their capacity as orthogonal functions *on a line interval*, $-1 \leq \beta \leq 1$ or $0 \leq \beta \leq 1$, respectively.)

Finally, (8–150) gives the density contrast $\Delta\rho$ corresponding to isostatic compensation. The plane approximation to the present theory may be found in (Dorman and Lewis, 1970). For practical results see, e.g., (Lewis and Dorman, 1970), (Bechtel et al., 1987) and (Hein et al., 1989).

8.3.2 The Inverse Vening Meinesz Problem

Here the density contrast $\Delta\rho$ is considered constant but the Moho depth T is to be determined from the condition (8–113) or, equivalently, from the given attraction A_C which the compensating masses exert at sea level, cf. (8–114).

Let us thus compute the attraction A_C of the compensating masses, bounded by the sphere $r = R - T_0$ representing the "normal Moho" (corresponding to a normal crustal thickness around $T_0 = 30\,\text{km}$ as mentioned in sec. 8.1.2) and the actual Moho, assuming constant density contrast:

$$\Delta\rho = \text{const.} \tag{8–165}$$

The corresponding potential is expressed by

$$V_C(P) = G\Delta\rho \iint_\sigma \int_{r=R-T}^{R-T_0} \frac{1}{l}\, r^2 dr\, d\sigma \quad , \tag{8–166}$$

again using (8–123), without primes, for the volume element dv. Further, by Fig. 8.15, we have

$$l^2 = R^2 + r^2 - 2Rr\cos\psi \quad . \tag{8–167}$$

The attraction is

$$A_C = -\frac{\partial V_C}{\partial R} = -G\Delta\rho \iint_\sigma \int_{r=R-T}^{R-T_0} \frac{\partial}{\partial R}\left(\frac{1}{l}\right) r^2 dr\, d\sigma \quad , \tag{8–168}$$

considering (for one moment only!) R in the integrand as variable. The limits of integration remain unchanged because, as the point P can be imagined to move in conformity with $\partial/\partial R$, the layer between $r = R - T$ and $r = R - T_0$ stays in place.

Changing the upper limit to R only implies the addition of a constant since

$$\int_{R-T}^{R-T_0} = \int_{R-T}^{R} - \int_{R-T_0}^{R} \quad ,$$

and the last integral is easily seen to be a global constant over the sphere $r = R$: it represents the attraction of a spherical shell of constant density bounded by the two

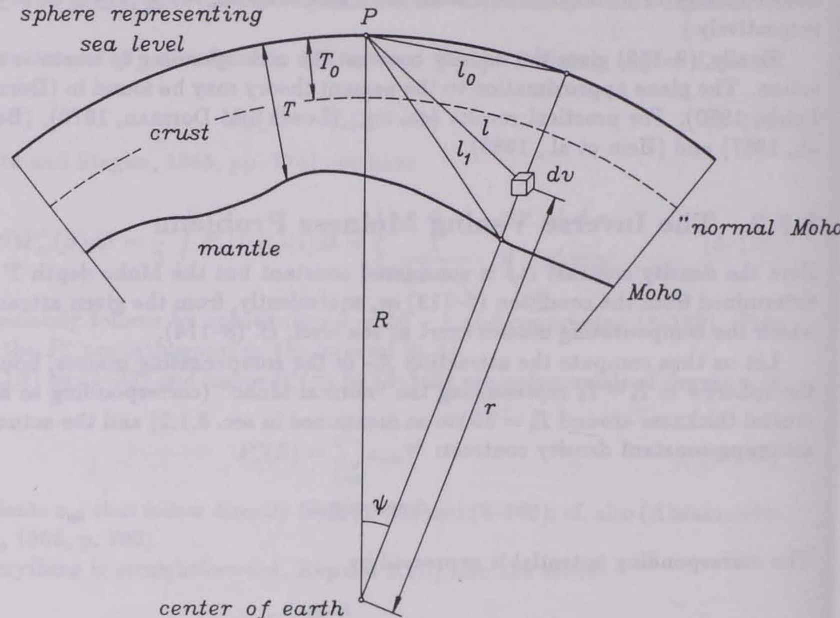

FIGURE 8.15: Notations for the inverse Vening Meinesz problem

concentric spheres $r = R - T_0$ and $r = R$. Disregarding this constant, which will be justified later, we may thus replace (8–168) by

$$A_C = -G\Delta\rho \iint_\sigma \int_{r=R-T}^{R} \frac{\partial}{\partial R}\left(\frac{1}{l}\right) r^2 dr d\sigma \quad . \tag{8–169}$$

Now, to a very good approximation

$$\frac{\partial}{\partial R}\left(\frac{1}{l}\right) = -\frac{\partial}{\partial r}\left(\frac{1}{l}\right) \quad . \tag{8–170}$$

This can be seen because if the sphere is replaced by a plane, the xy-plane, then the distance l between two points (x, y, z) and (x', y', z') is given by

$$l = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \quad ,$$

and

$$\frac{\partial l}{\partial z} = -\frac{\partial l}{\partial z'}$$

is immediately verified by direct computation. In the spherical case, (8–170) holds as a "planar approximation" (sec. 8.2.1); to the same approximation we may replace r^2

by R^2, in view of (8–44). Thus (8–169) becomes

$$A_C = G\Delta\rho R^2 \iint_\sigma \int_{r=R-T}^{R} \frac{\partial}{\partial r}\left(\frac{1}{l}\right) dr\, d\sigma \quad , \qquad (8\text{–}171)$$

and the integration with respect to r can be performed immediately, giving

$$A_C = G\Delta\rho R^2 \iint_\sigma \left(\frac{1}{l_0} - \frac{1}{l_1}\right) d\sigma \quad , \qquad (8\text{–}172)$$

l_0 and l_1 being shown in Fig. 8.15; cf. also eq. (8–78).

Now, by (1–53) we have

$$\frac{1}{l_1} = \sum_{n=0}^{\infty} \frac{(R-T)^n}{R^{n+1}} P_n(\cos\psi) \qquad (8\text{–}173)$$

and, formally, since now $r = R$,

$$\frac{1}{l_0} = \sum_{n=0}^{\infty} \frac{R^n}{R^{n+1}} P_n(\cos\psi) \quad . \qquad (8\text{–}174)$$

Introducing the auxiliary quantities

$$H^{(n)} = \frac{R^n - (R-T)^n}{R^n} = 1 - \left(1 - \frac{T}{R}\right)^n \quad , \qquad (8\text{–}175)$$

we may thus write (8–172) as

$$A_C = G\Delta\rho R \iint_\sigma \sum_{n=0}^{\infty} H^{(n)} P_n(\cos\psi) d\sigma \quad . \qquad (8\text{–}176)$$

We expand the function $H^{(n)}$ as a series of Laplace spherical harmonics:

$$H^{(n)}(\theta, \lambda) = \sum_{n'=0}^{\infty} H_{n'}^{(n)}(\theta, \lambda) \quad , \qquad (8\text{–}177)$$

with the degree now denoted by n'. Then the terms with $n' \neq n$ in (8–176) are removed by orthogonality, and by the integral formula (1–49) we get with the only remaining terms for which $n' = n$:

$$A_C = 4\pi G\Delta\rho R \sum_{n=0}^{\infty} \frac{H_n^{(n)}(\theta, \lambda)}{2n+1} \quad . \qquad (8\text{–}178)$$

Since

$$\frac{T}{R} < \frac{60\,\text{km}}{6000\,\text{km}} = 0.01 \quad ,$$

the binomial series for $(1 - T/R)^n$ in (8–175) will converge, and $H^{(n)}$ becomes

$$\begin{aligned} H^{(n)} &= n\frac{T}{R} - \binom{n}{2}\left(\frac{T}{R}\right)^2 + \binom{n}{3}\left(\frac{T}{R}\right)^3 - + \dots \\ &= n\tau - \binom{n}{2}\tau^2 + \binom{n}{3}\tau^3 \dots \quad , \end{aligned} \tag{8–179}$$

putting

$$\tau = \frac{T}{R} = \sum_{n=0}^{\infty} \tau_n(\theta, \lambda) \quad . \tag{8–180}$$

Thus (8–178) assumes the form (there is no term $n = 0$):

$$A_C = 4\pi G \Delta\rho R \left[\sum_{n=1}^{\infty} \frac{n}{2n+1} \tau_n - \sum_{n=1}^{\infty} \frac{n(n-1)}{2(2n+1)} (\tau^2)_n + \right. \\ \left. + \sum_{n=1}^{\infty} \frac{n(n-1)(n-2)}{6(2n+1)} (\tau^3)_n \cdots \right] \quad . \tag{8–181}$$

This will be our basic formula. Its meaning is the following. Take the Moho depth T and divide by R to get

$$\tau = f_1(\theta, \lambda) \quad . \tag{8–182}$$

Raise this function to the second, third, etc., powers:

$$\tau^2 = [\tau(\theta, \lambda)]^2 = f_2(\theta, \lambda) \quad , \tag{8–183}$$

$$\tau^3 = [\tau(\theta, \lambda)]^3 = f_3(\theta, \lambda) \quad , \tag{8–184}$$

$$\dots\dots\dots$$

all being functions of θ and λ. Now τ_n $[= \tau_n(\theta, \lambda)]$ is the n-th Laplace surface harmonic, given by (1–49), of the function (8–182), $(\tau^2)_n$ is the Laplace surface harmonic of the function (8–183), $(\tau^3)_n$ of (8–184), and so on.

Expand also A_C as a series of Laplace harmonics of type (1–48):

$$\frac{A_C}{4\pi G \Delta\rho R} = \sum_{n=1}^{\infty} a_n(\theta, \lambda) \quad . \tag{8–185}$$

This expression starts with $n = 1$: there must be no constant term for which $n = 0$. This means that any non-zero global average must be subtracted. This procedure also removes the constant introduced by the transition from (8–168) to (8–169), which finally justifies this transition.

Then (8–181) shows that

$$a_n(\theta, \lambda) = \frac{n}{2n+1} \tau_n - \frac{n(n-1)}{2(2n+1)} (\tau^2)_n + \frac{n(n-1)(n-2)}{6(2n+1)} (\tau^3)_n \cdots \quad , \tag{8–186}$$

relating the known attraction $A_C(\theta, \lambda)$ to the unknown Moho depth $T(\theta, \lambda)$.

This equation can be solved iteratively, writing it as

$$\tau_n(\theta, \lambda) = \frac{2n+1}{n} a_n + \frac{n-1}{2} (\tau^2)_n - \frac{(n-1)(n-2)}{6} (\tau^3)_n \cdots \quad , \tag{8–187}$$

and

$$\tau(\theta, \lambda) = \sum_{n=1}^{\infty} \left[\frac{2n+1}{n} a_n + \frac{n-1}{2} (\tau^2)_n - \frac{(n-1)(n-2)}{6} (\tau^3)_n \cdots \right] \quad . \tag{8–188}$$

As a first approximation we disregard τ^2, τ^3, ..., obtaining

$$\tau_{\text{approx}} = \sum_{n=1}^{\infty} \frac{2n+1}{n} a_n \quad . \tag{8–189}$$

This approximate value is applied to compute approximate functions τ^2, τ^3, ... These functions are expanded into series of Laplace harmonics which are then used on the right-hand side of (8–188) to compute a better left-hand side $\tau(\theta, \lambda)$. This procedure can be repeated as necessary, hoping that it converges .

An integral formula for the principal term. As the series in (8–188) converge slowly, it is preferable to convert them to integral formulas.

Since by (8–185)

$$\sum_{n=1}^{\infty} a_n(\theta, \lambda) = \frac{A_C}{4\pi G \Delta\rho R} \equiv a(\theta, \lambda) \quad , \tag{8–190}$$

we have

$$\sum \frac{2n+1}{n} a_n = 2 \sum a_n + \sum \frac{a_n}{n} = 2a(\theta, \lambda) + \sum \frac{a_n}{n} \quad . \tag{8–191}$$

Now, by (1–49),

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{a_n}{n} &= \sum \frac{1}{n} \frac{2n+1}{4\pi} \iint_{\sigma} a(\theta', \lambda') P_n(\cos\psi) d\sigma \\ &= \iint_{\sigma} a(\theta', \lambda') M(\psi) d\sigma \quad , \end{aligned} \tag{8–192}$$

where

$$M(\psi) = \frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{2n+1}{n} P_n(\cos\psi) \quad . \tag{8–193}$$

Putting, according to (Moritz, 1980, p. 182)

$$\frac{1}{L} = \frac{1}{\sqrt{1 - 2qt + q^2}} = \sum_{n=0}^{\infty} q^n P_n(t) \tag{8–194}$$

with $q < 1$ and

$$t = \cos\psi \quad , \tag{8–195}$$

as well as

$$M(q,\psi) = \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{2n+1}{n}\,q^n P_n(t) \quad , \tag{8–196}$$

we get

$$\begin{aligned} M(q,\psi) &= \frac{1}{4\pi}\left[2\sum_{n=1}^{\infty} q^n P_n(t) + \sum_{n=1}^{\infty}\frac{1}{n}\,q^n P_n(t)\right] \\ &= \frac{1}{4\pi}\left[-2 + 2\sum_{n=0}^{\infty} q^n P_n(t) + \sum_{n=1}^{\infty}\frac{1}{n} q^n P_n(t)\right] \end{aligned} \tag{8–197}$$

or, by (eqs. (23–29) and (23–31), *ibid.*, p. 185),

$$M(q,\psi) = \frac{1}{4\pi}\left(-2 + \frac{2}{L} + \ln\frac{2}{N}\right) \tag{8–198}$$

with (*ibid.*, eq. (23–32))

$$N = 1 + L - q\cos\psi \quad . \tag{8–199}$$

In these formulas we may put $q = 1$ ($q < 1$ has served only as an auxiliary "convergence factor") to obtain

$$L_0 = 2\sin\frac{\psi}{2} \quad , \tag{8–200}$$

$$N_0 = 2\left(\sin^2\frac{\psi}{2} + \sin\frac{\psi}{2}\right) \quad , \tag{8–201}$$

so that (8–198) and hence (8–193) become

$$M(\psi) = \frac{1}{4\pi}\left[\frac{1}{\sin\frac{\psi}{2}} - 2 - \ln\left(\sin^2\frac{\psi}{2} + \sin\frac{\psi}{2}\right)\right] \quad , \tag{8–202}$$

which shows some similarity to Stokes' function (Heiskanen and Moritz, 1967, eq. (2–164)).

Secondary terms. Consider now the second term on the right-hand side of (8–188)

$$\mathrm{II} = \frac{1}{2}\sum_{n=1}^{\infty}(n-1)(\tau^2)_n = \frac{1}{2}\sum_{n=1}^{\infty} n(\tau^2)_n - \frac{1}{2}\sum_{n=1}^{\infty}(\tau^2)_n \quad . \tag{8–203}$$

This is equivalent to (the sum may start with zero now)

$$\mathrm{II} = -\frac{1}{2}\tau^2 + \frac{1}{2}\sum_{n=0}^{\infty} n(\tau^2)_n \quad . \tag{8–204}$$

Now the integral formula (1–102) of (Heiskanen and Moritz, 1967, p. 39) comes in handy. With $V = \tau^2$, $R = 1$, and $l_0 = 2\sin\frac{\psi}{2}$ we thus get

$$-\sum_{n=0}^{\infty} n(\tau^2)_n = \frac{1}{16\pi}\iint_{\sigma}\frac{\tau^2 - \tau_P^2}{\sin^3\frac{\psi}{2}}\,d\sigma \equiv L_1(\tau^2) \quad , \tag{8–205}$$

where (in the integral only) τ_P^2 refers to the point P at which II is to be computed, and τ^2 to the surface element $d\sigma$; ψ is the spherical distance between P and $d\sigma$. Thus (8–204) becomes simply

$$\mathrm{II} = -\frac{1}{2}\tau^2 - \frac{1}{32\pi}\iint_\sigma \frac{\tau^2 - \tau_P^2}{\sin^3\frac{\psi}{2}}\, d\sigma \quad . \tag{8–206}$$

Finally we consider the last term in (8–188):

$$\mathrm{III} = -\sum_{n=1}^{\infty} \frac{(n-1)(n-2)}{6}(\tau^3)_n \quad . \tag{8–207}$$

This term being very small, we may retain the highest power of n only, so that, to a sufficient approximation,

$$\mathrm{III} = -\frac{1}{6}\sum_{n=0}^{\infty} n^2(\tau^3)_n \quad . \tag{8–208}$$

Now we perform a particularly insidious trick, which, however, is familiar to some people in physical geodesy. Multiplication of the spectrum by n corresponds to the (negative) integral operator L_1 defined by (8–205). Multiplication of the spectrum by n^2 thus means applying the operator L_1 twice. Thus, with

$$L_2 = \frac{1}{2}L_1^2 \tag{8–209}$$

(Moritz, 1980, p. 385, eq. (45–37)) we get

$$\mathrm{III} = -\frac{1}{3}L_2(\tau^3) \quad , \tag{8–210}$$

which, by (*ibid.*, eqs. (45–36), (45–35), and (45–34)), becomes with $\theta = 90° - \phi$ and $R = 1$

$$\mathrm{III} = \frac{1}{6}\left(\frac{\partial^2\tau^3}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2\tau^3}{\partial\lambda^2} + \frac{\partial\tau^3}{\partial\theta}\cot\theta\right) \tag{8–211}$$

in spherical coordinates (θ, λ), or simply

$$\mathrm{III} = \frac{R^2}{6}\left(\frac{\partial^2\tau^3}{\partial x^2} + \frac{\partial^2\tau^3}{\partial y^2}\right) \tag{8–212}$$

in a local system xy in the tangential plane. The reader will, of course, recognize the Laplacian surface operator in the plane (8–212) and on the sphere (8–211).

By the way, the simplifications involved in the transition from (8–207) to (8–208) precisely correspond to the "planar approximation", as the reader may verify.

Using (8–191), (8–192), (8–206), and (8–212), eq. (8–188) becomes

$$\tau(\theta, \lambda) = 2a(\theta, \lambda) + \iint_\sigma a(\theta', \lambda')M(\psi)d\sigma - \frac{1}{2}\tau^2 - \\ - \frac{1}{32\pi}\iint_\sigma \frac{\tau^2 - \tau_P^2}{\sin^3\frac{\psi}{2}}\, d\sigma + \frac{R^2}{6}\left(\frac{\partial^2\tau^3}{\partial x^2} + \frac{\partial^2\tau^3}{\partial y^2}\right) \quad , \tag{8–213}$$

as our final equation (which may be new) for determining the Moho depth $\tau = T/R$ from the attraction A_C of a regional isostatic compensation reaching with constant density contrast $\Delta\rho$ to depth T. Eq. (8–213) is dimensionless; the quantity $a(\theta, \lambda)$ is related to the attraction A_C by (8–190), and the function $M(\psi)$ is given by (8–202).

Eq. (8–213) lends itself to an iterative solution which can be described as follows. Given A_C, we compute $a(\theta, \lambda)$ by (8–190). A first approximation for $\tau(\theta, \lambda)$ is obtained by disregarding in (8–213) the terms τ^2 and τ^3. These terms can then be approximately computed by raising the approximate function $\tau(\theta, \lambda)$ to the second and third powers. The functions $\tau^2(\theta, \lambda)$ and $\tau^3(\theta, \lambda)$ may be used on the right-hand side of (8–213) to compute a better approximation to $\tau(\theta, \lambda)$. The iteration may be repeated as necessary.

Since already the last term in (8–213) is very local and, above all, extremely sensitive to data noise, a further approximation (to τ^4, etc.), although possible in principle, probably will not make much sense.

The convergence behavior seems to be similar to that of the Molodensky series well known in physical geodesy: although the series may not be convergent in a mathematical sense, it is probably "practically convergent" in the sense that the first few terms give a good approximation provided the data are suitably smoothed. For a general discussion of such cases see (Moritz 1980, pp. 413–414).

Note that neither (8–188) nor (8–193) contain a term $n = 0$, so that the present method defines the Moho depth T only up to an additive global constant or, geometrically speaking, up to a constant vertical shift. This shift can obviously be determined from seismic observations.

Finally we note that the plane approximation of this problem with the geoid or terrestrial sphere replaced by a plane, is well known, especially in applied geophysics (cf. Parker, 1972; Oldenburg, 1974; Granser, 1986, 1987), and has also been applied to the determination of the Moho (Granser, 1988). The present approach is spherical, corresponding to a global inverse problem.

8.3.3 Concluding Remarks

Some isostatic compensation exists without any doubt whatsoever. This is plausible physically and has recently been confirmed on a global scale by Sünkel (1985; 1986b, p. 450), who has shown that the "degree variances" (cf. Heiskanen and Moritz, 1967, p. 259), which describe the average power of the gravitational spectrum, from degree 15 or 20 onwards can almost completely be explained by the combined effect of topography and compensation; cf. also (Rummel et al., 1988). The lower harmonics, of course, come almost exclusively from the mantle; and harmonics of the very highest frequencies are due to uncompensated local topography.

Besides this global result, it is surprising that even the Alps seem to be relatively well compensated: isostatic reduction considerably diminishes the size of gravity anomalies and deflections of the vertical, cf. (Sünkel, 1987, p. 62); see also (Wagini et al., 1988) and (Steinhauser and Pustizek, 1987).