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Analysis of Boundary Element Methods for Laplacian Eigenvalue Problems

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Abstract

The aim of the book is to provide an analysis of the boundary element method for the numerical solution of Laplacian eigenvalue problems. The representation of Laplacian eigenvalue problems in the form of boundary integral equations leads to nonlinear eigenvalue problems for related boundary integral operators. The solution of boundary element discretizations of such eigenvalue problems requires appropriate methods for algebraic nonlinear eigenvalue problems. Although the numerical solution of eigenvalue problems for partial differential operators using boundary element methods has a long tradition, a rigorous numerical analysis has not been established so far. One of the main goals of this work is to develop a convergence and error analysis of the Galerkin boundary element approximation of Laplacian eigenvalue problems. To this end, the concept of eigenvalue problems for so–called holomorphic Fredholm operator functions is used. This concept is a generalization of the theory for eigenvalue problems of bounded linear operators. The analysis of the approximation of eigenvalue problems for holomorphic Fredholm operator functions is usually done in the framework of the concept of regular approximation schemes. In this work convergence results and error estimates are derived for Galerkin discretizations of such eigenvalue problems. These results are then applied to the discretizations of Laplacian boundary integral operator eigenvalue problems. Furthermore, numerical methods for the solution of algebraic nonlinear eigenvalue problems are reviewed. The little–known Kummer's method is presented and its convergence behavior for algebraic holomorphic eigenvalue problems is analyzed by using the concept of holomorphic operator functions. Finally, a numerical example is considered and results of a boundary element and a finite element approximation of the eigenvalues are presented which confirm the theoretical results.

Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit der Analysis der Randelementmethode zur numerischen Lösung von Eigenwertproblemen für den Laplace-Operator. Die Darstellung von Eigenwertproblemen des Laplace-Operators in Form von Randintegralgleichungen führt auf nichtlineare Eigenwertprobleme für entsprechende Randintegraloperatoren. Die Lösung der Randelementdiskretisierung dieser Eigenwertprobleme erfordert Algorithmen für algebraische nichtlineare Eigenwertprobleme. Obwohl die Verwendung von Randelementmethoden zur numerischen Lösung von Eigenwertproblemen eine lange Tradition aufweist, existiert dafür keine vollständige numerische Analysis. Ziel dieser Arbeit ist es, eine Konvergenz- und Fehleranalysis für die Galerkin–Randelementmethode für Eigenwertprobleme des Laplace-Operators durchzuführen. Es wird gezeigt, dass dafür die Theorie der holomorphen Fredholm Operatorfunktionen ein geeignetes theoretisches Konzept ist. Diese Theorie stellt eine natürliche Erweiterung der Spektraltheorie für beschränkte lineare Operatoren dar. Die Untersuchung von Diskretisierungen von Eigenwertproblemen

für holomorphe Fredholm Operatorfunktionen wird gewöhnlich im Rahmen der Theorie regulärer Approximationen durchgeführt, in welche sich auch die Galerkin-Methode einordnen lässt. In dieser Arbeit werden Konvergenzaussagen und Fehlerabschätzungen für Galerkin-Diskretisierungen solcher Eigenwertprobleme hergeleitet. Diese Ergebnisse werden dann auf die Diskretisierung von Randintegraleigenwertproblemen des Laplace-Operators angewandt. Des Weiteren werden Verfahren zur numerischen Lösung von algebraischen nichtlinearen Eigenwertproblemen untersucht. Dabei wird das wenig bekannte Kummersche Verfahren vorgestellt und die Konvergenz des Verfahrens für algebraische holomorphe Eigenwertprobleme unter Verwendung der Theorie von holomorphen Operatorfunktionen nachgewiesen. Am Ende der Arbeit wird ein numerisches Beispiel betrachtet und die Genauigkeit der Approximationen der Randelementmethode und der Finiten Elemente Methode für die Eigenwerte verglichen. Dabei werden die theoretischen Resultate der Arbeit bestätigt.

CONTENTS

1 INTRODUCTION

Laplacian eigenvalue problems provide a mathematical model for the description of different phenomena in science and engineering [18]. Eigenvalues and eigenfunctions characterize the response of physical systems which are subject to forces. In acoustics the eigensolutions describe the resonance behavior of mechanical systems. Their knowledge plays an important role for the design of objects which are subject to oscillations. In quantum mechanics the eigenvalues represent energy levels of certain quantum mechanical systems. For the design of waveguides in fiber optics also an analysis of Laplacian eigenvalue problems is needed. Moreover, Laplacian eigenvalue problems can be considered as model problems for more general second order partial differential operators in elasticity and electromagnetics.

The computation of eigenvalues and eigenfunctions of Laplacian eigenvalue problems can be done only in a few cases analytically. As for source problems for partial differential equations the most commonly used numerical method is the finite element method (FEM). For this method, a profound functional analytical framework and a rigorous error analysis have been established for eigenvalue problems [85, 10, 89, 12, 8]. The finite element method for eigenvalue problems is formally equivalent to source problems. Based on a variational formulation of the eigenvalue problem in an appropriate function space, approximate solutions are sought in finite-dimensional subspaces. The main idea of the discretization is to decompose the computational domain into geometrically simple subdomains, the so–called finite elements, on which finite–dimensional subspaces are defined, e.g., polynomials. This approach finally leads in the case of Laplacian eigenvalue problems to algebraic generalized eigenvalue problems with sparse matrices. Efficient solution techniques for such problems are presented in [34, 11, 54, 59, 68]. Small eigenvalues and corresponding eigenfunctions can be approximated very well by the finite element method, however an accurate approximation of large eigenvalues requires a very fine discretization of the computational domain and leads to prohibitive computational costs.

An alternative approach for the numerical solution of Laplacian eigenvalue problems is the boundary element method (BEM) [22, 81, 60, 45, 15, 51] which is also used for eigenvalue problems in elastodynamics [7, 52] and for plate problems [25, 52]. The boundary element method for Laplacian eigenvalue problems is based on equivalent boundary integral formulations which are nonlinear eigenvalue problems for related boundary integral operators. Different to finite element approaches which require a discretization of the computational domain, the use of boundary integral formulations and boundary element methods for the numerical solution of the eigenvalue problems requires only a discretization

of the boundary. The discretized eigenvalue problems are algebraic nonlinear eigenvalue problems with fully populated matrices, where the matrix entries are transcendental functions with respect to the eigenparameter. Usually, these eigenvalue problems are solved by using iterative schemes to determine the roots of the corresponding characteristic equations [81, 52, 60, 22, 23]. In several publications different approaches are suggested to approximate the nonlinear boundary integral operator eigenvalue problem by a polynomial one. A Taylor polynomial approximation of the fundamental solution with respect to the wave number is suggested by so-called multiple reciprocity methods [45, 46, 16, 69, 15]. In [51] an interpolation of the fundamental solution is considered. The discretization of polynomial boundary integral operator eigenvalue problems leads to algebraic polynomial eigenvalue problems which can be transformed into equivalent generalized eigenvalue problems.

To our knowledge, a rigorous numerical analysis for the approximations of boundary integral operator eigenvalue problems has not be done so far. Only in few works [22, 23, 84] the issue of the numerical analysis has been addressed. One of the main goals of this work is to give a convergence and error analysis of the Bubnov–Galerkin boundary element approximation of Laplacian eigenvalue problems. We use the concept of eigenvalue problems for so–called holomorphic Fredholm operator functions [27, 97, 26, 55]. This concept is a generalization of the theory for eigenvalue problems of bounded linear operators and provides important tools for the numerical analysis of approximations of such eigenvalue problems. The most important result is the representation of the resolvent close to an eigenvalue as a Laurent series with finite principle part [55, 26]. The analysis of the approximation of eigenvalue problems for holomorphic Fredholm operator functions has a long tradition [31, 44, 91, 90, 47, 48] and is usually done in the framework of the concepts of the discrete approximation scheme [86] and the regular approximation of operator functions [30]. In this framework a complete convergence analysis and asymptotic error estimates for eigenvalues are given by Karma in [47, 48]. These results are also valid for the Bubnov–Galerkin method since this approach fulfills the required assumptions. Nevertheless, we do a numerical analysis for the Bubnov–Galerkin method using main ideas of [47, 48], and in addition we present error estimates for eigenvectors, which has not be done in [47, 48].

The solution of discretized boundary integral operator eigenvalue problems for the Laplacian requires numerical algorithms for algebraic nonlinear eigenvalue problems. The construction of robust and efficient nonlinear eigenvalue solvers is a subject of ongoing research and there is a lot of literature on this topic, see the review article [63] and references therein. The numerical analysis of nonlinear eigenvalue solvers is available only for special classes of nonlinear eigenvalue problems. In particular, the focus lies on polynomial eigenvalue problems. The numerical analysis of algorithms for more general nonlinear eigenvalue problems is usually restricted to simple eigenvalues. This is mainly due to the fact that there is no standard theory for general nonlinear eigenvalue problems. Although most algebraic nonlinear eigenvalue problems which are considered in the literature would fit in

the concept of holomorphic Fredholm operator functions, this concept is usually not used for the analysis of the algorithms. In this work the little–known Kummer's method [57, 58] is presented and an error analysis for simple and multiple eigenvalues is given by using the theory of holomorphic Fredholm operator functions.

This book is organized as follows. In Chapter 2, different formulations of Laplacian eigenvalue problems are presented and their properties are described. First, the standard variational formulations of Laplacian eigenvalue problems are considered and they are characterized in terms of compact selfadjoint operators. The well–known properties of the Laplacian eigenvalue problems follow directly from the spectral theory of compact selfadjoint operators. We briefly address the finite element approximation of eigenvalue problems and present some error estimates. In the second part of this chapter, boundary integral representations of Laplacian eigenvalue problems are derived and the properties of the related boundary integral operators are presented.

The concept of eigenvalue problems for holomorphic Fredholm operator functions is introduced in Chapter 3. The basic definitions are provided and important results of the spectral theory are summarized.

A numerical analysis for the Bubnov–Galerkin approximation of eigenvalue problems for holomorphic Fredholm operator functions is done in Chapter 4. First, the convergence of the approximations of the eigenvalues and eigenvectors to the continuous ones is proven. Then, asymptotic error estimates are derived. Finally, the stability of the algebraic multiplicity of the approximations of the eigenvalues is shown.

In Chapter 5, Galerkin boundary element approximations of the Dirichlet and Neumann Laplacian eigenvalue problem are analyzed. It is shown that the boundary integral operator eigenvalue problems which are derived in Chapter 2 are eigenvalue problems of holomorphic Fredholm operator functions. The results of Chapter 4 are applied to the boundary element approximations of the boundary integral operator eigenvalue problems. Convergence of the boundary element approximations for the eigenvalues and the eigenfunctions are shown and asymptotic error estimates are given.

Numerical algorithms for algebraic nonlinear eigenvalue problems are discussed in Chapter 6. The well–known inverse iteration for nonlinear eigenvalue problems and two variants of it, the two–sided Rayleigh functional iteration and the residual inverse iteration are reviewed. The convergence behavior and the computational costs of these algorithms are compared. In the second part of this chapter we introduce Kummer's method for algebraic holomorphic eigenvalue problems and analyze its convergence properties.

In Chapter 7, a numerical example for the Dirichlet Laplacian eigenvalue problem is considered and results of a boundary element and a finite element approximation of the eigenvalues are presented. The numerical results of the boundary element approximations confirm the theoretical error estimate. Moreover, a high accuracy of the boundary element approximations is noticeable by a comparatively small number of boundary elements.

2 FORMULATIONS AND PROPERTIES OF LAPLACIAN EIGENVALUE PROBLEMS

In this chapter we present different formulations of the Laplacian eigenvalue problem on bounded Lipschitz domains with either Dirichlet or Neumann boundary conditions. We first consider the standard variational formulation of the Laplacian eigenvalue problems which can be characterized in terms of compact selfadjoint operators. We use the spectral theory for compact selfadjoint operator to analyze the properties of the Laplacian eigenvalue problems. Further, we briefly address the finite element method for eigenvalue problems for compact selfadjoint operators and present some error estimates for the eigenvalues and eigenelements.

In the second part of this chapter we derive boundary integral representations of Laplacian eigenvalue problems. These formulations lead to nonlinear eigenvalue problems for related boundary integral operators. We provide an analysis of the boundary integral operators such that the theory of eigenvalue problems for holomorphic Fredholm operator functions can be applied in the following chapters.

2.1 Sobolev spaces

In this section we introduce the relevant function spaces for the formulations of the Laplacian eigenvalue problems. The main references of this section are the textbooks [61] and [41].

Definition 2.1.1. *Let Ω be an open subset of $\mathbb{R}^d$. For $k \in \mathbb{N}_0$ the Sobolev space $W_2^k(\Omega)$ is defined by*

$$W_2^k(\Omega) := \{u \in L_2(\Omega) : \partial^\alpha u \in L_2(\Omega) \text{ for } |\alpha| \leq k\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, $|\alpha| = \alpha_1 + \ldots + \alpha_d$, and $\partial^\alpha u(x) = \dfrac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \ldots \dfrac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} u(x)$ are to be understood as weak partial derivatives.

The Sobolev space $W_2^k(\Omega)$ is equipped with the norm

$$\|u\|_{W_2^k(\Omega)} := \left(\sum_{|\alpha| \leq k} \int_\Omega |\partial^\alpha u(x)|^2 dx \right)^{1/2}$$

and it is a Hilbert space with the inner product

$$(u,v)_{W_2^k(\Omega)} := \sum_{|\alpha|\le k} \int_\Omega \partial^\alpha u(x)\overline{\partial^\alpha v(x)}dx,$$

see [61, p.75]. The definition of Sobolev spaces $W_2^s(\Omega)$ can be extended for any arbitrary $s > 0$.

Definition 2.1.2. *Let Ω be an open subset of $\mathbb{R}^d$. For $s = k+\mu$ with $k \in \mathbb{N}_0$ and $\mu \in (0,1)$, the Sobolev space $W_2^s(\Omega)$ is defined by*

$$W_2^s(\Omega) = \{u \in W_2^k(\Omega) : |\partial^\alpha u|_{\mu,\Omega} < \infty \text{ for } |\alpha| = k\},$$

where the Sobolev-Slobodeckii semi–norm $|\cdot|_{\mu,\Omega}$ is given by

$$|u|_{\mu,\Omega} := \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{d+2\mu}} dxdy \right)^{1/2}.$$

The Sobolev space $W_2^s(\Omega)$ for $s = k+\mu$ with $k \in \mathbb{N}_0$ and $\mu \in (0,1)$ is equipped with the norm

$$\|u\|_{W_2^s(\Omega)} := \left(\|u\|^2_{W_2^k(\Omega)} + \sum_{|\alpha|=k} |\partial^\alpha u|^2_{\mu,\Omega} \right)^{1/2}.$$

Again, $W_2^s(\Omega)$ is a Hilbert space with respect to the inner product

$$(u,v)_{W_2^s(\Omega)} := (u,v)_{W_2^k(\Omega)} + \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{[\partial^\alpha u(x) - \partial^\alpha u(y)][\overline{\partial^\alpha v(x) - \partial^\alpha v(y)}]}{|x-y|^{d+2\mu}} dxdy,$$

see [61, Chapter 3, p.75].

A second family of Sobolev spaces $H^s(\mathbb{R}^d)$ can be introduced by using the Fourier transform

$$\hat{u}(\xi) = \int_{\mathbb{R}^d} e^{-i2\pi x\cdot\xi} u(x)dx$$

for $u \in L_1(\Omega)$. The Sobolev space $H^s(\mathbb{R}^d)$ for $s \in \mathbb{R}$ is defined by

$$H^s(\mathbb{R}^d) := \{u \in \mathcal{S}^*(\mathbb{R}^d) : \mathcal{J}^s u \in L_2(\mathbb{R}^d)\},$$

where $\mathcal{S}^*(\mathbb{R}^d)$ is the space of the continuous linear functionals on the Schwartz space of rapidly decreasing functions in $C^\infty(\mathbb{R}^d)$,

$$\mathcal{S}(\mathbb{R}^d) := \{\varphi \in C^\infty(\mathbb{R}^d) : \sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta \varphi(x)| < \infty \text{ for all multi-indices } \alpha \text{ and } \beta\},$$

and where $\mathcal{J}^s$ is the Bessel potential of order s,

$$\mathcal{J}^s u(x) := \int_{\mathbb{R}^d} (1+|\xi|^2)^{s/2} \hat{u}(\xi) e^{i2\pi\xi\cdot x} d\xi \quad \text{for } x \in \mathbb{R}^d.$$

The Sobolev spaces $H^s(\mathbb{R}^d)$ and $W_2^s(\mathbb{R}^d)$ coincide for each $s \geq 0$, see for example [61, Theorem 3.16].

For general domains $\Omega \subset \mathbb{R}^d$ the following Sobolev spaces $H^s(\Omega)$ are defined.

Definition 2.1.3. *Let Ω be an open subset of $\mathbb{R}^d$ and $s \in \mathbb{R}$. We define*

$$H^s(\Omega) := \{u = \widetilde{u}_{|\Omega} : \widetilde{u} \in H^s(\mathbb{R}^d)\},$$

with the norm

$$\|u\|_{H^s(\Omega)} := \inf_{\widetilde{u}\in H^s(\mathbb{R}^d),\, \widetilde{u}_{|\Omega}=u} \|\widetilde{u}\|_{H^s(\mathbb{R}^d)}.$$

Further,

$$\widetilde{H}^s(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^s(\mathbb{R}^d)}},$$
$$H_0^s(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^s(\Omega)}}.$$

So far we have considered arbitrary non-empty subsets Ω in $\mathbb{R}^d$. In order to relate the above defined Sobolev spaces to each other we have to make some regularity assumptions for the boundary $\Gamma := \partial\Omega$. First of all consider the set

$$\Omega = \{x = (x', x_d) \in \mathbb{R}^d : x_d < f(x') \text{ for all } x' = (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}\}, \tag{2.1}$$

where $f : \mathbb{R}^{d-1} \to \mathbb{R}$ is a bounded function which is k times differentiable, and where the derivatives $\partial^\alpha f$ with $|\alpha| = k$ satisfy

$$|\partial^\alpha f(x') - \partial^\alpha f(y')| \leq M|x' - y'|^\mu \quad \text{for all } x', y' \in \mathbb{R}^{n-1}$$

with some $\mu \in [0,1]$. Such a set Ω as defined in (2.1) is called a $C^{k,\mu}$ hypograph.

Definition 2.1.4. *The open set $\Omega \subset \mathbb{R}^d$ is called a $C^{k,\mu}$ domain if its boundary Γ is compact and if there exist finite families $\{W_j\}$ and $\{\Omega_j\}$ which have the following properties:*

i) *The family $\{W_j\}$ is a finite open cover of Γ.*

ii) *Each Ω_j can be transformed to a $C^{k,\mu}$ hypograph by a rigid motion.*

iii) *For each j the equality $W_j \cap \Omega = W_j \cap \Omega_j$ is satisfied.*

If Ω is a $C^{k,\mu}$ domain, then the boundary can be parameterized by k times differentiable functions. Therefore we call the boundary of a $C^{k,\mu}$ domain k times differentiable. If this property is only locally satisfied, then we call the boundary piecewise smooth.

A $C^{0,1}$ domain is called a Lipschitz domain. For instance, any polygonal bounded domain in $\mathbb{R}^2$ and any domain in $\mathbb{R}^3$ which is bounded by a polyhedron is a Lipschitz domain. Note that a Lipschitz domain may be unbounded. For example, if Ω is a bounded Lipschitz domain, then its complement $\mathbb{R}^d \setminus \overline{\Omega}$ is also a Lipschitz domain.

The following theorem quotes results about the relations of the above defined Sobolev spaces for Lipschitz domains.

Theorem 2.1.5. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. Then, we have for $s \geq 0$:*

i) $W_2^s(\Omega) = H^s(\Omega)$.

ii) $\widetilde{H}^s(\Omega) \subset H_0^s(\Omega)$.

iii) $\widetilde{H}^s(\Omega) = H_0^s(\Omega)$ *for* $s \notin \left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\right\}$.

Moreover, for all $s \in \mathbb{R}$,

$$\widetilde{H}^s(\Omega) = \left[H^{-s}(\Omega)\right]^* \quad \text{and} \quad H^{-s}(\Omega) = \left[\widetilde{H}^s(\Omega)\right]^*.$$

Proof. See [61, Chapter 3]. □

Sobolev spaces on the boundary

In the following we assume that $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain. The L_2-norm on the boundary $\Gamma = \partial\Omega$ is defined by

$$\|u\|_{L_2(\Gamma)} := \left(\int_\Gamma |u(x)|^2 ds_x \right)^{1/2}.$$

For $s \in (0,1)$ the Sobolev-Slobodeckii-norm is defined by

$$\|u\|_{H^s(\Gamma)} := \left(\|u\|_{L_2(\Gamma)}^2 + \int_\Gamma \int_\Gamma \frac{|u(x)-u(y)|^2}{|x-y|^{d-1+2s}} ds_x ds_y \right)^{1/2}.$$

Definition 2.1.6. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain with the boundary $\Gamma = \partial\Omega$. The spaces $L_2(\Gamma)$ and $H^s(\Gamma)$ are defined as closures,*

$$L_2(\Gamma) := \overline{C^0(\Gamma)}^{\|\cdot\|_{L_2(\Gamma)}},$$
$$H^s(\Gamma) := \overline{C^0(\Gamma)}^{\|\cdot\|_{H^s(\Gamma)}} \quad \text{for } s \in (0,1).$$

The spaces $L_2(\Gamma)$ and $H^s(\Gamma)$ for $s \in (0,1)$ are Hilbert spaces equipped with the inner products

$$(u,v)_{L_2(\Gamma)} := \int_\Gamma u(x)\overline{v(x)}ds_x,$$

$$(u,v)_{H^s(\Gamma)} := (u,v)_{L_2(\Gamma)} + \int_\Gamma \int_\Gamma \frac{[u(x)-u(y)]\overline{[v(x)-v(y)]}}{|x-y|^{d-1+2s}} ds_x ds_y \quad \text{for } s \in (0,1),$$

see [41, p. 172].

For indices $s \geq 1$ also Sobolev spaces $H^s(\Gamma)$ can be defined, see, e.g., [41, Section 4.2]. This requires for $s > 1$ stronger regularity assumptions for the boundary than the Lipschitz property, i.e., the boundary must be of class $C^{k,\kappa}$ and $s \leq k+\kappa$. For definitions and details see [41, Section 4.2].

For negative indices s the Sobolev spaces $H^s(\Gamma)$ are defined by duality with respect to the $L_2(\Gamma)$-inner product. More precisely, for $s < 0$ we define the norm

$$\|t\|_{H^s(\Gamma)} := \sup_{0 \neq u \in H^{-s}(\Gamma)} \frac{|(u,t)_{L_2(\Gamma)}|}{\|u\|_{H^{-s}(\Gamma)}}. \tag{2.2}$$

The closure of $L_2(\Gamma)$ with respect to (2.2) is denoted by $H^s(\Gamma)$. Note that for $\ell \in L_2(\Gamma) \subset H^s(\Gamma)$, $s < 0$, and $v \in H^{-s}(\Gamma)$, we can identify the duality product by using the inner product in $L_2(\Gamma)$,

$$\ell(v) = \langle v, \ell \rangle_{H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)} = (\overline{v}, \ell)_{L_2(\Gamma)},$$

see [41, p. 175]. Further, for $s < 0$ we define for functionals $\ell \in H^s(\Gamma)$ and $v \in H^{-s}(\Gamma)$ the sesquilinear form

$$(v,\ell)_\Gamma := \langle v, \overline{\ell} \rangle_{H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)} := \overline{\ell}(v) := \overline{\langle \overline{v}, \ell \rangle}_{H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)}. \tag{2.3}$$

For an open subset $\Gamma_0 \subset \Gamma$ and for a sufficiently smooth boundary Γ we define the Sobolev spaces for $s \geq 0$,

$$H^s(\Gamma_0) := \{v = \tilde{v}_{|\Gamma_0} : \tilde{v} \in H^s(\Gamma)\},$$
$$\widetilde{H}^s(\Gamma_0) := \{v = \tilde{v}_{|\Gamma_0} : \tilde{v} \in H^s(\Gamma), \operatorname{supp} \tilde{v} \subset \Gamma_0\},$$

with the norm

$$\|v\|_{H^s(\Gamma_0)} := \inf_{\tilde{v} \in H^s(\Gamma) \wedge \tilde{v}_{|\Gamma_0} = v} \|\tilde{v}\|_{H^s(\Gamma)}.$$

For $s < 0$ the Sobolev spaces are defined as dual spaces

$$H^s(\Gamma_0) := \left[\widetilde{H}^{-s}(\Gamma_0)\right]^* \quad \text{and} \quad \widetilde{H}^s(\Gamma_0) := \left[H^{-s}(\Gamma_0)\right]^*.$$

Let us now assume that Γ is closed and piecewise smooth,

$$\Gamma = \bigcup_{i=1}^{J} \overline{\Gamma}_i, \quad \Gamma_i \cap \Gamma_j = \emptyset \quad \text{for } i \neq j.$$

The Sobolev space $H^s_{\mathrm{pw}}(\Gamma)$ for $s > 0$ is defined by

$$H^s_{\mathrm{pw}}(\Gamma) := \{v \in L_2(\Gamma) : v_{|\Gamma_i} \in H^s(\Gamma_i), i = 1, \ldots, J\}$$

with the norm

$$\|v\|_{H^s_{\mathrm{pw}}(\Gamma)} := \left(\sum_{i=1}^{J} \|v_{|\Gamma_i}\|^2_{H^s(\Gamma_i)} \right)^{1/2}.$$

For $s < 0$ we define

$$H^s_{\mathrm{pw}}(\Gamma) := \prod_{i=1}^{J} \widetilde{H}^s(\Gamma_i)$$

with the norm

$$\|w\|_{H^s_{\mathrm{pw}}(\Gamma)} := \sum_{i=1}^{J} \|w_{|\Gamma_i}\|_{\widetilde{H}^s(\Gamma_i)}.$$

Lemma 2.1.7. *If $w \in H^s_{\mathrm{pw}}(\Gamma)$ and $s < 0$, then*

$$\|w\|_{H^s(\Gamma)} \leq \|w\|_{H^s_{\mathrm{pw}}(\Gamma)}. \tag{2.4}$$

Proof. See [83, Lemma 2.20]. □

Remark 2.1.8. *If Ω is a Lipschitz domain, then for all definitions and statements above concerning Sobolev spaces on subsets on the boundary $\Gamma = \partial\Omega$ we have to assume that $|s| \leq 1$. The results for $|s| > 1$ are only valid if stronger regularity conditions for the boundary Γ are assumed, see [41, Section 4.3].*

Trace operators and normal derivatives

The trace operators relate the Sobolev spaces on a domain Ω to the Sobolev spaces on its boundary $\Gamma = \partial\Omega$.

Theorem 2.1.9. *Let Ω be a bounded domain in $\mathbb{R}^d$. Define the interior trace operator $\gamma_0^{\mathrm{int}} : C^\infty(\overline{\Omega}) \to C^\infty(\Gamma)$ by*

$$\gamma_0^{\mathrm{int}} u := u|_\Gamma.$$

If Ω is a $C^{k-1,1}$ domain then the operator γ_0^{int} has a unique extension to a bounded linear operator

$$\gamma_0^{\mathrm{int}} : H^s(\Omega) \to H^{s-1/2}(\Gamma) \tag{2.5}$$

for $\frac{1}{2} < s \leq k$. This extension has a continuous right inverse $\mathcal{E} : H^{s-1/2}(\Gamma) \to H^s(\Omega)$.

Proof. See [61, Theorem 3.37]. □

If Ω is a bounded Lipschitz domain, i.e., $k = 1$, then (2.5) implies that the interior trace operator is a continuous linear map

$$\gamma_0^{\text{int}} : H^s(\Omega) \to H^{s-1/2}(\Gamma)$$

for $s \in (\frac{1}{2}, 1]$. This result can be extended to $s \in (\frac{1}{2}, \frac{3}{2})$, see [61, Theorem 3.38].

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ there exists a unique outward unit normal vector $n : \partial\Omega \to \mathbb{R}^d$ almost everywhere [61, p. 96f]. We define for a function $u \in C^\infty(\overline{\Omega})$ the interior normal derivative

$$\gamma_1^{\text{int}} u := \nabla u|_\Gamma \cdot n.$$

The interior normal derivative γ_1^{int} can be extended to a bounded linear operator

$$\gamma_1^{\text{int}} : H^1(\Omega, \Delta) \to H^{-1/2}(\Gamma),$$

see [17, Lemma 3.2], where

$$H^1(\Omega, \Delta) := \{u \in H^1(\Omega) : \Delta u \in L_2(\Omega)\}.$$

In a similar way an exterior trace operator and an exterior normal derivative with respect to a bounded Lipschitz domain Ω can be defined. Set $\Omega^c := \mathbb{R}^d \setminus \overline{\Omega}$ and let n be again the outward unit normal vector of Ω. Then we define for a function $u \in C^\infty(\overline{\Omega^c})$

$$\begin{aligned} \gamma_0^{\text{ext}} u &:= u|_\Gamma, \\ \gamma_1^{\text{ext}} u &:= \nabla u|_\Gamma \cdot n. \end{aligned}$$

For the exterior trace operator γ_0^{ext} there exists a bounded linear extension to

$$\gamma_0^{\text{ext}} : H^1_{\text{loc}}(\Omega^c) \to H^{1/2}(\Gamma),$$

see [17, Lemma 3.2], where

$$H^s_{\text{loc}}(\Omega^c) := \{u : \Omega^c \to \mathbb{C} : u|_{\Omega^c \cap K} \in H^s(\Omega^c \cap K) \text{ for any compact } K \subset \mathbb{R}^d\},$$

for $s \geq 0$. Also the exterior normal derivative γ_1^{ext} can be extended to a bounded linear operator

$$\gamma_1^{\text{ext}} : H^1_{\text{loc}}(\Omega^c, \Delta) \to H^{-1/2}(\Gamma),$$

see [17, Lemma 3.2], where

$$H^1_{\text{loc}}(\Omega^c, \Delta) := \{u \in H^1_{\text{loc}}(\Omega^c, \Delta) : \Delta u \in L_{2,\text{loc}}(\Omega^c)\}.$$

2.2 Variational formulation of Laplacian eigenvalue problems

In this section we analyze the standard variational formulation of the Laplacian eigenvalue problem with either Dirichlet or Neumann boundary conditions on bounded Lipschitz domains in $\mathbb{R}^d$. We will show that these eigenvalue problems can be formulated in terms of compact selfadjoint operators. Therefore we use the spectral theory for compact self-adjoint operators to characterize the properties of these eigenvalue problems. The main references of this section are [4, 8].

First, we consider the Dirichlet Laplacian eigenvalue problem in the classical form:
Find $\lambda \in \mathbb{C}$ and $0 \neq u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that

$$-\Delta u = \lambda u \quad \text{on } \Omega, \qquad \gamma_0^{\text{int}} u = 0 \quad \text{on } \Gamma = \partial\Omega. \tag{2.6}$$

A solution (λ, u) of (2.6) is called an eigenpair of the Dirichlet Laplacian eigenvalue problem in the classical sense. Multiplying the first equation by the complex conjugate of a test function $v \in H_0^1(\Omega)$, integrating over the domain Ω and using integration by parts, we obtain the variational formulation of (2.6):
Find $\lambda \in \mathbb{C}$ and $0 \neq u \in H_0^1(\Omega)$ such that

$$a(u,v) = \lambda b(u,v) \quad \text{for all } u, v \in H_0^1(\Omega) \tag{2.7}$$

with the sesquilinear form $a(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$ defined by

$$a(u,v) := \int_\Omega \nabla u \cdot \overline{\nabla v} dx, \tag{2.8}$$

and the sesquilinear form $b(\cdot,\cdot) : L_2(\Omega) \times L_2(\Omega) \to \mathbb{C}$ defined by

$$b(u,v) := \int_\Omega u \bar{v} dx. \tag{2.9}$$

A solution $(\lambda, u) \in \mathbb{C} \times H_0^1(\Omega) \setminus \{0\}$ of (2.7) is called an eigenpair of (2.6) in the weak sense. Every eigenpair (λ, u) which fulfills the eigenvalue problem (2.6) in the classical sense is obviously a solution of the variational problem (2.7).
On the other hand, if $(\lambda, u) \in \mathbb{C} \times H_0^1(\Omega) \setminus \{0\}$ is a solution of the variational eigenvalue problem (2.7) and if we assume that $u \in C^2(\overline{\Omega})$, then we can apply Green's formula to (2.7) and obtain with the DuBois-Reymond lemma [61, Theorem 3.7] that (λ, u) is a solution of the eigenvalue problem in the classical sense. However, the assumptions $u \in C^2(\overline{\Omega})$ requires in general stronger regularity conditions on the domain Ω than the Lipschitz property. From the regularity result in [2, Theorem 9.8] it follows that an eigenfunction u of the variational problem is in $C^2(\overline{\Omega})$ if the domain is of class C^5.

Let us now consider the sesquilinear form $a(\cdot,\cdot)$ as defined in (2.8). By the Cauchy–Schwarz inequality, $a(\cdot,\cdot)$ is bounded on $H_0^1(\Omega)$. From the Poincaré's inequality

$$\|u\|_{L_2(\Omega)} \leq c\|\nabla u\|_{L_2(\Omega)} \quad \text{for all } u \in H_0^1(\Omega),$$

see, e.g., [19], [87, Lemma 10.2], it follows that $a(\cdot,\cdot)$ is $H_0^1(\Omega)$-elliptic, i.e.,

$$a(u,u) \geq c\|u\|^2_{H^1(\Omega)} \quad \text{for all } u \in H_0^1(\Omega).$$

In addition, the sesquilinear form $a(\cdot,\cdot)$ is Hermitian. Therefore we may consider $a(\cdot,\cdot)$ as inner product of the Hilbert space $H_0^1(\Omega)$. Define

$$[H_0^1(\Omega)]^\times := \{f : H_0^1(\Omega) \to \mathbb{C},\ f \text{ is continuous and conjugate-linear}\},$$

then as a consequence of the Riesz representation theorem, see, e.g., [99, p. 105], there exists an isomorphism $R : [H_0^1(\Omega)]^\times \to H_0^1(\Omega)$ such that

$$f(v) = a(Rf, v) \tag{2.10}$$

is satisfied for all $f \in [H_0^1(\Omega)]^\times$ and $v \in H_0^1(\Omega)$.
Further let us define the operator $S : H_0^1(\Omega) \to [H_0^1(\Omega)]^\times$ by

$$(Su)(\cdot) := b(u,\cdot) \quad \text{for } u \in H_0^1(\Omega). \tag{2.11}$$

The operator S is obviously linear and bounded. Since $b(\cdot,\cdot)$ is the inner product in $L_2(\Omega)$, it follows by Rellich's embedding theorem [1] that $S : H_0^1(\Omega) \to [H_0^1(\Omega)]^\times$ is compact. Using the definitions (2.10) and (2.11) of R and S, respectively, we have the representation

$$a(RSu, v) = (Su)(v) = b(u,v)$$

for all $u, v \in H_0^1(\Omega)$. Hence, the variational problem

$$a(u,v) = \lambda b(u,v) = \lambda a(RSu, v) \quad \text{for all } v \in H_0^1(\Omega)$$

is equivalent to

$$u = \lambda RSu. \tag{2.12}$$

Consequently, $(\lambda, u) \in \mathbb{C} \setminus \{0\} \times H_0^1(\Omega)$ is an eigenpair of the variational problem (2.7) if and only if $(\frac{1}{\lambda}, u)$ is an eigenpair of the eigenvalue problem

$$\frac{1}{\lambda} u = RSu \qquad \text{in } H_0^1(\Omega). \tag{2.13}$$

Let us define the operator

$$T := RS : H_0^1(\Omega) \to H_0^1(\Omega), \tag{2.14}$$

then from the properties of the operators R and S it follows immediately that T is linear, bounded and compact. Furthermore, T is selfadjoint with respect to the inner product $a(\cdot,\cdot)$, since

$$a(RSu,v) = b(u,v) = \overline{b(v,u)} = \overline{a(RSv,u)} = a(u,RSv)$$

is satisfied for all $u,v \in H_0^1(\Omega)$. Finally, notice that all eigenvalues of the variational eigenvalue problem (2.7) are larger than zero, because otherwise we would have an eigenvalue $\lambda \leq 0$ and a corresponding eigenfunction $0 \neq u \in H_0^1(\Omega)$ such that

$$a(u,u) = \lambda b(u,u) \leq 0, \tag{2.15}$$

which is a contradiction to the fact that the sesquilinear form $a(\cdot,\cdot)$ is $H_0^1(\Omega)$-elliptic. In the next theorem we summarize the above results.

Theorem 2.2.1. *$(\lambda,u) \in \mathbb{C} \times H_0^1(\Omega)$ is an eigenpair of the eigenvalue problem (2.7), if and only if $(\frac{1}{\lambda},u)$ is an eigenpair of the eigenvalue problem*

$$Tu = \frac{1}{\lambda}u, \tag{2.16}$$

where $T : H_0^1(\Omega) \to H_0^1(\Omega)$ is defined by (2.14). The operator T is linear, compact and selfadjoint.

So we can use the spectral theory of compact selfadjoint operators for the analysis of the variational eigenvalue problem (2.7).

Theorem 2.2.2. *Let $T : X \to X$ be a linear, compact and selfadjoint operator on a Hilbert space X and let $\sigma(T)$ the spectrum of T, i.e.,*

$$\sigma(T) = \{\lambda \in \mathbb{C} : (\lambda I_X - T) \text{ is not invertible}\}.$$

Then:

i) *$\sigma(T) \setminus \{0\}$ consists of countably many eigenvalues with zero as the only possible accumulation point.*

ii) *All eigenvalues are real.*

iii) *The eigenelements are orthogonal in X.*

iv) *The geometric multiplicity of each eigenvalue λ is finite, i.e.,* $\dim\ker(\lambda I_X - T) < \infty$.

v) *The ascent*

$$\varkappa(T,\lambda) := \max\{n \in \mathbb{N} : \ker(\lambda I_X - T)^{n-1} \neq \ker(\lambda I_X - T)^n\}$$

of each eigenvalue λ is equal to 1.

vi) *The algebraic multiplicity*

$$m(T,\lambda) := \dim\ker\Big((\lambda I_X - T)^{\varkappa(T,\lambda)}\Big) \tag{2.17}$$

of each eigenvalue λ is equal to its geometric multiplicity.

Proof. The assertions follow from the Riesz–Schauder theory for compact selfadjoint operator, see, e.g., [4, Satz 9.6 and Satz 10.12]. □

Using Theorem 2.2.2, we can summarize the properties of the variational eigenvalue problem (2.7).

Corollary 2.2.3. *Consider the variational formulation of the Dirichlet Laplacian eigenvalue problem (2.7). Then:*

i) *All eigenvalues are real and strictly positive.*

ii) *The set of eigenvalues is a countable infinite sequence diverging to $+\infty$.*

iii) *The dimension of the geometric eigenspace of each eigenvalue λ is finite, i.e.,*

$$\dim\operatorname{span}\{u \in H_0^1(\Omega) : a(u,v) = \lambda b(u,v) \quad \textit{for all } v \in H_0^1(\Omega)\} < \infty. \tag{2.18}$$

iv) *The eigenfunctions are orthogonal in $H_0^1(\Omega)$.*

Let us now consider the Neumann Laplacian eigenvalue problem. The formulation in classical form reads as follows:
Find $\lambda \in \mathbb{C}$ and $0 \neq u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that

$$-\Delta u = \lambda u \quad \text{on } \Omega, \qquad \gamma_1^{\text{int}} u = 0 \quad \text{on } \partial\Omega = \Gamma. \tag{2.19}$$

The variational problem is derived in the same way as for the Dirichlet case and leads to: Find $\lambda \in \mathbb{C}$ and $0 \neq u \in H^1(\Omega)$ such that

$$a(u,v) = \lambda b(u,v) \quad \text{for all } v \in H^1(\Omega), \tag{2.20}$$

where $a(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ and $b(\cdot,\cdot) : L_2(\Omega) \times L_2(\Omega) \to \mathbb{C}$ are defined as in (2.8) and (2.9), respectively. An eigenpair (λ,u) of (2.19) is obviously also an eigenpair of the variational formulation (2.20). On the other hand, we get with the same arguments as for the Dirichlet case that a solution $(\lambda,u) \in \mathbb{C} \times H^1(\Omega) \setminus \{0\}$ of the variational problem (2.20) is a solution of the eigenvalue problem (2.19) in the classical sense if $u \in C^2(\overline{\Omega})$.

Before we can show that the Neumann variational eigenvalue problem is equivalent to an eigenvalue problem for a compact selfadjoint operator, we have to make two remarks. First, if $u \neq 0$ is a constant function on Ω, then $(0,u)$ is an eigenpair of the Neumann

eigenvalue problem in the classical and in the variational sense. If $u \in H^1(\Omega)$ is an eigenfunction of (2.20) corresponding to a nonzero eigenvalue, and if we choose as test function $v \equiv 1$ on Ω, then from

$$0 = a(u,1) = \lambda \int_\Omega u dx$$

it follows that

$$\int_\Omega u dx = 0.$$

Therefore, we can restrict the corresponding space for the variational formulation of the Neumann eigenvalue problem (2.20) for nonzero eigenvalues to

$$H^1_*(\Omega) := \{u \in H^1(\Omega) : \int_\Omega u dx = 0\}. \tag{2.21}$$

The Poincaré's inequality (2.10) holds also on $H^1_*(\Omega)$, see, e.g., [87, Lemma 10.2], and so the sesquilinear form $a(\cdot,\cdot)$ is $H^1_*(\Omega)$-elliptic. Since $a(\cdot,\cdot)$ is Hermitian and bounded, it defines an inner product of $H^1_*(\Omega)$. Therefore, we can conclude with the same arguments as in the Dirichlet case that there exists a linear, compact selfadjoint operator

$$W : H^1_*(\Omega) \to H^1_*(\Omega)$$

such that $(\lambda, u) \in \mathbb{C} \setminus \{0\} \times H^1_*(\Omega)$ is an eigenpair of the Neumann variational eigenvalue problem (2.20) if and only if $(\frac{1}{\lambda}, u)$ is an eigenpair of the eigenvalue problem

$$Wu = \frac{1}{\lambda} u \quad \text{in } H^1_*(\Omega).$$

Hence, the properties of the variational Neumann eigenvalue problem follows from Theorem 2.2.2.

Remark 2.2.4. *Theorem 2.2.2 shows that all eigenvalues of a compact selfadjoint operator are real. Moreover, the eigenelements may be taken to be real, see [8, Remark 4.1]. Therefore eigenvalue problems in Hilbert spaces for compact selfadjoint operators may be formulated in terms of real Hilbert spaces.*

For the numerical approximations of the eigenvalue problems (2.7) and (2.20) regularity properties of the eigenfunctions are important. There are different results with respect to the assumptions on the boundary, see, e.g., [8, Theorem 4.1], [32, Theorem 2.4.2.7] or [35, Chapter 9 and 11]. Here we quote a general result, see [35, Korollar 9.1.19].

Theorem 2.2.5. *Let Ω be a bounded domain of class $C^{1+k,\mu}$ with $k \in \mathbb{N}_0$. Let $\mu \in [0,1]$ and let (λ, u) be an eigenpair of the variational eigenvalue problem (2.7) or (2.20). Let $s \geq 0$ and*

$$\begin{aligned} s &\leq k+\mu \quad \text{if } \mu \in \{0,1\}, \\ s &< k+\mu \quad \text{if } \mu \in (0,1), \end{aligned}$$

then

$$u \in H^{1+s}(\Omega).$$

It is important to mention that the conditions on the boundary in Theorem 2.2.5 for the regularity of the eigenfunctions are sufficient but not necessary conditions.

2.3 FEM for eigenvalue problems of compact selfadjoint operators

In this section we consider variational posed eigenvalue problems which can be reduced to eigenvalue problems for compact selfadjoint operators. We briefly sketch a finite element approximation of such problems and present an apriori error estimate for the eigenpairs. The main reference of this section is the review article [8] of Babuška and Osborn.

Let X and W be real Hilbert spaces where X is compactly imbedded in W. We consider the following eigenvalue problem: Find $(\lambda, u) \in \mathbb{R} \times X \setminus \{0\}$ such that

$$a(u,v) = \lambda b(u,v) \quad \text{for all } v \in X, \tag{2.22}$$

where

$$a(\cdot,\cdot) : X \times X \to \mathbb{R}$$

is a bounded symmetric and X-elliptic bilinear form, and where

$$b(\cdot,\cdot) : W \times W \to \mathbb{R}$$

is a bounded and symmetric bilinear form which satisfies

$$b(u,u) > 0 \quad \text{for all } 0 \neq u \in X.$$

From the assumptions on the spaces and the bilinear forms, it follows with the same arguments as for the variational eigenvalue problems (2.7) and (2.20) that there exists a compact selfadjoint operator $T : X \to X$ such that

$$a(Tu,v) = b(u,v) \quad \text{for all } u,v \in X. \tag{2.23}$$

Further, $(\lambda, u) \in \mathbb{R} \times X$ is an eigenpair of (2.22) if and only if $(\frac{1}{\lambda}, u)$ is an eigenpair of the eigenvalue problem

$$Tu = \frac{1}{\lambda} u \quad \text{in } X.$$

By Theorem 2.2.2, the eigenvalue problem (2.22) has a countable sequence of eigenvalues. Let $\{\lambda_i\}_{i=1}^{\infty}$ denote the ordered sequence of eigenvalues with $\lambda_i \leq \lambda_{i+1}$ and where the λ_i are repeated according to their multiplicities. Then, the corresponding sequence $\{u_i\}_{i=1}^{\infty}$ of eigenelements can be chosen such that

$$a(u_i, u_j) = \lambda_i b(u_i, u_j) = \delta_{ij},$$

see Theorem 2.2.2.

The finite element method for eigenvalue problems is formally equal to the finite element method for source problems. Here we present a conforming finite element method for eigenvalue problems. A discontinuous Galerkin method is presented in [6]. In order to approximate the eigenvalues and eigenelements of the eigenvalue problem (2.22) a sequence of finite dimensional subspaces $\{X_N\}_{N\in\mathbb{N}} \subset X$ is chosen which has the approximation property

$$\lim_{N\to\infty} \inf_{x_N \in X_N} \|u - x_N\|_X = 0 \quad \text{for all } u \in X.$$

Approximate solutions of the eigenvalue problem (2.22) are solutions of the Galerkin variational problem which reads as follows: Find $(\lambda_N, u_N) \in \mathbb{R} \times X_N \setminus \{0\}$ such that

$$a(u_N, v_N) = \lambda_N b(u_N, v_N) \quad \text{for all } v_N \in X_N. \tag{2.24}$$

Let $\{\varphi_1, \ldots, \varphi_{n_N}\}$ be a basis of X_N, then the variational problem is equivalent to the algebraic generalized eigenvalue problem: Find $(\lambda_N, \underline{z}) \in \mathbb{R} \times \mathbb{R}^{n_N} \setminus \{0\}$ such that

$$A\underline{z} = \lambda_N B\underline{z},$$

with $\underline{z} = (z_1, \ldots, z_{n_N})$,

$$A[i,j] = a(\varphi_i, \varphi_j), \qquad B[i,j] = b(\varphi_i, \varphi_j) \qquad \text{for } 1 \leq i, j \leq n_N,$$

and where $u_N = \sum_{i=1}^{n_N} z_i \varphi_i$. The Galerkin eigenvalue problem (2.24) has a finite sequence of eigenvalues

$$\lambda_{1,N} \leq \lambda_{2,N} \leq \ldots \leq \lambda_{n_N,N}$$

where the corresponding sequence of eigenelements $\{u_{i,N}\}_{i=1}^{n_N}$ can be taken to satisfy

$$a(u_{i,N}, u_{j,N}) = \lambda_{i,N} b(u_{i,N}, u_{j,N}) = \delta_{ij}. \tag{2.25}$$

The analysis of the Galerkin variational eigenvalue problem can be again reduced to a corresponding eigenvalue problem of a compact selfadjoint operator $T_N : X \to X_N \subset X$ which is defined by

$$a(T_N u, v_N) = b(u, v_N) \quad \text{for all } u \in X,\, v_N \in X_N.$$

The operator T_N can be written as $P_N T$, where $P_N : X \to X$ denotes the projection of X onto X_N defined by

$$a(P_N u, v_N) = a(u, v_N) \quad \text{for all } u \in X,\, v_N \in X_N.$$

From the approximation property of X_N and the compactness of T, the operator convergence $T_n \to T$ follows, see [8, Section 2.8]. These considerations are the basis for the error estimates which are given in the next theorem.

Theorem 2.3.1. *Let λ_k be an eigenvalue of (2.22) with the geometric multiplicity q and assume that $\lambda_k = \lambda_{k+1} = \ldots = \lambda_{k+q-1}$. Then:*

i) *There exists a constant $C > 0$ such that*

$$\lambda_k \le \lambda_{j,N} \le \lambda_k + C d_N^2(\lambda_k) \qquad \text{for } j = k, \ldots, k+q-1, \tag{2.26}$$

where

$$d_N(\lambda_k) := \sup_{u \in E(\lambda_k)} \inf_{\varphi \in X_N} \|u - \varphi\|_X,$$

and where

$$E(\lambda_k) := \{u \text{ is an eigenvector corresponding to } \lambda_k \text{ with } \|u\| = 1\}.$$

ii) *Let $u_{j,N}$ be an eigenelement corresponding to $\lambda_{j,N}$ for $j = k, \ldots, k+q-1$, then there exists a constant $c > 0$ such that*

$$\inf_{u \in E(\lambda_k)} \|u - u_{j,N}\|_X \le c d_n(\lambda_k). \tag{2.27}$$

Proof. See [8, p. 699]. □

Theorem 2.3.1 shows that as for source problems a quasi optimal error estimate for eigenfunctions for compact selfadjoint operators can be achieved by using finite element approximations. For additional results concerning the error analysis we refer to [8, Section 7 and 8] and references therein.

2.4 Boundary integral formulations of Laplacian eigenvalue problems

The numerical solution of Laplacian eigenvalue problems with boundary element methods is based on equivalent boundary integral representations [81, 51, 60, 45, 22, 15]. These representations differ in the choice of the fundamental solution and in the choice of the boundary integral equations. In all cases these formulations lead to nonlinear eigenvalue problems for related boundary integral operators.

In this section we derive boundary integral formulations for the Dirichlet and Neumann Laplacian eigenvalue problem on bounded Lipschitz domains $\Omega \subset \mathbb{R}^3$ with piecewise smooth boundary. For this purpose we consider the Helmholtz equation

$$-\Delta u - \kappa^2 u = 0 \quad \text{on } \Omega, \tag{2.28}$$

since every eigenpair (κ^2, u) of a Laplacian eigenvalue problem is a solution of (2.28). In the following we will give an introduction to boundary integral equations for the Helmholtz equation and provide a review of important properties of the corresponding boundary integral operators. The main references for this section are [61, 17, 41, 83].

A fundamental solution U^*_κ for $\kappa \in \mathbb{C}$ of the Helmholtz equation (2.28) in $\mathbb{R}^3$ is given by

$$U^*_\kappa(x,y) = \frac{1}{4\pi}\frac{e^{i\kappa|x-y|}}{|x-y|} \quad \text{for } x \neq y, \tag{2.29}$$

see, e.g., [61, Theorem 9.4]. The single layer potential for a function $w \in H^{-1/2}(\Gamma)$ and $\kappa \in \mathbb{C}$ is defined by

$$(\widetilde{V}(\kappa)w)(x) := \int_\Gamma U^*_\kappa(x,y)w(y)ds_y \quad \text{for } x \in \Omega \cup \Omega^c, \tag{2.30}$$

which provides a solution of the Helmholtz equation (2.28), see [61, p. 202]. For a function $v \in H^{1/2}(\Gamma)$ and $\kappa \in \mathbb{C}$ the double layer potential is defined by

$$(W(\kappa)v)(x) := \int_\Gamma \gamma^{\text{int}}_{1,y} U^*_\kappa(x,y)v(y)ds_y \quad \text{for } x \in \Omega \cup \Omega^c, \tag{2.31}$$

which is also a solution of the Helmholtz equation (2.28), cf. [61, p.202]. Applying the interior trace operator γ^{int}_0 and the normal derivative γ^{int}_1 to the potentials yields

$$\gamma^{\text{int}}_0 \widetilde{V}(\kappa) = V(\kappa), \tag{2.32}$$

$$\gamma^{\text{int}}_0 W(\kappa) = -\frac{1}{2}I + K(\kappa), \tag{2.33}$$

$$\gamma^{\text{int}}_1 \widetilde{V}(\kappa) = \frac{1}{2}I + K'(\kappa), \tag{2.34}$$

$$\gamma^{\text{int}}_1 W(\kappa) = -D(\kappa), \tag{2.35}$$

almost everywhere on Γ, see [61, p. 218], with the single layer potential operator $V(\kappa) : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$, the double layer potential operator $K(\kappa) : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$, the adjoint double layer potential operator $K'(\kappa) : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ and the hypersingular boundary integral operator $D(\kappa) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$. Note that it is sufficient, due to their use within a variational framework, to consider the above representations on smooth

parts of the surface. These boundary integral operators are continuous linear operators and admit the following representations [17, p.615]:

$$(V(\kappa)w)(x) := \int_\Gamma U^*_\kappa(x,y)w(y)ds_y \qquad \text{for } x \in \Gamma, \tag{2.36}$$

$$(K(\kappa)v)(x) := \int_\Gamma \gamma^{\text{int}}_{1,y} U^*_\kappa(x,y)v(y)ds_y \qquad \text{for } x \in \Gamma,$$

$$\left(K'(\kappa)w\right)(x) := \int_\Gamma \gamma^{\text{int}}_{1,x} U^*_\kappa(x,y)w(y)ds_y \qquad \text{for } x \in \Gamma,$$

$$(D(\kappa)v)(x) := -\gamma^{\text{int}}_{1,x} \int_\Gamma \gamma^{\text{int}}_{1,y} U^*_\kappa(x,y)v(y)ds_y \quad \text{for } x \in \Gamma. \tag{2.37}$$

The integral representation of $V(\kappa)$ and $D(\kappa)$ are to be understood as weakly singular and as hypersingular surface integral, respectively. The integrals for $K(\kappa)$ and $K'(\kappa)$ are in general Cauchy singular integrals.

Using the single layer potential and the double layer potential, any weak solution u of (2.28) can be represented by

$$u(x) = (\widetilde{V}(\kappa)\gamma^{\text{int}}_1 u)(x) - (W(\kappa)\gamma^{\text{int}}_0 u)(x) \quad \text{for } x \in \Omega, \tag{2.38}$$

see, e.g., [61, Theorem 7.5]. Applying the trace operators γ^{int}_0 and γ^{int}_1 to (2.38) leads to the boundary integral equations

$$\gamma^{\text{int}}_0 u(x) = (V(\kappa)\gamma^{\text{int}}_1 u)(x) + \frac{1}{2}\gamma^{\text{int}}_0 u(x) - (K(\kappa)\gamma^{\text{int}}_0 u)(x) \quad \text{for } x \in \Gamma, \tag{2.39}$$

$$\gamma^{\text{int}}_1 u(x) = \frac{1}{2}\gamma^{\text{int}}_1 u(x) + (K'(\kappa)\gamma^{\text{int}}_1 u)(x) + (D(\kappa)\gamma^{\text{int}}_0 u)(x) \quad \text{for } x \in \Gamma. \tag{2.40}$$

Let us now consider the exterior Helmholtz equation

$$-\Delta u = \kappa^2 u \quad \text{on } \Omega^c = \mathbb{R}^3 \setminus \overline{\Omega}. \tag{2.41}$$

Here we assume that $\kappa > 0$ and introduce the following radiation condition

$$\lim_{R\to\infty} \int_{\partial B_R(0)} \left| \frac{\partial}{\partial y} u(y) - i\kappa u(y) \right|^2 ds_y = 0. \tag{2.42}$$

The single layer potential (2.30) and the double layer potential (2.31) provide solutions of (2.41), see [61, p. 202]. Any solution $u \in H^1_{\text{loc}}(\Omega^c)$ of (2.41) which fulfills the radiation condition (2.42) can be represented by

$$u(x) = -(\widetilde{V}(\kappa)\gamma^{\text{ext}}_1 u)(x) + (W(\kappa)\gamma^{\text{ext}}_0 u)(x) \quad \text{for } x \in \Omega^c, \tag{2.43}$$

see [61, Theorem 6.10]. Applying the operators γ_0^{ext} and γ_1^{ext} to (2.43) leads to the boundary integral equations [61, p. 218]

$$\gamma_0^{\text{ext}} u(x) = -(V(\kappa)\gamma_1^{\text{ext}} u)(x) + \frac{1}{2}\gamma_0^{\text{ext}} u(x) + (K(\kappa)\gamma_0^{\text{ext}} u)(x) \quad \text{for } x \in \Gamma, \tag{2.44}$$

$$\gamma_1^{\text{ext}} u(x) = \frac{1}{2}\gamma_1^{\text{ext}} u(x) - (K'(\kappa)\gamma_1^{\text{ext}} u)(x) - (D(\kappa)\gamma_0^{\text{ext}} u)(x) \quad \text{for } x \in \Gamma. \tag{2.45}$$

The potentials $\widetilde{V}(\kappa)$ and $W(\kappa)$ satisfy the jump relations on the boundary

$$\gamma_0^{\text{ext}}\widetilde{V}(\kappa)w - \gamma_0^{\text{int}}\widetilde{V}(\kappa)w = 0, \quad \gamma_1^{\text{ext}}\widetilde{V}(\kappa)w - \gamma_1^{\text{int}}\widetilde{V}(\kappa)w = -w \quad \text{for } w \in H^{-1/2}(\Gamma) \tag{2.46}$$

$$\gamma_0^{\text{ext}} W(\kappa)v - \gamma_0^{\text{int}} W(\kappa)v = v, \quad \gamma_1^{\text{ext}} W(\kappa)v - \gamma_1^{\text{int}} W(\kappa)v = 0 \quad \text{for } v \in H^{1/2}(\Gamma). \tag{2.47}$$

The following two lemmas address the uniqueness of the solutions of exterior boundary value problems for the Helmholtz equation.

Lemma 2.4.1. *Let $\kappa \in \mathbb{R}_+$. Let $u \in H^1_{\text{loc}}(\Omega^c)$ be a solution of the exterior boundary value problem*

$$-\Delta u - \kappa^2 u = 0 \quad \text{on } \Omega^c$$

with either a homogeneous Dirichlet boundary condition

$$\gamma_0^{\text{ext}} u = 0 \quad \text{on } \Gamma,$$

or with a homogeneous Neumann boundary condition

$$\gamma_1^{\text{ext}} u = 0 \quad \text{on } \Gamma.$$

If u satisfies the radiation condition (2.42), then $u = 0$ on Ω^c.

Proof. See, e.g., [13, Theorem 7.6.1, Theorem 7.6.2]. □

Lemma 2.4.2. *Let $\kappa \in \mathbb{R}_+$.*

i) *If $w \in H^{-1/2}(\Gamma)$ with $V(\kappa)w = 0$ on Γ, then $\widetilde{V}(\kappa)w = 0$ on Ω^c.*

ii) *If $v \in H^{1/2}(\Gamma)$ with $D(\kappa)v = 0$ on Γ, then $W(\kappa)v = 0$ on Ω^c.*

Proof. i) Let $\kappa \in \mathbb{R}_+$ and let $w \in H^{-1/2}(\Gamma)$ with $V(\kappa)w = 0$ on Γ. The single layer potential $\widetilde{V}(\kappa)w \in H^1_{\text{loc}}(\Omega^c)$ is a solution of the Helmholtz equation on Ω^c and fulfills the radiation condition (2.42), see, e.g., [61, Theorem 7.15, Theorem 9.6]. From the jump relation (2.46) of $\widetilde{V}(\kappa)$ and from (2.32) we obtain

$$\gamma_0^{\text{ext}}\widetilde{V}(\kappa)w = \gamma_0^{\text{int}}\widetilde{V}(\kappa)w = V(\kappa)w = 0 \quad \text{on } \Gamma.$$

Thus, by Lemma 2.4.1, $\widetilde{V}(\kappa)w = 0$ on Ω^c.

ii) Let $\kappa \in \mathbb{R}_+$ and let $v \in H^{1/2}(\Gamma)$ with $D(\kappa)v = 0$ on Γ. Then $W(\kappa)v \in H^1_{\mathrm{loc}}(\Omega^c)$ is a solution of the Helmholtz equation on Ω^c and fulfills the radiation condition (2.42), see [61, Theorem 7.15, Theorem 9.6]. Using the jump relation (2.47) of $W(\kappa)$ and (2.35), we get

$$\gamma_1^{\mathrm{ext}} W(\kappa)v = \gamma_1^{\mathrm{int}} W(\kappa)v = -D(\kappa)v = 0 \quad \text{on } \Gamma.$$

Hence, by Lemma 2.4.1, the assertion follows. □

Now we can state the following equivalence between the weak formulation of the Dirichlet Laplacian eigenvalue problem (2.7) and the boundary integral formulation (2.39).

Theorem 2.4.3. *Consider the Dirichlet Laplacian eigenvalue problem*

$$-\Delta u = \kappa^2 u \quad \textit{on } \Omega, \qquad \gamma_0^{\mathrm{int}} u = 0 \quad \textit{on } \Gamma. \tag{2.48}$$

i) *If $(\kappa, u) \in \mathbb{R}_+ \times H^1_0(\Omega)$ is an eigenpair of (2.48) in a weak sense, then the normal derivative $\gamma_1^{\mathrm{int}} u \neq 0$ and $w = \gamma_1^{\mathrm{int}} u$ fulfills the boundary integral equation*

$$V(\kappa)w = 0 \quad \textit{on } \Gamma. \tag{2.49}$$

The eigenfunction u admits the representation

$$u = \widetilde{V}(\kappa)w \quad \textit{on } \Omega.$$

ii) *If $(\kappa, w) \in \mathbb{R}_+ \times H^{-1/2}(\Gamma) \setminus \{0\}$ fulfills the boundary integral equation (2.49), then $u = \widetilde{V}(\kappa)w$ is an eigenfunction of the eigenvalue problem (2.48) in the weak sense corresponding to the eigenvalue κ^2 .*

Proof. i) Let $(\kappa, u) \in \mathbb{R}_+ \times H^1_0(\Omega)$ be an eigenpair of the Dirichlet Laplacian eigenvalue problem (2.48) in the weak sense. Then the boundary integral equation (2.39) gives

$$0 = V(\kappa)\gamma_1^{\mathrm{int}} u \quad \text{on } \Gamma.$$

Using the representation formula (2.38), we can write the eigenfunction u as

$$u = \widetilde{V}(\kappa)\gamma_1^{\mathrm{int}} u \quad \text{on } \Omega.$$

Since $u \neq 0$ on Ω and since $\widetilde{V}(\kappa)$ is linear, it follows that $\gamma_1^{\mathrm{int}} u \neq 0$ on Γ.

ii) Let $(\kappa, w) \in \mathbb{R}_+ \times H^{-1/2}(\Gamma) \setminus \{0\}$ be a solution of the boundary integral equation (2.49). The function u defined by

$$u = \widetilde{V}(\kappa)w \quad \text{on } \Omega$$

is a weak solution of the Helmholtz equation (2.48), cf. (2.30). The boundary condition of (2.48) is fulfilled, since by (2.32) and by assumption on w, we have

$$\gamma_0^{\mathrm{int}} u = \gamma_0^{\mathrm{int}} \widetilde{V}(\kappa)w = V(\kappa)w = 0.$$

It remains to show that $u \neq 0$ on Ω. From $V(\kappa)w = 0$ on Γ it follows by Lemma 2.4.2 that

$$\widetilde{V}(\kappa)w = 0 \quad \text{on } \Omega^c,$$

and therefore $\gamma_1^{\text{ext}}\widetilde{V}(\kappa)w = 0$ on Γ. Using the jump relation (2.46) of $\widetilde{V}(\kappa)$, we get

$$\gamma_1^{\text{int}}\widetilde{V}(\kappa)w = w \neq 0 \quad \text{on } \Gamma. \tag{2.50}$$

Hence, $u = \widetilde{V}(\kappa)w \neq 0$ on Ω. Thus, u is an eigenfunction of the eigenvalue problem (2.48) in the weak sense corresponding to the eigenvalue κ^2 . □

Theorem 2.4.3 provides an equivalent boundary integral formulation of the Dirichlet Laplacian eigenvalue problem and it reads as follows:
Find $(\kappa, w) \in \mathbb{R}_+ \times H^{-1/2}(\Gamma) \setminus \{0\}$ such that

$$V(\kappa)w = 0. \tag{2.51}$$

Obviously, the eigenvalue problem (2.51) is a nonlinear eigenvalue problem.

Note that an alternative boundary integral formulation for the Dirichlet Laplacian eigenvalue problem is possible by using the boundary integral equation (2.40). This yields the following eigenvalue problem [23]: Find $(\kappa, w) \in \mathbb{R}_+ \times H^{-1/2}(\Gamma) \setminus \{0\}$ such that

$$-\frac{1}{2}w + K'(\kappa)w = 0.$$

Also for the Neumann Laplacian eigenvalue (2.20) problem equivalent boundary integral formulations can be stated. First, we consider the boundary integral equation (2.40).

Theorem 2.4.4. *Consider the Neumann eigenvalue problem*

$$-\Delta u = \kappa^2 u \quad \text{on } \Omega, \qquad \gamma_1^{\text{int}} u = 0 \quad \text{on } \Gamma. \tag{2.52}$$

i) *If $(\kappa, u) \in \mathbb{R}_+ \times H^1(\Omega)$ is an eigenpair of (2.52) in the weak sense, then $\gamma_0^{\text{int}} u \neq 0$ and $v = \gamma_0^{\text{int}} u$ fulfills the boundary integral equation*

$$D(\kappa)v = 0 \quad \text{on } \Gamma. \tag{2.53}$$

The eigenfunction u admits the representation

$$u = -W(\kappa)v \quad \text{on } \Omega.$$

ii) *If $(\kappa, v) \in \mathbb{R}_+ \times H^{1/2}(\Gamma) \setminus \{0\}$ fulfills the boundary integral equation (2.53), then $u = -W(\kappa)v$ is an eigenfunction of the eigenvalue problem (2.52) in the weak sense corresponding to the eigenvalue κ^2 .*

Proof. i) Let $(\kappa, u) \in \mathbb{R}_+ \times H^1(\Omega)$ be an eigenpair of the Neumann Laplacian eigenvalue problem (2.52) in the weak sense. The boundary integral equation (2.40) shows that

$$0 = D(\kappa)\gamma_0^{\text{int}} u \quad \text{on } \Gamma.$$

Using the representation formula (2.38), we can write the eigenfunction u by

$$u = -W(\kappa)\gamma_0^{\text{int}} u \quad \text{on } \Omega.$$

Since $u \neq 0$ on Ω and since $W(\kappa)$ is linear, it follows that $\gamma_0^{\text{int}} u \neq 0$ on Γ.

ii) Let $(\kappa, v) \in \mathbb{R}_+ \times H^{1/2}(\Gamma) \setminus \{0\}$ be a solution of the boundary integral equation (2.53). The function u defined by

$$u = -W(\kappa)v \quad \text{on } \Omega$$

is a weak solution of the Helmholtz equation (2.52), cf. (2.31). The boundary condition of (2.52) is fulfilled, since by (2.32), we have $\gamma_1^{\text{int}} u = -\gamma_1^{\text{int}} W(\kappa)v = D(\kappa)v = 0$. It remains to show that $u \neq 0$ on Ω. From $D(\kappa)v = 0$ on Γ it follows by Lemma 2.4.2 that $u = W(\kappa)v = 0$ on Ω^c and therefore $\gamma_0^{\text{ext}} W(\kappa)v = 0$ on Γ. Using the jump relation (2.47) of $W(\kappa)$, we get

$$-\gamma_0^{\text{int}} W(\kappa)v = v \neq 0 \quad \text{on } \Gamma.$$

Hence, $u = -W(\kappa)v \neq 0$ on Ω. Thus, u is an eigenfunction of the Neumann eigenvalue problem (2.52) in the weak sense corresponding to the eigenvalue κ^2 . □

An alternative formulation of the Neumann Laplacian eigenvalue is obtained by using the boundary integral equation (2.39). This yields the eigenvalue problem [23]:
Find $(\kappa, v) \in \mathbb{R}_+ \times H^{1/2}(\Gamma) \setminus \{0\}$ such that

$$\frac{1}{2}v + K(\kappa)v = 0.$$

In the following we analyze the properties of the boundary integral operators $V(\kappa)$ and $D(\kappa)$.

Lemma 2.4.5. *Let $\kappa \in \mathbb{C}$. Then the operators*

$$V(\kappa) - V(0) : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$$
$$D(\kappa) - D(0) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

are compact.

Proof. In [77, Lemma 3.9.8] the assertions are proven for real κ. The proof there remains valid also for complex κ because it relies on the regularity of the kernel of the corresponding Newton potential from which required mapping properties are derived. □

Lemma 2.4.6. *The operator $V(0) : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is $H^{-1/2}(\Gamma)$-elliptic, i.e., there exists a constant $c_V > 0$ such that*

$$(V(0)w, w)_\Gamma \geq c_V \|w\|^2_{H^{-1/2}(\Gamma)} \tag{2.54}$$

holds for all $w \in H^{-1/2}(\Gamma)$.

Proof. See, e.g., [40], [61, Corollary 8.13]. □

A direct consequence of Lemma 2.4.5 and Lemma 2.4.6 is the following result for $V(\kappa)$.

Theorem 2.4.7. *The boundary integral operator $V(\kappa) : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is Fredholm with zero index.*

Proof. The operator $V(\kappa) : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is a compact perturbation of the $H^{-1/2}(\Gamma)$-elliptic operator $V(0)$,

$$V(\kappa) = V(0) + V(\kappa) - V(0).$$

Therefore $V(\kappa)$ is Fredholm and $\operatorname{ind} V(\kappa) = 0$, see, e.g., [61, Theorem 2.38]. □

In the following we will show that the hypersingular operator $D(\kappa)$ is also Fredholm operator with zero index. However, $D(0)$ is not $H^{1/2}(\Gamma)$-elliptic, since

$$\ker D(0) = \operatorname{span}\{1_\Gamma\},$$

where $1_\Gamma \equiv 1$ on Γ. But, if we consider the subspace

$$H^{1/2}_{**}(\Gamma) := \{v \in H^{1/2}(\Gamma) : (v, \tilde{1}_\Gamma)_\Gamma = 0\},$$

where $\tilde{1}_\Gamma \in H^{-1/2}(\Gamma)$ is defined by

$$\tilde{1}_\Gamma(v) = \int_\Gamma v(x) ds_x \quad \text{for } v \in H^{1/2}(\Gamma),$$

then there exists a constant $c_D > 0$ such that

$$(v, D(0)v)_\Gamma \geq c_D \|v\|^2_{H^{1/2}(\Gamma)} \quad \text{for all } v \in H^{1/2}_{**}(\Gamma), \tag{2.55}$$

see, [83, p. 147], [61, Theorem 8.21]. Let us define the stabilized boundary integral operator

$$\widetilde{D}(0) := D(0) + \xi(\cdot, \tilde{1}_\Gamma)_\Gamma \tilde{1}_\Gamma, \tag{2.56}$$

with $\xi \in \mathbb{R}_+$. Then the operator $\widetilde{D}(0)$ is $H^{1/2}(\Gamma)$-elliptic, see [83, p. 177]. Hence, we can state the desired result for $D(\kappa)$.

Lemma 2.4.8. *Let $\kappa \in \mathbb{C}$. The boundary integral operator $D(\kappa) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is Fredholm with* $\operatorname{ind} D(\kappa) = 0$.

Proof. We can write

$$D(\kappa) = \widetilde{D}(0) + D(\kappa) - D(0) + \xi\,(\cdot, \tilde{1}_\Gamma)_\Gamma \tilde{1}_\Gamma.$$

The operator

$$D(\kappa) - D(0) + \xi\,(\cdot, \tilde{1}_\Gamma)_\Gamma \tilde{1}_\Gamma : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

is compact, since $D(0) - D(\kappa)$ and the operator defined by $v \mapsto (v, \tilde{1}_\Gamma)_\Gamma \tilde{1}_\Gamma$ are compact. Further, the operator $\widetilde{D}(0) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is $H^{1/2}(\Gamma)$-elliptic. Thus, $D(\kappa)$ is Fredholm and $\operatorname{ind} D(\kappa) = 0$, see [61, Theorem 2.38]. □

3 EIGENVALUE PROBLEMS OF HOLOMORPHIC FREDHOLM OPERATOR FUNCTIONS

Boundary integral formulations of Laplacian eigenvalue problems lead to nonlinear eigenvalue problems for related boundary integral operators. A standard theory for general nonlinear eigenvalue problem is not available. However, for eigenvalue problems for so-called holomorphic Fredholm operator functions a generalization of the theory of linear eigenvalue problems has been developed [50, 28, 27, 76, 97, 26, 55]. This theory will be essential for the analysis and the discretization of boundary integral operator eigenvalue problems in the subsequent chapters.

3.1 Holomorphic operator functions

In this section we give a short introduction to holomorphic functions which map into Banach spaces. We will restrict ourselves to the basic definitions and some important results which are needed for our purpose. For a detailed presentation and analysis of this topic we refer to [39].

Definition 3.1.1. *Let Λ be an open and connected subset of $\mathbb{C}$ and let B be a Banach space. A function $u : \Lambda \to B$ is called holomorphic on Λ if it can be represented as the sum of a power series*

$$u(\lambda) = \sum_{k=0}^{\infty} (\lambda - \mu)^k a_k, \quad a_k \in B,$$

which is convergent in B in a neighborhood of any point $\mu \in \Lambda$.

The following theorem gives useful equivalent characterizations of holomorphic functions.

Theorem 3.1.2. *Let Λ be an open and connected subset of $\mathbb{C}$, let B be a Banach space, and let $u : \Lambda \to B$. The following statements are equivalent:*

i) *u is holomorphic on Λ.*

ii) *u is differentiable for every $\lambda \in \Lambda$, i.e., there exists a $u'(\lambda) \in B$ such that*

$$\lim_{h \to 0} \left\| \frac{u(\lambda + h) - u(\lambda)}{h} - u'(\lambda) \right\|_B = 0.$$

iii) *The function defined by* $\lambda \mapsto \langle u(\lambda), g\rangle_{B\times B^*}$ *is holomorphic on* Λ *for all* $g \in B^*$*, where* $\langle\cdot,\cdot\rangle_{B\times B^*}$ *denotes the duality pairing of* B *and its dual* B^**.*

Proof. See, e.g., [101, Chapter V.3] or [39, Chapter 3.2]. □

Let X and Y be Banach spaces and let $\mathcal{L}(X,Y)$ be the space of the bounded linear operators which map from X into Y. Let

$$\|A\|_{\mathcal{L}(X,Y)} := \sup_{x\in X,\,\|x\|_X\leq 1} \|Ax\|_Y$$

be the induced operator norm, then $\mathcal{L}(X,Y)$ is a Banach space and the above definition and characterizations of holomorphic functions can be applied to so-called operator functions $A : \Lambda \to \mathcal{L}(X,Y)$. In the next corollary we specify the property iii) in Theorem 3.1.2 for operator functions.

Corollary 3.1.3. *Let* $A : \Lambda \to \mathcal{L}(X,Y)$ *be an operator function on* Λ*. Then the following statements are equivalent:*

i) *A is holomorphic on* Λ*.*

ii) *The function defined by* $\lambda \mapsto \langle A(\lambda)x, g\rangle_{Y\times Y^*}$ *is holomorphic on* Λ *for all* $x \in X$ *and all* $g \in Y^*$*.*

Proof. See [49, Theorem 3.12]. □

In the next corollary we show that the maximum modulus principle for holomorphic functions $f : \Lambda \to \mathbb{C}$ is also valid for holomorphic operator functions.

Corollary 3.1.4. *Let* $A : \Lambda \to \mathcal{L}(X,Y)$ *be a holomorphic operator function on* Λ *and let* Λ_0 *be a bounded and closed subset of* Λ*. Then*

$$\max_{\lambda\in\Lambda_0} \|A(\lambda)\|_{\mathcal{L}(X,Y)} = \max_{\lambda\in\partial\Lambda_0} \|A(\lambda)\|_{\mathcal{L}(X,Y)}. \tag{3.1}$$

Proof. Using that

$$\|y\|_Y = \sup_{g\in Y^*,\,\|g\|_{Y^*}\leq 1} \langle y, g\rangle_{Y\times Y^*}$$

for every $y \in Y$, we can write

$$\|A(\lambda)\|_{\mathcal{L}(X,Y)} = \sup_{x\in X,\,\|x\|_X\leq 1} \|A(\lambda)x\|_Y = \sup_{\substack{x\in X,\,\|x\|_X\leq 1\\ g\in Y^*,\,\|g\|_{Y^*}\leq 1}} |\langle A(\lambda)x, g\rangle_{Y\times Y^*}|.$$

By Corollary 3.1.3, the function defined by $\lambda \to \langle A(\lambda)x, g\rangle_{Y\times Y^*}$ is holomorphic on Λ_0 for all $x \in X$ and $g \in Y^*$. Hence, the assertion follows from the maximum modulus principle in $\mathbb{C}$ [72, Theorem 10.24]. □

Definition 3.1.5. *Let $A : \Lambda \to \mathcal{L}(X,Y)$ be a holomorphic operator function on Λ. The set*

$$\rho(A) = \{\lambda \in \Lambda : \exists A(\lambda)^{-1} \in \mathcal{L}(Y,X)\}$$

is called resolvent set of A. The complement of the resolvent set in Λ is called spectrum $\sigma(A)$ of A.
A number $\lambda_0 \in \Lambda$ is called eigenvalue of A if there exists a non trivial $x^0 \in X$ such that

$$A(\lambda_0)x^0 = 0.$$

x^0 is called eigenelement of A corresponding to the eigenvalue λ_0.

In the next lemma we present an important result for the resolvent of a holomorphic operator function.

Lemma 3.1.6. *Let $A : \Lambda \to \mathcal{L}(X,Y)$ be a holomorphic operator function on Λ and assume that $\lambda_0 \in \rho(A)$. Then there exists a neighborhood $U_\delta(\lambda_0) \subset \Lambda$ of λ_0, $\delta > 0$, such that $A(\lambda)^{-1} \in \mathcal{L}(Y,X)$ for all $\lambda \in U_\delta(\lambda_0)$. Moreover, the function $A(\cdot)^{-1} : U_\delta(\lambda_0) \to \mathcal{L}(Y,X)$ is holomorphic and its derivative admits the representation*

$$\frac{d}{d\lambda}\left(A(\lambda)^{-1}\right) = -A(\lambda)^{-1}A'(\lambda)A(\lambda)^{-1}.$$

Proof. For $\lambda_0 \in \rho(A)$ and $\lambda \in \Lambda$ we can write

$$A(\lambda) = [A(\lambda)A(\lambda_0)^{-1}]A(\lambda_0) = \left[I_Y - [A(\lambda_0) - A(\lambda)]A(\lambda_0)^{-1}\right]A(\lambda_0). \tag{3.2}$$

The holomorphy of the operator function A implies that there exists a $\delta > 0$ such that

$$\left\|[A(\lambda_0) - A(\lambda)]A(\lambda_0)^{-1}\right\|_{\mathcal{L}(Y,Y)} \leq \|A(\lambda_0) - A(\lambda)\|_{\mathcal{L}(X,Y)} \left\|A(\lambda_0)^{-1}\right\|_{\mathcal{L}(Y,X)} < 1$$

for all $\lambda \in U_\delta(\lambda_0)$. By the Neumann series theorem we have

$$\left[I_Y - [A(\lambda_0) - A(\lambda)]A(\lambda_0)^{-1}\right]^{-1} = \sum_{k=0}^{\infty} \left[(A(\lambda_0) - A(\lambda))A(\lambda_0)^{-1}\right]^k$$

for all $\lambda \in U_\delta(\lambda_0)$, where the series converges in the operator norm and defines a bounded linear operator which maps from Y into itself. Using (3.2) we obtain for $\lambda \in U_\delta(\lambda)$

$$A(\lambda)A(\lambda_0)^{-1} \sum_{k=0}^{\infty} \left[(A(\lambda_0) - A(\lambda))A(\lambda_0)^{-1}\right]^k = I_Y. \tag{3.3}$$

Thus, the inverse $A(\lambda)^{-1} \in \mathcal{L}(Y,X)$ exists for all $\lambda \in U_\delta(\lambda_0)$ and admits the representation

$$A(\lambda)^{-1} = A(\lambda_0)^{-1} \sum_{k=0}^{\infty} \left[(A(\lambda_0) - A(\lambda))A(\lambda_0)^{-1}\right]^k. \tag{3.4}$$

Next, we can also write

$$A(\lambda+h)^{-1} = A(\lambda)^{-1} \sum_{k=0}^{\infty} \left[(A(\lambda) - A(\lambda+h)) A(\lambda)^{-1} \right]^k \tag{3.5}$$

for $\lambda \in U_\delta(\lambda_0)$ and sufficiently small $h > 0$. From the continuity of the operator function A the continuity of $A(\cdot)^{-1}$ on $U_\delta(\lambda_0)$ follows with (3.5). Therefore we can conclude

$$\frac{A(\lambda+h)^{-1} - A(\lambda)^{-1}}{h} = A(\lambda)^{-1} \frac{A(\lambda) - A(\lambda+h)}{h} A(\lambda+h)^{-1} \to -A(\lambda)^{-1} A'(\lambda) A(\lambda)^{-1}$$

as $h \to 0$. Thus, by Theorem 3.1.2, $A(\cdot)^{-1}$ is holomorphic on $U_\delta(\lambda_0)$. □

As a consequence of the last lemma we see that the resolvent set of a holomorphic operator function is open and that its spectrum is closed.

3.2 Basics of eigenvalue problems of holomorphic Fredholm operator functions

The study of eigenvalue problems for holomorphic operator functions with Fredholm operators has a long tradition [50, 28, 27, 76, 97, 26, 55]. With different concepts and approaches a wide range of results has been derived. For the numerical analysis of approximations of such eigenvalue problems [38, 31, 44, 90, 47, 48] these results are essential, in particular for the error analysis. In this section we provide the basic definitions and concepts of the theory for eigenvalue problems for holomorphic Fredholm operator functions and present the main results. For a detailed presentation and analysis we refer to [55].

Definition 3.2.1. *We call an operator function $A : \Lambda \to \mathcal{L}(X,Y)$ Fredholm if the operator*

$$A(\lambda) : X \to Y$$

is Fredholm for all $\lambda \in \Lambda$.

Theorem 3.2.2. *Let $A : \Lambda \to \mathcal{L}(X,Y)$ be a holomorphic Fredholm operator function and let the resolvent set $\rho(A)$ of A be not empty. Then:*

i) *The index* $\operatorname{ind} A(\lambda) = 0$ *for all $\lambda \in \Lambda$.*

ii) *The spectrum $\sigma(A)$ has no cluster points in Λ.*

iii) *Each $\lambda \in \sigma(A)$ is an eigenvalue of A.*

Proof. For i), see [31]. For ii), see [26, Corollary 8.4]. iii) follows from i), because if $A(\lambda)$ is not surjective then $A(\lambda)$ is not injective. Hence, if $\lambda \in \sigma(A)$ then $A(\lambda)$ is not injective and thus there exists a $x \neq 0$ with $A(\lambda)x = 0$. □

The concept of Jordan chains for linear eigenvalue problems can be extended to holomorphic eigenvalue problems.

Definition 3.2.3. *Let $A : \Lambda \to \mathcal{L}(X,Y)$ be a holomorphic operator function on Λ. Let (λ_0, x^0) be an eigenpair of the eigenvalue problem $A(\lambda)x = 0$. Elements $x^0, x^1, \ldots, x^{m-1}$ in X are called generalized eigenelements if they satisfy*

$$\sum_{j=0}^{n} \frac{1}{j!} A^{(j)}(\lambda_0) x^{n-j} = 0 \quad \text{for } n = 0, 1, \ldots, m-1. \tag{3.6}$$

The ordered collection $x^0, x^1, \ldots, x^{m-1}$ is called Jordan chain of length m corresponding to λ_0.

For linear eigenvalue problems with compact operators the length of any Jordan chain of a nonzero eigenvalue is finite [4, Satz 9.6]. In the case of eigenvalue problems for holomorphic Fredholm operator functions this result is true for any eigenvalue.

Lemma 3.2.4. *Let $A : \Lambda \to \mathcal{L}(X,Y)$ be a holomorphic Fredholm operator function on Λ and let $\rho(A) \neq \emptyset$. Let $\lambda_0 \in \Lambda$ be an eigenvalue of the eigenvalue problem $A(\lambda)x = 0$, then the length of any Jordan chain corresponding to λ_0 is finite.*

Proof. See [55, Lemma A.8.3.]. □

With the last lemma we can define the maximal length of the Jordan chain of an eigenelement and of an eigenvalue.

Definition 3.2.5. *Let $A : \Lambda \to \mathcal{L}(X,Y)$ be a holomorphic Fredholm operator function on Λ and let $\rho(A) \neq \emptyset$. Let $\lambda_0 \in \Lambda$ be an eigenvalue of the eigenvalue problem $A(\lambda)x = 0$.*

i) *Let x^0 be an eigenelement corresponding to λ_0. The maximal length of a Jordan chain beginning with x^0 is called the order $m(A, \lambda_0, x^0)$ of the eigenelement x^0.*

ii) *By*

$$\varkappa(A, \lambda_0) := \max_{x \in \ker A(\lambda_0) \setminus \{0\}} m(A, \lambda_0, x)$$

we denote the maximal length of Jordan chains corresponding to λ_0.

iii) *The closed linear hull of all generalized eigenelements of A corresponding to λ_0 is called the generalized eigenspace $G(A, \lambda_0)$ of A corresponding to λ_0.*

For the numerical analysis of the discretization of eigenvalue problems for holomorphic operator functions it is essential that the dimension of the generalized eigenspace for all eigenvalues is finite.

Lemma 3.2.6. *Let $A : \Lambda \to \mathcal{L}(X,Y)$ be a holomorphic Fredholm operator function on Λ and let $\rho(A) \neq \emptyset$. Let $\lambda_0 \in \Lambda$ be an eigenvalue of the eigenvalue problem $A(\lambda)x = 0$. Then,*

$$\dim G(A,\lambda_0) \leq \varkappa(A,\lambda_0) \cdot \dim \ker A(\lambda_0). \tag{3.7}$$

Proof. See [44, pp. 7]. □

The generalized eigenspace of an eigenvalue can be described by an ordered collection of eigenelements [55, Proposition A.4.5.], this motivates the following definition.

Definition 3.2.7. *Let $A : \Lambda \to \mathcal{L}(X,Y)$ be a holomorphic Fredholm operator function on Λ and let $\rho(A) \neq \emptyset$. Let $\lambda_0 \in \Lambda$ be an eigenvalue of the eigenvalue problem $A(\lambda)x = 0$. A system of eigenelements $x_1^0, \ldots, x_J^0$ corresponding to λ_0 is called canonical if*

i) *$x_1^0, \ldots, x_J^0$ is a basis of $\ker A(\lambda_0)$,*

ii) *$m(A,\lambda_0,x_1^0) = \varkappa(A,\lambda_0)$,*

iii) *x_j^0 is an eigenelement of the maximal possible order belonging to some direct complement M_j in $\ker A(\lambda_0)$ to the linear hull $\operatorname{span}\{x_1^0, \ldots, x_{j-1}^0\}$, i.e.,*

$$m(A,\lambda_0,x_j^0) = \max_{x \in M_j \setminus \{0\}} m(A,\lambda_0,x) \quad \text{for } j = 2,\ldots,J.$$

Obviously, a canonical system of eigenelements of an eigenvalue is not unique, but the order of the eigenelements of two canonical systems coincides.

Lemma 3.2.8. *Let $A : \Lambda \to \mathcal{L}(X,Y)$ be a holomorphic Fredholm operator function on Λ and let $\rho(A) \neq \emptyset$. Let $\lambda_0 \in \sigma(A)$ and suppose that $x_1^0, \ldots, x_J^0$ and $u_1^0, \ldots, u_J^0$ are two canonical systems of the eigenvalue λ_0. Then,*

$$m(A,\lambda_0,x_i^0) = m(A,\lambda_0,u_i^0) \quad \text{for } i = 1,\ldots,J. \tag{3.8}$$

Proof. See [55, Proposition A.4.6.]. □

With the last lemma we can define partial multiplicities of an eigenvalue and extend the concept of algebraic multiplicity of linear eigenvalue problems to eigenvalue problems for holomorphic Fredholm operator functions.

Definition 3.2.9. *Let $A : \Lambda \to \mathcal{L}(X,Y)$ be a holomorphic Fredholm operator function on Λ and let $\rho(A) \neq \emptyset$. Let $\lambda_0 \in \Lambda$ be an eigenvalue of the eigenvalue problem $A(\lambda)x = 0$ and $x_1^0, \ldots, x_J^0$ be a corresponding canonical basis of the eigenspace. The numbers*

$$m_i(A,\lambda_0) := m(A,\lambda_0,x_i^0) \quad \text{for } i = 1,\ldots J$$

are called partial multiplicities of A corresponding to λ_0. The number

$$m(\lambda_0) = \sum_{i=1}^{J} m_i(A, \lambda_0)$$

is called the algebraic multiplicity of A at λ_0.

The next technical result is needed later for the numerical analysis of the discretization of eigenvalue problems of holomorphic Fredholm operator functions.

Lemma 3.2.10. *Let $A : \Lambda \to \mathcal{L}(X,Y)$ be a holomorphic Fredholm operator function, let $\rho(A) \neq \emptyset$ and let $\lambda_0 \in \sigma(A)$. Let $x_1^0, \ldots, x_J^0$ be a canonical system of eigenelements of A corresponding to λ_0. If $m(A, \lambda_0, x_k^0) = m(A, \lambda_0, x_j^0) = m$ for some $1 \leq k < j \leq J$ and if $(\alpha_k, \alpha_j) \neq (0,0)$, then*

$$m(A, \lambda_0, \alpha_k x_k^0 + \alpha_j x_j^0) = m. \tag{3.9}$$

Proof. If $\alpha_k = 0$ or $\alpha_j = 0$, then (3.9) is obviously fulfilled. Therefore let us assume that $\alpha_k \neq 0$ and $\alpha_j \neq 0$. Since $x_1^0, \ldots, x_J^0$ is a canonical system of eigenelements and $k < j$, it follows that $x_j^0 \notin \operatorname{span}\{x_1^0, \ldots, x_{j-1}^0\}$ and therefore

$$(\alpha_k x_k^0 + \alpha_j x_j^0) \notin \operatorname{span}\{x_1^0, \ldots, x_{j-1}^0\}.$$

Hence by iii) of Definition 3.2.7, $m(A, \lambda_0, \alpha_k x_k^0 + \alpha_j x_j^0) \leq m(A, \lambda_0, x_j^0) = m$. Let $x_k^0, \ldots, x_k^{m-1}$ and $x_j^0, \ldots, x_j^{m-1}$ some Jordan chains of x_k^0 and x_j^0, respectively. Then it can be seen very easily that

$$\alpha_k x_k^0 + \alpha_j x_j^0,\ \alpha_k x_k^1 + \alpha_j x_j^1, \ldots, \alpha_k x_k^{m-1} + \alpha_j x_j^{m-1}$$

is a Jordan chain of A corresponding to λ_0 of length m. Thus,

$$m(A, \lambda_0, \alpha_k x_k^0 + \alpha_j x_j^0) = m.$$

□

For the investigation of the length of the Jordan chains and of the partial multiplicities of an eigenvalue the concept of Jordan functions / root functions has been used in [90, 64, 47].

Definition 3.2.11. *Let $A : \Lambda \to \mathcal{L}(X,Y)$ be a holomorphic operator function and let $\lambda_0 \in \sigma(A)$. A holomorphic function $u : U_\delta(\lambda_0) \to X$ in a neighborhood of λ_0 is called a Jordan function of order m for A corresponding to λ_0 if*

i) $u(\lambda_0) \neq 0$ *and*

ii) λ_0 *is a zero of multiplicity* m *of the function* $f : \lambda \mapsto A(\lambda)u(\lambda)$, *i.e.,*

$$\frac{d^j}{d\lambda^j}[A(\lambda)u(\lambda)]_{\lambda=\lambda_0} = 0 \quad \text{for } j = 0,1,\ldots,m-1 \quad \text{and}$$
$$\frac{d^m}{d\lambda^m}[A(\lambda)u(\lambda)]_{\lambda=\lambda_0} \neq 0. \tag{3.10}$$

Note, if u is a Jordan function for A corresponding to λ_0, then $u(\lambda_0)$ is an eigenelement of A corresponding to λ_0. The following lemma shows that for every Jordan function of order m a corresponding Jordan chain of length m can be constructed.

Lemma 3.2.12. *Let* $A : \Lambda \to \mathcal{L}(X,Y)$ *be a holomorphic operator function, let* $\lambda_0 \in \sigma(A)$ *and let* $u : U_\delta(\lambda_0) \to X$ *be a Jordan function of order* m. *Then*

$$u(\lambda_0), \frac{1}{1!}u'(\lambda_0), \frac{1}{2!}u^{(2)}(\lambda_0), \ldots, \frac{1}{(m-1)!}u^{(m-1)}(\lambda_0) \tag{3.11}$$

is a Jordan chain of A *corresponding to* λ_0.

Proof. Since the function $f : \lambda \mapsto A(\lambda)u(\lambda)$ has a zero of multiplicity m, we have

$$0 = \frac{d^n}{dx^n}[A(\lambda)u(\lambda)]_{\lambda=\lambda_0} = \sum_{j=0}^{n} \frac{n!}{(n-j)!\,j!}A^{(j)}(\lambda_0)u^{(n-j)}(\lambda_0)$$
$$= n! \sum_{j=0}^{n} \frac{1}{j!}A^{(j)}(\lambda_0)\frac{1}{(n-j)!}u^{(n-j)}(\lambda_0) \quad \text{for } n = 0,\ldots,m-1.$$

□

If a Jordan chain of length m is given, then a corresponding Jordan function can be easily constructed by a polynomial.

Lemma 3.2.13. *Let* $A : \Lambda \to \mathcal{L}(X,Y)$ *be a holomorphic Fredholm operator function with* $\rho(A) \neq \emptyset$ *and let* (λ_0, x^0) *be an eigenpair of* A *with* $m = m(A,\lambda_0,x^0)$. *Let*

$$x^0, \ldots, x^{m-1}$$

be some Jordan chain of x^0 *of maximal order. Then the polynomial*

$$u(\lambda) = x^0 + (\lambda - \lambda_0)x^1 + \ldots + (\lambda - \lambda_0)^{m-1}x^{m-1}$$

is a Jordan function of A *corresponding to* λ_0 *of order* m.

Proof. Since x^0 is an eigenelement, we have $u(\lambda_0) = x^0 \neq 0$. By Definition 3.2.3 of a Jordan chain we have

$$\sum_{j=0}^{n} \frac{1}{j!} A^{(j)}(\lambda_0) u^{n-j} = 0, \quad n = 0, 1, \ldots, m-1.$$

With $u^{(k)}(\lambda_0) = k! x^k$ for $k = 0, 1, \ldots, m-1$ we obtain

$$\begin{aligned} \frac{d^n}{dx^n} [A(\lambda)u(\lambda)]_{\lambda=\lambda_0} &= \sum_{j=0}^{n} \frac{n!}{(n-j)!\, j!} A^{(j)}(\lambda_0) u^{(n-j)}(\lambda_0) \\ &= n! \sum_{j=0}^{n} \frac{1}{j!} A^{(j)}(\lambda_0) x^{n-j} = 0, \qquad n = 0, \ldots, m-1, \end{aligned}$$

which implies that the function $f : \lambda \mapsto A(\lambda)u(\lambda)$ has a zero of multiplicity at least m. The function f must have a zero of multiplicity equals m because otherwise,

$$u(\lambda_0) = x^0, \frac{1}{1!} u'(\lambda_0) = x^1, \ldots, \frac{1}{(m-1)!} u^{(m-1)}(\lambda_0) = x^{m-1}, \frac{1}{m!} u^{(m)}(\lambda_0) = 0$$

would be, by Lemma 3.2.12, a Jordan chain of x^0 of length $m+1 > m = m(A, \lambda_0, x^0)$. But this is a contradiction to the fact that $m(A, \lambda_0, x^0)$ is the maximal length of a Jordan chain beginning with x^0. □

At the end of this section we cite the Keldysh theorem [50, 55]. It is the key tool of the numerical analysis of the discretization of eigenvalue problems for holomorphic Fredholm operator functions. Moreover, we use it for the construction of eigenvalue solvers for algebraic holomorphic eigenvalue problems. The theorem shows that the resolvent admits a representation as Laurent series in a neighborhood of each eigenvalue, where the principal part of the Laurent series is a finite sum.

Theorem 3.2.14. *Let $A : \Lambda \to \mathcal{L}(X, Y)$ be a holomorphic Fredholm operator function and let $\rho(A) \neq \emptyset$. Let $\lambda_0 \in \sigma(A)$, then for $\lambda \in \Lambda \setminus \{\lambda_0\}$ sufficiently close to λ_0 we have*

$$A(\lambda)^{-1} = \sum_{k=r}^{1} (\lambda - \lambda_0)^{-k} B_k + F(\lambda),$$

where $r = \varkappa(A, \lambda_0)$, $B_k \in \mathcal{L}(X, Y)$ are operators of finite rank with $B_r \neq 0$, and where F is a holomorphic operator function.

Proof. See [27], [55, Theorem A.10.2.]. □

3.3 Adjoint eigenvalue problems

For the numerical analysis of approximations of holomorphic eigenvalue problems we have to consider also the adjoint eigenvalue problem. Since there are different definitions of the adjoint of an operator, we want to specify these definitions in detail.

Definition 3.3.1. *Let X be a Banach space. A continuous map $x \mapsto \overline{x}$ which maps X into itself is called a conjugation on X if*

$$\overline{x+y} = \overline{x} + \overline{y}, \quad \overline{\alpha x} = \overline{\alpha}\,\overline{x} \quad \text{and} \quad \overline{(\overline{x})} = x$$

is satisfied for all $x, y \in X$ and $\alpha \in \mathbb{C}$.

Notice that a conjugation is bounded, conjugate-linear, and has a bounded inverse. A conjugation on X induces also a conjugation on the dual space $X^* = \mathcal{L}(X, \mathbb{C})$ by

$$\langle x, \overline{f} \rangle_{X \times X^*} = \overline{\langle \overline{x}, f \rangle_{X \times X^*}} \quad \text{for } x \in X,\ f \in X^*, \tag{3.12}$$

where $\langle \cdot, \cdot \rangle_{X \times X^*}$ is the duality pairing of $X \times X^*$, i.e.,

$$\langle x, f \rangle_{X \times X^*} = f(x) \quad \text{for } x \in X,\ f \in X^*.$$

Further we can define a bounded sesquilinear form $[\cdot, \cdot]_{X \times X^*} : X \times X^*$ by

$$[x, f]_{X \times X^*} = \langle x, \overline{f} \rangle_{X \times X^*} \quad \text{for } x \in X,\ f \in X^*.$$

If X is reflexive and if X^{**} is identified with X, then we have

$$[x, f]_{X \times X^*} = \overline{[f, x]}_{X^* \times X}. \tag{3.13}$$

If conjugations are defined on a Hilbert space X and on its dual X^* by (3.12), then from

$$|\langle y, \overline{f} \rangle_{X \times X^*}| = |\overline{\langle \overline{y}, f \rangle}_{X \times X^*}| = |\langle \overline{y}, f \rangle_{X \times X^*}| \tag{3.14}$$

it follows that

$$\|f\|_{X^*} = \sup_{0 \neq \overline{x} \in X} \frac{|\langle \overline{x}, f \rangle_{X \times X^*}|}{\|\overline{x}\|_X} = \sup_{0 \neq x \in X} \frac{|\langle x, \overline{f} \rangle_{X \times X^*}|}{\|x\|_X} = \|\overline{f}\|_{X^*}.$$

Thus, the conjugation on X^* is a conjugate-linear bijective isometry on X^*. By the Riesz representation theorem, see e.g. [99, p. 105], there exists a conjugate-linear bijective isometry $J : X \to X^*$ such that

$$\langle x, Jy \rangle_{X \times X^*} = (x, y)_X \tag{3.15}$$

is satisfied for all $x, y \in X$. Therefore we can define a linear bijective isometry $\iota : X \to X^*$ by $x \mapsto \overline{J(x)}$. Thus, we can represent the inner product of X by the sesquilinear form $[\cdot,\cdot]_{X\times X^*}$ in the following way,

$$[x, \iota y]_{X\times X^*} = \langle x, \overline{\iota y}\rangle_{X\times X^*} = \langle x, Jy\rangle_{X\times X^*} = (x,y)_X. \tag{3.16}$$

If X and Y are both equipped with a conjugation and if $A \in \mathcal{L}(X,Y)$, then we define the adjoint $A^* : Y^* \to X^*$ of A such that

$$[x, A^* g]_{X\times X^*} = [Ax, g]_{Y\times Y^*}$$

is satisfied for all $x \in X$ and $g \in Y^*$.

If X and Y are Hilbert spaces and if $A \in \mathcal{L}(X,Y)$, then the Hilbert space adjoint $A^\star : Y \to X$ is defined such that

$$(x, A^\star y)_X = (Ax, y)_Y \tag{3.17}$$

is satisfied for all $x \in X$ and $y \in Y$. The relation between the adjoint $A^* : Y^* \to X^*$ and the Hilbert space adjoint $A^\star : Y \to X$ is

$$\iota_X^{-1} A^* \iota_Y = A^\star.$$

Definition 3.3.2. *Let $A : \Lambda \to \mathcal{L}(X,Y)$ be a holomorphic Fredholm operator function. The operator function*

$$A^* : \{\lambda : \overline{\lambda} \in \Lambda\} \to \mathcal{L}(Y^*, X^*)$$

defined by

$$A^*(\lambda) = A(\overline{\lambda})^*$$

is called adjoint operator function of A.

In the next lemma we show that the properties of a holomorphic Fredholm operator function remain valid for its adjoint.

Lemma 3.3.3. *Let $A : \Lambda \to \mathcal{L}(X,Y)$ be a holomorphic Fredholm operator function with $\rho(A) \neq \emptyset$ and let Y be reflexive. Then the adjoint operator function*

$$A^* : \{\lambda : \overline{\lambda} \in \Lambda\} \to \mathcal{L}(Y^*, X^*)$$

is holomorphic and $A^*(\overline{\lambda})$ is Fredholm with $\operatorname{ind} A^*(\overline{\lambda}) = 0$ *for all $\lambda \in \Lambda$.*

Proof. Let $\mu \in \Lambda$ be arbitrary but fixed. Since A is holomorphic on Λ, there exists a $\delta > 0$ such that $A(\lambda)$ admits a representation as power series

$$A(\lambda) = \sum_{k=0}^{\infty} (\lambda - \mu)^k B_k, \quad B_k \in \mathcal{L}(X,Y),$$

which is convergent in $\mathcal{L}(X,Y)$ for all $\lambda \in U_\delta(\mu)$. Then, by Theorem 3.1.2, we can write

$$\langle A(\lambda)x, \overline{g}\rangle_{Y\times Y^*} = [A(\lambda)x, g]_{Y\times Y^*} = \sum_{k=0}^{\infty} (\lambda - \mu)^k [B_k x, g]_{Y\times Y^*}$$

for $\lambda \in U_\delta(\mu)$, where the series is convergent for all $x \in X$ and $g \in Y^*$. Consequently, for $\lambda \in U_\delta(\overline{\mu})$ also

$$\sum_{k=0}^{\infty} (\lambda - \overline{\mu})^k \overline{[B_k x, g]}_{Y\times Y^*} \tag{3.18}$$

is convergent for all $x \in X$ and $g \in Y^*$.
Let us now consider the adjoint operator function of A which is given by $A^*(\lambda) = A(\overline{\lambda})^*$. We can write for $\lambda \in U_\delta(\overline{\mu})$, $x \in X$ and $g \in Y^*$,

$$\begin{aligned}\langle A^*(\lambda)g, \overline{x}\rangle_{X^*\times X} &= \langle A(\overline{\lambda})^* g, \overline{x}\rangle_{X^*\times X} = [A(\overline{\lambda})^* g, x]_{X^*\times X} \\ &= [g, A(\overline{\lambda})x]_{Y^*\times Y} = \overline{[A(\overline{\lambda})x, g]}_{Y\times Y^*} = \sum_{k=0}^{\infty} \overline{(\overline{\lambda} - \mu)^k [B_k x, g]}_{Y\times Y^*} \\ &= \sum_{k=0}^{\infty} (\lambda - \overline{\mu})^k \overline{[B_k x, g]}_{Y\times Y^*},\end{aligned}$$

where we used that Y is reflexive and (3.13). Thus, by (3.18), the function

$$\lambda \mapsto \langle A^*(\lambda)g, \overline{x}\rangle_{X^*\times X}$$

is holomorphic on $U_\delta(\overline{\mu})$ for all $x \in X$ and $g \in Y^*$ and we conclude with Corollary 3.1.3 that $A^* : \{\lambda : \overline{\lambda} \in \Lambda\} \to \mathcal{L}(Y^*, X^*)$ is holomorphic.

Since $\rho(A) \neq \emptyset$, it follows immediately from Theorem 3.2.2 that $\operatorname{ind} A(\lambda) = 0$ for all $\lambda \in \Lambda$. The adjoint operator of any Fredholm operator with index 0 is Fredholm with index 0, see, e.g., [61, Theorem 2.27]. This implies that $\operatorname{ind} A(\lambda)^* = \operatorname{ind} A^*(\overline{\lambda}) = 0$ for all $\lambda \in \Lambda$. □

The Fredholm alternative, see, e.g., [61, Theorem 2.27], shows that

$$\dim\ker A(\lambda_0) = \dim\ker A^*(\overline{\lambda}_0).$$

Hence, λ_0 is an eigenvalue of A if and only if $\overline{\lambda}_0$ is an eigenvalue of A^*. Further, the geometric multiplicities of λ_0 and $\overline{\lambda}_0$ coincide. Also the partial and algebraic multiplicities of λ_0 and $\overline{\lambda}_0$ are equal as we see in the next lemma.

Lemma 3.3.4. *Let $A : \Lambda \to \mathcal{L}(X,Y)$ be a holomorphic Fredholm operator function and let $\rho(A) \neq \emptyset$. Then, $\lambda_0 \in \sigma(A)$ if and only if $\overline{\lambda} \in \sigma(A^*)$ and the geometric, partial, and algebraic multiplicities coincide.*

Proof. See [55, Proposition A.9.2.]. □

If X is a Hilbert space and if $A : \Lambda \to \mathcal{L}(X,X)$ is a holomorphic Fredholm operator function, then we want to consider the Hilbert space adjoint

$$A(\lambda)^\star = \imath^{-1} A(\lambda)^* \imath \tag{3.19}$$

and define $A^\star : \{\lambda : \overline{\lambda} \in \Lambda\} \to \mathcal{L}(X,X)$ by

$$A^\star(\lambda) = \imath^{-1} A^*(\lambda) \imath.$$

Remark 3.3.5. *If X is a Hilbert space and if $X = Y$, then Lemma 3.3.3 and Lemma 3.3.4 remain valid when we replace A^* by $A^\star$, since $\imath : X \to X^*$ is an isomorphism.*

4 APPROXIMATION OF HOLOMORPHIC EIGENVALUE PROBLEMS

The approximation of eigenvalue problems for holomorphic Fredholm operator functions is in the most cases analyzed by using the concept of the so-called discrete approximation scheme [86] together with the concept of the regular approximation of operator functions [30]. Such approaches [31, 44, 91, 90, 47, 48] require assumptions on the approximation spaces as well as several assumptions on the approximations of the operator. The Galerkin approximation of eigenvalue problems for holomorphic Fredholm operator functions of the form $A(\lambda) = T + S(\lambda)$, where T is elliptic and $S(\lambda)$ is compact, fulfills those assumptions. Nevertheless, we will establish an alternative convergence and error analysis for the Galerkin discretization of such eigenvalue problems in this chapter. For that we only have to assume the standard approximation property of the sequence of the trial spaces.

4.1 Assumptions and basic properties

We consider eigenvalue problems

$$A(\lambda)x = 0 \tag{4.1}$$

for holomorphic operator functions

$$A : \Lambda \to \mathcal{L}(X,X),$$

where $\Lambda \subset \mathbb{C}$ is an open and connected subset of $\mathbb{C}$ and X is a Hilbert space over $\mathbb{C}$. We assume that the operator $A(\lambda)$ admits the representation

$$A(\lambda) = T + S(\lambda) \quad \text{for all } \lambda \in \Lambda, \tag{4.2}$$

where $T \in \mathcal{L}(X,X)$ is X-elliptic, i.e., there exists a constant $c_T > 0$ such that

$$(Tx,x)_X \geq c_T \|x\|_X^2 \quad \text{for all } x \in X, \tag{4.3}$$

and where $S(\lambda) \in \mathcal{L}(X,X)$ is compact for all $\lambda \in \Lambda$. These assumptions on the operator function A imply that $A(\lambda)$ is Fredholm with $\operatorname{ind} A(\lambda) = 0$ for all $\lambda \in \Lambda$. Indeed, by the Lax-Milgram theorem, see, e.g., [4, Satz 4.7], the operator T has a bounded inverse and therefore $\operatorname{ind} T = 0$. Since every compact perturbation of a Fredholm operator is a

Fredholm operator with the same index [61, Theorem 2.26], we conclude that the operator $A(\lambda)$ is Fredholm with $\operatorname{ind} A(\lambda) = 0$ for all $\lambda \in \Lambda$.

For the Galerkin approximation of the eigenvalue problem (4.1) we assume that there exists a sequence

$$\{X_n\}_{n\in\mathbb{N}} \tag{4.4}$$

of nested finite dimensional subspaces $X_n \subset X$ with $X_n \subset X_{n+1}$, which satisfies the approximation property

$$\lim_{n\to\infty} \inf_{x_n\in X_n} \|x - x_n\|_X = 0 \quad \text{for all } x \in X. \tag{4.5}$$

Theorem 4.1.1. *Let X be a Hilbert space and let W be a finite dimensional subspace of X.*

i) *$v \in W$ is a best approximation to $x \in X$ with respect to W, i.e.,*

$$\|x - v\|_X = \inf_{w\in W} \|x - w\|_X, \tag{4.6}$$

if and only if

$$(x - v, w)_X = 0 \quad \textit{for all } w \in W. \tag{4.7}$$

ii) *For every element $x \in X$ there exists a unique best approximation $v \in W$ with respect to W.*

Proof. See [61, Lemma 2.28, Lemma 2.29]. □

Using Theorem 4.1.1, we can define for every $n \in \mathbb{N}$ a map

$$P_n : X \to X_n \subset X \tag{4.8}$$

which maps each element of $x \in X$ to its unique best approximation in X_n. The operator $P_n : X \to X$ is a projection, since $\operatorname{Im} P_n = X_n$ and $P_n x_n = x_n$ for all $x_n \in X_n$. The operator P_n is linear, since by Theorem 4.1.1, we have

$$0 = \alpha(x - P_n x, z_n)_X + \beta(y - P_n y, z_n)_X = (\alpha x + \beta y - [\alpha P_n x + \beta P_n y], z_n)_X$$

for all $x, y \in X$ and $z_n \in X_n$. Thus, $\alpha P_n x + \beta P_n y = P_n(\alpha x + \beta y)$.

Further,

$$\|P_n\|_{\mathcal{L}(X,X)} = 1, \tag{4.9}$$

since we have on the hand

$$\|P_n\|_{\mathcal{L}(X,X)} = \|P_n P_n\|_{\mathcal{L}(X,X)} \le \|P_n\|_{\mathcal{L}(X,X)} \|P_n\|_{\mathcal{L}(X,X)},$$

i.e., $\|P_n\|_{\mathcal{L}(X,X)} = 0$ or $\|P_n\|_{\mathcal{L}(X,X)} \ge 1$. On the other hand, by i) of Theorem 4.1.1,

$$\|x\|_X^2 = \|P_n x + x - P_n x\|_X^2 = \|P_n x\|_X^2 + \|x - P_n x\|_X^2 \ge \|P_n x\|_X^2 \quad \text{for all } x \in X,$$

i.e., $\|P_n\|_{\mathcal{L}(X,X)} \leq 1$.

P_n is selfadjoint, because by using again i) of Theorem 4.1.1 we see that

$$\begin{aligned}(P_n x, y)_X &= (P_n x, P_n y)_X + (P_n x, y - P_n y)_X = (P_n x, P_n y)_X = (x - P_n x, P_n y)_X + (x, P_n y)_X \\ &= (x, P_n y)_X \end{aligned} \tag{4.10}$$

for all $x, y \in X$.

Finally, the approximation property (4.5) of $\{X_n\}_{n\in\mathbb{N}}$ implies that

$$\lim_{n\to\infty} \|x - P_n x\|_X = 0 \quad \text{for all } x \in X. \tag{4.11}$$

Lemma 4.1.2. *Let G be a finite dimensional subspace of X and let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of finite dimensional subspaces of X which has the approximation property (4.5). Then for any $c > 0$ we have*

$$\sup_{\substack{x\in G \\ \|x\|_X \leq c}} \inf_{x_n \in X_n} \|x - x_n\|_X \to 0 \quad \text{as } n \to \infty. \tag{4.12}$$

Proof. Let $x^1, \ldots, x^k$ be some orthonormal basis of G and let $c > 0$ be arbitrary but fixed. From the approximation property of the spaces X_n it follows that for every $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that for all $n \geq N$ there exists a subset $\{x_n^1, \ldots, x_n^k\} \subset X_n$ satisfying

$$\|x^i - x_n^i\|_X \leq \frac{\varepsilon}{kc} \quad \text{for } 1 \leq i \leq k. \tag{4.13}$$

Let $x \in G$ with $\|x\|_X \leq c$, then x admits a representation by

$$x = \sum_{i=1}^{k} \alpha_i x^i$$

with $|\alpha_i| \leq c$. So we conclude for $\tilde{x}_n = \sum_{i=1}^{k} \alpha_i x_n^i \in X_n$ with (4.13) that

$$\begin{aligned}\inf_{x_n \in X_n} \|x - x_n\|_X \leq \|x - \sum_{i=1}^{k} \alpha_i x_n^i\|_X &= \|\sum_{i=1}^{k} \alpha_i (x^i - x_n^i)\|_X \\ &\leq \sum_{i=1}^{k} |\alpha_i| \|x^i - x_n^i\|_X \leq \frac{\varepsilon}{k} \sum_{i=1}^{k} |\alpha_i| \leq \varepsilon,\end{aligned}$$

which implies that

$$\sup_{\substack{z\in G \\ \|z\| \leq c}} \inf_{x_n \in X_n} \|x - x_n\| \to 0 \quad \text{as } n \to \infty.$$

□

4.2 Convergence results for Galerkin approximations

In this section we consider eigenvalue problems

$$A(\lambda)x = 0 \tag{4.14}$$

for holomorphic Fredholm operator functions $A : \Lambda \to \mathcal{L}(X,X)$ with $A(\lambda) = T + S(\lambda)$ as given in (4.2). We use a Bubnov-Galerkin method with the test and trial spaces $\{X_n\}_{n\in\mathbb{N}}$ as given in (4.4) for the approximation of the eigenvalue problem (4.14).
A pair $(\lambda_0^n, x_n^0) \in \Lambda \times X_n \setminus \{0\}$ is an approximate solution of the eigenvalue problem (4.14) if it satisfies the Galerkin variational eigenvalue problem

$$(A(\lambda_0^n)x_n^0, v_n)_X = 0 \quad \text{for all } v_n \in X_n. \tag{4.15}$$

The orthogonality relation (4.7) gives

$$(A(\lambda_0^n)x_n^0 - P_nA(\lambda_0^n)x_n^0, v_n)_X = 0 \quad \text{for all } v_n \in X_n.$$

This implies that $(\lambda_0^n, x_n^0) \in \Lambda \times X_n \setminus \{0\}$ is a solution of the Galerkin variational eigenvalue problem (4.15) if and only if it is a solution of the projected eigenvalue problem

$$P_nA(\lambda_0^n)x_n^0 = 0. \tag{4.16}$$

The convergence analysis of the approximate solutions of the eigenvalue problem (4.14) follows [38].

Lemma 4.2.1. *Let $A : \Lambda \to \mathcal{L}(X,X)$ be as given in (4.2) and let $\{\lambda_n\}_{n\in\mathbb{N}} \subset \Lambda$ be a sequence with*

$$\lim_{n\to\infty} \lambda_n = \lambda_0 \in \Lambda. \tag{4.17}$$

Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is a sequence with $x_n \in X_n$ and $\|x_n\|_X = 1$ such that

$$\lim_{n\to\infty} P_nA(\lambda_n)x_n = 0. \tag{4.18}$$

Then there exists an element $x^0 \in X$ with $\|x^0\|_X = 1$ and

$$A(\lambda_0)x^0 = 0. \tag{4.19}$$

Further there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}} \subset \{x_n\}_{n\in\mathbb{N}}$ with

$$\lim_{k\to\infty} \|x^0 - x_{n_k}\|_X = 0. \tag{4.20}$$

Proof. Since $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence in the Hilbert space X, there exists a weakly convergent subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ [98, Theorem III.3.7], i.e., there exists a $x^0 \in X$ such that

$$\lim_{k\to\infty}(x_{n_k},v)_X = (x^0,v)_X \quad \text{for all } v\in X. \tag{4.21}$$

First we show that

$$\lim_{k\to\infty}(P_{n_k}A(\lambda_{n_k})x_{n_k},v)_X = \lim_{k\to\infty}(x_{n_k},A(\lambda_{n_k})^\star P_{n_k}v)_X = (x^0,A(\lambda^0)^\star v)_X = (A(\lambda_0)x^0,v)_X \tag{4.22}$$

holds for all $v\in X$. We have

$$\begin{aligned}\|A(\lambda_{n_k})^\star P_{n_k}v - A(\lambda_0)^\star v\|_X &\le \|[A(\lambda_{n_k})^\star - A(\lambda_0)^\star]P_{n_k}v\|_X + \|A(\lambda_0)^\star[P_{n_k}v - v]\|_X\\ &\le \|[A(\lambda_{n_k})^\star - A(\lambda_0)^\star]\|_{\mathcal{L}(X,X)}\|P_{n_k}v\|_X + \|A(\lambda_0)^\star\|_{\mathcal{L}(X,X)}\|[P_{n_k}v - v]\|_X\\ &\le \|[A(\lambda_{n_k})^\star - A(\lambda_0)^\star]\|_{\mathcal{L}(X,X)}\|v\|_X + \|A(\lambda_0)^\star\|_{\mathcal{L}(X,X)}\|[P_{n_k}v - v]\|_X \end{aligned}\tag{4.23}$$

for all $v\in X$, where we used that $\|P_n\|_{\mathcal{L}(X,X)} = 1$, see (4.9). The holomorphy of the operator function $A:\Lambda\to\mathcal{L}(X,X)$ implies

$$\|A(\lambda_{n_k})^\star - A(\lambda_0)^\star\|_{\mathcal{L}(X,X)} = \|[A(\lambda_{n_k}) - A(\lambda_0)]^\star\|_{\mathcal{L}(X,X)} \to 0$$

as $k\to\infty$ and together with the approximation property (4.11) of $\{X_n\}_{n\in\mathbb{N}}$ we get from (4.23)

$$\lim_{k\to\infty}\|A(\lambda_{n_k})^\star P_{n_k}v - A(\lambda_0)^\star v\|_X = 0 \quad \text{for all } v\in X. \tag{4.24}$$

Using (4.24) and (4.21) we obtain

$$\begin{aligned}&|(x_{n_k},A(\lambda_{n_k})^\star P_{n_k}v)_X - (x^0,A(\lambda_0)^\star v)_X|\\ &\qquad\le |(x_{n_k},A(\lambda_{n_k})^\star P_{n_k}v - A(\lambda_0)^\star v)_X| + |(x_{n_k}-x^0,A(\lambda_0)^\star v)_X|\\ &\qquad\qquad\le \|x_{n_k}\|_X\|A(\lambda_{n_k})^\star P_{n_k}v - A(\lambda_0)^\star v\|_X + |(x_{n_k}-x^0,A(\lambda_0)^\star v)_X| \to 0\end{aligned}$$

as $k\to\infty$ for all $v\in X$, thus we have shown (4.22),

$$\lim_{k\to\infty}(P_{n_k}A(\lambda_{n_k})x_{n_k},v)_X = (A(\lambda_0)x^0,v)_X \quad \text{for all } v\in X.$$

Therefore (4.18) implies that

$$(A(\lambda_0)x^0,v)_X = 0 \quad \text{for all } v\in X,$$

hence

$$A(\lambda_0)x^0 = 0. \tag{4.25}$$

Since $S(\lambda) : X \to X$ is compact for all $\lambda \in \Lambda$, the convergence $S(\lambda_0)x_{n_k} \to S(\lambda_0)x^0$ follows from the weak convergence $x_{n_k} \rightharpoonup x^0$. Together with the continuity of the operator function $S : \Lambda \to \mathcal{L}(X,X)$,

$$\lim_{k\to\infty} \|S(\lambda_{n_k}) - S(\lambda_0)\|_{\mathcal{L}(X,X)} = 0,$$

we obtain

$$\|S(\lambda_{n_k})x_{n_K} - S(\lambda_0)x^0\|_X \leq \|[S(\lambda_{n_k}) - S(\lambda_0)]x_{n_k}\|_X + \|S(\lambda_0)[x_{n_k} - x^0]\|_X \to 0 \quad \text{as } k \to \infty.$$

This yields

$$\begin{aligned} \|S(\lambda_0)x^0 - P_{n_k}S(\lambda_{n_k})x_{n_k}\|_X &\leq \|S(\lambda_0)x^0 - P_{n_k}S(\lambda_0)x^0\|_X + \|P_{n_k}[S(\lambda_0)x^0 - S(\lambda_{n_k})x_{n_k}]\|_X \\ &\leq \|(I_X - P_{n_k})S(\lambda_0)x^0\|_X + \|S(\lambda_0)x^0 - S(\lambda_{n_k})x_{n_k}\|_X \to 0 \end{aligned} \tag{4.26}$$

as $k \to \infty$. Since $A(\lambda) = T + S(\lambda)$, we get with (4.25), (4.18) and (4.26)

$$\begin{aligned} \|Tx^0 - P_{n_k}Tx_{n_k}\|_X &= \|[A(\lambda_0) - S(\lambda_0)]x^0 - P_{n_k}[A(\lambda_{n_k}) - S(\lambda_{n_k})]x_{n_k}\|_X \\ &\leq \|[A(\lambda_0)x^0 - P_{n_k}A(\lambda_{n_k})x_{n_k}\|_X + \|S(\lambda_0)x^0 - P_{n_k}S(\lambda_{n_k})x_{n_k}\|_X \to 0 \end{aligned} \tag{4.27}$$

as $k \to \infty$. Using that T is X-elliptic and that P_{n_k} is selfadjoint, it follows with (4.21) and (4.27) that

$$\begin{aligned} c_T\|x^0 - x_{n_k}\|_X^2 &\leq |(T(x^0 - x_{n_k}), x^0 - x_{n_k})_X| \\ &\leq |(Tx^0, x^0)_X - (Tx_{n_k}, x^0)_X| + |(Tx_{n_k}, x_{n_k})_X - (Tx^0, x_{n_k})_X| \\ &= |(x^0, T^\star x^0)_X - (x_{n_k}, T^\star x^0)_X| + |(Tx_{n_k}, P_{n_k}x_{n_k})_X - (Tx^0, x_{n_k})_X| \\ &= |(x^0, T^\star x^0)_X - (x_{n_k}, T^\star x^0)_X| + |(P_{n_k}Tx_{n_k}, x_{n_k})_X - (Tx^0, x_{n_k})_X| \\ &\leq |(x^0, T^\star x^0)_X - (x_{n_k}, T^\star x^0)_X| + \|P_{n_k}Tx_{n_k} - Tx^0\|_X\|x_{n_k}\|_X \to 0 \end{aligned}$$

as $k \to \infty$, thus

$$\lim_{k\to\infty} \|x^0 - x_{n_k}\|_X = 0$$

and $\|x^0\|_X = 1$, since $\|x_{n_k}\|_X = 1$. □

Note that the last lemma does not assert the existence of a converging sequence of eigenvalues of the projected eigenvalue problems.

Lemma 4.2.2. *Let $A : \Lambda \to \mathcal{L}(X,X)$ be given as in (4.2) and suppose that $\Lambda_0 \subset \rho(A)$ is a compact set in $\mathbb{C}$. Then there exist a constant $C(\Lambda_0) > 0$ and a $N(\Lambda_0) \in \mathbb{N}$ such that for all $n \geq N(\Lambda_0)$ and all $x_n \in X_n$ with $\|x_n\|_X = 1$ the following properties hold:*

i)

$$\|P_n A(\lambda)x_n\|_X \geq C(\Lambda_0) \quad \textit{for all } \lambda \in \Lambda_0, \tag{4.28}$$

ii) $[P_n A(\lambda)]^{-1} : X_n \to X_n$ *exists and is uniformly bounded,*

$$\|[P_n A(\lambda)]^{-1}\|_{\mathcal{L}(X_n,X_n)} \leq \frac{1}{C(\Lambda_0)} \quad \textit{for all } \lambda \in \Lambda_0. \tag{4.29}$$

Proof. i) Let us assume that the inequality (4.28) does not hold. Then there exists a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}} \subset \Lambda_0$ and a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ with $x_{n_k} \in X_{n_k}$ and $\|x_{n_k}\|_X = 1$ such that

$$\lim_{k\to\infty} P_{n_k} A(\lambda_{n_k})x_{n_k} = 0.$$

Since Λ_0 is compact, there exists a subsequence $\{\lambda_{\tilde{n}_k}\}_{k\in\mathbb{N}} \subset \{\lambda_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \lambda_{\tilde{n}_k} \to \lambda^* \in \Lambda_0.$$

Lemma 4.2.1 implies that there exists a $x^0 \in X$ with $\|x^0\|_X = 1$ and

$$A(\lambda^*)x^0 = 0,$$

which is a contradiction to the fact that $\lambda^* \in \Lambda_0 \subset \rho(A)$. Thus, inequality (4.28) holds.

ii) Part i) implies that $P_n A(\lambda) : X_n \to X_n$ is injective for all $\lambda \in \Lambda_0$ and all $n \geq N(\Lambda_0)$. Since X_n is finite dimensional, we conclude that $P_n A(\lambda) : X_n \to X_n$ is invertible for all $\lambda \in \Lambda_0$ and all $n \geq N(\Lambda_0)$. From the estimate (4.28) it follows for all $x_n \in X_n$ with $\|x_n\|_X = 1$ that

$$1 = \|x_n\|_X = \|P_n A(\lambda)[P_n A(\lambda)]^{-1}x_n\|_X \geq C(\Lambda_0)\|[P_n A(\lambda)]^{-1}x_n\|_X$$

for all $\lambda \in \Lambda_0$ and all $n \geq N(\Lambda_0)$, which proves the inequality (4.29). □

The next theorem shows that for every eigenvalue of A there exists a converging sequence of eigenvalues of the projected eigenvalue problems.

Theorem 4.2.3. *Let $A : \Lambda \to \mathcal{L}(X,X)$ be as given in (4.2).*

i) *For each eigenvalue $\lambda_0 \in \sigma(A)$ there exists a sequence $\{\lambda_0^n\}_{n=N_0}^{\infty}$ of eigenvalues of the projected eigenvalue problem $P_n A(\lambda_0^n)x_n^0 = 0$ such that*

$$\lim_{n\to\infty} \lambda_0^n = \lambda_0.$$

ii) *If $\{\lambda_0^n\}_{n\in\mathbb{N}}$ is a sequence of eigenvalues of the projected eigenvalue problem*

$$P_nA(\lambda_0^n)x_n^0 = 0,$$

and if $\{x_n^0\}_{n\in\mathbb{N}}$ is a sequence of corresponding eigenelements with $x_n^0 \in X_n$ and $\|x_n^0\|_X = 1$, then

$$\lim_{n\to\infty} \lambda_0^n = \lambda_0 \in \sigma(A).$$

Moreover, every limit point $\hat{x}^0$ of the sequence $\{x_n^0\}_{n\in\mathbb{N}}$ is an eigenelement of A corresponding to λ_0 with $\|\hat{x}^0\|_X = 1$.

Proof. i) Let us assume the contrary of assertion i) and let

$$\varepsilon_n := \inf\{|\lambda_0 - \lambda| : \lambda \in \sigma(P_nA)\} \quad \text{for } n \geq N_0.$$

Then there exists a subsequence $\{\varepsilon_{n_k}\}_{k\in\mathbb{N}} \subset \{\varepsilon_n\}_{n\in N}$ and a constant $\varepsilon > 0$ such that

$$\inf\{|\lambda_0 - \lambda| : \lambda \in \sigma(P_{n_k}A)\} = \varepsilon_{n_k} \geq \varepsilon \quad \text{for all } k \in \mathbb{N}.$$

Hence, we have

$$U_\varepsilon(\lambda_0) = \{\lambda : |\lambda - \lambda_0| < \varepsilon\} \subset \rho(P_{n_k}A) \quad \text{for all } k \in \mathbb{N}.$$

By Theorem 3.2.2, all eigenvalues of A are isolated, which implies that there exists a $\delta > 0$ with $\delta < \varepsilon$ such that

$$\Lambda_\delta := \{\lambda : |\lambda - \lambda_0| = \delta\} \subset \rho(A).$$

Since Λ_δ is a compact subset of $\mathbb{C}$ we can use Lemma 4.2.2 to conclude that there exits a $N(\Lambda_\delta) \in \mathbb{N}$ and a constant $c(\Lambda_\delta) > 0$ such that for all $n_k \geq N(\Lambda_\delta)$

$$\Lambda_\delta \subset \rho(P_{n_k}A) \qquad \text{and} \qquad \|[P_{n_k}A(\lambda)]^{-1}\|_{\mathcal{L}(X_{n_k},X_{n_k})} \leq c(\Lambda_\delta) \quad \text{for all } \lambda \in \Lambda_\delta.$$

By Theorem 3.1.6, the operator function $[P_{n_k}A](\cdot)^{-1} : \Lambda \to \mathcal{L}(X_{n_k}, X_{n_k})$ is holomorphic on the set

$$\{\lambda : |\lambda - \lambda_0| \leq \delta\} \subset \rho(P_{n_k}A),$$

therefore we can apply the principle of maximum of modulus (3.1) and obtain

$$\|[P_{n_k}A(\lambda_0)]^{-1}\|_{\mathcal{L}(X_{n_k},X_{n_k})} \leq c(\Lambda_\delta) \quad \text{for all } n_k \geq N(\Lambda_\delta).$$

Since for all $x \in X$

$$\begin{aligned}\|P_nA(\lambda_0)P_nx - A(\lambda_0)x\|_X &\leq \|P_nA(\lambda_0)P_nx - P_nA(\lambda_0)x\|_X + \|P_nA(\lambda_0)x - A(\lambda_0)x\|_X \\ &\leq \|A(\lambda_0)\|_{\mathcal{L}(X,X)}\|(P_n - I_X)x\|_X + \|(P_n - I_X)A(\lambda_0)x\|_X \to 0\end{aligned}$$

as $n \to \infty$, it follows for all $x \in X$ that

$$P_n A(\lambda_0) P_n x \to A(\lambda_0) x$$

as $n \to \infty$. Hence, for $x^0 \in \ker A(\lambda_0)$ with $\|x^0\|_X = 1$ and $n_k \geq N(\Lambda_\delta)$ we get

$$\begin{aligned} \|P_{n_k} x^0\|_X &= \|[P_{n_k} A(\lambda_0)]^{-1} P_{n_k} A(\lambda_0) P_{n_k} x^0\|_X \\ &\leq c(\Lambda_\delta) \|P_{n_k} A(\lambda_0) P_{n_k} x^0\|_X \to c(\Lambda_\delta) \|A(\lambda_0) x^0\|_X = 0, \end{aligned}$$

which is a contradiction to $\|P_{n_k} x^0\|_X \to \|x^0\|_X = 1$. Thus i) holds.

ii) follows immediately from Lemma 4.2.1. □

4.3 Asymptotic error estimates

For the error analysis of the Galerkin approximations of the eigenvalue problem (4.14) we use the approach of [47, 48]. There an error analysis is given for so–called regular approximations of eigenvalue problems for holomorphic Fredholm operator functions. The idea of that approach is to construct for the eigenvalue problems for A and $P_n A$ equivalent eigenvalue problems for matrix functions M and M_n. The error analysis is done then for the matrix functions M and M_n.

We follow [47,48] for the construction of the matrix functions M and M_n as well as for the error analysis. But since we have other assumptions for the approximations, we use partly other arguments. Moreover, we also give error estimates for the eigenelements which is not done in [47, 48].

The first result in this section provides the theoretical basis of this approach.

Lemma 4.3.1. *Let X, Y and Z be Banach spaces and let*

$$\begin{aligned} &A : \Lambda \to \mathcal{L}(X,Y), \quad R : \Lambda \to \mathcal{L}(X,Y), \\ &C : \Lambda \to \mathcal{L}(X,Z), \quad D : \Lambda \to \mathcal{L}(Z,X), \quad M : \Lambda \to \mathcal{L}(Z,Z) \end{aligned}$$

be holomorphic operator functions. Let $A(\lambda)$ be Fredholm for all $\lambda \in \Lambda$, let $\lambda_0 \in \sigma(A)$ and let $\rho(A) \neq \emptyset$. Let $\Lambda \subset \rho(R)$ and let the following relations

$$A(\lambda) = R(\lambda)\,[I_X - D(\lambda) C(\lambda)], \tag{4.30}$$

$$M(\lambda) = I_Z - C(\lambda) D(\lambda) \tag{4.31}$$

hold for all $\lambda \in \Lambda$.

i) *If $A(\lambda_0) x^0 = 0$ and $0 \neq x^0 \in X$, then*

$$x^0 = D(\lambda_0) C(\lambda_0) x^0, \quad C(\lambda_0) x^0 \neq 0 \quad \text{and} \quad M(\lambda_0) C(\lambda_0) x^0 = 0.$$

ii) *If* $M(\lambda_0)z^0 = 0$ *and* $0 \neq z^0 \in Z$, *then*

$$z^0 = C(\lambda_0)D(\lambda_0)z^0, \quad D(\lambda_0)z^0 \neq 0, \quad \text{and} \quad A(\lambda_0)D(\lambda_0)z^0 = 0.$$

iii) *If u is a Jordan function of order m for A corresponding to* λ_0, *then the function*

$$f : \lambda \mapsto C(\lambda)u(\lambda)$$

is a Jordan function of order $m' \geq m$ *for M corresponding to* λ_0.

iv) *If v is a Jordan function of order* m' *for M corresponding to* λ_0, *then the function*

$$g : \lambda \mapsto D(\lambda)v(\lambda)$$

is a Jordan function of order $m \geq m'$ *for A corresponding to* λ_0.

v) *If* $x^0, x^1, \ldots, x^m$ *is a Jordan chain of A corresponding to* λ_0 *and* $m(A, \lambda_0, x^0) = m+1$, *then* $m(M, \lambda_0, C(\lambda_0)x^0) = m+1$.

vi) *If* $z^0, z^1, \ldots, z^m$ *is a Jordan chain of M corresponding to* λ_0 *and* $m(M, \lambda_0, z^0) = m+1$, *then* $m(A, \lambda_0, D(\lambda_0)z^0) = m+1$.

vii) *If* $x_1^0, \ldots, x_J^0$ *is a canonical system of eigenelements of A corresponding to* λ_0, *then* $C(\lambda_0)x_1^0, \ldots, C(\lambda_0)x_J^0$ *is a canonical system of eigenelements of M corresponding to* λ_0 *and the partial and algebraic multiplicities of A and M coincide.*

Proof. i) If $A(\lambda_0)x^0 = 0$ and $x^0 \neq 0$, then the construction (4.30) of A and $\Lambda \subset \rho(R)$ imply that $[I_X - D(\lambda)C(\lambda)]x^0 = 0$ and $x^0 = D(\lambda)C(\lambda)x^0 \neq 0$. Thus, $C(\lambda)x^0 \neq 0$. Further, with $M(\lambda) = I_Z - C(\lambda)D(\lambda)$ it follows

$$M(\lambda_0)C(\lambda_0)x^0 = [I_Z - C(\lambda_0)D(\lambda_0)]C(\lambda_0)x^0 = C(\lambda_0)[I_X - D(\lambda_0)C(\lambda_0)]x^0 = 0.$$

ii) If $z^0 \neq 0$ and $M(\lambda_0)z^0 = 0$, then the definition of $M(\lambda) = I_Z - C(\lambda)D(\lambda)$ implies that $C(\lambda_0)D(\lambda_0)z^0 = z^0 \neq 0$. Further, with (4.30) we obtain

$$\begin{aligned} A(\lambda_0)D(\lambda_0)z^0 &= R(\lambda_0)[I_X - D(\lambda_0)C(\lambda_0)]D(\lambda_0)z^0 \\ &= R(\lambda_0)D(\lambda_0)[z^0 - C(\lambda_0)D(\lambda_0)z^0] = 0. \end{aligned}$$

iii) Let $u : \Lambda \to X$ be a Jordan function of order m, then the function $A(\lambda)u(\lambda)$ has a zero of multiplicity m and $u(\lambda_0) \neq 0$. Hence by part i), $C(\lambda_0)u(\lambda_0) \neq 0$. By using (4.31) and (4.30) we can write

$$\begin{aligned} M(\lambda)C(\lambda)u(\lambda) &= [I_Z - C(\lambda)D(\lambda)]C(\lambda)u(\lambda) = C(\lambda)[I_X - D(\lambda)C(\lambda)]u(\lambda) \\ &= [C(\lambda)R(\lambda)^{-1}]A(\lambda)u(\lambda). \end{aligned}$$

Since all occurring functions are holomorphic, we conclude that the multiplicity of λ_0 of the function $\lambda \mapsto M(\lambda)C(\lambda)u(\lambda)$ is greater than or equal to m. Thus, the function $f : \lambda \mapsto C(\lambda)u(\lambda)$ is a Jordan function of order greater than or equal to m.

iv) The proof can be done analogously as for part iii).

v) Let (λ_0, x^0) be an eigenpair of A and let $x^0, x^1, \ldots, x^m$ be a Jordan chain with $m+1 = m(A, \lambda_0, x^0)$. Then, by Lemma 3.2.13, the function

$$u : \lambda \mapsto x^0 + (\lambda - \lambda_0)x^1 + \ldots + (\lambda - \lambda_0)^m x^m$$

is a Jordan function of A corresponding to λ_0 of order $m+1$. From iii) we know that the function $f : \lambda \mapsto C(\lambda)u(\lambda)$ is a Jordan function of M corresponding to λ_0 of order at least $m+1$. By Lemma 3.2.12,

$$f(\lambda_0) = C(\lambda_0)x^0, \frac{1}{1!}f'(\lambda_0), \frac{1}{2!}f^{(2)}(\lambda_0), \ldots, \frac{1}{m!}f^{(m)}(\lambda_0)$$

is a Jordan chain of M corresponding to λ_0. Assume that $m(M, \lambda_0, C(\lambda_0)x^0) > m+1$, then there exists a Jordan chain

$$C(\lambda_0)x^0, z^1, z^2, \ldots, z^{m+1}$$

of M and by Lemma 3.2.13 there exists a Jordan function

$$w : \lambda \mapsto C(\lambda)x^0 + (\lambda - \lambda_0)z^1 + \ldots + (\lambda - \lambda_0)^{m+1} z^{m+1}$$

of M of order at least $m+2$. From iv) it follows that the function $g : \lambda \mapsto D(\lambda)w(\lambda)$ is a Jordan function of A of order at least $m+2$. Hence, by Lemma 3.2.12 and by i),

$$g(\lambda_0) = D(\lambda_0)w(\lambda_0) = D(\lambda_0)C(\lambda_0)x^0 = x^0, \frac{1}{1!}g'(\lambda_0), \frac{1}{2!}g^{(2)}(\lambda_0), \ldots, \frac{1}{(m+1)!}g^{(m+1)}$$

is a Jordan chain of A corresponding to λ_0 of length $m+2$ beginning with the eigenelement x^0. This is a contradiction to the fact that $m(A, \lambda_0, x^0) = m+1$. Thus, we have $m(M, \lambda_0, C(\lambda_0)x^0) = m+1$.

vi) The proof can be done analogously as for part v).

vii) Let $x_1^0, \ldots, x_J^0$ be a canonical system of eigenelements of A corresponding to λ_0. First we show that $C(\lambda_0)x_1^0, \ldots, C(\lambda_0)x_J^0$ is a basis of the eigenspace $\ker M(\lambda_0)$. From i) we know that $C(\lambda_0)x_j^0$ are eigenelements of M corresponding to λ_0 for $j = 1, \ldots, J$. Assume that

$$\alpha_1 C(\lambda_0)x_1^0 + \ldots + \alpha_J C(\lambda_0)x_J^0 = 0$$

for some $(\alpha_1, \ldots, \alpha_J)^\top \in \mathbb{C}^J$. Then by i) we have

$$\alpha_1 D(\lambda_0)C(\lambda_0)x_1^0 + \ldots + \alpha_J D(\lambda_0)C(\lambda_0)x_J^0 = \alpha_1 x_1^0 + \ldots + \alpha_J x_J^0 = 0,$$

implying that $\alpha_j = 0$ for $j = 1,\dots,J$. Thus $C(\lambda_0)x_1^0,\dots,C(\lambda_0)x_J^0$ are linearly independent in Z. Assume now that there exists a $z^0 \in Z$ such that $C(\lambda_0)x_1^0,\dots,C(\lambda_0)x_J^0,z^0$ are linear independent eigenelements of M corresponding to λ_0 in Z. Then from ii) it follows analogously as above that $x_1^0,\dots,x_J^0,D(\lambda_0)z^0$ are linear independent eigenelements of A corresponding to λ_0, which is a contradiction to the fact that $x_1^0,\dots,x_J^0$ is a basis of the eigenspace $\ker A(\lambda_0)$. Hence, $C(\lambda_0)x_1^0,\dots,C(\lambda_0)x_J^0$ is a basis of the eigenspace $\ker M(\lambda_0)$.

Next we show that $C(\lambda_0)x_1^0,\dots,C(\lambda_0)x_J^0$ is a canonical basis of $\ker M(\lambda_0)$. Let us consider Jordan chains of x_j^0

$$x_j^0, x_j^1, \dots, x_j^{m_j-1}$$

of maximal order $m_j = m(A,\lambda_0,x_j^0)$ for $j = 1,\dots J$. From v) we know that

$$m(A,\lambda_0,x_j^0) = m(M,\lambda_0,C(\lambda_0)x_j^0) \quad \text{for } j = 1,\dots,J.$$

First we show that $\varkappa(M,\lambda_0) = m(M,\lambda_0,C(\lambda_0)x_1^0)$. Assume the contrary, then there exists a Jordan chain $z_1^0,\dots,z_1^{\tilde{m}}$ of M corresponding to λ_0 with $\tilde{m} > m(M,\lambda_0,C(\lambda_0)x_1^0)$. By vi), there exists a Jordan chain of A beginning with $D(\lambda_0)z_1^0$ of length $\tilde{m} > \varkappa(A,\lambda_0)$, which gives a contradiction. Hence $\varkappa(M,\lambda_0) = m(M,\lambda_0,C(\lambda_0)x_1^0)$.

It remains to show iii) of the Definition 3.2.7 for M and the elements $C(\lambda_0)x_1^0,\dots,C(\lambda_0)x_J^0$. We do this by induction. Let us assume that the condition iii) of Definition 3.2.7 is fulfilled for $C(\lambda_0)x_1,\dots,C(\lambda_0)x_j$ for some $j \in 2,\dots,J-1$, i.e.,

$$\begin{aligned} &C(\lambda_0)x_j^0 \in \ker A(\lambda_0) \setminus \operatorname{span}\{C(\lambda_0)x_1^0,\dots,C(\lambda_0)x_{j-1}^0\} =: \tilde{M}_j \text{ and} \\ &m(M,\lambda_0,C(\lambda_0)x_j^0) = \max_{z\in\tilde{M}_j} m(M,\lambda_0,z). \end{aligned}$$

We show that this condition holds also for $C(\lambda_0)x_1^0,\dots,C(\lambda_0)x_j^0,C(\lambda_0)x_{j+1}^0$. Assume the contrary, then there exists a Jordan chain $\hat{z}^0,\hat{z}^1,\dots,\hat{z}^{\hat{m}}$ with

$$\hat{z}^0 \notin \operatorname{span}\{C(\lambda_0)x_1,\dots,C(\lambda_0)x_{j-1}\} \text{ and } \hat{m} > m(M,\lambda_0,C(\lambda_0)x_{j+1}) = m_{j+1}. \tag{4.32}$$

Suppose that

$$\alpha_1 x_1^0 + \dots + \alpha_j x_j^0 + \alpha_{j+1} D(\lambda_0)\hat{z}^0 = 0$$

for some $(\alpha_1,\dots,\alpha_{j+1})^\top \in \mathbb{C}^{j+1}$. From i), $D(\lambda_0)C(\lambda_0)x_i^0 = x_i^0$ for $i = 1,\dots,j$, we get then

$$\begin{aligned} 0 &= \alpha_1 D(\lambda_0)C(\lambda_0)x_1^0 + \dots + \alpha_j D(\lambda_0)C(\lambda_0)x_j^0 + \alpha_{j+1}D(\lambda_0)\hat{z}^0 \\ &= D(\lambda_0)\left[\alpha_1 C(\lambda_0)x_1^0 + \dots + \alpha_j C(\lambda_0)x_j^0 + \alpha_{j+1}\hat{z}^0\right]. \end{aligned}$$

With (4.32) and ii) we conclude that $\alpha_i = 0$ for $i = 1,\dots,j+1$ and hence

$$D(\lambda_0)\hat{z}^0 \notin \operatorname{span}\{x_1^0,\dots,x_j^0\} \text{ and } m(A,\lambda_0,D(\lambda_0)\hat{z}^0) > m_{j+1} = m(A,\lambda_0,x_{j+1}^0).$$

This is a contradiction to the fact that the partial multiplicities of a Jordan chain do not depend on the special choice of the canonical system of eigenelements, see Lemma 3.2.8. Thus, condition iii) of the Definition 3.2.7 of a canonical system of eigenelements is fulfilled for $C(\lambda_0)x_1^0,\ldots,C(\lambda_0)x_0^J$, which proves the assertion. □

In the following we will construct for the eigenvalue problem (4.14),

$$A(\lambda)x = 0,$$

a decomposition of $A(\lambda)$ according to Lemma 4.3.1,

$$A(\lambda) = R(\lambda)[I_X - D(\lambda)C(\lambda)],$$

with operator functions of the form

$$C : \Lambda \to \mathcal{L}(X,\mathbb{C}^J) \quad \text{and} \quad D : \Lambda \to \mathcal{L}(\mathbb{C}^J,X),$$

where $\mathbb{C}^J$ is the standard J-dimensional complex vectorspace. Then, the operator function $A : \Lambda \to \mathcal{L}(X,X)$ is equivalent to the matrix function $M : \Lambda \to \mathcal{L}(\mathbb{C}^J,\mathbb{C}^J)$,

$$M(\lambda) = I_J - C(\lambda)D(\lambda),$$

in the sense of Lemma 4.3.1.

Lemma 4.3.2. *Let $A : \Lambda \to \mathcal{L}(X,X)$ be a holomorphic Fredholm operator function with $\rho(A) \neq \emptyset$, $\lambda_0 \in \sigma(A)$ and* $\dim\ker A(\lambda_0) = J$.
Let $x_1^0,\ldots,x_J^0$ be some canonical system of eigenelements of A corresponding to λ_0 with $m_k = m(A,\lambda_0,x_k^0)$ for $k = 1,\ldots,J$. Let

$$x_k^0,\ldots,x_k^{m_k-1}$$

be some Jordan chain of maximal length of x_k^0 for $k = 1,\ldots,J$ and let

$$a_k(\lambda) = \sum_{i=0}^{m_k-1} (\lambda-\lambda_0)^i x_k^i \quad \text{for } k = 1,\ldots,J. \tag{4.33}$$

Then,

$$\frac{1}{j!}\frac{d^j}{d\lambda^j}\left[A(\lambda)a_k(\lambda)\right]_{|\lambda=\lambda_0} = \begin{cases} 0 & \text{for } j = 0,1,\ldots m_k-1 \\ u_k \neq 0 & \text{for } j = m_k, \end{cases} \tag{4.34}$$

for $k = 1,\ldots,J$, where $u_1,\ldots,u_J$ are linearly independent in X and constitute a basis in some direct complement of $\mathrm{Im}A(\lambda_0)$ *in X, i.e.,*

$$X = \mathrm{Im}A(\lambda_0) \oplus \mathrm{span}\{u_1,\ldots,u_J\}. \tag{4.35}$$

Proof. The property (4.34) follows immediately from Lemma 3.2.12, since the functions a_k are Jordan functions of order m_k.

By Theorem 3.2.2, $A(\lambda_0)$ is a Fredholm operator function with $\operatorname{ind} A(\lambda_0) = 0$ and therefore $\operatorname{codim} \operatorname{Im} A(\lambda_0) = \dim \ker A(\lambda_0) = J$. Assume that (4.35) does not hold. Then, either $u_1, \ldots, u_J$ are linearly dependent or $\operatorname{Im} A(\lambda_0) \cap \operatorname{span}\{u_1, \ldots, u_J\} \neq \{0\}$. In both cases there exist $\alpha_1, \ldots, \alpha_J \in \mathbb{C}$ such that

$$|\alpha_1| + \ldots + |\alpha_J| \neq 0 \quad \text{and} \quad \alpha_1 u_1 + \ldots + \alpha_J u_J = \tilde{u} \in \operatorname{Im} A(\lambda_0). \tag{4.36}$$

Let $\tilde{x} \in X$ with $A(\lambda_0)\tilde{x} = \tilde{u}$, and let

$$m = \max\{m_k : k = 1, \ldots, J;\ \alpha_k \neq 0\}.$$

Consider the function $a : \Lambda \to X$ defined by

$$a(\lambda) = \left(\sum_{\alpha_k \neq 0} \alpha_k (\lambda - \lambda_0)^{m - m_k} a_k(\lambda) \right) - (\lambda - \lambda_0)^m \tilde{u}. \tag{4.37}$$

Then

$$a(\lambda_0) = \sum_{m_k = m} \alpha_k a_k(\lambda_0) = \sum_{m_k = m} \alpha_k x_k^0 \neq 0$$

because x_k^0 are linearly independent eigenelements of A for $k = 1, \ldots, J$. The element $a(\lambda_0)$ is a nontrivial linear combination of eigenelements of order m of a canonical system of the eigenspace of A corresponding to the eigenvalue λ_0. Therefore, by Lemma 3.2.10, we have $m(A, \lambda_0, a(\lambda_0)) = m$.

Let us consider

$$\begin{aligned} &\frac{d^n}{d\lambda^n} [A(\lambda) a(\lambda)]_{|\lambda = \lambda_0} \\ &\quad = \sum_{\alpha_k \neq 0} \left\{ \sum_{j=0}^{n} \binom{n}{j} \frac{d^j}{d\lambda^j} \left[\alpha_k (\lambda - \lambda_0)^{m - m_k} \right]_{|\lambda = \lambda_0} \frac{d^{n-j}}{d\lambda^{n-j}} [A(\lambda) a_k(\lambda)]_{|\lambda = \lambda_0} \right\} \\ &\qquad - \frac{d^n}{d\lambda^n} [(\lambda - \lambda_0)^m A(\lambda) \tilde{u}]_{|\lambda = \lambda_0}. \end{aligned} \tag{4.38}$$

We have

$$\frac{d^j}{d\lambda^j} \left[\alpha_k (\lambda - \lambda_0)^{m - m_k} \right]_{|\lambda = \lambda_0} = \begin{cases} 0, & \text{if } j \neq m - m_k, \\ \alpha_k (m - m_k)!, & \text{if } j = m - m_k. \end{cases} \tag{4.39}$$

Note, if $j = m - m_k$, then

$$\begin{aligned} n - j &\leq m - 1 - (m - m_k) = m_k - 1, && \text{if } n \leq m - 1, \\ n - j &= m_k, && \text{if } n = m_k. \end{aligned}$$

Since the function a_k is a Jordan function of A corresponding to λ_0 of order m_k, and using the definition of u_k, we get

$$\frac{d^{n-j}}{d\lambda^{n-j}}\left[A(\lambda)a_k(\lambda)\right]_{|\lambda=\lambda_0} = \begin{cases} 0 & \text{if } j = m-m_k \text{ and } n \le m-1, \\ m_k! u_k & \text{if } j = m-m_k \text{ and } n = m. \end{cases} \tag{4.40}$$

Hence, from (4.38) we conclude with (4.39) and (4.40) that

$$\frac{d^n}{d\lambda^n}\left[A(\lambda)a(\lambda)\right]_{|\lambda=\lambda_0} = 0 \quad \text{for } 0 \le n \le m-1$$

and, by using that $\sum_{\alpha_k \neq 0} \alpha_k u_k = \tilde{v} = A(\lambda_0)\tilde{x}$,

$$\frac{d^m}{d\lambda^m}\left[A(\lambda)a(\lambda)\right]_{|\lambda=\lambda_0} = \sum_{\alpha_k \neq 0} \binom{m}{m-m_k} \alpha_k (m-m_k)! m_k! u_k - m! A(\lambda_0)\tilde{x} = 0.$$

Thus, the function a is a Jordan function of order at least $m+1$. By Lemma 3.2.12, we have $m(A,\lambda_0,a(\lambda_0)) \ge m+1$, which is a contradiction to the fact that $m(A,\lambda_0,a(\lambda_0)) = m$. Therefore assumption (4.36) gives a contradiction and consequently (4.35) holds. □

Define

$$u_i(\lambda) := (\lambda-\lambda_0)^{-m_i} A(\lambda)a_i(\lambda) \quad \text{for } \lambda \neq \lambda_0 \text{ and } i = 1,\dots,J, \tag{4.41}$$

where the function a_i is given by (4.33). Lemma 4.3.2 shows that the function $A(\lambda)a_i(\lambda)$ has at λ_0 a zero of multiplicity m_i, therefore the function u_i can be continued at $\lambda = \lambda_0$ by continuity by

$$u_i(\lambda_0) := \frac{1}{m_i!}\frac{d^{m_i}}{d\lambda^{m_i}}\left[A(\lambda)a_i(\lambda)\right]_{|\lambda=\lambda_0} = u_i. \tag{4.42}$$

Note that the function u_i is holomorphic on Λ and that by Lemma 4.3.2 we have

$$X = \operatorname{Im} A(\lambda_0) \oplus \operatorname{span}\{u_1(\lambda_0),\dots,u_J(\lambda_0)\}. \tag{4.43}$$

Let us now consider the adjoint eigenvalue problem $A^\star(\lambda)x = 0$. Lemma 3.3.4 and Remark 3.3.5 show that λ_0 is an eigenvalue of A if and only if $\overline{\lambda}_0$ is an eigenvalue of $A^\star$. Furthermore, the geometric and partial multiplicities of λ_0 and $\overline{\lambda}_0$ coincide. Let

$$y_1^0,\dots,y_J^0$$

be some canonical system of eigenelements of $A^\star$ corresponding to $\overline{\lambda}_0$ with

$$m_k = m(A^\star,\overline{\lambda}_0,y_k^0)$$

for $k = 1,\dots,J$. Let

$$y_k^0, y_k^1,\dots,y_k^{m_k-1} \tag{4.44}$$

be some Jordan chain of maximal order m_k of y_k^0 for $k = 1,\dots,J$. Finally, let us define the functions $b_k : \mathbb{C} \to X$ by

$$b_k(\lambda) = \sum_{j=0}^{m_k-1} (\lambda - \overline{\lambda}_0)^j y_k^j \quad \text{for } k = 1,...,J.$$

In an analogous way as for A and λ_0 we can define holomorphic functions

$$v_k : \{\lambda : \overline{\lambda} \in \Lambda\} \to X, \qquad k = 1,...,J,$$

for $A^\star$ by

$$v_k(\lambda) := \begin{cases} (\lambda - \overline{\lambda}_0)^{-m_k} A^\star(\lambda) b_k(\lambda) & \text{for } \lambda \neq \overline{\lambda}_0, \\ \dfrac{1}{m_k!} \dfrac{d^{m_k}}{d\lambda^{m_k}} [A^\star(\lambda) b_k(\lambda)]_{|\lambda=\overline{\lambda}_0} & \text{for } \lambda = \overline{\lambda}_0. \end{cases} \tag{4.45}$$

With Lemma 4.3.2 we get the decomposition

$$X = \mathrm{Im} A^\star(\overline{\lambda}_0) \oplus \mathrm{span}\{v_1(\overline{\lambda}_0), \dots, v_J(\overline{\lambda}_0)\}.$$

Let us further define the operator functions $K : \Lambda \to \mathcal{L}(X,X)$ and $K_n : \Lambda \to \mathcal{L}(X,X)$ for $n \in \mathbb{N}$ by

$$K(\lambda)x := \sum_{i=1}^{J} (x, v_i(\overline{\lambda}))_X u_i(\lambda) \quad \text{and} \quad K_n(\lambda)x := P_n K(\lambda). \tag{4.46}$$

Obviously, the operators $K(\lambda)$ and $K_n(\lambda)$ are compact for all $\lambda \in \Lambda$, since both have a finite dimensional range. Notice that $\lambda \mapsto v_i(\overline{\lambda})$ is antiholomorphic in λ_0. But it can be easily seen that $\lambda \mapsto (x, v_i(\overline{\lambda}))_X$ is holomorphic in λ_0. Since the function v_i is holomorphic in $\overline{\lambda}_0$, it admits a representation by

$$v_i(\lambda) = \sum_{i=0}^{\infty} (\lambda - \overline{\lambda}_0)^i \tilde{v}_i,$$

and by Theorem 3.1.2,

$$\lambda \mapsto (v_i(\lambda), x)_X = \sum_{i=0}^{\infty} (\lambda - \overline{\lambda}_0)^i (\tilde{v}_i, x)_X$$

is holomorphic in $\overline{\lambda}_0$ for all $x \in X$. This implies that

$$\lambda \mapsto (x, v_i(\overline{\lambda}))_X = (x, \sum_{i=0}^{\infty} (\overline{\lambda} - \overline{\lambda}_0)^i \tilde{v}_i)_X = \sum_{i=0}^{\infty} (\lambda - \lambda_0)^i \overline{(\tilde{v}_i, x)}_X.$$

is holomorphic in λ_0 for all $x \in X$. Hence, K and K_n are also holomorphic. Let us finally define the operator functions $R : \Lambda \to \mathcal{L}(X,X)$ and $R_n : \Lambda \to \mathcal{L}(X,X)$ for $n \in \mathbb{N}$ by

$$R(\lambda) := A(\lambda) + K(\lambda) \quad \text{and} \quad R_n(\lambda) := P_n R(\lambda). \tag{4.47}$$

The next lemma shows that $R(\lambda) : X \to X$ and $R_n(\lambda) : X_n \to X_n$ have a bounded inverse in a neighborhood $\Lambda_1 \subset \Lambda$ of λ_0. Moreover, we we can show that $R_n(\lambda)^{-1}$ is uniformly bounded in Λ_1 for sufficiently large n.

Lemma 4.3.3. *Let A be as given in (4.2). Further, let R and R_n be as defined in (4.47). Then there exist a neighborhood $\Lambda_1 \subset \Lambda$ of λ_0 and a $N_1(\Lambda_1) \in \mathbb{N}$ such that the following properties hold:*

i) *The operators*

$$R(\lambda) : X \to X \quad \text{and} \quad R_n(\lambda) : X_n \to X_n$$

are invertible for all $n \geq N(\Lambda_1)$ and all $\lambda \in \Lambda_1$.

ii) *For any compact $\Lambda_2 \subset \Lambda_1$ there exist a $N(\Lambda_2) \in \mathbb{N}$ and a constant $c(\Lambda_2) > 0$ satisfying*

$$\sup \left\{ \left\| R_n(\lambda)^{-1} \right\|_{\mathcal{L}(X_n, X_n)} : \lambda \in \Lambda_2, n \geq N(\Lambda_2) \right\} \leq c(\Lambda_2).$$

Proof. First we show that $R(\lambda_0)$ is invertible. Since $R(\lambda_0) = A(\lambda_0) + K(\lambda_0)$ is a compact perturbation of the Fredholm operator $A(\lambda_0)$ with $\operatorname{ind} A(\lambda_0) = 0$, it follows that $R(\lambda_0)$ is Fredholm and $\operatorname{ind} R(\lambda_0) = 0$, see [61, Theorem 2.26]. To prove the invertibility of $R(\lambda_0)$, it suffices therefore to show that $R(\lambda_0)$ is injective. Suppose that $R(\lambda_0)\tilde{x} = 0$ for some $\tilde{x} \in X$, then, by construction of $R(\lambda_0)$ and $K(\lambda_0)$, we have $A(\lambda_0)\tilde{x} = K(\lambda_0)\tilde{x} = 0$. From

$$K(\lambda_0)\tilde{x} = \sum_{i=1}^{J} (\tilde{x}, v_i(\overline{\lambda_0}))_X u_i(\lambda_0) = 0,$$

it follows that

$$(\tilde{x}, v_i(\overline{\lambda}_0))_X = 0 \quad \text{for } i \in 1, \ldots, J, \tag{4.48}$$

since $u_i(\lambda_0)$ are linearly independent in X by construction (4.41) and Lemma 4.3.2. From $\tilde{x} \in \ker A(\lambda_0)$ we get $(\tilde{x}, A^\star(\overline{\lambda}_0)y)_X = 0$ for all $y \in X$ and thus

$$(\tilde{x}, z)_X = 0 \quad \text{for all } z \in \operatorname{Im} A^\star(\overline{\lambda}_0). \tag{4.49}$$

Since $X = \operatorname{Im} A^\star(\overline{\lambda}_0) \oplus \operatorname{span}\{v_1(\overline{\lambda}_0), \ldots, v_J(\overline{\lambda}_0)\}$, we get with (4.48) and (4.49) that

$$(\tilde{x}, w)_X = 0 \quad \text{for all } w \in X,$$

which implies that $\tilde{x} = 0$. Consequently, $R(\lambda_0)$ is injective and thus invertible.

By Lemma 3.1.6, there exists a neighborhood $\Lambda_1 := U_\delta(\lambda_0)$ of λ_0 such that we have $R(\lambda)^{-1} \in \mathcal{L}(X, X)$ for all $\lambda \in \Lambda_1$. Using the definition (4.2) of A, we can write

$$R(\lambda) = A(\lambda) + K(\lambda) = T + S(\lambda) + K(\lambda) = T + \tilde{C}(\lambda) \quad \text{for all } \lambda \in \Lambda,$$

where T is X-elliptic and $\tilde{C}(\lambda)$ is compact. By Lemma 4.2.2, there exist for any compact $\Lambda_2 \subset \Lambda_1$ a $N(\Lambda_2) \in \mathbb{N}$ and a constant $c(\Lambda_2) > 0$ such that $R_n(\lambda)^{-1} \in \mathcal{L}(X_n, X_n)$ for all $n \geq N_2$ and all $\lambda \in \Lambda_2$ satisfying

$$\|R_n(\lambda)^{-1}\|_{\mathcal{L}(X_n,X_n)} \leq c(\Lambda_2).$$

□

Now we are able to define equivalent matrix functions M and M_n for A and P_nA, respectively, in the sense of Lemma 4.3.1. We can write for all $\lambda \in \Lambda_1$, where Λ_1 is given as in Lemma 4.3.3,

$$\begin{aligned} A(\lambda) &= R(\lambda) - K(\lambda) = R(\lambda)[I_X - R(\lambda)^{-1}K(\lambda)] \\ &= R(\lambda)[I_X - \sum_{i=1}^{J} (\cdot, v_i(\overline{\lambda}))_X R(\lambda)^{-1} u_i(\lambda)]. \end{aligned}$$

Define $C(\lambda) : X \to \mathbb{C}^J$ and $D(\lambda) : \mathbb{C}^J \to X$ by

$$C(\lambda) := ((\cdot, v_1(\overline{\lambda}))_X, \ldots, (\cdot, v_J(\overline{\lambda}))_X), \qquad D(\lambda)(\xi_1, \ldots, \xi_J) := \sum_{i=1}^{J} \xi_i R(\lambda)^{-1} u_i(\lambda),$$

then we can write

$$A(\lambda) = R(\lambda)[I_X - D(\lambda)C(\lambda)] \quad \text{for all } \lambda \in \Lambda_1. \tag{4.50}$$

Now we can define the matrix function $M : \Lambda_1 \to \mathcal{L}(\mathbb{C}^J, \mathbb{C}^J)$ by

$$M(\lambda) := I_J - C(\lambda)D(\lambda) \tag{4.51}$$

and have

$$\begin{aligned} (M(\lambda)(\xi_1, \ldots, \xi_J))_i &= \xi_i - (D(\lambda)\xi, v_i(\overline{\lambda}))_X \\ &= \xi_i - \sum_{i=j}^{J} \xi_j (R(\lambda)^{-1} u_j(\lambda), v_i(\overline{\lambda}))_X. \end{aligned} \tag{4.52}$$

Analogously we can derive a matrix function M_n for P_nA. For all $\lambda \in \Lambda_1$, where Λ_1 is defined as in Lemma 4.3.3, we can write

$$\begin{aligned} P_nA(\lambda) = R_n(\lambda) - K_n(\lambda) &= R_n(\lambda)[I_{X_n} - R_n(\lambda)^{-1}K_n(\lambda)] \\ &= R_n(\lambda)[I_X - \sum_{i=1}^{J} (\cdot, v_i(\overline{\lambda}))_X R_n(\lambda)^{-1} P_n u_i(\lambda)]. \end{aligned}$$

Define $C_n : \Lambda_1 \to \mathcal{L}(X_n, \mathbb{C}^J)$ and $D_n : \Lambda_1 \to \mathcal{L}(\mathbb{C}^J, X_n)$ by

$$C_n(\lambda) = ((\cdot, v_1(\overline{\lambda}))_X, \ldots, (\cdot, v_J(\overline{\lambda}))_X), \qquad D_n(\lambda)(\xi_1, ..., \xi_J) = \sum_{i=1}^{J} \xi_i R_n(\lambda)^{-1} P_n u_i(\lambda),$$

then

$$P_n A(\lambda) = R_n(\lambda)[I_{X_n} - D_n(\lambda) C_n(\lambda)].$$

Finally, let us define the matrix function $M_n : \Lambda_1 \to \mathcal{L}(\mathbb{C}^J, \mathbb{C}^J)$ by

$$M_n(\lambda) := I_J - C_n(\lambda) D_n(\lambda), \tag{4.53}$$

then

$$\begin{aligned}(M_n(\lambda)(\xi_1, ..., \xi_J))_i &= \xi_i - (D_n(\lambda)\xi, v_i(\overline{\lambda}))_X \\ &= \xi_i - \sum_{j=1}^{J} \xi_j (R_n(\lambda)^{-1} P_n u_j(\lambda), v_i(\overline{\lambda}))_X. \end{aligned} \tag{4.54}$$

With the above constructions of the matrix functions M and M_n we can apply Lemma 4.3.1 which shows that the eigenvalue problems for A and M and for $P_n A$ and M_n are equivalent in Λ_1.

Corollary 4.3.4. *Let M and M_n be defined by (4.51) and (4.53), respectively.*

i) *$\lambda_0 \in \Lambda_1$ is an eigenvalue of A if and only if λ_0 is an eigenvalue of M. For any eigenvalue $\lambda_0 \in \sigma(A) \cap \Lambda_1$ the geometric, partial, and algebraic multiplicities are equal for A and M.*

ii) *$\lambda_0 \in \Lambda_1$ is an eigenvalue of $P_n A$ if and only if λ_0 is an eigenvalue of M_n. For any $\lambda_0 \in \sigma(P_n A) \cap \Lambda_1$ the geometric, partial, and algebraic multiplicities are equal for $P_n A$ and M_n.*

In the next lemma we consider the matrix functions M and M_n and give an asymptotic error estimate for the matrix entries which depends on the approximation property of the trial spaces X_n with respect to the generalized eigenspaces of A and $A^\star$. This estimate is essential for the derivation of the error estimate for the eigenvalues of the Galerkin approximation.

Lemma 4.3.5. *Let M and M_n be defined by (4.51) and (4.53), respectively. Then for every compact $\Lambda_2 \subset \Lambda_1$ there exist a constant $c(\Lambda_2) > 0$ and a $N \in \mathbb{N}$ such that the estimate*

$$\begin{aligned}&\sup\left\{|m^{ij}(\lambda) - m_n^{ij}(\lambda)| : \lambda \in \Lambda_2, 1 \le i, j \le J\right\} \\ &\qquad \le c(\Lambda_2) \sup_{\substack{z \in G(A, \lambda_0) \\ \|z\|_X \le 1}} \inf_{x_n \in X_n} \|z - x_n\|_X \sup_{\substack{z \in G(A^\star, \overline{\lambda}_0) \\ \|z\|_X \le 1}} \inf_{y_n \in X_n} \|z - y_n\|_X \end{aligned} \tag{4.55}$$

holds for all $n \ge N$.

Proof. Let Λ_2 be an arbitrary but fixed compact subset of Λ_1. For $\lambda \in \Lambda_2$ we can write

$$\begin{aligned} m^{ij}(\lambda) - m_n^{ij}(\lambda) &= (R_n(\lambda)^{-1}P_n u_j(\lambda), v_i(\overline{\lambda}))_X - (R(\lambda)^{-1}u_j(\lambda), v_i(\overline{\lambda}))_X \\ &= ([R_n(\lambda)^{-1}P_n - R(\lambda)^{-1}]u_j(\lambda), v_i(\overline{\lambda}))_X \\ &= ([R_n(\lambda)^{-1}P_n R(\lambda) - I_X]R(\lambda)^{-1}u_j(\lambda), v_i(\overline{\lambda}))_X. \end{aligned} \tag{4.56}$$

We have

$$[R_n(\lambda)^{-1}P_n R(\lambda) - I_X]x_n = 0$$

for all $x_n \in X_n$ and therefore

$$[R_n(\lambda)^{-1}P_n R(\lambda) - I_X]x = [R_n(\lambda)^{-1}P_n R(\lambda) - I_X](x - x_n) \tag{4.57}$$

for all $x \in X$ and all $x_n \in X_n$. Using that P_n is selfadjoint, we obtain

$$\begin{aligned} 0 &= (P_n[R(\lambda)R_n(\lambda)^{-1} - I_X]P_n x, y)_X \\ &= (x, P_n\{[R_n(\lambda)^{-1}]^\star R(\lambda)^\star - I_X\}P_n y)_X \\ &= (x, \{P_n[R_n(\lambda)^{-1}]^\star R(\lambda)^\star - I_X\}P_n y)_X \end{aligned}$$

for all $x, y \in X$ from which we get

$$0 = (x, \{P_n[R_n(\lambda)^{-1}]^\star R(\lambda)^\star - I_X\}y_n)_X \tag{4.58}$$

for all $x \in X$ and all $y_n \in X_n$. Since

$$R_n(\lambda)^{-1}P_n R(\lambda) - I_X = R(\lambda)^{-1}[R(\lambda)R_n(\lambda)^{-1}P_n - I_X]R(\lambda),$$

we obtain with (4.57) and (4.58)

$$\begin{aligned} &([R_n(\lambda)^{-1}P_n R(\lambda) - I_X]x, y)_X \\ &= (R(\lambda)^{-1}[R(\lambda)R_n(\lambda)^{-1}P_n - I_X]R(\lambda)(x - x_n), y)_X \\ &= (R(\lambda)(x - x_n), \{P_n[R_n(\lambda)^{-1}]^\star R(\lambda)^\star - I_X\}[R(\lambda)^{-1}]^\star y)_X \\ &= (R(\lambda)(x - x_n), \{P_n[R_n(\lambda)^{-1}]^\star R(\lambda)^\star - I_X\}\{[R(\lambda)^{-1}]^\star y - y_n\})_X \\ &= ([R(\lambda)R_n(\lambda)^{-1}P_n - I_X]R(\lambda)(x - x_n), [R(\lambda)^{-1}]^\star y - y_n)_X \end{aligned} \tag{4.59}$$

for all x, $y \in X$ and all x_n, $y_n \in X_n$. From (4.56) we get

$$\begin{aligned} &|m^{ij}(\lambda) - m_n^{ij}(\lambda)| = \\ &= |([R_n(\lambda)^{-1}P_n R(\lambda) - I_X]R(\lambda)^{-1}u_j(\lambda), v_i(\overline{\lambda}))_X| \\ &= |([R(\lambda)R_n(\lambda)^{-1}P_n - I_X]R(\lambda)[R(\lambda)^{-1}u_j(\lambda) - x_n], [R(\lambda)^{-1}]^\star v_i(\overline{\lambda}) - y_n)_X| \\ &\le \|[R(\lambda)R_n(\lambda)^{-1}P_n - I_X]R(\lambda)\|_{\mathcal{L}(X,X)}\|R(\lambda)^{-1}u_j(\lambda) - x_n\|_X\|[R(\lambda)^{-1}]^\star v_i(\overline{\lambda}) - y_n\|_X \end{aligned} \tag{4.60}$$

for all x_n, $y_n \in X_n$. By Lemma 4.3.3, the inverse $R_n(\lambda)^{-1}$ is uniformly bounded on Λ_2, i.e., there exists a constant $C_2 > 0$ and a $N_2 \in \mathbb{N}$ such that

$$\|R_n(\lambda)^{-1}\|_{\mathcal{L}(X_n,X_n)} \leq C_2 \quad \text{for all } \lambda \in \Lambda_2 \text{ and all } n \geq N_2.$$

With the continuity of R and using that $\|P_n\|_{\mathcal{L}(X_n,X_n)} = 1$, we get

$$\begin{aligned}\|[R(\lambda)R_n(\lambda)^{-1}P_n - I_X]R(\lambda)\|_{\mathcal{L}(X,X)} \\ \leq \left[\|R(\lambda)\|_{\mathcal{L}(X,X)}\|R_n(\lambda)^{-1}\|_{\mathcal{L}(X_n,X_n)} + 1\right]\|R(\lambda)\|_{\mathcal{L}(X,X)} \leq c_2\end{aligned}$$

for all $\lambda \in \Lambda_2$ and all $n \geq N_2$. From (4.60) it follows that

$$|m^{ij}(\lambda) - m_n^{ij}(\lambda)| \leq c_2 \inf_{x_n \in X_n} \left\|R(\lambda)^{-1}u_j(\lambda) - x_n\right\|_X \inf_{y_n \in X_n} \|[R(\lambda)^{-1}]^\star v_i(\overline{\lambda}) - y_n\|_X \quad (4.61)$$

for all $\lambda \in \Lambda_2$ and all $n \geq N_2$.

Next we show that $R(\lambda)^{-1}u_j(\lambda)$ is an element of the generalized eigenspace $G(A,\lambda_0)$ for all $\lambda \in \Lambda_2$ and for $j = 1,\ldots,J$. Recall the definition (4.41) of the function u_j,

$$u_j(\lambda) = \begin{cases}(\lambda - \lambda_0)^{-m_j}A(\lambda)a_j(\lambda) & \text{for } \lambda \neq \lambda_0, \\ u_j & \text{for } \lambda = \lambda_0,\end{cases}$$

where

$$a_j(\lambda) = \sum_{k=0}^{m_j-1}(\lambda - \lambda_0)^k x_j^k,$$

and where x_j^k are generalized eigenelements of A corresponding to λ_0. By Lemma 4.3.2 and by the construction of the operator function R, we see that $R(\lambda_0)^{-1}u_j$ is an element of the eigenspace $\ker A(\lambda_0)$ and therefore it is also an element of the generalized eigenspace $G(A,\lambda_0)$. For $\lambda_0 \neq \lambda \in \Lambda_2$ we can write

$$R(\lambda)^{-1} = A(\lambda)^{-1}[A(\lambda) + K(\lambda) - K(\lambda)]R(\lambda)^{-1} = A(\lambda)^{-1}[I_X - K(\lambda)R^{-1}(\lambda)]. \quad (4.62)$$

Using the definition of the function u_j we have

$$A(\lambda)^{-1}u_j(\lambda) = (\lambda - \lambda_0)^{-m_j}A(\lambda)^{-1}A(\lambda)a_j(\lambda) = (\lambda - \lambda_0)^{-m_j}a_j(\lambda)$$

and we see that

$$u_j(\lambda) \in G(A,\lambda_0), \quad (4.63)$$

since $a_j(\lambda)$ is a linear combination of generalized eigenelements. We can write

$$A(\lambda)^{-1}K(\lambda)x = \sum_{i=1}^{J}(x, v_i(\overline{\lambda}))_X A(\lambda)^{-1}u_i(\lambda) \quad \text{for all } x \in X,$$

and with (4.63) we have $A(\lambda)^{-1}K(\lambda)u_j(\lambda) \in G(A,\lambda_0)$. Hence, with (4.62) we conclude $R(\lambda)^{-1}u_j(\lambda) \in G(A,\lambda_0)$.

In a similar way we show now that $[R(\lambda)^{-1}]^\star v_i(\overline{\lambda})$ is an element of the generalized eigenspace $G(A^\star,\overline{\lambda}_0)$ for all $\lambda \in \Lambda_2$ and $i = 1,\dots,J$. Recall the definition (4.45) of the function v_i,

$$v_i(\lambda) = \begin{cases} (\lambda-\overline{\lambda}_0)^{-m_i}A^\star(\lambda)b_i(\lambda) & \text{for } \lambda \neq \overline{\lambda}_0, \\ \dfrac{1}{m_i!}\dfrac{d^{m_i}}{d\lambda^{m_i}}[A^\star(\lambda)b_i(\lambda)]_{\lambda=\overline{\lambda}_0} & \text{for } \lambda = \overline{\lambda}_0, \end{cases}$$

where

$$b_i(\lambda) = \sum_{k=0}^{m_i-1}(\lambda-\overline{\lambda}_0)^k y_i^k$$

and where y_i^k are generalized eigenelements of $A^\star$ corresponding to $\overline{\lambda}_0$. By Lemma 4.3.2 and by the construction of the operator function R, we see that $[R(\lambda_0)^{-1}]^\star v_i(\overline{\lambda}_0)$ is an element of the eigenspace $\ker A^\star(\overline{\lambda}_0) \subset G(A^\star,\overline{\lambda}_0)$. For $\lambda_0 \neq \lambda \in \Lambda_2$ we can write

$$\begin{aligned}[R(\lambda)^{-1}]^\star &= \left\{R(\lambda)^{-1}\left[A(\lambda)+K(\lambda)-K(\lambda)\right]A(\lambda)^{-1}\right\}^\star \\ &= \left\{\left[I_X - R(\lambda)^{-1}K(\lambda)\right]A(\lambda)^{-1}\right\}^\star \\ &= [A(\lambda)^{-1}]^\star\{I_X - K(\lambda)^\star[R(\lambda)^{-1}]^\star\}. \end{aligned} \tag{4.64}$$

Note that for $K(\lambda)^\star$ we have

$$\begin{aligned}(K(\lambda)x,y)_X &= (\sum_{k=1}^{J}(x,v_k(\overline{\lambda}))_X u_k(\lambda),y)_X = \sum_{k=1}^{J}(x,v_k(\overline{\lambda}))_X(u_k(\lambda),y)_X \\ &= \sum_{k=1}^{J}(x,(y,u_k(\lambda))_X v_k(\overline{\lambda}))_X = (x,K(\lambda)^\star y)_X \end{aligned} \tag{4.65}$$

for all $x,y \in X$. Using that $[A(\lambda)^{-1}]^\star = A^\star(\overline{\lambda})^{-1}$, we can write

$$[A(\lambda)^{-1}]^\star v_i(\overline{\lambda}) = (\overline{\lambda}-\overline{\lambda}_0)^{-m_i}A^\star(\overline{\lambda})^{-1}A^\star(\overline{\lambda})v_i(\overline{\lambda}_0) = (\overline{\lambda}-\overline{\lambda}_0)^{-m_i}v_i(\overline{\lambda}),$$

and with the definition of the function v_i we have

$$[A(\lambda)^{-1}]^\star v_i(\overline{\lambda}) \in G(A^\star,\overline{\lambda}_0). \tag{4.66}$$

Moreover with (4.65) we get

$$[A(\lambda)^{-1}]^\star K(\lambda)^\star y = \sum_{k=1}^{J}(y,u_k(\lambda))_X A^\star(\overline{\lambda})^{-1}v_k(\overline{\lambda}) \quad \text{for all } y \in X,$$

and with (4.66) we have $[A(\lambda)^{-1}]^\star K(\lambda)^\star v_i(\overline{\lambda}) \in G(A^\star,\overline{\lambda}_0)$. From (4.64) we conclude that $[R(\lambda)^{-1}]^\star v_i(\overline{\lambda}) \in G(A^\star,\overline{\lambda}_0)$.

Since the functions $R(\lambda)^{-1}$, $[R(\lambda)^{-1}]^\star$, $u_j(\lambda)$, and $v_i(\lambda)$ are continuous in Λ_2, there exists a constant $c_3 > 0$ such that

$$\|R(\lambda)^{-1}u_j(\lambda)\|_X \le c_3 \quad \text{and} \quad \|[R(\lambda)^{-1}]^\star v_i(\lambda)\|_X \le c_3$$

for all $\lambda \in \Lambda_2$ and for $1 \le i, j \le J$. Hence, we get the estimates

$$\sup_{\lambda\in\Lambda_2} \inf_{x_n\in X_n} \|R(\lambda)^{-1}u_j(\lambda) - x_n\|_X \le \sup_{\substack{z\in G(A,\lambda_0)\\ \|z\|_X\le c_3}} \inf_{x_n\in X_n} \|z - x_n\|_X = c_3 \sup_{\substack{z\in G(A,\lambda_0)\\ \|z\|_X\le 1}} \inf_{x_n\in X_n} \|z - x_n\|_X$$

for $1 \le j \le J$ and

$$\sup_{\lambda\in\Lambda_2} \inf_{y_n\in X_n} \|[R(\lambda)^{-1}]^\star v_i(\overline{\lambda}) - y_n\|_X \le \sup_{\substack{z\in G(A^\star,\overline{\lambda}_0)\\ \|z\|_X\le c_3}} \inf_{y_n\in X_n} \|z - y_n\|_X = c_3 \sup_{\substack{z\in G(A^\star,\overline{\lambda}_0)\\ \|z\|_X\le 1}} \inf_{y_n\in X_n} \|z - y_n\|_X$$

for $1 \le i \le J$. Therefore we finally obtain from (4.61) the estimate

$$\begin{aligned}\sup\{|m^{ij}(\lambda) - m_n^{ij}(\lambda)| : \lambda \in \Lambda_2, 1 \le i, j \le J\} \\ \le c_2 c_3^2 \sup_{\substack{z\in G(A,\lambda_0)\\ \|z\|_X\le 1}} \inf_{x_n\in X_n} \|z - x_n\|_X \sup_{\substack{z\in G(A^\star,\overline{\lambda}_0)\\ \|z\|_X\le 1}} \inf_{y_n\in X_n} \|z - y_n\|_X\end{aligned}$$

for all $n \ge N_2$. □

Let us define d_n and $d_n^\star$ by

$$d_n = \sup_{\substack{z\in G(A,\lambda_0)\\ \|z\|_X\le 1}} \inf_{x_n\in X_n} \|z - x_n\|_X \quad \text{and} \quad d_n^\star = \sup_{\substack{z\in G(A^\star,\overline{\lambda}_0)\\ \|z\|_X\le 1}} \inf_{y_n\in X_n} \|z - y_n\|_X. \tag{4.67}$$

Lemma 4.1.2 shows that

$$d_n \to 0 \qquad \text{and} \qquad d_n^\star \to 0$$

as $n \to \infty$, since $G(A,\lambda_0)$ and $G(A^\star,\overline{\lambda}_0)$ are finite dimensional subspaces of X, see Lemma 3.2.6. Using the matrix norm

$$\|M\|_{\mathcal{L}(\mathbb{C}^J,\mathbb{C}^J)} := \sup_{1\le i,j\le J} |m^{ij}| \quad \text{for } M \in \mathbb{C}^{J\times J},$$

we get from (4.55) the following convergence result

$$\sup\{\|M(\lambda) - M_n(\lambda)\|_{\mathcal{L}(\mathbb{C}^J,\mathbb{C}^J)} : \lambda \in \Lambda_2\} \le c d_n d_n^\star \to 0 \tag{4.68}$$

as $n \to \infty$. Hence, the sequence $\{M_n\}_{n\in\mathbb{N}}$ of matrix functions is uniformly convergent in every compact subset Λ_2 of Λ_1.

With the estimate (4.68) we are now able to prove an asymptotic error estimate for the eigenvalues of the projected eigenvalue problem (4.16).

Theorem 4.3.6. *Let $A : \Lambda \to \mathcal{L}(X,X)$ be as given in (4.2). Let $\Lambda_c \subset \Lambda$ be a compact set such that $\partial\Lambda_c \subset \rho(A)$ and $\Lambda_c \cap \sigma(A) = \{\lambda_0\}$. Then there exist a constant $C > 0$ and a $N \in \mathbb{N}$ such that*

$$\sigma(P_nA) \cap \Lambda_c \neq \emptyset \tag{4.69}$$

and

$$|\lambda_0^n - \lambda_0| \leq C(d_n d_n^\star)^{1/\varkappa} \quad \text{for all } \lambda_0^n \in \sigma(P_nA) \cap \Lambda_c \tag{4.70}$$

hold for all $n \geq N$, with d_n and $d_n^\star$ given as in (4.67) and $\varkappa = \varkappa(A, \lambda_0)$.

Proof. The assertion, $\sigma(P_nA) \cap \Lambda_c \neq \emptyset$ for sufficiently large n, is a direct consequence of i) of Theorem 4.2.3.

Let us choose a neighborhood $\Lambda_1 \subset \Lambda_c$ of λ_0 such that $M(\lambda)$ and $M_n(\lambda)$ are defined for all $\lambda \in \Lambda_1$ and all $n \geq N_1$. Then, by Corollary 4.3.4 and Theorem 4.2.3 we have

$$\begin{aligned} \{\lambda_0\} = \sigma(A) \cap \Lambda_1 = \sigma(M) \cap \Lambda_1, \\ \sigma(P_nA) \cap \Lambda_C = \sigma(P_nA) \cap \Lambda_1 = \sigma(M_n) \cap \Lambda_1 \end{aligned} \tag{4.71}$$

for all $n \geq N_1$. By Theorem 3.2.14, we can represent the inverse $M(\lambda)^{-1}$ by

$$M(\lambda)^{-1} = \sum_{k=-r}^{\infty} (\lambda - \lambda_0)^k \tilde{M}_k \quad \text{for all } \lambda \in U_{\hat{\delta}}(\lambda_0) \setminus \{\lambda_0\}, \tag{4.72}$$

for sufficiently small $\hat{\delta} > 0$ with $U_{\hat{\delta}}(\lambda_0) \subset \Lambda_1$ and where $r = \varkappa(M, \lambda_0)$.
Choose a $\delta > 0$ with $\overline{U_\delta(\lambda_0)} \subset U_{\hat{\delta}}(\lambda_0)$. By Theorem 4.2.3, there exists a $N_\delta \geq N_1$ such that

$$\sigma(P_nA) \cap \Lambda_c \subset \overline{U_\delta(\lambda_0)} \quad \text{for all } n \geq N_\delta.$$

With (4.71) and $\overline{U_\delta(\lambda_0)} \subset \Lambda_1$ it follows that

$$\sigma(P_nA) \cap \Lambda_c = \sigma(P_nA) \cap \overline{U_\delta(\lambda_0)} = \sigma(M_n) \cap \overline{U_\delta(\lambda_0)} \quad \text{for all } n \geq N_\delta. \tag{4.73}$$

So it is sufficient to consider the eigenvalues of M and M_n in $\overline{U_\delta(\lambda_0)}$.

Using the representation (4.72) of the inverse $M(\lambda)^{-1}$, we can define the matrix function $H : \overline{U_\delta(\lambda_0)} \to \mathcal{L}(\mathbb{C}^J, \mathbb{C}^J)$ by

$$H(\lambda) = (\lambda - \lambda_0)^r M(\lambda)^{-1}.$$

The matrix function H is continuous in $\overline{U_\delta(\lambda_0)}$ and therefore there exists a $c_1 > 0$ such that

$$\|H(\lambda)\|_{\mathcal{L}(\mathbb{C}^J, \mathbb{C}^J)} \leq c_1 \quad \text{for all } \lambda \in \overline{U_\delta(\lambda_0)}. \tag{4.74}$$

Obviously, we have

$$\|[(\lambda - \lambda_0)^{-r} M(\lambda)]^{-1}\|_{\mathcal{L}(\mathbb{C}^J, \mathbb{C}^J)} = \|H(\lambda)\|_{\mathcal{L}(\mathbb{C}^J, \mathbb{C}^J)} \leq c_1$$

for all $\lambda \in \overline{U_\delta(\lambda_0)} \setminus \{\lambda_0\}$. If $\lambda \in \overline{U_\delta(\lambda_0)} \setminus \{\lambda_0\}$ satisfies

$$\|(\lambda-\lambda_0)^{-r}M(\lambda)-(\lambda-\lambda_0)^{-r}M_n(\lambda)\|_{\mathcal{L}(\mathbb{C}^J,\mathbb{C}^J)} < \frac{1}{c_1}, \tag{4.75}$$

then by the Neumann series theorem [21, Lemma 8.3], the matrix function $(\lambda-\lambda_0)^{-r}M_n(\lambda)$ is invertible. Using the estimate (4.68), we have for sufficiently large $n \in \mathbb{N}$,

$$\|(\lambda-\lambda_0)^{-r}[M(\lambda)-M_n(\lambda)]\|_{\mathcal{L}(\mathbb{C}^J,\mathbb{C}^J)} \leq c(\delta)|\lambda-\lambda_0|^{-r}d_n d_n^\star$$

for all $\lambda \in \overline{U_\delta(\lambda_0)} \setminus \{\lambda_0\}$. Hence, for all $\lambda \in \overline{U_\delta(\lambda_0)} \setminus \{\lambda_0\}$ with

$$c(\delta)|\lambda-\lambda_0|^{-r}d_n d_n^\star < \frac{1}{c_1},$$

i.e.,

$$|\lambda-\lambda_0|^r > c_1 c(\delta) d_n d_n^\star,$$

it follows with (4.75) that the inverse $M_n(\lambda)^{-1}$ exists. Consequently, $\lambda_0^n \in \sigma(M_n) \cap \overline{U_\delta(\lambda_0)}$ holds only if

$$|\lambda_0^n-\lambda_0|^r \leq c_1 c(\delta) d_n d_n^\star$$

is satisfied. This gives with (4.73) and $r = \varkappa(M,\lambda_0) = \varkappa(A,\lambda_0)$ the error estimate (4.70). □

In the next theorem we give an asymptotic error estimate for the eigenelements of the projected eigenvalue problem (4.16). The error estimate depends on the error of the eigenvalue of the projected eigenvalue problem and on the approximation property of the trial spaces with respect to the eigenspace.

Theorem 4.3.7. *Let $A : \Lambda \to \mathcal{L}(X,X)$ be as given in (4.2) and let λ_0 be an eigenvalue of A. Let $\{\lambda_0^n\}_{n=n_0}^\infty$ be a sequence of eigenvalues of the projected eigenvalue problems $P_n A(\lambda)x_n = 0$ which converges to λ_0, and let $\{x_n^0\}_{n=n_0}^\infty$ be a sequence of corresponding eigenelements with $x_n^0 \in X_n$ and $\|x_n^0\|_X = 1$. Then there exist a constant $c > 0$ and a $N \in \mathbb{N}$ such that*

$$\inf_{x^0 \in \ker A(\lambda_0)} \|x_n^0 - x^0\|_X \leq c\Big(|\lambda_n^0-\lambda_0| + \sup_{\substack{y^0 \in \ker A(\lambda_0) \\ \|y^0\|_X \leq 1}} \inf_{x_n \in X_n} \|y^0 - x_n\|_X\Big) \tag{4.76}$$

holds for all $n \geq N$.

Proof. Let $\{\lambda_0^n\}_{n=n_0}^\infty$ be a sequence of eigenvalues of the projected eigenvalue problems $P_n A(\lambda)x_n = 0$ which converges to an eigenvalue λ_0 of A, and let $\{x_n^0\}_{n=n_0}^\infty$ be a corresponding sequence of eigenelements with $x_n^0 \in X_n$ and $\|x_n^0\|_X = 1$. Let us first define $\hat{x}^{0,n} \in \ker A(\lambda_0)$ by

$$\|x_n^0 - \hat{x}^{0,n}\|_X = \min_{x \in \ker A(\lambda_0)} \|x_n^0 - x\|_X. \tag{4.77}$$

The above minima are attained, since $\ker A(\lambda_0)$ is a finite dimensional subspace of X. Using Lemma 4.2.1, we have

$$\lim_{n\to\infty} \|x_n^0 - \hat{x}^{0,n}\|_X = 0. \tag{4.78}$$

With the approximation property of $\{X_n\}_{n\in\mathbb{N}}$ we get

$$\|x_n^0 - P_n\hat{x}^{0,n}\|_X \leq \|x_n^0 - \hat{x}^{0,n}\|_X + \|\hat{x}^{0,n} - P_n\hat{x}^{0,n}\|_X \to 0 \tag{4.79}$$

as $n \to \infty$. Let ε_n and $x^{0,n} \in \ker(A,\lambda_0)$ defined by

$$\varepsilon_n = \|x_n^0 - P_n x^{0,n}\|_X = \min_{x\in P_n \ker(A,\lambda_0)} \|x_n^0 - x\|_X. \tag{4.80}$$

The above minima are attained, since $P_n \ker A(\lambda_0)$ is a finite dimensional subspace of X. Let us consider the estimate

$$\inf_{x\in\ker A(\lambda_0)} \|x_n^0 - x\|_X \leq \|x_n^0 - x^{0,n}\|_X \leq \|x_n^0 - P_n x^{0,n}\|_X + \|P_n x^{0,n} - x^{0,n}\|_X. \tag{4.81}$$

Using (4.80) and (4.79) we get

$$\|x_n^0 - P_n x^{0,n}\|_X \leq \|x_n^0 - P_n\hat{x}^{0,n}\|_X \leq \|x_n^0 - \hat{x}^{0,n}\|_X + \|\hat{x}^{0,n} - P_n\hat{x}^{0,n}\|_X \to 0 \tag{4.82}$$

as $n \to \infty$, from which it follows that $\lim_{n\to\infty} \|P_n x^{0,n}\|_X = 1$. Therefore the sequence $\{x^{0,n}\}_{n=n_0}^{\infty}$ is bounded by a constant $c_0 > 0$. Hence, using Lemma 4.1.2 we get the estimate

$$\|P_n x^{0,n} - x^{0,n}\|_X \leq \sup_{\substack{z^0\in\ker A(\lambda_0)\\ \|z^0\|\leq c_0}} \|P_n z^0 - z^0\|_X = c_0 \sup_{\substack{y^0\in\ker A(\lambda_0)\\ \|y^0\|_X\leq 1}} \inf_{x_n\in X_n} \|x_n - y^0\|_X. \tag{4.83}$$

Let us consider now $\|x_n^0 - P_n x^{0,n}\|_X$. First we show that there exists a constant $c_1 > 0$ such that

$$\|x_n^0 - P_n x^{0,n}\|_X \leq c_1 \|P_n A(\lambda_0^n)[x_n^0 - P_n x^{0,n}]\|_X \tag{4.84}$$

holds for sufficiently large $n \in \mathbb{N}$. Assume the contrary, then there exists a subsequence $\{x_{n_k}^0 - P_{n_k} x^{0,n_k}\}_{k\in\mathbb{N}} \subset \{x_n^0 - P_n x^{0,n}\}_{n=n_0}^{\infty}$ such that

$$\|x_{n_k}^0 - P_{n_k} x^{0,n_k}\|_X > \|P_{n_k} A(\lambda_0^{n_k})[x_{n_k}^0 - P_{n_k} x^{0,n_k}]\|_X \tag{4.85}$$

holds for all $k \in \mathbb{N}$. With the continuity of A and with (4.82) we have

$$\|P_{n_k} A(\lambda_0^{n_k})[x_{n_k}^0 - P_{n_k} x^{0,n_k}]\|_X \leq c\|x_{n_k}^0 - P_{n_k} x^{0,n_k}\|_X \to 0$$

as $k \to \infty$. Assumption (4.85) implies then that

$$\lim_{k\to\infty} \left\| P_{n_k} A(\lambda_0^{n_k}) \frac{x_{n_k}^0 - P_{n_k} x^{0,n_k}}{\|x_{n_k}^0 - P_{n_k} x^{0,n_k}\|_X} \right\|_X = 0. \tag{4.86}$$

By Lemma 4.2.1, there exist a subsequence

$$\left\{ \frac{x^0_{\hat{n}_k} - P_{\hat{n}_k} x^{0,\hat{n}_k}}{\|x^0_{\hat{n}_k} - P_{\hat{n}_k} x^{0,\hat{n}_k}\|_X} \right\}_{k\in\mathbb{N}} \subset \left\{ \frac{x^0_{n_k} - P_{n_k} x^{0,n_k}}{\|x^0_{n_k} - P_{n_k} x^{0,n_k}\|_X} \right\}_{k\in\mathbb{N}}$$

and a $y^0 \in \ker A(\lambda_0)$ such that

$$\lim_{k\to\infty} \frac{x^0_{\hat{n}_k} - P_{\hat{n}_k} x^{0,\hat{n}_k}}{\|x^0_{\hat{n}_k} - P_{\hat{n}_k} x^{0,\hat{n}_k}\|_X} = y^0.$$

So we conclude

$$\begin{aligned}
\varepsilon_{\hat{n}_k} &\le \left\| x^0_{\hat{n}_k} - P_{\hat{n}_k} x^{0,\hat{n}_k} - \left\| x^0_{\hat{n}_k} - P_{\hat{n}_k} x^{0,\hat{n}_k} \right\|_X P_{\hat{n}_k} y^0 \right\|_X \\
&\le \left\| x^0_{\hat{n}_k} - P_{\hat{n}_k} x^{0,\hat{n}_k} \right\|_X \left\| \frac{x^0_{\hat{n}_k} - P_{\hat{n}_k} x^{0,\hat{n}_k}}{\|x^0_{\hat{n}_k} - P_{\hat{n}_k} x^{0,\hat{n}_k}\|_X} - P_{\hat{n}_k} y^0 \right\|_X \\
&\le \left\| x^0_{\hat{n}_k} - P_{\hat{n}_k} x^{0\hat{n}_k} \right\|_X \left(\left\| \frac{x^0_{\hat{n}_k} - P_{\hat{n}_k} x^{0,\hat{n}_k}}{\|x^0_{\hat{n}_k} - P_{\hat{n}_k} x^{0,\hat{n}_k}\|_X} - y^0 \right\|_X + \left\| y^0 - P_{\hat{n}_k} y^0 \right\|_X \right) \qquad (4.87) \\
&= \varepsilon_{\hat{n}_k} o(1),
\end{aligned}$$

which is a contradiction. Consequently, estimate (4.84) holds.

Using the continuity of A, there exist a constant $c_2 > 0$ and a $N \in \mathbb{N}$ such that

$$\|P_n A(\lambda_n) - P_n A(\lambda_0)\|_{\mathcal{L}(X,X)} \le c_2 |\lambda_0^n - \lambda_0| \qquad (4.88)$$

for all $n \ge N$. Since $P_n A(\lambda_0^n) x_n^0 = 0$, we can write

$$P_n A(\lambda_0^n)[x_n^0 - P_n x^{0,n}] = [P_n A(\lambda_0) - P_n A(\lambda_0^n)] P_n x^{0,n} - P_n A(\lambda_0) P_n x^{0,n},$$

and with (4.84) and (4.88) we get

$$\begin{aligned}
\left\| x_n^0 - P_n x^{n,0} \right\|_X &\le c_1 \left\{ \left\| [P_n A(\lambda_0) - P_n A(\lambda_0^n)] P_n x^{0,n} \right\|_X + \left\| P_n A(\lambda_0) P_n x^{0,n} \right\|_X \right\} \\
&\le c_1 c_2 \left| \lambda_n - \lambda_0 \right| \left\| P_n x^{0,n} \right\|_X + c_1 \left\| P_n A(\lambda_0) P_n x^{0,n} \right\|_X \qquad (4.89)
\end{aligned}$$

for sufficiently large $n \in \mathbb{N}$. Since $x^{0,n} \in \ker A(\lambda_0)$, we have

$$\begin{aligned}
\left\| P_n A(\lambda_0) P_n x^{0,n} \right\|_X &= \left\| P_n A(\lambda_0) P_n x^{0,n} - P_n A(\lambda_0) x^{0,n} \right\|_X \\
&\le \|A(\lambda_0)\|_{\mathcal{L}(X,X)} \left\| P_n x^{0,n} - x^{0,n} \right\|_X .
\end{aligned}$$

Using that the sequence $\{P_n x^{0,n}\}_{n\in\mathbb{N}}$ is bounded, we get from (4.89)

$$\left\| x_n^0 - P_n x^{n,0} \right\|_X \le c \left(|\lambda_0^n - \lambda_0| + \left\| P_n x^{0,n} - x^{0,n} \right\|_X \right)$$

for sufficiently large n. Hence, we finally obtain from (4.81) with (4.83) the estimate

$$\inf_{x\in\ker A(\lambda_0)} \|x_n^0 - x\|_X \le c \left(|\lambda_0^n - \lambda_0| + \sup_{\substack{y^0\in\ker A(\lambda_0)\\ \|y^0\|_X\le 1}} \inf_{x_n\in X_n} \|y^0 - x_n\|_X \right) \tag{4.90}$$

for sufficiently large n. □

Using the error estimate (4.70) for the eigenvalues of the projected eigenvalue problem, we get from (4.76) the following asymptotic error estimate for the eigenelements

$$\inf_{x\in\ker A(\lambda_0)} \|x_n^0 - x\|_X \le c[(d_n d_n^\star)^{1/\varkappa(A,\lambda_0)} + d_n], \tag{4.91}$$

where we used that the eigenspace $\ker A(\lambda_0)$ is a subset of the generalized eigenspace $G(A,\lambda_0)$.

4.4 Stability of the algebraic multiplicities

Approximations of linear and nonlinear eigenvalue problems affect the geometric and algebraic multiplicities of the eigenvalues [75, 35, 44, 27]. In general, a multiple eigenvalue of the continuous problem splits into several discrete eigenvalues. In this section we show that for eigenvalue problems for holomorphic Fredholm operator functions the algebraic multiplicity is stable under Galerkin discretization, i.e., the algebraic multiplicity of a continuous eigenvalue is equal to the sum of the algebraic multiplicities of its discretizations. For the proof we use again the equivalence of the eigenvalue problems for the operator functions A and P_nA to the eigenvalue problems of the matrix functions M and M_n, see [47]. Another approach for the proof is chosen in [100] where an alternative equivalent characterization of the algebraic multiplicity from [27] is used.

For our approach we need the following essential result which shows how the perturbation of a holomorphic operator function effects the algebraic multiplicities of its eigenvalues.

Theorem 4.4.1. *Let $\Lambda\subset\mathbb{C}$ be open and connected with a simple rectifiable boundary. Let $A:\overline{\Lambda}\to\mathcal{L}(X,Y)$ be holomorphic on Λ and continuous on $\overline{\Lambda}$. Let $\sigma(A)\cap\Lambda=\{\lambda_1,\dots,\lambda_n\}$ and let $\overline{\Lambda}\setminus\{\lambda_1,\dots,\lambda_n\}\subset\rho(A)$. Then there exists a $\delta>0$ such that for each function $B:\overline{\Lambda}\to\mathcal{L}(X,Y)$ which is holomorphic on Λ and continuous on $\overline{\Lambda}$, and which satisfies*

$$\max_{\lambda\in\partial\Lambda} \|B(\lambda)-A(\lambda)\|_{\mathcal{L}(X,Y)} < \delta,$$

it follows that $\sigma(B)\cap\Lambda=\{\mu_1,\dots,\mu_r\}$, $\overline{\Lambda}\setminus\{\mu_1,\dots,\mu_r\}\subset\rho(B)$ and

$$\sum_{k=1}^{n} m(A,\lambda_i) = \sum_{k=1}^{r} m(B,\mu_k).$$

Proof. See [24]. □

Theorem 4.4.2. *Let $A : \Lambda \to \mathcal{L}(X,X)$ be given as in (4.2) and let $\Lambda_c \subset \Lambda$ be compact and connected with a simple rectifiable boundary. Let $\partial\Lambda_c \subset \rho(A)$ and $\Lambda_c \cap \sigma(A) = \{\lambda_0\}$. Then there exists a $N(\Lambda_c) \in \mathbb{N}$ such that for all $n \geq N(\Lambda_c)$ we have*

$$m(A,\lambda_0) = \sum_{\lambda_0 \in \sigma(P_nA)\cap\Lambda_c} m(P_nA,\lambda_0). \tag{4.92}$$

Proof. Consider the matrix functions M and M_n as defined in (4.51). Choose $\varepsilon > 0$ sufficiently small such that the matrix functions M and M_n are defined in $\overline{U_\varepsilon(\lambda_0)}$. For sufficiently large $n \in \mathbb{N}$ we have by Theorem 4.2.3

$$\sigma(P_nA) \cap \Lambda_c = \sigma(P_nA) \cap \overline{U_\varepsilon(\lambda_0)}. \tag{4.93}$$

Corollary 4.3.4 shows that

$$\sigma(A) \cap \overline{U_\varepsilon(\lambda_0)} = \sigma(M) \cap \overline{U_\varepsilon(\lambda_0)} = \{\lambda_0\} \tag{4.94}$$

and that the algebraic multiplicities of λ_0 coincide for A and M. Further, again by Corollary 4.3.4, we have

$$\sigma(M_n) \cap \overline{U_\varepsilon(\lambda_0)} = \sigma(P_nA) \cap \overline{U_\varepsilon(\lambda_0)},$$

and

$$\sum_{\lambda_0 \in \sigma(P_nA)\cap\overline{U_\varepsilon(\lambda_0)}} m(P_nA,\lambda_0) = \sum_{\lambda_0 \in \sigma(M_n)\cap\overline{U_\varepsilon(\lambda_0)}} m(M_n,\lambda_0) \tag{4.95}$$

for sufficiently large $n \in \mathbb{N}$. Using (4.68), we have

$$\max_{\lambda \in \partial U_\varepsilon(\lambda_0)} \|M(\lambda) - M_n(\lambda)\|_{\mathcal{L}(\mathbb{C}^J,\mathbb{C}^J)} \to 0 \quad \text{as } n \to \infty.$$

So we may apply Theorem 4.4.1 and obtain

$$m(M,\lambda_0) = \sum_{\lambda_0 \in \sigma(M_n)\cap\overline{U_\varepsilon(\lambda_0)}} m(M_n,\lambda_0)$$

for sufficiently large $n \in \mathbb{N}$. From (4.95), (4.94) and (4.93) the assertion follows. □

5 GALERKIN APPROXIMATION OF BOUNDARY INTEGRAL OPERATOR EIGENVALUE PROBLEMS

Discretizations of boundary integral formulations of Laplacian eigenvalue problems by using boundary elements are considered in many works, see [23, 46, 51, 52, 3, 14, 84] and references therein. However, only in a few works [22, 23, 84] the issue of the numerical analysis is addressed. To our knowledge, a rigorous numerical analysis of the discretizations of boundary integral operator eigenvalue problems including error estimates for the eigenvalues and eigenelements has not be done so far.

In this chapter we show that the boundary integral formulations of the Dirichlet and Neumann Laplacian eigenvalue problem which we derived in Chapter 2 are eigenvalue problems for holomorphic Fredholm operator functions. Therefore we can apply the results of Chapter 4 to Galerkin boundary element discretizations of the boundary integral operator eigenvalue problems. We prove the convergence of the boundary element approximations for the eigenvalues and eigenelements and give asymptotic error estimates. Furthermore, we show that the algebraic multiplicity of the eigenvalues are stable under Galerkin discretizations.

5.1 Properties of boundary integral operator eigenvalue problems

First we consider the boundary integral formulation of the Dirichlet Laplacian eigenvalue problem (2.51): Find $(\kappa, w) \in \mathbb{R}_+ \times H^{-1/2}(\Gamma) \setminus \{0\}$ such that

$$V(\kappa)w = 0. \tag{5.1}$$

By Theorem 2.4.7, the single layer potential operator $V(\kappa)$ is Fredholm for all $\kappa \in \mathbb{C}$. Next we show that $V(\cdot)$ defines a holomorphic operator function.

Lemma 5.1.1. *The operator function*

$$\begin{aligned} V : \mathbb{C} &\to \mathcal{L}(H^{-1/2}(\Gamma), H^{1/2}(\Gamma)), \\ \kappa &\mapsto V(\kappa), \end{aligned}$$

where $V(\kappa)$ is the single layer potential operator as given in (2.36), is holomorphic.

Proof. According to Corollary 3.1.3 it is sufficient to show that the function

$$f_{t,w}(\kappa) := \langle V(\kappa)t, w\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)}$$

is holomorphic on $\mathbb{C}$ for every $t, w \in H^{-1/2}(\Gamma)$. For $\kappa \in \mathbb{C}$ we can write

$$\begin{aligned} f_{t,w}(\kappa) &= \langle V(\kappa)t, w\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} = \langle \frac{1}{4\pi}\int_\Gamma \frac{e^{i\kappa|\cdot - y|}}{|\cdot - y|} t(y)ds_y, w\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} \\ &= \sum_{n=0}^{\infty} \kappa^n \langle \frac{1}{4\pi}\int_\Gamma \frac{i^n|\cdot - y|^{n-1}}{n!} t(y)ds_y, w\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)}. \end{aligned} \tag{5.2}$$

Here we used that the operator $A_n : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ defined by

$$(A_n t)(x) := \frac{1}{4\pi}\int_\Gamma \frac{i^n|x-y|^{n-1}}{n!} t(y)ds_y$$

is linear and bounded for every $n \in \mathbb{N}_0$. This property of A_n can be shown for $n \geq 1$ in a similar way as it is done for the case $n = 0$ in [83, Chapter 6], since the kernel of A_n for $n \geq 1$ is more regular than the kernel of A_0. Note that A_0 is the single layer potential operator of the Laplace equation.
The representation (5.2) of the function $f_{t,w}$ shows that $f_{t,w} : \mathbb{C} \to \mathbb{C}$ is holomorphic for every $t, w \in H^{-1/2}(\Gamma)$ and we conclude with Corollary 3.1.3 that the operator function $V : \mathbb{C} \to \mathcal{L}(H^{-1/2}(\Gamma), H^{1/2}(\Gamma))$ is holomorphic. □

Next we consider the boundary integral formulation of the Neumann Laplacian eigenvalue problem (2.53): Find $(\kappa, u) \in \mathbb{R}_+ \times H^{1/2}(\Gamma) \setminus \{0\}$ such that

$$D(\kappa)u = 0. \tag{5.3}$$

Lemma 5.1.2. *The operator function*

$$\begin{aligned} D : \mathbb{C} &\to \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma)), \\ \kappa &\mapsto D(\kappa), \end{aligned}$$

where $D(\kappa)$ is the hypersingular boundary integral operator as given in (2.37), is holomorphic.

Proof. The proof is done in an analogous manner as in the case of the single layer operator. We show that the function

$$g_{u,v}(\kappa) := \langle D(\kappa)u, v\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \tag{5.4}$$

is holomorphic on $\mathbb{C}$ for every $u, v \in H^{1/2}(\Gamma)$. For $\kappa \in \mathbb{C}$ we can write

$$g_{u,v}(\kappa) = \langle D(\kappa)u, v\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} = -\langle \gamma_{1,\cdot}^{\text{int}} \frac{1}{4\pi} \int_\Gamma \gamma_{1,y}^{\text{int}} \frac{e^{i\kappa|\cdot - y|}}{|\cdot - y|} u(y) ds_y, v\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}$$

$$= -\sum_{n=0}^{\infty} \kappa^n \langle \gamma_{1,\cdot}^{\text{int}} \frac{1}{4\pi} \int_\Gamma \gamma_{1,y}^{\text{int}} \frac{i^n |\cdot - y|^{n-1}}{n!} u(y) ds_y, v\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}. \tag{5.5}$$

Here we used that the trace operator γ_1^{int} is linear and that $B_n : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ defined by

$$(B_n u)(x) := \frac{1}{4\pi} \int_\Gamma \gamma_{1,y}^{\text{int}} \frac{i^n |x-y|^{n-1}}{n!} u(y) ds_y \tag{5.6}$$

is linear and bounded for every $n \in \mathbb{N}_0$. The operator B_0 is the hypersingular operator of the Laplace equation. In [83, Chapter 6] it is proven that $B_0 \in \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$. In a similar way this can be shown for B_n for $n \geq 1$, since the kernel of B_n for $n \geq 1$ is more regular than the kernel of B_0.
Because $g_{u,v} : \mathbb{C} \to \mathbb{C}$ is holomorphic for every $u, v \in H^{1/2}(\Gamma)$, it follows by Corollary 3.1.3 that the operator function $D : \mathbb{C} \to \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$ is also holomorphic. □

In order to apply the results of Chapter 4 to the eigenvalue problems (5.1) and (5.3), we have to introduce additional operators such that we get eigenvalue problems of the required form as in (4.2). Consider the Riesz map $J : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, then the operator $\imath_{H^{1/2}(\Gamma)} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ defined by

$$\imath_{H^{1/2}(\Gamma)} v := \overline{Jv} \quad \text{for } v \in H^{1/2}(\Gamma)$$

is an isomorphism, see Section 3.3. Recalling the definition (2.3) of the sesquilinear form

$$(u, w)_\Gamma = \langle u, \overline{w}\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)},$$

we can write

$$(u, \imath_{H^{1/2}(\Gamma)} v)_\Gamma = \langle u, \overline{\imath_{H^{1/2}(\Gamma)} v}\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} = \langle u, Jv\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} = (u, v)_{H^{1/2}(\Gamma)}$$

for all $u, v \in H^{1/2}(\Gamma)$. Consequently, we have

$$(u, w)_\Gamma = (u, \imath_{H^{1/2}(\Gamma)} \imath_{H^{1/2}(\Gamma)}^{-1} w)_\Gamma = (u, \imath_{H^{1/2}(\Gamma)}^{-1} w)_{H^{1/2}(\Gamma)} \tag{5.7}$$

for all $u \in H^{1/2}(\Gamma)$ and $w \in H^{-1/2}(\Gamma)$. Using the Hilbert space adjoint $[\imath_{H^{1/2}(\Gamma)}^{-1}]^\star$ we get

$$(u, \imath_{H^{1/2}(\Gamma)}^{-1} w)_{H^{1/2}(\Gamma)} = ([\imath_{H^{1/2}(\Gamma)}^{-1}]^\star u, w)_{H^{-1/2}(\Gamma)} \quad \text{for all } u \in H^{1/2}(\Gamma), w \in H^{-1/2}(\Gamma). \tag{5.8}$$

Define

$$\mathcal{I} := [\iota^{-1}_{H^{1/2}(\Gamma)}]^\star : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma), \tag{5.9}$$

then we can write using (5.7) and (5.8)

$$(u,w)_\Gamma = (\mathcal{I}u,w)_{H^{-1/2}(\Gamma)} \quad \text{for all } u \in H^{1/2}(\Gamma),\, w \in H^{-1/2}(\Gamma). \tag{5.10}$$

Note that $\mathcal{I} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is an isomorphism. Finally, using (5.10) we have the representations

$$(V(\kappa)t,w)_\Gamma = (\mathcal{I}V(\kappa)t,w)_{H^{-1/2}(\Gamma)} \quad \text{for all } t,w \in H^{-1/2}(\Gamma), \tag{5.11}$$

and

$$(u,D(\kappa)v)_\Gamma = (u,\mathcal{I}^\star D(\kappa)v)_{H^{1/2}(\Gamma)} \quad \text{for all } u,v \in H^{1/2}(\Gamma). \tag{5.12}$$

Theorem 5.1.3. *Consider the operator function*

$$\begin{aligned} \mathcal{I}V : \mathbb{C} &\to \mathcal{L}(H^{-1/2}(\Gamma),H^{-1/2}(\Gamma)),\\ \kappa &\mapsto \mathcal{I}V(\kappa), \end{aligned} \tag{5.13}$$

where $\mathcal{I} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is given as in (5.9). Then:

i) *The operator function $\mathcal{I}V : \mathbb{C} \to \mathcal{L}(H^{-1/2}(\Gamma),H^{-1/2}(\Gamma))$ is holomorphic and the operator $\mathcal{I}V(\kappa)$ is a compact perturbation of the $H^{-1/2}(\Gamma)$-elliptic operator $\mathcal{I}V(0)$ for all $\kappa \in \mathbb{C}$.*

ii) *The spectra of V and $\mathcal{I}V$ coincide and*

$$\ker V(\kappa) = \ker \mathcal{I}V(\kappa)$$

for any $\kappa \in \mathbb{C}$. Further, for any eigenvalue $\kappa \in \sigma(V)$ the maximal length of a Jordan chain and the algebraic multiplicity are equal for V and $\mathcal{I}V$,

$$\varkappa(V,\kappa) = \varkappa(\mathcal{I}V,\kappa), \qquad m(V,\kappa) = m(\mathcal{I}V,\kappa).$$

Proof. i) The holomorphy of $\mathcal{I}V$ follows directly from the holomorphy of V. Next, we can write

$$\mathcal{I}V(\kappa) = \mathcal{I}V(0) + \mathcal{I}(V(\kappa) - V(0)),$$

where $\mathcal{I}(V(\kappa)-V(0)) : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is compact, since, by Lemma 2.4.5, the operator $V(\kappa)-V(0) : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is compact. The $H^{-1/2}(\Gamma)$-ellipticity of $\mathcal{I}V(0)$ follows from the ellipticity of $V(0)$, see Lemma 2.4.6. With (5.11) we have

$$(\mathcal{I}V(0)t,t)_{H^{-1/2}(\Gamma)} = (V(0)t,t)_\Gamma \geq C_V \|t\|^2_{H^{-1/2}(\Gamma)} \quad \text{for } t \in H^{-1/2}(\Gamma).$$

ii) The assertions are a direct consequence of the fact that $\mathcal{I} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is an isomorphism. □

Also the eigenvalue problems for the operator functions $D : \mathbb{C} \to \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$ and $\mathcal{I}^\star D : \mathbb{C} \to \mathcal{L}(H^{1/2}(\Gamma), H^{1/2}(\Gamma))$ are equivalent.

Theorem 5.1.4. *Consider the operator function*

$$\begin{aligned} \mathcal{I}^\star D : \mathbb{C} &\to \mathcal{L}(H^{1/2}(\Gamma), H^{1/2}(\Gamma)), \\ \kappa &\mapsto \mathcal{I}^\star D(\kappa), \end{aligned} \tag{5.14}$$

where $\mathcal{I}^\star : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is given as in (5.12). Then:

i) *The operator function $\mathcal{I}^\star D : \mathbb{C} \to \mathcal{L}(H^{1/2}(\Gamma), H^{1/2}(\Gamma))$ is holomorphic and for any $\kappa \in \mathbb{C}$ the operator $\mathcal{I}^\star D(\kappa)$ is a compact perturbation of the $H^{1/2}(\Gamma)$-elliptic operator $\mathcal{I}^\star \widetilde{D}(0)$, where $\widetilde{D}(0)$ is given as in (2.56).*

ii) *The spectra of D and $\mathcal{I}^\star D$ coincide and*

$$\ker D(\kappa) = \ker \mathcal{I}^\star D(\kappa)$$

for any $\kappa \in \mathbb{C}$. Further, for any eigenvalue $\kappa \in \sigma(D)$ the maximal length of a Jordan chain and the algebraic multiplicity are equal for D and $\mathcal{I}^\star D$,

$$\varkappa(D, \kappa) = \varkappa(\mathcal{I}^\star D, \kappa), \qquad m(D, \kappa) = m(\mathcal{I}^\star D, \kappa).$$

Proof. The proof can be done in a similar way as for Theorem 5.1.3. □

For the error estimates of the Galerkin approximations of the boundary integral operator eigenvalue problems we have to consider the Hilbert space adjoint operators of $\mathcal{I}V(\kappa)$ and $\mathcal{I}^\star D(\kappa)$.

Lemma 5.1.5. *Let $\kappa \in \mathbb{R}$, then*

$$[\mathcal{I}V(\kappa)]^\star = \mathcal{I}V(-\kappa).$$

Proof. Let $\kappa \in \mathbb{R}$ and $t, w \in H^{-1/2}(\Gamma)$, then

$$\begin{aligned} (\mathcal{I}V(\kappa)w, t)_{H^{-1/2}(\Gamma)} &= \langle V(\kappa)w, \overline{t} \rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} = \frac{1}{4\pi} \int_\Gamma \int_\Gamma \frac{e^{i\kappa|x-y|}}{|x-y|} w(y) ds_y \overline{t(x)} ds_x \\ &= \frac{1}{4\pi} \int_\Gamma w(y) \int_\Gamma \overline{\frac{e^{-i\kappa|x-y|}}{|x-y|} t(x)} ds_x ds_y = \langle w, \overline{V(-\kappa)t} \rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \\ &= \overline{\langle V(-\kappa)t, \overline{w} \rangle}_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} = \overline{(V(-\kappa)t, w)}_\Gamma \\ &= \overline{(\mathcal{I}V(-\kappa)t, w)}_{H^{-1/2}(\Gamma)} = (w, \mathcal{I}V(-\kappa)t)_{H^{-1/2}(\Gamma)}. \end{aligned}$$

Hence, $[\mathcal{I}V(\kappa)]^\star = \mathcal{I}V(-\kappa)$. □

Next we consider the adjoint of the hypersingular operator $D(\kappa)$. The following representation of the duality pairing of the hypersingular operator holds for piecewise smooth functions $u, v \in H^{1/2}(\Gamma) \cap C(\Gamma)$,

$$\langle D(\kappa)u, v\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} = \frac{1}{4\pi}\int_\Gamma\int_\Gamma \frac{e^{i\kappa|x-y|}}{|x-y|}(\underline{\mathrm{curl}}_\Gamma u(y), \underline{\mathrm{curl}}_\Gamma v(y))ds_y ds_x - \kappa^2\frac{1}{4\pi}\int_\Gamma\int_\Gamma \frac{e^{i\kappa|x-y|}}{|x-y|}u(y)v(x)(\underline{n}(x), \underline{n}(y))ds_y ds_x, \quad (5.15)$$

see [66, Theorem 3.4.2], where

$$\underline{\mathrm{curl}}_\Gamma u(x) = \underline{n}(x) \times \nabla\tilde{u}(x) \quad \text{for } x \in \Gamma,$$

and where $\underline{n}$ is the outward unit normal vector and $\tilde{u}$ is some (locally defined) extension of u into the neighborhood of Γ.

Lemma 5.1.6. *Let $\kappa \in \mathbb{R}$, then*

$$[\mathcal{I}^\star D(\kappa)]^\star = \mathcal{I}^\star D(-\kappa).$$

Proof. Using the representation (5.15), we get for $\kappa \in \mathbb{R}$ and piecewise smooth functions $u, v \in H^{1/2}(\Gamma) \cap C(\Gamma)$

$$\begin{aligned}
(v, \mathcal{I}^\star D(\kappa)u)_{H^{1/2}(\Gamma)} &= (v, D(\kappa)u)_\Gamma = \langle v, \overline{D(\kappa)u}\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} \\
&= \langle \overline{D(\kappa)u}, v\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} = \langle D(-\kappa)v, \overline{u}\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \\
&= \overline{\langle u, \overline{D(-\kappa)v}\rangle}_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} = \overline{(u, D(-\kappa)v)}_\Gamma \\
&= \overline{(u, \mathcal{I}^\star D(-\kappa)v)}_{H^{1/2}(\Gamma)} = (\mathcal{I}^\star D(-\kappa)v, u)_{H^{1/2}(\Gamma)}.
\end{aligned}$$

Thus, $[\mathcal{I}^\star D(\kappa)]^\star = \mathcal{I}^\star D(-\kappa)$. □

5.2 Boundary elements

Recall that we have assumed that $\Omega \subset \mathbb{R}^3$ is a Lipschitz domain with piecewise smooth boundary $\Gamma = \partial\Omega$. We consider a family $\{\Gamma_h\}$ of decompositions of the boundary Γ,

$$\Gamma_h = \bigcup_{\ell=1}^{n_h} \overline{\tau}_\ell, \quad (5.16)$$

with boundary elements τ_ℓ. We restrict ourselves to plane triangles for the choice of the boundary elements. The errors which may occur by this approximation of the boundary

Γ are not considered here. For an analysis of these errors, see [65]. For each boundary decomposition (5.16) we assume that two neighboring boundary elements share either a node or an edge. We define the local mesh size of a boundary element τ_ℓ

$$h_\ell := \left(\int_{\tau_\ell} ds_x \right)^{1/2}$$

and the global mesh sizes of a boundary decomposition Γ_h

$$h = h_{\max} := \max_{\ell=1,\ldots,n_h} h_\ell, \qquad h_{\min} := \min_{\ell=1,\ldots,n_h} h_\ell.$$

The diameter of a boundary element τ_ℓ is defined by

$$d_\ell := \sup_{x,y\in\tau_\ell} |x-y|.$$

We assume that the family $\{\Gamma_h\}$ is uniformly shape regular, that is, there exists a constant $c > 0$ which is independent of the boundary decomposition such that

$$d_\ell \le c h_\ell \quad \text{for all } \ell = 1,\ldots,n_h.$$

For the Galerkin discretization of the boundary integral operator eigenvalue problems we consider finite dimensional trial spaces with respect to the boundary decompositions Γ_h. A conforming trial space of $H^{-1/2}(\Gamma)$ is $S_h^0(\Gamma)$, the space of piecewise constant functions. We use $\{\psi_\ell^h\}_{\ell=1}^{n_h}$ as basis functions of $S_h^0(\Gamma)$ with respect to the boundary decomposition Γ_h, where ψ_ℓ^h is constant one on the boundary element τ_ℓ and elsewhere zero. The space $S_h^1(\Gamma)$ of continuous piecewise linear functions is a conforming trial space of $H^{1/2}(\Gamma)$. We use nodal basis functions $\{\varphi_j^h\}_{j=1}^{m_h}$ for $S_h^1(\Gamma)$, where the vertices of the boundary decomposition Γ_h are the nodes. Let $\{x_j\}_{j=1}^{m_h}$ be the set of vertices of Γ_h, then the basis functions of $S_h^1(\Gamma)$ are given by

$$\varphi_j^h(x) = \begin{cases} 1 & \text{for } x = x_j, \\ 0 & \text{for } x = x_i \neq x_j, \\ \text{piecewise linear} & \text{elsewhere} \end{cases}$$

for $j = 1,\ldots,m_h$.

The trial spaces $S_h^0(\Gamma)$ and $S_h^1(\Gamma)$ have the following approximation properties.

Theorem 5.2.1. *Let $\eta \in \{0,1\}$, $\sigma \in [-1+\eta,\eta]$, and $s \in [\sigma,\eta+1]$. Then there exists a constant $c > 0$ such that for any $v \in H^s(\Gamma)$*

$$\inf_{v_h \in S_h^\eta(\Gamma)} \|v - v_h\|_{H^\sigma(\Gamma)} \le c h^{s-\sigma} \|v\|_{H^s(\Gamma)}. \tag{5.17}$$

Proof. See, e.g., [78, p. 252], [83, Section 10.2]. □

Similar approximation results are also valid for open parts Γ_i of Γ. For a piecewise smooth boundary we have for $v \in H^s_{\mathrm{pw}}(\Gamma)$

$$\inf_{v_h \in S_h^\eta(\Gamma)} \|v - v_h\|_{H^{\eta-1/2}(\Gamma)} \leq c h^{s-\eta+1/2} \|v\|_{H^s_{\mathrm{pw}}(\Gamma)}, \tag{5.18}$$

where $\eta \in \{0,1\}$ and $s \in [\eta - 1/2, \eta + 1]$, see [70, Theorem 2.1, Theorem 2.3].

5.3 Convergence, asymptotic error estimates and stability

Dirichlet Laplacian eigenvalue problem

The boundary integral formulation (2.51) of the Dirichlet Laplacian eigenvalue problem is by Theorem 5.1.3 equivalent to the boundary integral operator eigenvalue problem:
Find $(\kappa, w) \in \mathbb{R}_+ \times H^{-1/2}(\Gamma) \setminus \{0\}$ such that

$$\mathcal{I}V(\kappa)w = 0. \tag{5.19}$$

Using a family of finite dimensional subspaces $S_h^0(\Gamma)$ spanned by piecewise constant basis functions $\{\psi_\ell^h\}_{\ell=1}^{n_h}$, the Galerkin variational eigenvalue problem reads as follows: Find $(\kappa_h, w_h) \in \mathbb{C} \times S_h^0(\Gamma) \setminus \{0\}$ such that

$$(\mathcal{I}V(\kappa_h)w_h, v_h)_{H^{-1/2}(\Gamma)} = (V(\kappa_h)w_h, v_h)_\Gamma = 0 \tag{5.20}$$

is satisfied for all $v_h \in S_h^0(\Gamma)$. Setting

$$w_h = \sum_{\ell=1}^{n_h} w_\ell \psi_\ell^h,$$

then the variational problem (5.20) is equivalent to the algebraic nonlinear eigenvalue problem: Find $(\kappa_h, \underline{w}) \in \mathbb{C} \times \mathbb{C}^{n_h} \setminus \{\underline{0}\}$ such that

$$V_h(\kappa_h)\underline{w} = 0, \tag{5.21}$$

where

$$V_h(\kappa_h)[k,\ell] := \frac{1}{4\pi} \int_{\tau_\ell} \int_{\tau_k} \frac{e^{i\kappa_h|x-y|}}{|x-y|} ds_y ds_x$$

for $k, \ell = 1, \ldots, n_h$.

The following theorem shows the convergence of the Galerkin approximations (κ_h, w_h) to an eigenpair (κ, w) of the continuous eigenvalue problem (5.19).

Theorem 5.3.1.

i) *Let $\{(\kappa_h, w_h)\}$ be a sequence of eigenpairs of the Galerkin variational problem (5.20). If*

$$\lim_{h\to 0} \kappa_h = \kappa,$$

then κ is an eigenvalue of $\mathcal{I}V$.

ii) *For each eigenvalue κ of $\mathcal{I}V$ there exists a sequence of eigenpairs $\{(\kappa_h, w_h)\}$ of the Galerkin variational problem (5.20) with $\|w_h\|_{H^{-1/2}(\Gamma)} = 1$ such that*

$$\lim_{h\to 0} |\kappa_h - \kappa| = 0$$

and

$$\lim_{h\to 0} \inf_{w\in \ker \mathcal{I}V(\kappa)} \|w - w_h\|_{H^{-1/2}(\Gamma)} = 0.$$

Proof. We show that the assumptions of Theorem 4.2.3 are fulfilled. By Theorem 5.1.3, $\mathcal{I}V : \mathbb{C} \to \mathcal{L}(H^{-1/2}(\Gamma), H^{-1/2}(\Gamma))$ is a holomorphic operator function and for any $\kappa \in \mathbb{C}$ the operator $\mathcal{I}V(\kappa)$ is a compact perturbation of a $H^{-1/2}(\Gamma)$-elliptic operator. Further, Theorem 5.2.1 shows that the family of trial spaces $\{S_h^0(\Gamma)\}$ approximates $H^{-1/2}(\Gamma)$,

$$\lim_{h\to 0} \inf_{v_h \in S_h^0(\Gamma)} \|u - v_h\|_{H^{-1/2}(\Gamma)} \to 0 \quad \text{for all } u \in H^{-1/2}(\Gamma).$$

Thus, the assertions follow from Theorem 4.2.3. □

Next we give an error estimate for the discretizations (κ_h, w_h).

Theorem 5.3.2. *Let $\kappa \in \sigma(\mathcal{I}V) \cap \mathbb{R}$ and let $\sigma(\mathcal{I}V) \cap U_\delta(\kappa) = \{\kappa\}$, where*

$$U_\delta(\kappa) = \{\mu \in \mathbb{C} : |\kappa - \mu| < \delta\}.$$

Then there exists a $h_0 > 0$ such that for all $h \in (0, h_0)$

$$|\kappa - \kappa_h| \le c d_h^{2/\varkappa(\mathcal{I}V,\kappa)} \quad \text{for all } \kappa_h \in \sigma(V_h) \cap U_\delta(\kappa) \tag{5.22}$$

is satisfied, where

$$d_h = \sup_{\substack{t\in G(\mathcal{I}V,\kappa) \\ \|t\|_{H^{-1/2}(\Gamma)}\le 1}} \inf_{t_h \in S_h^0(\Gamma)} \|t - t_h\|_{H^{-1/2}(\Gamma)},$$

and where $\varkappa(\mathcal{I}V, \kappa)$ is the maximal length of a Jordan chain and $G(\mathcal{I}V, \kappa)$ is the generalized eigenspace of κ as defined in Definition 3.2.5. Further, for any $w_h \in \ker V_h(\kappa_h)$ with $\|w_h\|_{H^{-1/2}(\Gamma)} = 1$,

$$\inf_{w\in \ker \mathcal{I}V(\kappa)} \|w - w_h\|_{H^{-1/2}(\Gamma)} \le c\,(|\kappa - \kappa_h| + d_h) \tag{5.23}$$

is satisfied.

Proof. The error estimate (5.23) for the eigenelements follows immediately from Theorem 4.3.7. For the error estimate (5.22) of the eigenvalues we have to consider the adjoint of $\mathcal{I}V(\kappa)$. Let $\kappa \in \sigma(\mathcal{I}V)$ be real, then Lemma 5.1.5 shows that

$$[\mathcal{I}V(\kappa)]^\star = \mathcal{I}V(-\kappa).$$

Since $\ker \mathcal{I}V(\kappa) = \ker \mathcal{I}V(-\kappa)$, we conclude

$$\ker \mathcal{I}V(\kappa) = \ker[\mathcal{I}V(\kappa)]^\star = \ker[\mathcal{I}V]^\star(\overline{\kappa})$$

and

$$G(\mathcal{I}V,\kappa) = G(\mathcal{I}V,-\kappa) = G([\mathcal{I}V]^\star,\overline{\kappa}). \tag{5.24}$$

Applying Theorem 4.3.6 the error estimate (5.22) follows with (5.24). □

Using the approximation property of $S_h^0(\Gamma)$, we can give the following error estimates.

Corollary 5.3.3. *Let $\kappa \in \sigma(\mathcal{I}V) \cap \mathbb{R}$ and let $\sigma(\mathcal{I}V) \cap U_\delta(\kappa) = \{\kappa\}$. Let $k = \dim G(\mathcal{I}V,\kappa)$ and let $\{t_1,\ldots,t_k\}$ be an orthonormal basis of the generalized eigenspace $G(\mathcal{I}V,\kappa)$. Assume that $\{t_i\}_{i=1}^k \subset H^s_{\mathrm{pw}}(\Gamma)$ for some $s \in [-1/2,1]$, then there exists a $h_0 > 0$ such that for all $h \in (0,h_0)$*

$$|\kappa - \kappa_h| \le c(h^{2s+1})^{1/\varkappa(\mathcal{I}V,\kappa)} \sum_{i=1}^k \|t_i\|_{H^s_{\mathrm{pw}}(\Gamma)} \quad \text{for all } \kappa_h \in \sigma(V_h) \cap U_\delta(\kappa) \tag{5.25}$$

is satisfied. Further, for any $w_h \in \ker V_h(\kappa_h)$ with $\|w_h\|_{H^{-1/2}(\Gamma)} = 1$

$$\inf_{w \in \ker V(\kappa)} \|w - w_h\|_{H^{-1/2}(\Gamma)} \le c\left(|\kappa - \kappa_h| + h^{s+1/2} \sum_{i=1}^k \|t_i\|_{H^s_{\mathrm{pw}}(\Gamma)}\right) \tag{5.26}$$

is satisfied.

Proof. Let $\kappa \in \sigma(\mathcal{I}V)$ be real and let $\{t_i\}_{i=1}^k$ be an orthonormal basis of the generalized eigenspace $G(\mathcal{I}V,\kappa)$. Assume that $\{t_i\}_{i=1}^k \subset H^s_{\mathrm{pw}}(\Gamma)$ for some $s \in [-1/2,1]$. We show that

$$\sup_{\substack{t \in G(V,\kappa) \\ \|t\|_{H^{-1/2}(\Gamma)} \le 1}} \inf_{t_h \in S_h^0(\Gamma)} \|t - t_h\|_{H^{-1/2}(\Gamma)} \le c h^{s+1/2} \sum_{i=1}^k \|t_i\|_{H^s_{\mathrm{pw}}(\Gamma)}. \tag{5.27}$$

Since $\{t_i\}_{i=1}^k \subset H^s_{\mathrm{pw}}(\Gamma)$ and $S_h^0(\Gamma)$ is a finite dimensional subspace of $H^{-1/2}(\Gamma)$, there exists an element $t_{i,h} \in S_h^0(\Gamma)$ and a constant $c_i > 0$ such that

$$\|t_i - t_{i,h}\|_{H^{-1/2}(\Gamma)} = \min_{t_h \in S_h^0(\Gamma)} \|t_i - t_h\|_{H^{-1/2}(\Gamma)} \le c_i h^{s+1/2} \|t_i\|_{H^s_{\mathrm{pw}}(\Gamma)}$$

for $i = 1, \dots, k$, see (5.18). Let $t \in G(V, \kappa)$ with $\|t\|_{H^{-1/2}(\Gamma)} \leq 1$, then t admits a representation

$$t = \sum_{i=1}^{k} \alpha_i t_i, \quad |\alpha_i| \leq 1. \tag{5.28}$$

Set $\tilde{t}_h = \sum_{i=1}^{k} \alpha_i t_{i,h}$, then we obtain

$$\begin{aligned} \inf_{t_h \in S_h^0(\Gamma)} \|t - t_h\|_{H^{-1/2}(\Gamma)} &\leq \|t - \tilde{t}_h\|_{H^{-1/2}(\Gamma)} = \|\sum_{i=1}^{k} \alpha_i (t_i - t_{i,h})\|_{H^{-1/2}(\Gamma)} \\ &\leq \sum_{i=1}^{k} c_i h^{s+1/2} \|t_i\|_{H^s_{\mathrm{pw}}(\Gamma)} \leq c h^{s+1/2} \sum_{i=1}^{k} \|t_i\|_{H^s_{\mathrm{pw}}(\Gamma)}, \end{aligned}$$

which shows that (5.27) holds. Hence, the error estimates (5.25) and (5.26) follow from (5.22) and (5.23). □

In the case $\varkappa(\mathcal{I}V, \kappa) = 1$, i.e., the algebraic multiplicity of κ is equal to its geometric multiplicity, the generalized eigenspace $G(\mathcal{I}V, \kappa)$ coincides with $\ker \mathcal{I}V(\kappa)$. If in addition $\ker \mathcal{I}V(\kappa) \subset H^1_{\mathrm{pw}}(\Gamma)$, then we get with Corollary 5.3.3 the following error estimates for the Galerkin approximations (κ_h, w_h),

$$|\kappa - \kappa_h| \leq c h^3 \sum_{i=0}^{k} \|t_i\|_{H^1_{\mathrm{pw}}(\Gamma)}, \tag{5.29}$$

$$\inf_{w \in \ker V(\kappa)} \|w - w_h\|_{H^{-1/2}(\Gamma)} \leq c \left(|\kappa - \kappa_h| + h^{3/2} \sum_{i=0}^{k} \|t_i\|_{H^1_{\mathrm{pw}}(\Gamma)} \right),$$

where $\{t_1, \dots, t_k\}$ is some orthonormal basis of $\ker \mathcal{I}V(\kappa)$ in $H^{-1/2}(\Gamma)$.

Next we show that the algebraic multiplicity of an eigenvalue of $\mathcal{I}V$ is stable under the Galerkin discretization (5.20), that is, the sum of the algebraic multiplicities of the discrete eigenvalues corresponding to a continuous eigenvalue κ is equal to the algebraic multiplicity of κ.

Theorem 5.3.4. *Let $\kappa \in \sigma(\mathcal{I}V) \cap \mathbb{R}$ and let $\sigma(\mathcal{I}V) \cap U_\delta(\kappa) = \{\kappa\}$. Let $m(\mathcal{I}V, \kappa)$ the algebraic multiplicity of κ. Then there exists a $h_0 > 0$ such that for all $h < h_0$*

$$m(\mathcal{I}V, \kappa) = \sum_{\kappa_h \in \sigma(V_h) \cap U_\delta(\kappa)} m(V_h, \kappa_h).$$

If $\varkappa(\mathcal{I}V, \kappa) = 1$, then we have for all $h < h_0$

$$\dim \ker \mathcal{I}V(\kappa) = \dim\{t_h \in H^{-1/2}(\Gamma) : V(\kappa_h) t_h = 0 \text{ and } \kappa_h \in \sigma(V_h) \cap U_\delta(\kappa_h)\}.$$

Proof. The assertions follow immediately from Theorem 4.4.2. □

Neumann Laplacian eigenvalue problem

The boundary integral formulation (2.53) of the Neumann Laplacian eigenvalue problem is by Theorem 5.1.4 equivalent to the boundary integral operator eigenvalue problem: Find $(\kappa,u)\in\mathbb{R}_+\times H^{1/2}(\Gamma)\setminus\{0\}$ such that

$$\mathcal{I}^\star D(\kappa)u=0. \tag{5.30}$$

Using a sequence of finite dimensional subsaces $S_h^1(\Gamma)$ spanned by continuous piecewise linear basis functions $\{\varphi_k^h\}_{k=1}^{m_h}$, the Galerkin variational formulation reads as follows: Find $(\kappa_h,u_h)\in\mathbb{C}\times S_h^1(\Gamma)\setminus\{0\}$ such that

$$(v_h,\mathcal{I}^\star D(\kappa)u_h)_{H^{1/2}(\Gamma)}=(v_h,D(\kappa)u_h)_\Gamma=0 \tag{5.31}$$

is satisfied for all $v_h\in S_h^1(\Gamma)$. Set $u_h=\sum_{k=1}^{m_h}u_k\varphi_k^h$, then the variational problem (5.31) is equivalent to the algebraic nonlinear eigenvalue problem: Find $(\kappa_h,\underline{u})\in\mathbb{C}\times\mathbb{C}^{m_h}\setminus\{\underline{0}\}$ such that

$$D_h(\kappa_h)\underline{u}=0, \tag{5.32}$$

where

$$D_h(\kappa_h)[k,\ell]:=(\varphi_\ell,D(\kappa_h)\varphi_k)_\Gamma. \tag{5.33}$$

For an appropriate integral representation of the matrix entries (5.33) see (5.15).

Again, the convergence and error analysis of Chapter 4 can be applied, since $\mathcal{I}^\star D$ is a holomorphic Fredholm operator function and $\{S_h^1(\Gamma)\}$ is a sequence of conforming trial spaces with

$$\lim_{h\to0}\inf_{v_h\in S_h^1(\Gamma)}\|v-v_h\|_{H^{1/2}(\Gamma)}\to0\quad\text{for all } v\in H^{1/2}(\Gamma).$$

Hence, convergence results and error estimates can be derived from Theorem 4.2.3, Theorem 4.3.6 and Theorem 4.3.7. Also the stability result of Theorem 4.4.2 concerning the algebraic multiplicities remains valid. Here we want to give only an error estimate for the discretizations (κ_h,u_h).

Theorem 5.3.5. *Let $\kappa\in\sigma(\mathcal{I}^\star D)$ be real and let $\sigma(\mathcal{I}^\star D)\cap U_\delta(\kappa)=\{\kappa\}$.*
Let $k=\dim G(\mathcal{I}^\star D,\kappa)$ and let $\{w_1,\dots,w_k\}$ be an orthonormal basis of the generalized eigenspace $G(\mathcal{I}^\star D,\kappa)$. Assume that $\{w_i\}_{i=1}^k\subset H^s_{\mathrm{pw}}(\Gamma)$ for some $s\in[1/2,2]$, then there exists a $h_0>0$ such that for all $h<h_0$

$$|\kappa-\kappa_h|\le c(h^{2s-1})^{1/\varkappa(\mathcal{I}^\star D,\kappa)}\sum_{i=1}^k\|w_i\|_{H^s_{\mathrm{pw}}(\Gamma)}\quad\textit{for all }\kappa_h\in\sigma(D_h)\cap U_\delta(\kappa). \tag{5.34}$$

Further, for any $u_h\in\ker D_h(\kappa_h)$ with $\|u_h\|_{H^{1/2}(\Gamma)}=1$

$$\inf_{u\in\ker\mathcal{I}^\star D(\kappa)}\|u-u_h\|_{H^{1/2}(\Gamma)}\le c\left(|\kappa-\kappa_h|+h^{s-1/2}\sum_{i=1}^k\|w_i\|_{H^s_{\mathrm{pw}}(\Gamma)}\right) \tag{5.35}$$

is satisfied.

Proof. Using that $[\mathcal{I}^\star D(\kappa)]^\star = \mathcal{I}^\star D(-\kappa)$ and using the approximation property (5.18) of $S_h^1(\Gamma)$, the error estimates (5.34) and (5.35) follow from Theorem 4.3.6 and Theorem 4.3.7. □

If $\varkappa(\mathcal{I}^\star D, \kappa) = 1$ and $\ker \mathcal{I}^\star D(\kappa) \subset H^2_{\mathrm{pw}}(\Gamma)$, then we obtain the error estimates

$$|\kappa - \kappa_h| \leq c h^3 \sum_{i=0}^{k} \|w_i\|_{H^2_{\mathrm{pw}}(\Gamma)},$$

$$\inf_{u \in \ker \mathcal{I}^\star D(\kappa)} \|u - u_h\|_{H^{1/2}(\Gamma)} \leq c \left(|\kappa - \kappa_h| + h^{3/2} \sum_{i=0}^{k} \|w_i\|_{H^2_{\mathrm{pw}}(\Gamma)} \right),$$

where $\{w_1, \ldots, w_k\}$ is some orthonormal basis of $\ker \mathcal{I}^\star D(\kappa)$ in $H^{1/2}(\Gamma)$.

6 NUMERICAL METHODS FOR ALGEBRAIC NONLINEAR EIGENVALUE PROBLEMS

The Galerkin discretization of the boundary integral formulations of the Laplacian eigenvalue problems (5.19) and (5.30) leads to algebraic nonlinear eigenvalue problems of the form: Find $(\lambda, \underline{x}) \in \mathbb{C} \times \mathbb{C}^n \setminus \{\underline{0}\}$ such that

$$A(\lambda)\underline{x} = 0 \tag{6.1}$$

is fulfilled, where $A : \mathbb{C} \to \mathbb{C}^{n\times n}$ is a holomorphic matrix function. The subject of algebraic nonlinear eigenvalue problems is an active and open field in the numerical analysis. There is a lot of literature on numerical methods, see the review work [63] and references therein. However, black-box solvers as for linear eigenvalue problems are not available for general nonlinear eigenvalue problems. Polynomial eigenvalue problems are a special case of nonlinear eigenvalue problems because they can be transformed into equivalent linear eigenvalue problems [63]. Therefore they can be treated differently than general nonlinear eigenvalue problems. In the following we will not discuss methods for polynomial eigenvalue problems but focus on general nonlinear eigenvalue problems.

The classical and standard approach for problems with moderate size is either to consider the nonlinear eigenvalue problem as system of nonlinear equations and use a variant of Newton's method, see [56,42,67,5,71,80], or to reduce the nonlinear eigenvalue problem to a sequence of linear eigenvalue problems and use appropriate linear eigenvalue solvers, see [73,96,94]. The first class of methods can be characterized as shift-and-invert methods and they are generalizations of methods for linear eigenvalue problems as the inverse iteration or the Rayleigh quotient iteration.

In many applications the size of the nonlinear eigenvalue problems gets very large and therefore projection methods as Arnoldi/Krylov type methods [92, 43, 62, 74] and the Jacobi-Davidson type method [93, 82] have been introduced. These methods project large problems into appropriate subspaces whereby the size of the problems is reduced. For the solution of the projected problems the above mentioned standard methods are used.

The crucial point of all methods for nonlinear eigenvalue is that they converge in general only locally. Appropriate approximations of the eigenpairs are needed to guarantee convergence of the methods. In particular there is in general no guarantee to find all eigenvalues in a specified domain. There are some techniques suggested as deflation [33, 62] or the use of nonlinear Rayleigh functionals [96, 94] to overcome this problem. However,

these techniques works only either for polynomial problems or for problems with certain structure.

The convergence and error analysis of methods for nonlinear eigenvalue problems is in almost all cases restricted to simple eigenvalues. This is mainly due to the fact that a standard theory for nonlinear eigenvalue problems has been not established so far. Although most algebraic nonlinear eigenvalue problems which are considered in the literature would fit into the concept of holomorphic Fredholm operator functions, this concept is not used for the analysis of the standard algorithms.

In the following we will first review the standard Newton type methods for nonlinear eigenvalue problems and then present the little–known Kummer's method [57, 58]. For Kummer's method we will give a convergence analysis for simple and multiple eigenvalues. Methods which reduce the nonlinear eigenvalue problems to a sequence of linear eigenvalue problems are not considered here nor projection methods.

6.1 Standard Newton type methods

6.1.1 Inverse iteration

One of the classical approaches for the solution of the nonlinear eigenvalue problem (6.1) is to apply Newton's method to the extended system

$$F(\underline{x},\lambda) := \begin{pmatrix} A(\lambda)\underline{x} \\ \underline{v}^H\underline{x}-1 \end{pmatrix} = \begin{pmatrix} \underline{0} \\ 0 \end{pmatrix}, \tag{6.2}$$

where the second equation is a normalization constraint with some chosen vector $\underline{v} \in \mathbb{C}^n$. The Newton iteration is given by

$$F'(\underline{x}_i,\lambda_i)\begin{pmatrix} \underline{x}_{i+1}-\underline{x}_i \\ \lambda_{i+1}-\lambda_i \end{pmatrix} = -F(\underline{x}_i,\lambda_i), \tag{6.3}$$

where

$$F'(\underline{x},\lambda) = \begin{pmatrix} A(\lambda) & A'(\lambda)\underline{x} \\ \underline{v}^H & 0 \end{pmatrix}.$$

The derivative F' exists, since the matrix function A is holomorphic. The system (6.3) can be written as

$$\begin{aligned} A(\lambda_i)(\underline{x}_{i+1}-\underline{x}_i) + (\lambda_{i+1}-\lambda_i)A'(\lambda_i)\underline{x}_i &= -A(\lambda_i)\underline{x}_i, \\ \underline{v}^H(\underline{x}_{i+1}-\underline{x}_i) &= -\underline{v}^H\underline{x}_i + 1 \end{aligned}$$

which is equivalent to

$$\begin{aligned} A(\lambda_i)\underline{x}_{i+1} &= (\lambda_i - \lambda_{i+1})A'(\lambda_i)\underline{x}_i, \\ \underline{v}^H\underline{x}_{i+1} &= 1. \end{aligned} \tag{6.4}$$

Let $\underline{u}_{i+1} \in \mathbb{C}^n$ be a solution of

$$A(\lambda_i)\underline{u}_{i+1} = A'(\lambda_i)\underline{x}_i,$$

then, by using that $\underline{x}_{i+1} = (\lambda_i - \lambda_{i+1})\underline{u}_{i+1}$, we get from the second equation of (6.4) that $\underline{v}^H(\lambda_i - \lambda_{i+1})\underline{u}_{i+1} = 1$ and finally

$$\lambda_{i+1} = \lambda_i - \frac{1}{\underline{v}^H\underline{u}_{i+1}} \quad \text{and} \quad \underline{x}_{i+1} = \frac{\underline{u}_{i+1}}{\underline{v}^H\underline{u}_{i+1}}.$$

The described method as summarized in Algorithm 1 is a nonlinear version of the inverse iteration and was introduced in [88] for nonlinear eigenvalue problems.

Algorithm 1 Inverse iteration

1: **Input:** $\lambda_0, \underline{x}_0, \underline{v}$ such that $\underline{v}^H\underline{x}_0 = 1$
2: **for** $i = 0, 1, 2, \ldots$ until convergence **do**
3: solve $A(\lambda_i)\underline{u}_{i+1} = A'(\lambda_i)\underline{x}_i$ for $\underline{u}_{i+1}$
4: $\lambda_{i+1} = \lambda_i - (\underline{v}^H\underline{x}_i)/(\underline{v}^H\underline{u}_{i+1})$
5: $\underline{x}_{i+1} = \underline{u}_{i+1}/\underline{v}^H\underline{u}_{i+1}$
6: **end for**

The inverse iteration has the following convergence property.

Theorem 6.1.1. *Let λ_* be an algebraically simple eigenvalue of (6.1) and $\underline{x}_*$ a corresponding eigenvector with $\underline{v}^H\underline{x}_* = 1$. Then the inverse iteration converges locally quadratically to $(\underline{x}_*, \lambda_*)$.*

Proof. Since the inverse iteration is Newton's method applied to the nonlinear system (6.2), $F(\underline{x}, \lambda) = 0$, it suffices to show that F' is Lipschitz continuous in a neighborhood of $(\underline{x}_*, \lambda_*)$ and that $F'(\underline{x}_*, \lambda_*)$ is a nonsingular matrix, see [20, Theorem 2.1]. The function F' is locally Lipschitz continuous because the matrix function A is holomorphic. It remains to show that $F'(\underline{x}_*, \lambda_*)$ is a nonsingular matrix. Assume that

$$F'(\underline{x}_*, \lambda_*)\begin{pmatrix} \underline{z} \\ \mu \end{pmatrix} = \begin{pmatrix} A(\lambda_*)\underline{z} + \mu A'(\lambda_*)\underline{x}_* \\ \underline{v}^H\underline{z} \end{pmatrix} = \begin{pmatrix} \underline{0} \\ 0 \end{pmatrix} \tag{6.5}$$

for some $\underline{z} \in \mathbb{C}^n$ and $\mu \in \mathbb{C}$. Since λ_* is an algebraically simple eigenvalue, it follows by Definition 3.2.9 that $\ker A(\lambda_*) = \operatorname{span}\{\underline{x}_*\}$ and that the maximal length of a Jordan chain corresponding to λ_* is one, i.e., there exists no $\underline{s} \in \mathbb{C}^n$ such that

$$A'(\lambda_*)\underline{x}_* + A(\lambda_*)\underline{s} = \underline{0}.$$

Consequently, $A'(\lambda_*)\underline{x}_* \notin \mathrm{Im}A(\lambda_*)$. Together with the first equation of (6.5) it follows that

$$\underline{0} = A(\lambda_*)\underline{z} = \mu A'(\lambda_*)\underline{x}_*$$

and $\mu = 0$. Since λ_* is a geometrically simple eigenvalue, there exists some $\alpha \in \mathbb{C}$ such that $\underline{z} = \alpha\underline{x}_*$. Using the second equation of (6.5) and the normalization condition $\underline{v}^H\underline{x}_* = 1$, we get $0 = \underline{v}^H\underline{z} = \alpha\underline{v}^H\underline{x}_* = \alpha$. Hence, $(\underline{z}, \mu) = (\underline{0}, 0)$ and therefore $F'(\underline{x}_*, \lambda_*)$ is nonsingular. □

In the case of linear eigenvalue problems $A(\lambda) = B - \lambda I$, Algorithm 1 is the classical inverse iteration, if no updates for λ are computed, that is if step 4 is neglected. If λ is updated as in step 4, then Algorithm 1 is a variant of Rayleigh quotient iteration for linear problems with the two-sided Rayleigh quotient

$$\lambda_{i+1} = \frac{\underline{v}^H B\underline{u}_{i+1}}{\underline{v}^H\underline{u}_{i+1}}.$$

The classical Rayleigh quotient iteration for linear problems is obtained, if for the update in step 4 the one-sided Rayleigh quotient $\lambda_{i+1} = \dfrac{\underline{u}_{i+1}^H B\underline{u}_{i+1}}{\underline{u}_{i+1}^H\underline{u}_{i+1}}$ is used.

6.1.2 Rayleigh functional iterations

As for linear eigenvalue problems also for nonlinear eigenvalue problems several different updates for λ in Algorithm 1 are suggested in order to improve the convergence behavior. These updates use in addition approximations of the left eigenvector and different types of nonlinear Rayleigh functionals, which are generalizations of the Rayleigh quotient for linear problems. A vector $\underline{y}_* \in \mathbb{C}^n \setminus \{\underline{0}\}$ is called a left eigenvector of the eigenvalue λ_* of the eigenvalue problem (6.1) if it is a solution of the adjoint equation $A(\lambda_*)^H\underline{y}_* = \underline{0}$.

In this section we present the two-sided Rayleigh functional iteration [71], [80, Section 4.2] and the complex symmetric Rayleigh functional iteration [80, Chapter 7]. Both methods have a higher convergence rate than the inverse iteration. For a comprehensive discussion, an error analysis, and comparison of different methods using Rayleigh functionals we refer to [80].

Two–sided Rayleigh functional iteration

The two-sided Rayleigh functional $p : \mathbb{C}^n \times \mathbb{C}^n \supset I \to \mathbb{C}$ for a holomorphic matrix function $A : \mathbb{C} \to \mathbb{C}^{n\times n}$ is defined implicitly by

$$\underline{w}^H A\big(p(\underline{u}, \underline{w})\big)\,\underline{u} = 0, \tag{6.6}$$

see [80, p. 38]. If λ_* is an algebraically simple eigenvalue of A and $\underline{x}_*$ and $\underline{y}_*$ are corresponding right and left eigenvectors, respectively, then the functional p is locally uniquely defined [80, Theorem 3.5]. For a given approximation $(\underline{u},\underline{w})$ for the eigenvectors $(\underline{x}_*,\underline{y}_*)$ the two-sided Rayleigh functional $p(\underline{u},\underline{w})$ provides an appropriate approximation for the eigenvalue and p is stationary at $(\underline{x}_*,\underline{y}_*)$ [80, Chapter 3]. The use of the two-sided Rayleigh functional as updates for the eigenvalues in the inverse iteration requires approximations of the left eigenvector. Therefore an additional iteration for the approximation of the left eigenvector has to be implemented, which leads, combined with the use of the Rayleigh functional, to Algorithm 2.

Algorithm 2 Two-sided Rayleigh functional iteration

1: **Input:** $\lambda_0,\underline{x}_0,\underline{y}_0$ such that $\underline{x}_0^H\underline{x}_0 = \underline{y}_0^H\underline{y}_0 = 1$
2: **for** $i = 0,1,2,\ldots$ until convergence **do**
3: solve $A(\lambda_i)\underline{u}_{i+1} = A'(\lambda_i)\underline{x}_i$ for $\underline{u}_{i+1}$
4: solve $A(\lambda_i)^H\underline{w}_{i+1} = A'(\lambda_i)^H\underline{y}_i$ for $\underline{w}_{i+1}$
5: $\underline{x}_{i+1} = \underline{u}_{i+1}/\|\underline{u}_{i+1}\|$
6: $\underline{y}_{i+1} = \underline{w}_{i+1}/\|\underline{w}_{i+1}\|$
7: solve $\underline{y}_{i+1}^H A(\lambda_{i+1})\underline{x}_{i+1} = 0$ for λ_{i+1}
8: **end for**

The costs for the two-sided Rayleigh functional iteration are higher compared with the inverse iteration, since in addition a second linear system and a nonlinear equation have to be solved in every iteration step. If the linear system in step 3 is solved by factorizing the matrix $A(\lambda_i)$, then the same factorization can be used for the conjugate transpose $A(\lambda_i)^H$ for solving the second linear system in step 4. However, large problems require in general a preconditioned iterative solver. In this case at least only one preconditioner is needed to solve both linear systems. If the problem is Hermitian, then the right and the left eigenvector coincide and only one linear system has to be solved. The computation of the Rayleigh functional in step 7 requires the solution of a nonlinear equation and it can be tricky. In general some iterative solver has to be used. The costs for it can be expensive if the computation of the corresponding matrix is complex.

Local cubic convergence of the two-sided Rayleigh functional iteration for algebraically simple eigenvalues is shown in [80, Theorem 4.13]. To our knowledge no analysis has be done for multiple eigenvalues.

Complex symmetric Rayleigh functional iteration

The Galerkin discretization of the boundary integral formulations of the Laplacian eigenvalue problems (5.19) and (5.30) leads to complex symmetric eigenvalue problems (5.21)

and (5.32). For such eigenvalue problems a one–sided Rayleigh functional can be used for the updates of the eigenvalues in the inverse iteration [80, Chapter 7]. The equivalence

$$A(\lambda_*)\underline{x}_* = \underline{0} \qquad \Leftrightarrow \qquad \underline{x}_*^\top A(\lambda_*) = \underline{0}^\top$$

motivates to define the so–called complex symmetric Rayleigh functional p_S by

$$\underline{u}^\top A(p_S(\underline{u}))\underline{u} = 0.$$

For complex symmetric eigenvalue problems the use of the Rayleigh functional p_S as update for the eigenvalues in the inverse iteration yields a local cubic convergence [80, Theorem 7.7].

6.1.3 Residual inverse iteration

A simplified version of the inverse iteration is the so–called residual inverse iteration, which was introduced by Neumaier [67]. The idea can be described as follows: The equation (6.4) of the Newton iteration can be written as

$$\begin{aligned}\underline{x}_i - \underline{x}_{i+1} &= \underline{x}_i + (\lambda_{i+1} - \lambda_i)A(\lambda_i)^{-1}A'(\lambda_i)\underline{x}_i \\ &= A(\lambda_i)^{-1}\left[A(\lambda_i) + (\lambda_{i+1} - \lambda_i)A'(\lambda_i)\right]\underline{x}_i \\ &= A(\lambda_i)^{-1}A(\lambda_{i+1})\underline{x}_i + \mathcal{O}(|\lambda_{i+1} - \lambda_i|^2).\end{aligned}$$

Neglecting the second order term gives

$$\underline{x}_{i+1} = \underline{x}_i - A(\lambda_i)^{-1}A(\lambda_{i+1})\underline{x}_i.$$

Neumaier showed that if λ_i in $A(\lambda_i)^{-1}$ is replaced by a fixed shift σ, then the iteration still converges [67]. However, λ_{i+1} has to be updated in each iteration step by the solution of the nonlinear equation

$$\underline{v}^H A(\sigma)^{-1}A(\lambda_{i+1})\underline{x}_i = 0.$$

Using this update for the approximation of the eigenvalues yields the residual inverse iteration, which is summarized in Algorithm 3.

The advantage of the residual inverse iteration compared with the inverse iteration is that the system matrix which has to be inverted remains the same during the iteration. Moreover, the computation of the derivative $A'(\lambda_i)$ is no more longer needed. If the problem size is small a factorization of $A(\sigma)$ can be computed in advance which allows an efficient realization of $A(\sigma)^{-1}$ in step 3 and step 4 of Algorithm 3. For large problems an iterative solver is needed twice in one iteration step which is a disadvantage compared with the inverse iteration where only once in one iteration step an iterative solver is needed. Besides, for the residual inverse iteration a nonlinear equation has to be solved for the updates of λ.

Algorithm 3 Residual inverse iteration

1: **Input:** $\sigma, \underline{x}_0, \underline{v}$ such that $\underline{v}^H \underline{x}_0 = 1$
2: **for** $i = 0, 1, 2, \ldots$ until convergence **do**
3: solve $\underline{v}^H A(\sigma)^{-1} A(\lambda_{i+1}) \underline{x}_i = 0$ for λ_{i+1}
4: $\underline{r}_i = A(\lambda_{i+1}) \underline{x}_i$
5: solve $A(\sigma) \underline{s}_i = \underline{r}_i$ for $\underline{s}_i$
6: $\underline{u}_{i+1} = \underline{x}_i - \underline{s}_i$
7: $\underline{x}_{i+1} = \underline{u}_{i+1} / \underline{v}^H \underline{u}_{i+1}$
8: **end for**

A convergence result for the residual inverse iteration is given for the case if λ_* is a simple zero of $\det A(\lambda_*) = 0$, see [67]. If $\underline{x}_*$ is a corresponding eigenvector to λ_* with $\underline{v}^H \underline{x}_* = 1$, then the residual inverse iteration converges for all $(\sigma, \underline{x}_0)$ sufficiently close to $(\lambda_*, \underline{x}_0)$ with the error estimates

$$\frac{\|x_{i+1} - \underline{x}_*\|}{\|x_i - \underline{x}_*\|} = \mathcal{O}(|\sigma - \lambda_*|) \quad \text{and} \quad |\lambda_{i+1} - \lambda_*| = \mathcal{O}(\|x_i - \underline{x}_*\|).$$

In [80, Section 4.2] it is shown that a quadratic convergence order for the residual inverse iteration is obtained if the two-sided Rayleigh functionals is used for the updates of λ.

6.2 Kummer's method

In this section we want to derive Kummer's method for algebraic holomorphic eigenvalue problems, where we follow the work of Langer [58]. Kummer introduced in [57] an iterative method for polynomial eigenvalue problems in arbitrary dimensional Hilbert spaces. In his approach he constructed a scalar holomorphic function which has the eigenvalues as zeros. Langer showed in [58] that this approach can be extended to general holomorphic eigenvalue problems. Langer required the assumption that the spectrum of the holomorphic operator function consists only of isolated eigenvalues which are poles of the corresponding resolvent. This assumption is always fulfilled for eigenvalue problems for holomorphic Fredholm operator functions, provided that the resolvent set of the operator function is not empty, see Theorem 3.2.2 and Theorem 3.2.14. In particular, we have the following representation of the resolvent of a holomorphic matrix function.

Lemma 6.2.1. *Let Λ be an open and connected subset of $\mathbb{C}$ and let $A : \Lambda \to \mathbb{C}^{n \times n}$ be a holomorphic matrix function with a non-empty resolvent set $\rho(A)$. Let $\lambda_* \in \sigma(A)$, then there exists a $\delta > 0$ such that for all $\lambda \in \{\mu \in \Lambda : |\mu - \lambda_*| < \delta \text{ and } \mu \neq \lambda_*\}$ the resolvent admits the representation*

$$A(\lambda)^{-1} = \sum_{k=-r}^{-1} (\lambda - \lambda_*)^k B_k + F(\lambda), \tag{6.7}$$

where $B_k \in \mathbb{C}^{n\times n}$ for $k = -r,\dots,-1$ with $B_{-r} \neq 0$, F is a holomorphic matrix function, and $r = \varkappa(A,\lambda_*)$ is the maximal length of a Jordan chain of A corresponding to λ_*.

Proof. Since $A(\lambda) \in \mathcal{L}(\mathbb{C}^n,\mathbb{C}^n)$ is obviously a Fredholm operator for all $\lambda \in \Lambda$, the representation of the resolvent $A(\lambda)^{-1}$ follows immediately from Theorem 3.2.14. □

Let us in the following assume that $A : \Lambda \to \mathbb{C}^{n\times n}$ is a holomorphic matrix function with $\rho(A) \neq \emptyset$. If λ_* is an eigenvalue of A, then, by Lemma 6.2.1, we have the representation

$$A(\lambda)^{-1} = \sum_{k=-r}^{-1} (\lambda-\lambda_*)^k B_k + \sum_{k=0}^{\infty} (\lambda-\lambda_*)^k B_k \quad \text{for } \lambda \in U_\delta \setminus \{\lambda_*\}, \tag{6.8}$$

with $B_{-r} \neq 0$. Therefore there exist vectors $\underline{w},\underline{z} \in \mathbb{C}^n$ such that

$$(\underline{z}, B_{-r}\underline{w})_2 \neq 0. \tag{6.9}$$

Define the function $\varphi : \rho(A) \to \mathbb{C}$ by

$$\varphi(\lambda) := (\underline{z}, A(\lambda)^{-1}\underline{w})_2, \tag{6.10}$$

then φ is holomorphic on $U_\delta(\lambda_*) \setminus \{\lambda_*\}$, because φ admits the representation

$$\varphi(\lambda) = (\underline{z}, A(\lambda)^{-1}\underline{w})_2 = \sum_{k=-r}^{-1} (\lambda-\lambda_*)^k (\underline{z}, B_k\underline{w})_2 + \sum_{k=0}^{\infty} (\lambda-\lambda_*)^k (\underline{z}, B_k\underline{w})_2.$$

Since $|\varphi(\lambda)| \to \infty$ as $\lambda \to \lambda_0$, there exist a $\delta_1 > 0$ with $\delta_1 \leq \delta$ and a constant $K > 0$ such that

$$K \leq |\varphi(\lambda)| \quad \text{for all } \lambda \in U_{\delta_1}(\lambda_*) \setminus \{\lambda_*\}. \tag{6.11}$$

Hence, we may define the function $\psi : U_{\delta_1}(\lambda_*) \to \mathbb{C}$ by

$$\psi(\lambda) := \begin{cases} \dfrac{1}{\varphi(\lambda)} & \text{for } \lambda \neq \lambda_*, \\ 0 & \text{for } \lambda = \lambda_*. \end{cases} \tag{6.12}$$

The function ψ is holomorphic on $U_{\delta_1}(\lambda_*)$ and allows the Taylor series expansion

$$\psi(\lambda) = \frac{(\lambda-\lambda_*)^r}{(\underline{z}, B_{-r}\underline{w})_2} - (\lambda-\lambda_*)^{r+1} \frac{(\underline{z}, B_{-r+1}\underline{w})_2}{(\underline{z}, B_{-r}\underline{w})_2^2} + \mathcal{O}\left((\lambda-\lambda_*)^{r+2}\right).$$

We see that λ_* is a zero of ψ with multiplicity r. From

$$\psi'(\lambda) = \frac{\varphi'(\lambda)}{\varphi(\lambda)^2} \quad \text{for all } \lambda \in U_{\delta_1}(\lambda_*) \setminus \{\lambda_*\}$$

it follows that $\psi'(\lambda) = 0$ in $U_{\delta_1}(\lambda_*) \setminus \{\lambda_*\}$ if and only if $\varphi'(\lambda) = 0$. Since $|\varphi'(\lambda)| \to \infty$ as $\lambda \to \lambda_*$, there exists a $\delta_2 > 0$ such that

$$|\varphi'(\lambda)| > 0 \quad \text{for all } \lambda \in U_{\delta_2}(\lambda_*) \setminus \{\lambda_*\}.$$

Hence, λ_* is the only zero of ψ' in $U_{\delta_2}(\lambda_*)$ and its multiplicity is $(r-1)$. Thus, we may define the function $\eta : U_{\delta_2}(\lambda_*) \to \mathbb{C}$ by

$$\eta(\lambda) := \begin{cases} \dfrac{\psi(\lambda)}{\psi'(\lambda)} & \text{for } \lambda \neq \lambda_*, \\ 0 & \text{for } \lambda = \lambda_*, \end{cases} \tag{6.13}$$

which is holomorphic and admits the Taylor series expansion

$$\eta(\lambda) = \frac{1}{r}(\lambda - \lambda_*) + \frac{(\underline{z}, B_{-r+1}\underline{w})_2}{r^2(\underline{z}, B_{-r}\underline{w})_2}(\lambda - \lambda_*)^2 + \mathcal{O}\left((\lambda - \lambda_*)^3\right). \tag{6.14}$$

Further, we have

$$\eta(\lambda_*) = 0 \qquad \text{and} \qquad \eta'(\lambda_*) = \frac{1}{r}. \tag{6.15}$$

Thus, we have described the eigenvalue λ_* as zero of the functions ψ and η, which is the essential idea of Kummer's approach. The use of Newton's method to determine the zero of the function ψ yields finally Kummer's method [57, 58]. In the following theorem we show its convergence property.

Theorem 6.2.2 ([58, Satz 3]). *Let $s \in \mathbb{N}$, where $s \leq r = \varkappa(A, \lambda_*)$. There exists a $R > 0$ with $R \leq \delta_2$ such that the iteration*

$$\lambda_{i+1} = \lambda_i - s\eta(\lambda) = \lambda_i - s\frac{\psi(\lambda_i)}{\psi'(\lambda_i)} \quad \text{for } i = 0, 1, 2, \ldots \tag{6.16}$$

converges for any $\lambda_0 \in U_R(\lambda_*)$ to λ_*. If $s = r$, then the convergence rate is quadratic and we have

$$\frac{\lambda_{i+1} - \lambda_*}{(\lambda_i - \lambda_*)^2} \to \frac{(\underline{z}, B_{-r+1}\underline{w})_2}{r(\underline{z}, B_{-r}\underline{w})_2} \quad \text{as } i \to \infty. \tag{6.17}$$

If $s < r$, then the convergence rate is linear and we have

$$\frac{\lambda_{i+1} - \lambda_*}{\lambda_i - \lambda_*} \to \frac{r-s}{r} \quad \text{as } i \to \infty. \tag{6.18}$$

Proof. We use the Banach fixed point theorem for the function

$$f(\lambda) := \lambda - s\eta(\lambda)$$

to prove the convergence of the iteration (6.16). Since f is holomorphic on $U_{\delta_2}(\lambda_*)$ and since

$$f'(\lambda_*) = 1 - s\eta'(\lambda_*) = 1 - s\frac{1}{r} < 1, \tag{6.19}$$

there exists a $R > 0$, where we choose $R \leq \delta_2$, such that

$$|f'(\lambda)| < 1 \quad \text{for all } \lambda \in U_R(\lambda_*).$$

Hence, by the Banach fixed point theorem the iteration (6.16) converges to λ_* for all initial values $\lambda_0 \in U_R(\lambda_*)$.

Using the Taylor series expansion (6.14) of η we obtain from formula (6.16)

$$\begin{aligned}\lambda_{i+1} - \lambda_* &= \lambda_i - s\eta(\lambda_i) - \lambda_* \\ &= \lambda_i - \lambda_* - s\left[\frac{1}{r}(\lambda_i - \lambda_*) + (\lambda_i - \lambda_*)^2 \frac{(\underline{z}, B_{-r+1}\underline{w})_2}{r^2(\underline{z}, B_{-r}\underline{w})_2} + \mathcal{O}\left((\lambda - \lambda_*)^3\right)\right], \end{aligned} \tag{6.20}$$

which shows the convergence rates (6.17) and (6.18). □

In practical computations $\varkappa(A, \lambda_*)$ is not known a priori and therefore $s = 1$ is chosen for the iteration (6.16), which gives the classical Newton's method for $\psi(\lambda) = 0$. Then, a quadratic convergence rate is obtained, if $\varkappa(A, \lambda_*) = 1$.

Let us now consider the implementation of Kummer's method. Recalling the definition (6.12) of ψ,

$$\psi(\lambda) = \frac{1}{\varphi(\lambda)} = \frac{1}{(\underline{z}, A(\lambda)^{-1}\underline{w})_2} \quad \text{for } \lambda \in U_{\delta_1}(\lambda_*) \setminus \{\lambda_*\},$$

we get

$$\frac{\psi(\lambda)}{\psi'(\lambda)} = -\frac{(\underline{z}, A(\lambda)^{-1}\underline{w})_2}{\frac{d}{d\lambda}(\underline{z}, A(\lambda)^{-1}\underline{w})_2} = -\frac{(\underline{z}, A(\lambda)^{-1}\underline{w})_2}{(\underline{z}, \frac{d}{d\lambda}A(\lambda)^{-1}\underline{w})_2}. \tag{6.21}$$

Using the representation (3.2) for the derivative of $A(\cdot)^{-1}$, we can write

$$\frac{d}{d\lambda}A(\lambda)^{-1} = -A(\lambda)^{-1}A'(\lambda)A(\lambda)^{-1}$$

and obtain

$$\frac{\psi(\lambda)}{\psi'(\lambda)} = \frac{(\underline{z}, A(\lambda)^{-1}\underline{w})_2}{(\underline{z}, A(\lambda)^{-1}A'(\lambda)A(\lambda)^{-1}\underline{w})_2} = \frac{(\underline{z}, A(\lambda)^{-1}\underline{w})_2}{([A(\lambda)^{-1}]^H\underline{z}, A'(\lambda)A(\lambda)^{-1}\underline{w})_2}.$$

Let $\underline{x}_i \in \mathbb{C}^n$ and $\underline{y}_i \in \mathbb{C}^n$ denote the solutions of

$$A(\lambda_i)\underline{x}_i = \underline{w} \qquad \text{and} \qquad A(\lambda_i)^H\underline{y}_i = \underline{z}, \tag{6.22}$$

then we can write Kummer's iteration (6.16) as

$$\lambda_{i+1} = \lambda_i - \frac{(\underline{z}, \underline{x}_i)_2}{(\underline{y}_i, A'(\lambda_i)\underline{x}_i)_2},$$

where we have set $s = 1$.

Kummer's method also approximates a right and a left eigenvector by $\underline{x}_i$ and $\underline{y}_i$, respectively. In the following theorem we give a convergence result for the right eigenvector.

Theorem 6.2.3 ([58, Satz 4]). *Let $\underline{x}_i$ be defined by (6.22), then there exists a $i_0 \in \mathbb{N}$ such that*

$$\inf_{\underline{x} \in \ker A(\lambda_*)} \left\| \underline{x} - \frac{\underline{x}_i}{\|\underline{x}_i\|_2} \right\|_2 \leq c|\lambda_i - \lambda_*| \tag{6.23}$$

for all $i \geq i_0$, where $c > 0$ is a constant which is independent of i.

Proof. Let $\underline{x}_i$ be the solution of

$$A(\lambda_i)\underline{x}_i = \underline{w}.$$

Then, using the representation (6.8) of $A(\lambda)^{-1}$, we can write

$$\underline{x}_i = A(\lambda_i)^{-1}\underline{w} = \sum_{k=-r}^{\infty} (\lambda_i - \lambda_*)^k B_k \underline{w} = \frac{B_{-r}\underline{w}}{(\lambda_i - \lambda_*)^r} + \frac{C(\lambda_i)\underline{w}}{(\lambda_i - \lambda_*)^{r-1}}, \tag{6.24}$$

where

$$C(\lambda) := \sum_{k=0}^{\infty} (\lambda - \lambda_*)^k B_{k-r+1}.$$

The operator function $C : U_\delta(\lambda_*) \to \mathbb{C}$ is holomorphic, since $\sum_{k=0}^{\infty} (\lambda - \lambda_*)^k B_k$ is holomorphic on $U_\delta(\lambda_*)$ by assumption (6.8). For the norm of $\underline{x}_i$ we get

$$\begin{aligned} \|\underline{x}_i\|_2^2 = (\underline{x}_i, \underline{x}_i) &= \frac{\|B_{-r}\underline{w}\|_2^2}{(\lambda_i - \lambda_*)^{2r}} + 2\mathrm{Re}\frac{(B_{-r}\underline{w}, C(\lambda_i)\underline{w})_2}{(\lambda_i - \lambda_*)^{2r-1}} + \frac{\|C(\lambda_i)\underline{w}\|_2^2}{(\lambda_i - \lambda_*)^{2r-2}} \\ &= \frac{\|B_{-r}\underline{w}\|_2^2}{(\lambda_i - \lambda_*)^{2r}} \chi(\lambda_i), \end{aligned} \tag{6.25}$$

with

$$\chi(\lambda) := 1 + \frac{2\mathrm{Re}[(\lambda - \lambda_*)(B_{-r}\underline{w}, C(\lambda)\underline{w})_2]}{\|B_{-r}\underline{w}\|_2^2} + \frac{|\lambda - \lambda_*|^2 \|C(\lambda)\underline{w}\|_2}{\|B_{-r}\underline{w}\|_2^2}.$$

The function χ is well defined, since $\|B_{-r}\underline{w}\|_2^2 \neq 0$ by assumption (6.9). Note that χ is real valued, continuous on $U_\delta(\lambda_*)$, and

$$\chi(\lambda_*) = 1.$$

Using (6.24) and (6.25) we can write

$$\frac{\underline{x}_i}{\|\underline{x}_i\|_2} = \left(\frac{B_{-r}\underline{w}}{(\lambda_i-\lambda_*)^r} + \frac{C(\lambda_i)\underline{w}}{(\lambda_i-\lambda_*)^{r-1}} \right) \frac{|\lambda_i-\lambda_*|^r}{\|B_{-r}\underline{w}\|_2} \chi(\lambda_i)^{-1/2}. \tag{6.26}$$

The vector $B_{-r}\underline{w}$ is an eigenvector of the operator function A corresponding to λ_*, since from

$$\begin{aligned}(\lambda-\lambda_*)^r\underline{w} &= (\lambda-\lambda_*)^r A(\lambda)A(\lambda)^{-1}\underline{w} \\ &= A(\lambda_*)B_{-r}\underline{w} + \sum_{k=1}^{\infty}(\lambda-\lambda_*)^k A_k \sum_{k=1}^{\infty}(\lambda-\lambda_*)^k B_{-r+k}\end{aligned}$$

for $\lambda \in U_\delta(\lambda_*)\setminus\{\lambda_*\}$, it follows that

$$0 = \lim_{\lambda\to\lambda_*}(\lambda-\lambda_*)^r\underline{w} = A(\lambda_*)B_{-r}\underline{w}.$$

Therefore also

$$\tilde{\underline{x}}_i := \frac{B_{-r}\underline{w}}{(\lambda_i-\lambda_*)^r}\frac{|\lambda_i-\lambda_*|^r}{\|B_{-r}\underline{w}\|_2}\chi(\lambda_i)^{-1/2} \in \ker A(\lambda_*).$$

Thus, we get from (6.26)

$$\inf_{\underline{x}\in\ker A(\lambda_*)} \left\| \underline{x} - \frac{\underline{x}_i}{\|\underline{x}_i\|_2} \right\|_2 \le \left\| \tilde{\underline{x}}_i - \frac{\underline{x}_i}{\|\underline{x}_i\|_2} \right\|_2 = \frac{\|C(\lambda_i)\underline{w}\|_2}{\|B_{-r}\underline{w}\|_2}\chi(\lambda_i)^{-1/2}|\lambda_i-\lambda_*|. \tag{6.27}$$

Since $C(\lambda_i)\to B_{-r+1}$ and $\chi(\lambda_i)\to 1$ as $i\to\infty$, there exists a $i_0\in\mathbb{N}$ such that

$$\|C(\lambda_i)\underline{w}\|_2 \le \|B_{-r+1}\underline{w}\|_2 + 1 \quad \text{and} \quad \chi(\lambda_i)^{-1/2} \le 2$$

for all $i \ge i_0$. Hence, we get the estimate

$$\frac{\|C(\lambda_i)\underline{w}\|_2}{\|B_{-r}\underline{w}\|_2}\chi(\lambda_i)^{-1/2} \le 2\frac{\|B_{-r+1}\underline{w}\|_2+1}{\|B_{-r}\underline{w}\|_2} =: c,$$

which proves with (6.27) the estimate (6.23). □

Remark 6.2.4. *Kummer's method relies essentially on the representation of the resolvent as given in Theorem 3.2.14. Therefore the whole analysis can be done literally for eigenvalue problems for holomorphic Fredholm operator functions in arbitrary dimensional Hilbert spaces, provided that the corresponding resolvent set is not empty.*

The costs of Kummer's method as presented in Algorithm 4 lie in between the costs of the inverse iteration and the costs of the two-sided Rayleigh functional iteration. If the problem is small and a factorization of $A(\lambda_i)$ is used for the solution of the linear system in step 3 of Algorithm 4, then this factorization can be used again for the linear system in

Algorithm 4 Kummer's method

1: **Input:** $\lambda_0, \underline{w}, \underline{z}$
2: **for** $i = 0, 1, 2, \ldots$ until convergence **do**
3: solve $A(\lambda_i)\underline{x}_i = \underline{w}$ for $\underline{x}_i$
4: solve $A(\lambda_i)^H \underline{y}_i = \underline{z}$ for $\underline{y}_i$
5: $\lambda_{i+1} = \lambda_i - (\underline{z}, \underline{x}_i)_2 / (\underline{y}_i, A'(\lambda_i)\underline{x}_i)_2$
6: **end for**

step 4. For large problems an iterative solver is needed twice in Algorithm 4 and the costs are then significantly higher than for the inverse iteration.

For complex symmetric eigenvalue problems the costs of Kummer's method can be reduced, if input vectors $\underline{w}$ and $\underline{z}$ are chosen such that $\underline{z} = \overline{\underline{w}}$. Then, $\underline{y}_i = \overline{\underline{x}_i}$ is the solution of the linear equation in step 4 in Algorithm 4, since

$$A(\lambda_i)\underline{x}_i = \underline{w} \quad \Leftrightarrow \quad A(\lambda_i)^\top \underline{x}_i = \underline{w} \quad \Leftrightarrow \quad \overline{A(\lambda_i)^\top \underline{x}_i} = \overline{\underline{w}} \quad \Leftrightarrow \quad A(\lambda_i)^H \overline{\underline{x}_i} = \overline{\underline{w}}.$$

Hence, for complex symmetric eigenvalue problems only one linear system has to be solved, if we choose $\underline{z} = \overline{\underline{w}}$. The costs for Kummer's method are then approximately the same as for the inverse iteration and as for the complex symmetric Rayleigh functional iteration.

7 NUMERICAL EXPERIMENTS

In this chapter we present some numerical results of the boundary element approximation of the Dirichlet Laplacian eigenvalue problem (2.6). In addition, we compare these results with the results of a finite element approximation.

As domain Ω for the eigenvalue problem we choose the cube $\Omega = (0, \frac{1}{2})^3$. The eigenvalues are given by

$$\lambda_k = 4\pi^2 \left[k_1^2 + k_2^2 + k_3^2\right]$$

and the associated eigenfunctions are

$$u_k(x) = (\sin 2\pi k_1 x_1)(\sin 2\pi k_2 x_2)(\sin 2\pi k_3 x_3).$$

It turns out that the first eigenvalue ($k_1 = k_2 = k_3 = 1$)

$$\lambda_1 = 12\pi^2, \quad \kappa_1 = \sqrt{\lambda_1} = 2\sqrt{3}\pi$$

is simple, while the second eigenvalue ($k_1 = 2, k_2 = k_3 = 1$)

$$\lambda_2 = 24\pi^2, \quad \kappa_2 = \sqrt{\lambda_2} = 2\sqrt{6}\pi$$

is multiple.

Let us first consider the boundary element approximation of the eigenvalue problem. We use the boundary integral formulation (5.19) for the Galerkin discretization with piecewise constant basis functions as described in Chapter 5. The boundary $\Gamma = \partial\Omega$ is decomposed into N uniform plane triangular boundary elements with mesh size h. We use the inverse iteration, the complex symmetric Rayleigh functional iteration, and Kummer's method to solve the algebraic nonlinear eigenvalue problem (5.21),

$$V_h(\kappa_h)\underline{w} = 0.$$

Each method locally converges to the desired eigenvalues regardless of whether the eigenvalue is simple or multiple. Kummer's method seems not to be superior concerning the convergence behavior for multiple eigenvalues than the other methods. The convergence region of the three methods differs. The largest convergence region has the Rayleigh functional iteration followed by the inverse iteration.

The numerical results of the boundary element approximations for the eigenvalue κ_1 and κ_2 are presented in Table 1 and Table 2. A cubic convergence order $\mathcal{O}(h^3)$ can be observed, which confirms the theoretical error estimate in (5.29).

L	h	N	$\kappa_{1,h}^{\text{BEM}}$	$\lvert\kappa_1 - \kappa_{1,h}^{\text{BEM}}\rvert$	eoc
2	2^{-3}	384	10.8768	5.986e-03	-
3	2^{-4}	1536	10.8821	6.962e-04	3.1
4	2^{-5}	6144	10.8827	8.619e-05	3.0

Table 7.1: BEM approximation of $\kappa_1 = 2\sqrt{3}\pi \approx 10.8828$, simple eigenvalue.

L	h	N	$\kappa_{2_1,h}^{\text{BEM}}$	$\lvert\kappa_2 - \kappa_{2_1,h}^{\text{BEM}}\rvert$	eoc
2	2^{-3}	384	15.373851	1.7e-02	-
3	2^{-4}	1536	15.3887048	1.9e-03	3.1
4	2^{-5}	6144	15.39037160	2.3e-04	3.1
L	h	N	$\kappa_{2_2,h}^{\text{BEM}}$	$\lvert\kappa_2 - \kappa_{2_2,h}^{\text{BEM}}\rvert$	eoc
2	2^{-3}	384	15.37364	1.7e-02	-
3	2^{-4}	1536	15.3887060	1.9e-03	3.1
4	2^{-5}	6144	15.39037171	2.3e-04	3.1
L	h	N	$\kappa_{2_3,h}^{\text{BEM}}$	$\lvert\kappa_2 - \kappa_{2_3,h}^{\text{BEM}}\rvert$	eoc
2	2^{-3}	384	15.373876	1.7e-02	-
3	2^{-4}	1536	15.3887071	1.9e-03	3.1
4	2^{-5}	6144	15.39037180	2.3e-04	3.1

Table 7.2: BEM approximation of $\kappa_2 = 2\sqrt{6}\pi \approx 15.3906$, multiple eigenvalue.

The number of iterations which have to be performed to reach a tolerance for the residual norm of 10^{-10} are presented in Table 3. Note that for the Rayleigh functional iteration in every iteration step the Rayleigh functional has to be determined, which we have approximated by using three Newton steps.

	κ_1	κ_{2_1}	κ_{2_2}	κ_{2_3}
Inverse iteration	10	11	12	10
Rayleigh functional iteration	3	4	4	3
Kummer's method	11	11	12	11

Table 7.3: Number of iterations for BEM approximations, N=384.

For the finite element approximation we have used linear tetrahedral elements with respect to an uniform discretization of Ω with mesh size h. The FEM matrices are generated by Netgen/NGSolve [79]. As eigenvalue solver we use LOBPCG [53] with a two-level preconditioner. The numerical results of the finite element discretization to approximate

the first and second eigenvalue are listed in Tables 4 and 5, where M is the number of interior nodes which is equal to the number of degrees of freedom.

L	h	M	$\kappa_{1,M}^{\text{FEM}}$	$\lvert\kappa_1 - \kappa_{1,M}^{\text{FEM}}\rvert$
3	2^{-4}	343	11.3693	4.9e-01
4	2^{-5}	3375	11.0038	1.2e-01
5	2^{-6}	29791	10.9132	3.0e-02
6	2^{-7}	250047	10.8903	7.6e-03

Table 7.4: FEM approximation of $\kappa_1 = 2\sqrt{3}\pi \approx 10.8828$, simple eigenvalue.

L	h	M	$\kappa_{2_1,M}^{\text{FEM}}$	$\lvert\kappa_2 - \kappa_{2_1,M}^{\text{FEM}}\rvert$	$\kappa_{2_2,M}^{\text{FEM}}$	$\lvert\kappa_2 - \kappa_{2_2,M}^{\text{FEM}}\rvert$	$\kappa_{2_3,M}^{\text{FEM}}$	$\lvert\kappa_2 - \kappa_{2_3,M}^{\text{FEM}}\rvert$
3	2^{-4}	343	16.27	8.8e-01	16.28	8.9e-01	17.59	2.2
4	2^{-5}	3375	15.60	2.1e-01	15.60	2.4e-01	16.12	7.3e-01
5	2^{-6}	29791	15.44	5.1e-02	15.44	5.3e-02	15.63	2.4e-01
6	2^{-7}	250047	15.40	1.3e-02	15.40	1.4e-02	15.47	8.0e-02

Table 7.5: FEM approximation of $\kappa_2 = 2\sqrt{6}\pi \approx 15.3906$, multiple eigenvalue.

The numerical results reflect the different convergence rates of both methods for the Dirichlet Laplacian eigenvalue problem. The convergence order for finite element approximations with linear elements is quadratic, see (2.26). For boundary element approximations with piecewise constant elements at the best a cubic convergence order can be achieved. Note that the BEM approximations of the coarsest mesh on level $L = 2$ with matrix size $N = 384$ are approximately the same as the FEM approximations of the finest mesh on level $L = 6$ with matrix size $N = 250047$.

The disadvantage of the boundary element approach compared with the finite element approach is that in one run of the presented nonlinear eigenvalue algorithms only one eigenpair is approximated and that sufficiently good initial values are needed for the convergence to a desired eigenpair. In particular, there is no guarantee that all eigenvalues in a specified domain are found. This is in general a crucial point of nonlinear eigenvalue algorithms and a main topic of the research in this field [80, 95, 63].

The described boundary element discretization leads to fully populated matrices, therefore it is restricted to rather small problem size. Moreover, the costs of the computation of the matrix entries are considerably high. Hence, there is a need for so–called fast boundary element methods in order to reduce the memory requirements and the costs of the computations. Several concepts are available for this purpose as the fast multipole method [29], the adaptive cross approximation [9,70], panel clustering [37], or hierarchical matrices [36].

REFERENCES

[1] R. A. Adams and J. J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics*. Elsevier/Academic Press, Amsterdam, second edition, 2003.

[2] S. Agmon. *Lectures on elliptic boundary value problems*. D. Van Nostrand, Princeton, N.J.-Toronto-London, Princeton, 1965.

[3] A. Ali, C. Rajakumar, and S. M. Yunus. On the formulation of the acoustic boundary element eigenvalue problems. *Internat. J. Numer. Methods Engrg.*, 31:1271–1282, 1991.

[4] H. W. Alt. *Lineare Funktionalanalysis*. Springer, Berlin, 1992.

[5] P. M. Anselone and L. B. Rall. The solution of characteristic value-vector problems by Newton's method. *Numer. Math.*, 11:38–45, 1968.

[6] P. F. Antonietti, A. Buffa, and I. Perugia. Discontinuous Galerkin approximation of the Laplace eigenproblem. *Comput. Methods Appl. Mech. Engrg.*, 195(25-28):3483–3503, 2006.

[7] A. Ashraf and C Rajakumar. *Boundary Element Method: Applications in Sound and Vibration*. A.A. Balkema, Lisse, 2004.

[8] I. Babuška and J. Osborn. Eigenvalue problems. In *Handbook of numerical analysis, Vol. II*, Handb. Numer. Anal., II, pages 641–787. North-Holland, Amsterdam, 1991.

[9] M. Bebendorf and S. Rjasanow. Adaptive low-rank approximation of collocation matrices. *Computing*, 70(1):1–24, 2003.

[10] J. H. Bramble and J. E. Osborn. Rate of convergence estimates for nonselfadjoint eigenvalue approximations. *Math. Comp.*, 27:525–549, 1973.

[11] A. Brandt, S. McCormick, and J. Ruge. Multigrid methods for differential eigenproblems. *SIAM J. Sci. Statist. Comput.*, 4(2):244–260, 1983.

[12] F. Chatelin. *Spectral approximation of linear operators*. Computer Science and Applied Mathematics. Academic Press Inc., New York, 1983.

[13] G. Chen and J. Zhou. *Boundary element methods*. Computational Mathematics and Applications. Academic Press Ltd., London, 1992.

[14] J. T. Chen, I. L. Chen, and J. W. Lee. Eigensolutions of multiply connected membranes using the method of fundamental solutions. *Eng. Anal. Bound. Elem.*, 29(2):166–174, 2005.

[15] J. T. Chen, S. Y. Lin, I. L. Chen, and Y. T. Lee. Mathematical analysis and numerical study to free vibrations of annular plates using BIEM and BEM. *Internat. J. Numer. Methods Engrg.*, 65(2):236–263, 2006.

[16] J. T. Chen and F. C. Wong. Dual formulation of multiple reciprocity method for the acoustic mode of a cavity with a thin partition. *J. Sound Vibration*, 217(1):75–95, 1998.

[17] M. Costabel. Boundary integral operators on Lipschitz domains: elementary results. *SIAM J. Math. Anal.*, 19(3):613–626, 1988.

[18] R. Courant and D. Hilbert. *Methoden der mathematischen Physik. I.* Springer-Verlag, Berlin, 1968.

[19] R. Dautray and J.-L. Lions. *Mathematical analysis and numerical methods for science and technology. Vol. 1*. Springer-Verlag, Berlin, 1990.

[20] P. Deuflhard. *Newton methods for nonlinear problems*, volume 35 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2004. Affine invariance and adaptive algorithms.

[21] M. Dobrowolski. *Angewandte Funktionalanalysis*. Springer, Berlin, 2006.

[22] M. Durán, M. Miguez, and J.-C. Nédélec. Numerical stability in the calculation of eigenfrequencies using integral equations. *J. Comput. Appl. Math.*, 130(1-2):323–336, 2001.

[23] M. Durán, J.-C. Nédélec, and S Ossandón. An efficient Galerkin BEM to compute high acoustic eigenfrequencies. *J. Vib. Acoust.*, 131(3):(31001)1–9, 2009.

[24] V. M. Eni. Stability of the root-number of an analytic operator-function and on perturbations of its characteristic numbers and eigenvectors. *Soviet Math. Dokl.*, 8:542–545, 1967. (Translation from *Dokl. Akad. Nauk SSSR* 173:1251–1254, 1967).

[25] C. M. Fan, C. C. Tsai, and D. L. Young. The method of fundamental solutions for plate viabrations. *CMC Comput. Mater. Continua*, 4:235–245, 2006.

[26] I. C. Gohberg, S. Goldberg, and M. A. Kaashoek. *Classes of linear operators. Vol. I*, volume 49 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1990.

[27] I. C. Gohberg and E. I. Sigal. An operator generalization of the logarithmic residue theorem and Rouché's theorem. *Math. USSR-Sb. 13*, 13:603–625, 1971.

[28] B. Gramsch. Meromorphie in der Theorie der Fredholmoperatoren mit Anwendungen auf elliptische Differentialoperatoren. *Math. Ann.*, 188:97–112, 1970.

[29] L. Greengard and V. Rokhlin. A fast algorithm for particle simulations. *J. Comput. Phys.*, 73(2):325–348, 1987.

[30] R. D. Grigorieff. Zur Theorie linearer approximationsregulärer Operatoren. I, II. *Math. Nachr.*, 55:233–249; ibid. 55 (1973), 251–263, 1973.

[31] R. D. Grigorieff and H. Jeggle. Approximation von Eigenwertproblemen bei nichlinearer Parameterabhängigkeit. *Manuscripta Math.*, 10:245–271, 1973.

[32] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[33] J.-S. Guo, W. W. Lin, and C. S. Wang. Numerical solutions for large sparse quadratic eigenvalue problems. *Linear Algebra Appl.*, 225:57–89, 1995.

[34] W. Hackbusch. On the computation of approximate eigenvalues and eigenfunctions of elliptic operators by means of a multi-grid method. *SIAM J. Numer. Anal.*, 16(2):201–215, 1979.

[35] W. Hackbusch. *Theorie und Numerik elliptischer Differentialgleichungen*. Teubner Studienbücher Mathematik. [Teubner Mathematical Textbooks]. B. G. Teubner, Stuttgart, 1986.

[36] W. Hackbusch. A sparse matrix arithmetic based on $\mathcal{H}$-matrices. I. Introduction to $\mathcal{H}$-matrices. *Computing*, 62(2):89–108, 1999.

[37] W. Hackbusch and Z. P. Nowak. On the fast matrix multiplication in the boundary element method by panel clustering. *Numer. Math.*, 54(4):463–491, 1989.

[38] S. Hildebrandt. Über die Lösung nichtlinear Eigenwertaufgaben mit dem Galerkinverfahren. *Math. Z.*, 101:255–264, 1967.

[39] E. Hille and R. S. Phillips. *Functional analysis and semi-groups*. American Mathematical Society, Providence, R. I., 1974.

[40] G. C. Hsiao and W. L. Wendland. A finite element method for some integral equations of the first kind. *J. Math. Anal. Appl.*, 58(3):449–481, 1977.

[41] G. C. Hsiao and W. L. Wendland. *Boundary integral equations*, volume 164 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin, 2008.

[42] N. K. Jain and K. Singhal. On Kublanovskaya's approach to the solution of the generalized latent value problem for functional λ-matrices. *SIAM J. Numer. Anal.*, 20(5):1062–1070, 1983.

[43] E. Jarlebring and H. Voss. Rational Krylov for nonlinear eigenproblems, an iterative projection method. *Appl. Math.*, 50(6):543–554, 2005.

[44] H. Jeggle and W. L. Wendland. On the discrete approximation of eigenvalue problems with holomorphic parameter dependence. *Proc. Roy. Soc. Edinburgh Sect. A*, 78(1-2):1–29, 1977/78.

[45] N. Kamiya and E. Andoh. Robust boundary element scheme for Helmholtz eigenvalue equation. In *Boundary elements, XIII (Tulsa, OK, 1991)*, pages 839–850. Comput. Mech., Southampton, 1991.

[46] N. Kamiya, E. Andoh, and K. Nogae. Eigenvalue analysis by the boundary element method: new developments. *Engng. Anal. Bound. Elms.*, 12:151–162, 1993.

[47] O. Karma. Approximation in eigenvalue problems for holomorphic Fredholm operator functions. I. *Numer. Funct. Anal. Optim.*, 17(3-4):365–387, 1996.

[48] O. Karma. Approximation in eigenvalue problems for holomorphic Fredholm operator functions. II. (Convergence rate). *Numer. Funct. Anal. Optim.*, 17(3-4):389–408, 1996.

[49] T. Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995.

[50] M. V. Keldysh. On the eigenvalues and eigenfunctions of certain classes of non–selfadjoint operators. *Dokl. Akad. Nauk SSSR*, 77:11–14, 1951.

[51] S. M. Kirkup and S. Amini. Solution of the Helmholtz eigenvalue problem via the boundary element method. *Internat. J. Numer. Methods Engrg.*, 36:321–330, 1993.

[52] M. Kitahara. *Boundary integral equation methods in eigenvalue problems of elastodynamics and thin plates*, volume 10 of *Studies in Applied Mechanics*. Elsevier Scientific Publishing Co., Amsterdam, 1985.

[53] A. V. Knyazev. Toward the optimal preconditioned eigensolver: locally optimal block preconditioned conjugate gradient method. *SIAM J. Sci. Comput.*, 23(2):517–541, 2001.

[54] A. V. Knyazev and K. Neymeyr. Efficient solution of symmetric eigenvalue problems using multigrid preconditioners in the locally optimal block conjugate gradient method. *Electron. Trans. Numer. Anal.*, 15:38–55, 2003. Tenth Copper Mountain Conference on Multigrid Methods (Copper Mountain, CO, 2001).

[55] V. Kozlov and V. Maz′ya. *Differential equations with operator coefficients with applications to boundary value problems for partial differential equations*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1999.

[56] V. N. Kublanovskaya. On an approach to the solution of the generalized latent value problem for λ-matrices. *SIAM J. Numer. Anal.*, 7:532–537, 1970.

[57] H. Kummer. Zur praktischen Behandlung nichtlinearer Eigenwertaufgaben abgeschlossener linearer Operatoren. *Mitt. Math. Sem. Giessen*, 62, 1964.

[58] U. Langer. Untersuchungen zum Kummerschen Verfahren zur numerischen Behandlung nichtlinearer Eigenwertaufgaben. *Beiträge Numer. Math.*, 6:97–110, 1977.

[59] I. Livshits. An algebraic multigrid wave-ray algorithm to solve eigenvalue problems for the Helmholtz operator. *Numer. Linear Algebra Appl.*, 11(2-3):229–239, 2004.

[60] Y. Y. Lu and S.-T. Yau. Eigenvalues of the Laplacian through boundary integral equations. *SIAM J. Matrix Anal. Appl.*, 12(3):597–609, 1991.

[61] W. McLean. *Strongly elliptic systems and boundary integral equations*. Cambridge University Press, Cambridge, 2000.

[62] K. Meerbergen. Locking and restarting quadratic eigenvalue solvers. *SIAM J. Sci. Comput.*, 22(5):1814–1839 (electronic), 2000.

[63] V. Mehrmann and H. Voss. Nonlinear eigenvalue problems: a challenge for modern eigenvalue methods. *GAMM Mitt. Ges. Angew. Math. Mech.*, 27(2):121–152 (2005), 2004.

[64] R. Mennicken and M. Möller. Root functions, eigenvectors, associated vectors and the inverse of a holomorphic operator function. *Arch. Math. (Basel)*, 42(5):455–463, 1984.

[65] J.-C. Nédélec. Curved finite element methods for the solution of integral singular equations on surfaces in $\mathbf{R}^3$. In *Computing methods in applied sciences and engineering (Second Internat. Sympos., Versailles, 1975), Part 1*, pages 374–390. Lecture Notes in Econom. and Math. Systems, Vol. 134. Springer, Berlin, 1976.

[66] J.-C. Nédélec. *Acoustic and electromagnetic equations*, volume 144 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2001.

[67] A. Neumaier. Residual inverse iteration for the nonlinear eigenvalue problem. *SIAM J. Numer. Anal.*, 22(5):914–923, 1985.

[68] K. Neymeyr. A hierarchy of preconditioned eigensolvers for elliptic differential operators. Habilitation, University Tübingen, 2001.

[69] A. J. Nowak and A. C. Neves, editors. *The Multiple Reciprocity Boundary Element Method*. International Series on Computational Engineering. Computational Mechanics Publications, Southampton, 1994.

[70] S. Rjasanow and O. Steinbach. *The fast solution of boundary integral equations*. Springer, New York, 2007.

[71] K. Rothe. *Lösungsverfahren für nichtlineare Matrixeigenwertaufgaben mit Anwendungen auf die Ausgleichselementmethode*. Verlag an der Lottbek, Hamburg, 1989.

[72] W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[73] A. Ruhe. Algorithms for the nonlinear eigenvalue problem. *SIAM J. Numer. Anal.*, 10:674–689, 1973.

[74] A. Ruhe. A rational Krylov algorithm for nonlinear matrix eigenvalue problems. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 268(Chisl. Metody i Vopr. Organ. Vychisl. 14):176–180, 245–246, 2000.

[75] Y. Saad. *Numerical methods for large eigenvalue problems*. Algorithms and Architectures for Advanced Scientific Computing. Manchester University Press, Manchester, 1992.

[76] P. Sarreither. Über das Wachstum von Resolventen in der Nähe einer isolierten Singularität. *Manuscripta Math.*, 11:261–272, 1974.

[77] S. A. Sauter and C. Schwab. *Randelementmethoden. Analyse, Numerik und Implementierung schneller Algorithmen*. B. G. Teubner, Stuttgart, 2004.

[78] A. H. Schatz, V. Thomée, and W. L. Wendland. *Mathematical theory of finite and boundary element methods*. Birkhäuser Verlag, Basel, 1990.

[79] J. Schoeberl. Finite element package Netgen/NGSolve. www.hpfem.jku.at.

[80] K. Schreiber. *Nonlinear Eigenvale Problems: Newton-type Methods and Nonlinear Rayleigh Functionals*. Dissertation, TU Berlin, 2008.

[81] R. P. Shaw and G. R. C. Tai. Helmholtz–equation eigenvalues and eigenmodes for arbitrary domains. *J. Acoust. Soc. Am.*, 56(3):796–804, 1974.

[82] G. L. G. Sleijpen, A. G. L. Booten, D. R. Fokkema, and H. A. Van der Vorst. Jacobi-Davidson type methods for generalized eigenproblems and polynomial eigenproblems. *BIT*, 36(3):595–633, 1996.

[83] O. Steinbach. *Numerical approximation methods for elliptic boundary value problems. Finite and boundary elements*. Springer, New York, 2008.

[84] O. Steinbach and G. Unger. A boundary element method for the Dirichlet eigenvalue problem of the Laplace operator. *Numer. Math.*, 113:281–298, 2009.

[85] G. Strang and G. J. Fix. *An analysis of the finite element method*. Prentice-Hall Inc., Englewood Cliffs, N. J., 1973. Prentice-Hall Series in Automatic Computation.

[86] F. Stummel. Diskrete Konvergenz linearer Operatoren. I. *Math. Ann.*, 190:45–92, 1970/71.

[87] L. Tartar. *An introduction to Sobolev spaces and interpolation spaces*. Springer, Berlin, 2007.

[88] H. Unger. Nichtlineare Behandlung von Eigenwertaufgaben. *Z. Angew. Math. Mech.*, 30:281–282, 1950.

[89] G. M. Vainikko. On the rate of convergence of certain approximation methods of Galerkin type in an eigenvalue problem. *Amer. Math. Soc. Transl.*, 36:249–259, 1970.

[90] G. M. Vainikko. *Funktionalanalysis der Diskretisierungsmethoden*. B. G. Teubner Verlag, Leipzig, 1976.

[91] G. M. Vainikko and O. Karma. The rate of convergence of approximation methods for an eigenvalue problem in which the parameter occurs nonlinearly. *U.S.S.R. Computational Math. and Math. Phys.*, 14:23–39, 1974.

[92] H. Voss. An Arnoldi method for nonlinear eigenvalue problems. *BIT*, 44(2):387–401, 2004.

[93] H. Voss. A Jacobi-Davidson method for nonlinear eigenproblems. In *Computational science—ICCS 2004. Part II*, volume 3037 of *Lecture Notes in Comput. Sci.*, pages 34–41. Springer, Berlin, 2004.

[94] H. Voss. Numerical methods for sparse nonlinear eigenvalue problems. In *Proc. of the XV-th Summer School on Software and Algorithms of Numerical Mathematics, Hejnice, Czech Republic*, pages 133–160. University of West Bohemia, 2004.

[95] H. Voss. A minmax principle for nonlinear eigenvalue problems depending continuously on the eigenparameter. Technical Report 112, Arbeitsbereich Mathematik, TU Hamburg-Harburg, 2008.

[96] H. Voss and B. Werner. Solving sparse nonlinear eigenvalue problems. Technical Report 82/4, Inst. f. Angew. Mathematik, Universität Hamburg, 1982.

[97] W. L. Wendland. Bemerkungen zu Randelementmethoden bei Rissproblemen. In *Mathematical papers given on the occasion of Ernst Mohr's 75th birthday*, pages 307–322. Tech. Univ. Berlin, Berlin, 1985.

[98] D. Werner. *Funktionalanalysis*. Springer-Verlag, Berlin, 1997.

[99] J. Wloka. *Funktionalanalysis und Anwendungen*. Walter de Gruyter, Berlin-New York, 1971.

[100] R. Wolf. Zur Stabilität der algebraischen Vielfachheit von Eigenwerten von holomorphen Fredholm-Operatorfunktionen. *Applicable Anal.*, 9(3):165–177, 1979.

[101] K. Yosida. *Functional analysis*. Springer-Verlag, Berlin, 1995.

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