculate an interface problem without any function f on the right hand side for a tire, using two circles centered at the origin with radii $r = 0.25$ and $r = 0.1$, see Fig. 4.6. Again, it holds $\text{diam}(\Omega) < 1$.

Hence, domain Ω_1 has two separated boundaries, namely an interior boundary Γ_D and an exterior boundary Γ_C, which is simultaneously the boundary of the domain Ω_2. The

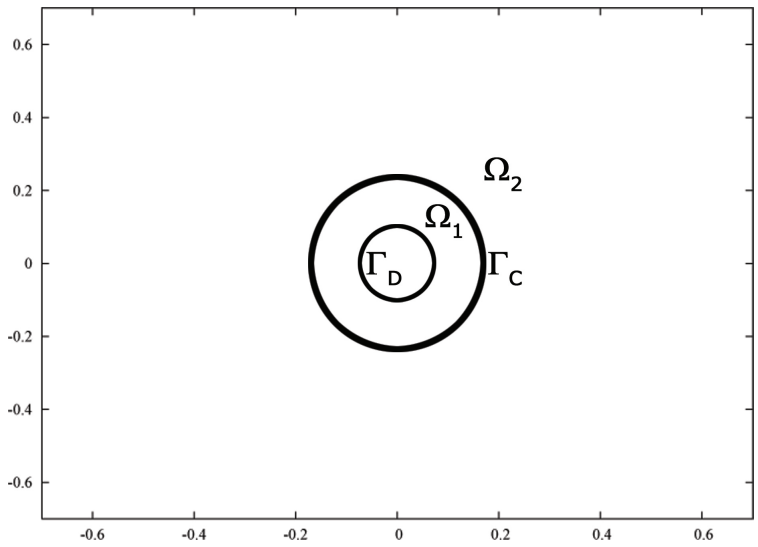

Figure 4.6: Domains for the third interface problem

changed interface problem then reads as

$$\begin{aligned} -\Delta u_1(x) &= 0 && \text{for } x \in \Omega_1, \\ \gamma_0^{int} u_1(x) &= \log|x_0 - x| && \text{for } x \in \Gamma_D \text{ and } x_0 = \begin{pmatrix} 0.01 \\ 0.01 \end{pmatrix}, \\ -\Delta u_2(x) &= 0 && \text{for } x \in \Omega_2, \end{aligned} \tag{4.10}$$

with the radiation condition (1.23), choosing $a = 1$, for the exterior solution u_2. The solution u_1 is separated into two parts on the boundary $\partial\Omega_1$, namely the unknown function on the coupled boundary Γ_C, and the given Dirichlet datum on Γ_D. Let us use a L_2 projection

$$g_h(x) = \sum_{l=1}^{N} g_l \varphi_l^1(x) \in S_h^1(\Gamma_D)$$

for the given function on the Dirichlet boundary, satisfying

$$\langle g_h, \varphi_k^1 \rangle_{\Gamma_D} = \langle \gamma_0^{int} u, \varphi_k^1 \rangle_{\Gamma_D} \quad \text{for } k = 1, \ldots, N_D.$$

Instead of the Newton potential in the variational formulation (4.5), we then have the Dirichlet part on the right hand side, i.e.

$$\left\langle (\alpha_1 \widetilde{S}^{int} + \alpha_2 \widetilde{S}^{ext}) \widetilde{u}_h, \varphi_k^1 \right\rangle_{\Gamma_C} = -\left\langle \alpha_1 \widetilde{S}^{int} g_h, \varphi_k^1 \right\rangle_{\Gamma_C} \quad \text{for } k = 1, \ldots, N_C.$$

Moreover, for the resulting system of linear equations we need a connectivity matrix A similar as introduced in (3.25), but now transforming a vector on the coupled boundary

	Interior domain		Exterior domain	
Elements	$Error_1$	eoc	$Error_1$	eoc
32	1,04E-04		5,78E-04	
64	1,50E-05	2,80	7,24E-05	3,00
128	1,93E-06	2,95	9,05E-06	3,00
256	2,45E-07	2,98	1,13E-06	3,00
512	3,08E-08	2,99	1,41E-07	3,00
1024	3,86E-09	3,00	1,77E-08	3,00
Theory:		3		3

Table 4.6: Accuracy for the third interface problem with two circles

Γ_C to the entire boundary $\partial\Omega_1 = \Gamma_D \cup \Gamma_C$:

$$\begin{aligned}\alpha_1\Big(A^\top D_{1,h} A\underline{\widetilde{u}} + A^\top(\frac{1}{2}M_{1,h}^\top + K_{1,h}^\top)\underline{t}_1\Big) + \alpha_2\Big(D_{2,h}\underline{\widetilde{u}} + (\frac{1}{2}M_{2,h}^\top - K_{2,h}^\top)\underline{t}_2\Big)\\ = -\alpha_1\Big(A^\top D_{1,h} A\underline{g} + A^\top(\frac{1}{2}M_{1,h}^\top + K_{1,h}^\top)\underline{t}^*\Big),\end{aligned} \tag{4.11}$$

with the temporary vectors $\underline{t}_1$ and $\underline{t}^*$ from the local systems on the entire boundary $\partial\Omega_1 = \Gamma_D \cup \Gamma_C$,

$$\begin{aligned} V_{1,h}^G \underline{t}_1 &= (\frac{1}{2}M_{1,h} + K_{1,h})A\underline{\widetilde{u}},\\ V_{1,h}^G \underline{t}^* &= (\frac{1}{2}M_{1,h} + K_{1,h})A\underline{g},\end{aligned} \tag{4.12}$$

and with $\underline{t}_2$ from the system given on the boundary $\partial\Omega_2 = \Gamma_C$,

$$V_{2,h}^G \underline{t}_2 = (\frac{1}{2}M_{2,h} - K_{2,h})\underline{\widetilde{u}}, \tag{4.13}$$

respectively. Once the solution has been found by solving (4.11) with a modified conjugate gradient scheme, the Neumann datum on the coupled boundary can be evaluated via the local systems (4.12) and (4.13). On the contrary to the two interface problems (4.4) and (4.8) above, now we are able to compare the approximate representation formula with an analytic solution, since we provide a Dirichlet datum on the interior boundary. Denoting

$$Error_1 = |u(x) - \tilde{u}(x)|, \quad x = (0.15, 0.03)^\top,$$

as well as

$$Error_2 = |u(x) - \tilde{u}(x)|, \quad x = (3, 3.5)^\top,$$

respectively, and since we use a L_2 projection of $\gamma_0^{int} u$, we expect cubic convergence of the point-wise errors. Finally, the results of the computations are reported in Table 4.6.

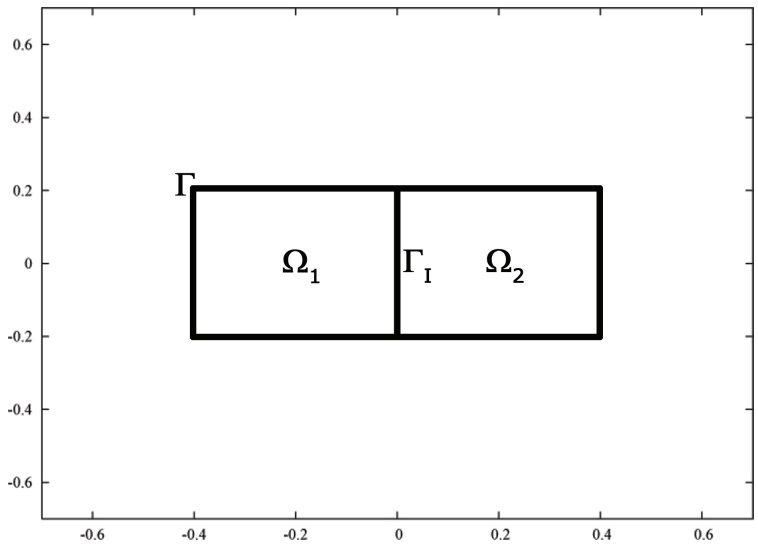

Figure 4.7: Rectangle for domain decomposition methods

4.2.4 Domain Decomposition Methods

We consider the local Dirichlet boundary value problems

$$\begin{aligned} -\Delta u_i(x) &= 0 && \text{for } x \in \Omega_i, \\ \gamma_0^{int} u_i(x) &= \log|x_0 - x| && \text{for } x \in \Gamma \cap \Gamma_i \text{ and } x_0 = \begin{pmatrix} 0.25 \\ 0.25 \end{pmatrix}, \end{aligned}$$

for $i = 1, 2$. The domain Ω is a rectangle with coordinates

$$(-0.4, -0.2), (0.4, 0.2), (0.4, 0.2), (-0.4, 0.2),$$

separated into two squares Ω_1 and Ω_2, see Fig. 4.7, and it fulfills $\text{diam}(\Omega) < 1$.
Via the transmission conditions

$$\gamma_{0,i}^{int} u_i(x) = \gamma_{0,j}^{int} u_j(x) \quad \text{and} \quad \gamma_{1,i}^{int} u_i(x) = -\gamma_{1,j}^{int} u_j(x) \quad \text{for } x \in \Gamma_{ij},$$

and using approximate Steklov-Poincaré operators, we obtain the perturbed Galerkin formulation (3.24) to find $\hat{u}_h \in S_h^1(\Gamma_S)$, such that

$$\sum_{i=1}^{p} \int_{\Gamma_i} (\widetilde{S}_i \hat{u}_{h|\Gamma_i})(x) v_{h|\Gamma_i}(x) ds_x = -\sum_{i=1}^{p} \int_{\Gamma_i} (\widetilde{S}_i \widetilde{g}_{|\Gamma_i})(x) v_{h|\Gamma_i}(x) ds_x$$

for all $v_h \in S_h^1(\Gamma_S)$. The equivalent system of equations (3.27) reads as

$$\widetilde{S}_h \underline{\hat{u}} = \sum_{i=1}^{p} A_i^\top \widetilde{S}_{i,h} A_i \underline{\hat{u}} = -\sum_{i=1}^{p} A_i^\top \widetilde{S}_{i,h} A_i \underline{g} = -\widetilde{S}_h \underline{g},$$

	Left square		Right square	
Elements	$Error_1$	eoc	$Error_1$	eoc
16	5,60E-06		7,83E-03	
32	1,45E-05	-1,37	3,07E-04	4,67
64	1,91E-06	2,92	3,33E-05	3,21
128	2,41E-07	2,99	3,82E-06	3,12
256	3,11E-08	2,96	4,56E-07	3,07
512	3,24E-09	3,26	5,47E-08	3,06
1024	9,27E-10	1,80	7,81E-09	2,81
Theory:		3		3

Table 4.7: Accuracy for domain decomposition methods for two squares

with connectivity matrices A_i for $A_i\underline{\hat{u}} = \underline{\hat{u}}_i$. Using a L_2 projection of the Dirichlet function on the global boundary Γ, we expect cubic convergence for the absolute error

$$Error_1 = |u(x) - \hat{u}(x)|, \quad x = (-0.3, 0.1)^\top,$$

calculating the solution in the left square Ω_1, as well as for

$$Error_2 = |u(x) - \hat{u}(x)|, \quad x = (0.3, 0.1)^\top$$

for the approximate representation formula for the right square Ω_2, respectively. We observe the numerical results in Table 4.7.
For domain decomposition methods, we noted in Subsection 3.6.1 that there is a possibility to use the inverse of the hypersingular operator D, evaluated on the coupled boundary, as a preconditioner for the conjugate gradient scheme to obtain $\hat{u}_h$ of system (3.27) on the coupled boundary. As we have seen in the remarks after Theorem 3.9, the number of iterations for the conjugate gradient scheme increases with a factor of $\sqrt{2}$. Using the preconditioning technique mentioned above, the iterations should be constant and therefore independent of the refinement level. Besides, a preconditioned conjugate gradient scheme has to be used. The results are presented in Table 4.8.

4.3 Outlook

As far as domain decomposition methods are concerned, a general theory was given in the previous chapters. For software implementations we used a simple model, but the next step would have been to solve these models in parallel. Looking ahead to computational costs, parallel solvers are nowadays a very interesting and efficient technique.
Here we mainly used the conjugate gradient scheme to solve the resulting systems of linear equations of the boundary value and free space transmission problems. We would need suitable preconditioning methods to get constant numbers of iterations, independent of the mesh size and therefore of the size of the system [3]. But what seems even more

		No precondition		Precondition
Elements	Coupled nodes	Iterations	Comparison	Iterations
16	3	3		3
32	7	7	2,3	7
64	15	11	1,6	10
128	31	16	1,5	11
256	63	24	1,5	11
512	127	35	1,5	11
1024	255	50	1,4	11
Theory:			$\sqrt{2}$	const

Table 4.8: CG-Iterations for (non-)preconditioned domain decomposition methods

important than that is to reduce the effort for the calculations of the boundary element method matrices. As we know, for a standard scheme the computational costs as well as the memory requirements grow at least quadratically in the number of boundary elements. For this purpose, there exist fast boundary element methods with nearly linear effort, meaning $O(N \log^{\alpha}(N))$ for $\alpha \in \mathbb{N}_0$. The main idea of these methods is to approximate the kernels of the appearing potential operators, namely the fundamental solution $U^*(x, y)$. Firstly, there is panel clustering [12], where the fundamental solution will be approximated with Taylor expansion. Secondly, in the fast multipole method (FMM) an alternative series expansion of $U^*(x, y)$ will be applied [23]. Thirdly, there is the adaptive cross approximation (ACA), based on matrix entries [26]. Furthermore, we could deal with hierarchical matrices, where the global matrix will be separated into smaller parts. Generally spoken, it is the idea to partition a matrix hierarchically. Other interesting and applicable topics are $\mathcal{H}$-matrices and wavelets [11].

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Erklärung

Ich erkläre an Eides statt, dass ich die vorliegende Arbeit selbstständig verfasst, andere als die angegebenen Quellen/Hilfsmittel nicht benutzt, und die den Quellen wörtlich und inhaltlich entnommenen Stellen als solche kenntlich gemacht habe.

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