3.3.3 A Remarkable Expression for the Density

Assume the body to consist of n layers bounded by surfaces S_k and S_{k+1} (Fig. 3.3). The density within each layer is constant, denoted in our case by ρ_{k+1}.

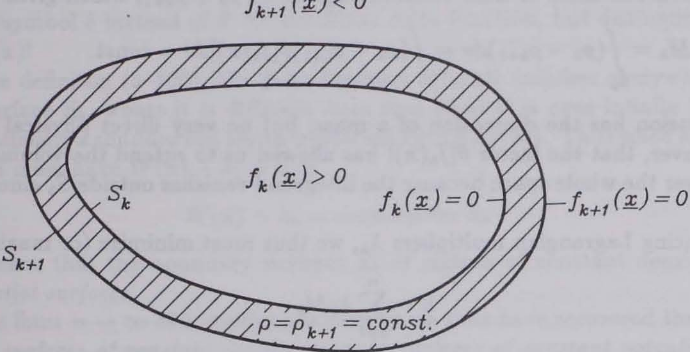

FIGURE 3.3: A layer of constant density ($\underline{x}$ denotes $\mathbf{x}$)

Let the surface S_k have the equation

$$f_k(\mathbf{x}) = 0 \quad , \tag{3–105}$$

and let f_k be monotonic with

$$f_k(\mathbf{x}) > 0 \qquad \text{inside } S_k \tag{3–106}$$

(otherwise change the sign of f_k!).

Then the density everywhere within the stratified body can be described by the single expression

$$\rho(\mathbf{x}) = \sum_{k=1}^{n} (\rho_k - \rho_{k+1}) \theta[f_k(\mathbf{x})] \quad . \tag{3–107}$$

The reader is invited to verify this on the basis of (3–103) and (3–106). Eq. (3–107) holds with the understanding that $\rho_{n+1} = 0$ since the density is zero outside the boundary surface $S = S_n$.

3.3.4 Variation of the Potential Energy

Let us find the extremum of the potential energy $E = E_W$ as given by (3–99):

$$E = \int \left(\frac{1}{2} V + \Phi \right) \rho dv \quad , \tag{3–108}$$

where ρ is expressed by (3–107); since $\rho = 0$ outside S, we may extend the integral formally over the whole space. The *side condition* is that the volume enclosed by