by (8–65) and (8–92), since $V \doteq V_C$ and hence $(V - V_C)/2R$ is very nearly zero; and B is associated with the factor 2π and not 4π, as (8–76) shows.

8.3 Inverse Problems in Isostasy

Consider Pratt's model (sec. 8.1.1). The compensation takes place along vertical columns; this is *local compensation.* There is a *variable* density contrast $\Delta\rho$ given in terms of elevation h by (8–3). The corresponding isostatic gravity anomaly Δg_I (8–37) will in general not be zero, partly because of imperfections in the model. The inverse problem consists in trying to make

$$\Delta g_I \equiv 0 \tag{8–113}$$

by *determining a suitable distribution* $\Delta\rho(z)$ *of the density anomaly* in each vertical column.

On the other hand, consider isostatic models of Airy and Vening Meinesz type. Here the density contrast $\Delta\rho$ is *constant*, but the Moho depth T is variable, depending on the topography locally (Airy) or regionally (Vening Meinesz) in a prescribed way (now T and T_0 are again used in the sense of sec. 8.1!). Here the inverse problem would consist in making Δg_I zero by *determining a suitable variable Moho depth* T for a prescribed constant density contrast $\Delta\rho$, which need not be $0.6\,\text{g/cm}^3$ but can be any given value between 0 and $0.7\,\text{g/cm}^3$ (say).

Rather than making Δg_I zero, we may also prescribe the Bouguer anomaly field. This amounts to the same since by (8–37), $\Delta g_I = 0$ implies

$$A_C = -\Delta g_B \quad . \tag{8–114}$$

So the problem is in fact: given A_C, to determine the compensating masses that produce it. In the inverse Pratt problem this is done by seeking an appropriate density contrast $\Delta\rho$, in the inverse Vening Meinesz problem this is achieved by suitably selecting the Moho depth T. Thus we have genuine inverse problems (with given constraints) in the sense of Chapter 7 (cf. also Barzaghi and Sansò, 1986).

8.3.1 The Inverse Pratt Problem

The basic paper is (Dorman and Lewis, 1970). Consider a column defined by fixing the spherical coordinates (θ, λ); the column extends from the earth's surface radially to the earth's center (theoretically: this corresponds to $D = R$ in sec. 8.1.1). In each column $\Delta\rho$ is a function of the radius vector r (or of depth), which accounts for the functional dependence

$$\Delta\rho = \Delta\rho(r, \theta, \lambda) \quad . \tag{8–115}$$

One assumes $\Delta\rho$ to be linearly related to the topography (height h) by a *"convolution"*

$$\Delta\rho(r', \theta', \lambda') = \iint_\sigma h(\theta'', \lambda'')K(r', \psi')d\sigma \quad , \tag{8–116}$$

where the "kernel" K, as far as dependence on θ, λ is concerned, is *isotropic*: it depends only on the spherical distance ψ', where

$$\cos\psi' = \cos\theta' \cos\theta'' + \sin\theta' \sin\theta'' \cos(\lambda'' - \lambda') \quad , \tag{8–117}$$

between the points (θ', λ') and (θ'', λ'') on the unit sphere (Fig. 8.14); the author

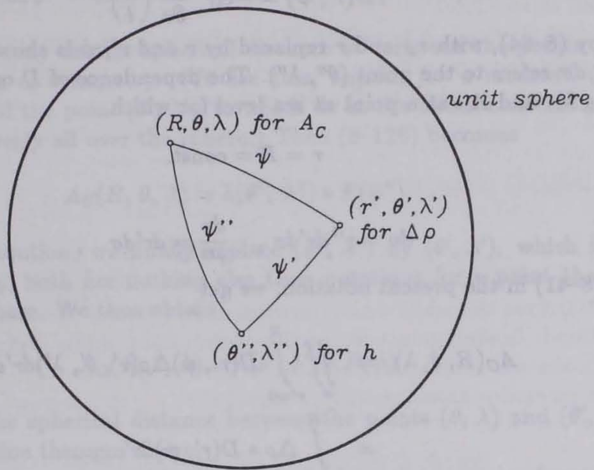

FIGURE 8.14: Various points on the sphere that play a role in the theory of Dorman and Lewis

apologizes for the clumsy notation with primes and double primes. Furthermore, K depends on depth through the radius vector r'. (The concept of "kernel" used here is, of course, completely different from that in sec. 7.2!)

Symbolically we may write the convolution (8–116) in a standard way as

$$\Delta\rho(r', \theta', \lambda') = h(\theta'', \lambda'') * K(r', \psi') \quad \text{or} \quad \Delta\rho = h * K \quad . \tag{8–118}$$

Eq. (8–116) is the exact spherical analogue of the familiar one-dimensional convolution on the line

$$f(x') = \int_{-\infty}^{\infty} h(x'')K(x' - x'')dx'' \quad \text{or} \quad f = h * K \quad ,$$

where $|x' - x''|$ denotes the distance between the points x' and x'' and thus corresponds to the spherical distance ψ'.

Now the potential of the compensating masses at a point (r, θ, λ) is represented by Newton's integral (1–1):

$$V_C(r, \theta, \lambda) = G \iiint_{earth} \frac{\Delta\rho(r', \theta', \lambda')}{l} \, dv \quad , \tag{8–119}$$

and the corresponding attraction by

$$A_C = -\frac{\partial V_C}{\partial r} = \iiint\limits_{earth} D(r', \psi)\Delta\rho(r', \theta', \lambda')\frac{dv}{r'^2} \quad , \tag{8–120}$$

where

$$D(r', \psi) = -Gr'^2\frac{\partial}{\partial r}\left(\frac{1}{l}\right) \tag{8–121}$$

as given by (8–64), with r_P and r replaced by r and r'; ψ is shown in Fig. 8.14, and, of course, $d\sigma$ refers to the point (θ'', λ''). The dependence of D on r is eliminated by computing V_C and A_C at a point at sea level for which

$$r = R = \text{const.} \tag{8–122}$$

With

$$dv = r'^2 dr' d\sigma \ , \quad \frac{dv}{r'^2} = dr' d\sigma \quad , \tag{8–123}$$

which is (8–41) in the present notation, we get

$$\begin{aligned} A_C(R, \theta, \lambda) &= \iint\limits_\sigma \int\limits_{r'=0}^{R} D(r', \psi)\Delta\rho(r', \theta', \lambda')dr'd\sigma \\ &= \int\limits_{r'=0}^{R} \Delta\rho * D(r', \psi)dr' \quad , \end{aligned} \tag{8–124}$$

using the convolution symbol; cf. (8–116) and (8–118).

Now we substitute (8–118):

$$A_C(R, \theta, \lambda) = \int\limits_{r'=0}^{R} h * K(r', \psi') * D(r', \psi)dr' \tag{8–125}$$

or, since $h = h(\theta'', \lambda'')$ does not depend on r',

$$A_C(R, \theta, \lambda) = h(\theta'', \lambda'') * \int\limits_{r'=0}^{R} K(r', \psi') * D(r', \psi)dr' \quad . \tag{8–126}$$

We define the *isostatic response function* F by

$$F(\psi'') = \int\limits_{r'=0}^{R} K(r', \psi') * D(r', \psi)dr' \quad . \tag{8–127}$$

Writing here the convolution integral explicitly, we have

$$F(\theta, \lambda; \theta'', \lambda'') = \int\limits_{r'=0}^{R} \iint\limits_\sigma K(r', \psi')D(r', \psi)d\sigma dr'$$

with (8–117) and

$$\cos\psi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\lambda' - \lambda) \quad .$$

Further, by Fig. 8.14,

$$\cos\psi'' = \cos\theta\cos\theta'' + \sin\theta\sin\theta''\cos(\lambda'' - \lambda) \quad .$$

In fact, F depends only on this spherical distance ψ'' between the points (θ, λ) and (θ'', λ'') for reasons of symmetry. (To see this, regard, *for a moment*, (θ, λ) and (θ'', λ'') as fixed and the point (θ', λ'), to which $d\sigma$ in the above convolution integral refers, as moving freely all over the sphere.) Then (8–126) becomes

$$A_C(R, \theta, \lambda) = h(\theta'', \lambda'') * F(\psi'') \quad .$$

To simplify the notation, we finally replace (θ'', λ'') by (θ', λ'), which is possible because, *ultimately*, both are nothing else than notations for a point that is freely variable on the sphere. We thus obtain

$$A_C(R, \theta, \lambda) = h(\theta', \lambda') * F(\psi) \quad , \tag{8–128}$$

since ψ denotes the spherical distance between the points (θ, λ) and (θ', λ') as expressed by the cosine theorem above.

Given A_C and the topographic height h, the isostatic response function can be determined by *"deconvolution"*.

Remember the two basic assumptions underlying the theory of Dorman and Lewis:

1. *linearity* in h; see eq. (8–116) but note that even Pratt's formula (8–3) is only approximately linear; cf. eq. (8–151) below;

2. *isotropy*; see again (8–116).

Let us also stress the difference with respect to the simple Pratt model: there, $\Delta\rho = 0$ from $r = 0$ to $r = R - D$ and constant in each column from $r = R - D$ to $r = R$, whereas now the density contrast $\Delta\rho$ within each column is a function of r.

Deconvolution. The problem is to solve the convolution equation for the isostatic response function $F(\psi)$. Dorman and Lewis (1970) perform this "deconvolution" first in the plane approximation, which is quite natural if the problem is considered local. This involves Bessel functions and Hankel transforms and would thus require mathematical tools not used in general in this book.

It is more in keeping with the spirit of the book to consider the original spherical global problem, which can be solved by means of our usual spherical harmonics.

In the spectral domain, convolution of two functions simply means *multiplication of the corresponding spectra*; cf. (Papoulis, 1968, p. 51; Hofmann-Wellenhof and Moritz, 1986, p. 236). This is very well known for the infinite line, where the spectral domain is also the infinite line. What one is often less aware of, is the fact that the spectral domain in the case of the sphere consists of the discrete points

$$\begin{aligned} n &= 0, 1, 2, 3, 4, 5, \ldots \\ m &= -n, -n+1, \ldots, -1, 0, 1, \ldots, n-1, n \ , \end{aligned} \tag{8–129}$$

and that the spectrum are the spherical harmonic coefficients a_{nm} and b_{nm}, or f_{nm} in the notation of sec. 7.6.1.

That a convolution on the sphere corresponds to the multiplication of the spherical harmonic coefficients by a factor (multiplication of the spectra!) is well known from many examples from physical geodesy. Poisson's integral (Heiskanen and Moritz, 1967, sec. 1–16) is a convolution: *ibid.*, eq. (1–89), which is equivalent to multiplying the spectrum by $(R/r)^{n+1}$, *ibid.*, eq. (1–87b). The same holds for Stokes' integral, *ibid.*, eq. (2–163a), whose spectral equivalent is

$$T_n = \frac{R}{n-1} \Delta g_n \tag{8–130}$$

(*ibid.*, p. 97); thus the spectrum is multiplied by $R/(n-1)$. Many other examples could be stated. In fact, a convolution is nothing else than an *isotropic linear integral operator* on the sphere; cf. also (Meissl, 1971).

Now let us return to our problem. Expand

$$A_C(R, \theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{nm} Y_{nm}(\theta, \lambda) \ , \tag{8–131}$$

$$h(\theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} H_{nm} Y_{nm}(\theta, \lambda) \ , \tag{8–132}$$

$$F(\psi) = \sum_{n=0}^{\infty} F_n P_n(\cos\psi) \ , \tag{8–133}$$

using the notation of sec. 7.6.1. The expansion (8–133) is purely "zonal" since it depends on ψ only. Let us verify the convolution theorem in the present case.

Eq. (8–128) may be written

$$A_C(R, \theta, \lambda) = \iint_\sigma h(\theta', \lambda') F(\psi) d\sigma \ . \tag{8–134}$$

We substitute (8–133) to get

$$A_C(R, \theta, \lambda) = \sum_{n=0}^{\infty} F_n \iint_\sigma h(\theta', \lambda') P_n(\cos\psi) d\sigma \ , \tag{8–135}$$

which by (1–49) equals

$$4\pi \sum_{n=0}^{\infty} \frac{F_n}{2n+1} Y_n(\theta, \lambda) = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{F_n}{2n+1} H_{nm} Y_{nm}(\theta, \lambda) \ , \tag{8–136}$$

expressing the Laplace harmonic $Y_n(\theta, \lambda)$ of h in terms of the base functions (7–24). The comparison with (8–131) gives

$$A_{nm} = \frac{4\pi}{2n+1} F_n H_{nm} \quad , \tag{8–137}$$

that is, *multiplication of the spectra*, up to a factor $4\pi/(2n+1)$ which is due to the fact that the base functions (7–24) are orthogonal but not normalized. Since A_C, h, and F are arbitrary functions, we have, by the way, *proved the spherical convolution theorem for the general case*!

Thus (8–137) gives

$$F_n = \frac{2n+1}{4\pi} \frac{A_{nm}}{H_{nm}} \quad , \tag{8–138}$$

which is independent of m. This condition must be satisfied by the elevation h and the attraction A_C (or the Bouguer anomaly), if the assumption of isotropy is justified. This already gives the isostatic response function by (8–133).

More difficult is the determination of the isostatic density anomaly $\Delta\rho$. For this we need the kernel K by (8–116): if

$$\Delta\rho(r, \theta, \lambda) = \sum_{n,m} \rho_{nm}(r) Y_{nm}(\theta, \lambda) \tag{8–139}$$

(in full analogy to (7–26)!) and

$$K(r, \psi) = \sum_{n=0}^{\infty} K_n(r) P_n(\cos\psi) \tag{8–140}$$

in analogy to (8–133), then by an appropriate application of (8–137) we have

$$\rho_{nm}(r) = \frac{4\pi}{2n+1} K_n(r) H_{nm} \quad . \tag{8–141}$$

There remains the determination of $K_n(r)$. The spectral equivalent (convolution corresponds to multiplication) of (8–127) is

$$F_n = \frac{4\pi}{2n+1} \int_{r'=0}^{R} K_n(r') D_n(r') dr' \quad . \tag{8–142}$$

Now what is $D_n(r')$? By (8–121) we have, using the standard Legendre series

$$\frac{1}{l} = \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n(\cos\psi) \quad , \tag{8–143}$$

$$D(r', \psi) = G r'^2 \sum_{n=0}^{\infty} (n+1) \frac{r'^n}{R^{n+2}} P_n(\cos\psi)$$

(putting $r = R$ after differentiation) or

$$D(r', \psi) = G \sum_{n=0}^{\infty} (n+1) \frac{r'^{n+2}}{R^{n+2}} P_n(\cos\psi) \quad . \tag{8–144}$$

This is written in the form (8–140), replacing r' by r

$$D(r, \psi) = \sum_{n=0}^{\infty} D_n(r) P_n(\cos\psi) \quad . \tag{8–145}$$

The comparison between (8–144) and (8–145) shows that

$$D_n(r) = G(n+1)\left(\frac{r}{R}\right)^{n+2} \quad . \tag{8–146}$$

Now we are almost through. Substituting (8–146) into (8–142) we get (with r' replaced by r)

$$F_n = \frac{4\pi}{2n+1} \int_{r=0}^{R} K_n(r) D_n(r) dr = 4\pi G \frac{n+1}{2n+1} \int_{0}^{R} K_n(r) \left(\frac{r}{R}\right)^{n+2} dr \quad . \tag{8–147}$$

Putting

$$\frac{r}{R} = \beta \tag{8–148}$$

we obtain

$$F_n = 4\pi G R \frac{n+1}{2n+1} \int_{0}^{1} K_n(\beta) \beta^{n+2} d\beta \quad , \tag{8–149}$$

where $K_n(\beta)$ is $K_n(r)$ expressed in terms of (8–148); it would have been more exact to write $K_n(R\beta)$.

Given F_n by (8–138), we can find functions $K_n(\beta)$ that satisfy (8–149). Obviously there are infinitely many possible solutions, since each $K_n(\beta)$ must satisfy only one condition (8–149), independently of the others.

Local compensation. To get a unique solution, Dorman and Lewis (1970) assume that the compensation is strictly local, which means that it takes place *immediately underneath the point at which the load is applied.* Thus the convolution (8–116) is replaced by multiplication by h (omitting the primes):

$$\Delta\rho(r, \theta, \lambda) = h(\theta, \lambda) K(r) \quad . \tag{8–150}$$

This exactly corresponds to (8–3) if in the denominator, $D + h$ is approximately replaced by D so as to obtain the *linear* relation

$$\Delta\rho = \frac{\rho_0}{D} h \quad , \tag{8–151}$$

except that Dorman and Lewis allow h to be multiplied by a factor variable with depth. This confirms the initial statement that we have an inverse problem for a compensation of Pratt type.

Retaining the original convolution (8–116) we should get a regional compensation, but then the mathematics would be more complicated.

We get (8–150) from (8–116) by formally putting

$$K(r, \psi) = K(r)\delta(\psi) \quad , \tag{8–152}$$

where the *delta function* $\delta(\psi)$ has the property

$$\delta(\psi) \equiv 0 \quad \text{except for} \quad \psi = 0 \quad , \tag{8–153}$$

$$\iint_\sigma \delta(\psi) d\sigma = 1 \quad . \tag{8–154}$$

This is the exact spherical analogue of (3–100) and (3–101).

What is the spectrum of this function $\delta(\psi)$? As usual, we write

$$\delta(\psi) = \sum_{n=0}^{\infty} \delta_n P_n(\cos\psi) \quad .$$

Then eq. (1–46), with $m = 0$, gives

$$\delta_n = \frac{2n+1}{4\pi} \iint_\sigma \delta(\psi) P_n(\cos\psi) d\sigma = \frac{2n+1}{4\pi} P_n(1)$$

by (8–153) and (8–154), or

$$\delta_n = \frac{2n+1}{4\pi} \tag{8–155}$$

since $P_n(1) = 1$ for all n. Hence, by (8–152) we get

$$K_n(r) = K(r)\delta_n = \frac{2n+1}{4\pi} K(r) \quad . \tag{8–156}$$

Then (8–149) becomes

$$F_n = GR(n+1) \int_0^1 K(\beta) \beta^{n+2} d\beta \quad . \tag{8–157}$$

Now, in contrast to (8–149), there is only *one* unknown function $K(\beta)$ which has to satisfy infinitely many conditions (8–158). The integral in (8–158), for various n, defines all "moments" of the function $K(\beta)$, F_n being known from (8–138). The determination of the function from its moments is called the *moment problem.*

One possible solution of the moment problem may be outlined as follows. Consider the moments

$$M_n = \int_0^1 K(\beta) \beta^n d\beta \quad . \tag{8–158}$$

If $K(\beta)$ were defined in the interval $[-1, 1]$, then an expansion into Legendre polynomials $P_n(\beta)$ (sec. 1.3) would offer itself: there is the basic orthogonality relation:

$$\int_{-1}^{1} P_n(\beta) P_{n'}(\beta) d\beta = \begin{cases} 0 & \text{if} \quad n' \neq n \quad , \\ \dfrac{2}{2n+1} & \text{if} \quad n' = n \quad ; \end{cases} \tag{8–159}$$

this follows from (1–41) and the first equation of (1–42), with $t = \cos\theta$ replaced by β.

Now, however, $K(\beta)$ is defined in the interval [0, 1]. Defining *shifted Legendre polynomials* by

$$P_n^*(\beta) = P_n(2\beta - 1) \tag{8–160}$$

(Abramowitz and Stegun, 1965, pp. 774), we have

$$\int_0^1 P_n^*(\beta)P_{n'}^*(\beta)d\beta = \frac{1}{2}\int_{-1}^1 P_n(t)P_{n'}(t)dt = \begin{cases} 0 & \text{if} \quad n' \neq n \quad , \\ \dfrac{1}{2n+1} & \text{if} \quad n' = n \quad , \end{cases} \tag{8–161}$$

which immediately follows by substituting $t = 2\beta - 1$ and considering (8–159). This shows that the P_n^* are orthogonal in the interval [0, 1].

Now, $P_n(\beta)$ by (1–33) and hence $P_n^*(\beta)$ by (8–160) are polynomials of degree n in β:

$$P_n^*(\beta) = \sum_{k=0}^{n} a_{nk}\beta^k \tag{8–162}$$

with coefficients a_{nk} that follow directly from (1–33) and (8–160); cf. also (Abramowitz and Stegun, 1965, p. 790).

Now everything is straightforward. Expand $K(\beta)$ into the series

$$K(\beta) = \sum_{n=0}^{\infty} k_n P_n^*(\beta) \quad . \tag{8–163}$$

Then we find the coefficients by multiplication by $P_{n'}^*(\beta)$ and integration from 0 to 1:

$$\int_0^1 K(\beta)P_{n'}^*(\beta)d\beta = \sum_{n=0}^{\infty} k_n \int_0^1 P_n^*(\beta)P_{n'}^*(\beta)d\beta \quad .$$

Orthogonality kills all terms except $n' = n$, and by (8–161) we have

$$k_n = (2n+1)\int_0^1 K(\beta)P_n^*(\beta)d\beta \quad .$$

The coefficients k_n, however, can be expressed in terms of the given moments (8–158), using (8–162):

$$k_n = (2n+1)\sum_{k=0}^{n} a_{nk}M_k \quad . \tag{8–164}$$

Hence the series (8–163) solves our problem. The moments M_k are defined by (8–158); the missing moments M_0 and M_1 may simply be put equal to zero. (Note that just now we have used the Legendre polynomials P_n, or their shifted counterparts (8–160), in a conceptually completely different sense than in sec. 1.3: there we considered orthogonal functions *on the sphere*; here the Legendre polynomials are used in their capacity as orthogonal functions *on a line interval*, $-1 \leq \beta \leq 1$ or $0 \leq \beta \leq 1$, respectively.)

Finally, (8–150) gives the density contrast $\Delta\rho$ corresponding to isostatic compensation. The plane approximation to the present theory may be found in (Dorman and Lewis, 1970). For practical results see, e.g., (Lewis and Dorman, 1970), (Bechtel et al., 1987) and (Hein et al., 1989).

8.3.2 The Inverse Vening Meinesz Problem

Here the density contrast $\Delta\rho$ is considered constant but the Moho depth T is to be determined from the condition (8–113) or, equivalently, from the given attraction A_C which the compensating masses exert at sea level, cf. (8–114).

Let us thus compute the attraction A_C of the compensating masses, bounded by the sphere $r = R - T_0$ representing the "normal Moho" (corresponding to a normal crustal thickness around $T_0 = 30$ km as mentioned in sec. 8.1.2) and the actual Moho, assuming constant density contrast:

$$\Delta\rho = \text{const.} \tag{8–165}$$

The corresponding potential is expressed by

$$V_C(P) = G\Delta\rho \iint_\sigma \int_{r=R-T}^{R-T_0} \frac{1}{l} r^2 dr\, d\sigma \quad , \tag{8–166}$$

again using (8–123), without primes, for the volume element dv. Further, by Fig. 8.15, we have

$$l^2 = R^2 + r^2 - 2Rr\cos\psi \quad . \tag{8–167}$$

The attraction is

$$A_C = -\frac{\partial V_C}{\partial R} = -G\Delta\rho \iint_\sigma \int_{r=R-T}^{R-T_0} \frac{\partial}{\partial R}\left(\frac{1}{l}\right) r^2 dr\, d\sigma \quad , \tag{8–168}$$

considering (for one moment only!) R in the integrand as variable. The limits of integration remain unchanged because, as the point P can be imagined to move in conformity with $\partial/\partial R$, the layer between $r = R - T$ and $r = R - T_0$ stays in place.

Changing the upper limit to R only implies the addition of a constant since

$$\int_{R-T}^{R-T_0} = \int_{R-T}^{R} - \int_{R-T_0}^{R} \quad ,$$

and the last integral is easily seen to be a global constant over the sphere $r = R$: it represents the attraction of a spherical shell of constant density bounded by the two