

4.1 Internal Potential

The gravitational potential at a point P in the interior of the body bounded by the surface S is

$$V(P) = \iiint G \frac{\rho}{l} dv = \iiint_{I_P} + \iiint_{E_P} = V_i(P) + V_e(P) \quad , \quad (4-6)$$

where I_P denotes the interior of the surface S_P of constant density labeled, as usual, by a parameter q , and E_P denotes its exterior, that is, the layer between S_P and S (Fig. 4.2).

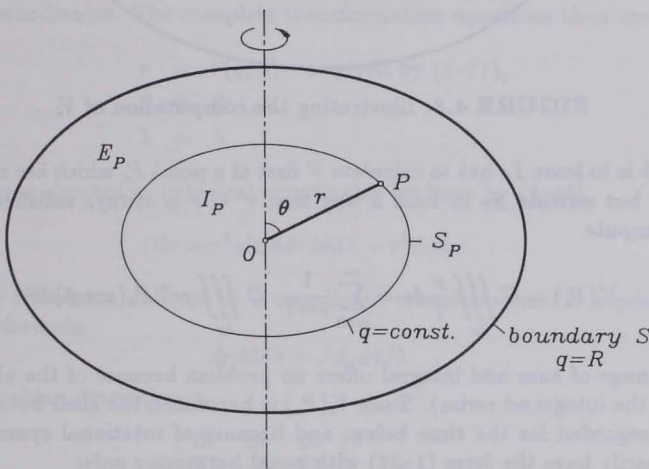


FIGURE 4.2: Illustrating the computation of $V(P)$

4.1.1 Potential of Interior I_P

Consider first only

$$V_i = G \iiint_{I_P} \frac{\rho}{l} dv \quad . \quad (4-7)$$

For $1/l$ we have the usual series

$$\frac{1}{l} = \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n(\cos \psi) \quad , \quad (4-8)$$

which converges if $r' < r$. The problem is that for $r = r_P = OP$ (Fig. 4.3), this convergence condition may be violated: r' may be greater than r .

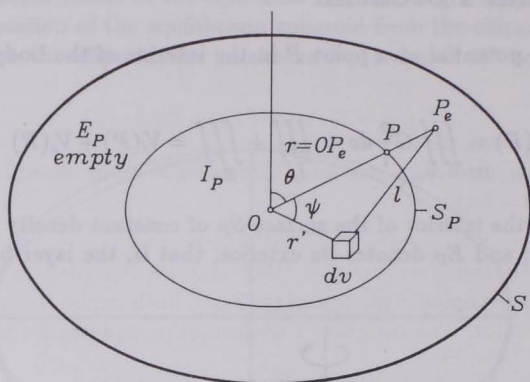


FIGURE 4.3: Illustrating the computation of V_i

The trick is to leave I_P but to calculate V first at a point P_e which lies on the radius vector of P but outside S_P in such a way that $r' < r$ is always satisfied (Fig. 4.3). Thus we compute

$$V_i(P_e) = G \iiint_{I_P} \frac{\rho}{l} dv = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \cdot G \iiint_{I_P} \rho r^n P_n(\cos \psi) dv \quad (4-9)$$

(the interchange of sum and integral offers no problem because of the absolute convergence of the integrand series). Since $V_i(P_e)$ is harmonic, the shell between S_P and S being disregarded for the time being, and because of rotational symmetry, (4-9) must necessarily have the form (1-37) with zonal harmonics only:

$$V_i(P_e) = \sum_{n=0}^{\infty} \frac{K_n}{r^{n+1}} P_n(\cos \theta)$$

or

$$V_i(P_e) = \frac{K_0(q)}{r} + \frac{K_2(q)}{r^3} P_2(\cos \theta) + \frac{K_4(q)}{r^5} P_4(\cos \theta) \quad , \quad (4-10)$$

neglecting higher-order terms. Here r , θ , λ are the spherical coordinates of P_e as usual; because of rotational symmetry there is no explicit dependence on longitude λ (no tesseral terms); and there are only even-degree zonal terms because of symmetry with respect to the equatorial plane. The coefficients K_n evidently depend on S_P and hence on its label q .

4.1.2 Change of Variable

The equation of any surface of constant density may be written as