This is permissible since u/l_0 in V'' is never greater than the terrain inclination, which is considered small. By substituting the series (8–57) into (8–53) and integrating with respect to u we find

$$V'' = V_1 + V_2 + V_3 + \cdots \tag{8–58}$$

with

$$\begin{aligned} V_1 &= G\rho R^2 \iint_\sigma \frac{h - h_P}{l_0}\, d\sigma \quad , \\ V_2 &= -\frac{1}{6}\, G\rho R^2 \iint_\sigma \frac{(h-h_P)^3}{l_0^3}\, d\sigma \quad , \\ &\ldots\ldots\ldots \end{aligned} \tag{8–59}$$

This method of expanding into a series of powers of $(h - h_P)/l_0$ was used by Molodensky in a different context (cf. Moritz, 1980, p. 360).

Thus we have from (8–47) and (8–52)

$$V = 4\pi G\rho h_P R + V_1 + V_2 + \cdots \tag{8–60}$$

Neglecting terms of higher order, we have as a *linear approximation*:

$$V = 4\pi G\rho h_P R + G\rho R^2 \iint_\sigma \frac{h - h_P}{l_0}\, d\sigma \quad . \tag{8–61}$$

This expression will be needed later.

8.2.2 Attraction of Topography

The vertical attraction A of the topographic masses is the negative vertical derivative of the potential:

$$A = -\frac{\partial V}{\partial r_P} = -G\rho \iiint \frac{\partial}{\partial r_P}\left(\frac{1}{l}\right) dv \quad , \tag{8–62}$$

in agreement with (8–40) and comparable to (8–31a). By differentiating (8–42) we find

$$\frac{\partial}{\partial r_P}\left(\frac{1}{l}\right) = -\frac{r_P - r\cos\psi}{l^3} \quad . \tag{8–63}$$

This can be written as

$$\frac{\partial}{\partial r_P}\left(\frac{1}{l}\right) = \frac{r^2 - r_P^2}{2r_P l^3} - \frac{1}{2r_P l} \quad . \tag{8–64}$$

This transformation, simple as it is, will be fundamental for what follows.

By substituting (8–64) into (8–62) we find

$$A = B + \frac{1}{2r_P} V \quad , \tag{8–65}$$

where V is the potential considered in the preceding section, and

$$B = -G\rho \iiint \frac{r^2 - r_P^2}{2r_P l^3} \, dv \quad . \tag{8-66}$$

The quantity B can be essentially simplified by the use of the planar approximation. With $r_P \doteq R$, $r + r_P \doteq 2R$ and with (8–44) and (8–46) we obtain

$$B = -G\rho R^2 \iint_{\sigma} \int_{\eta=0}^{h} \frac{\eta - h_P}{l^3} \, d\sigma d\eta \quad . \tag{8-67}$$

This expression is comparable to (8–45) and will be split up in an analogous way:

$$B = B' + B'' \tag{8-68}$$

with

$$B' = -G\rho R^2 \iint_{\sigma} \int_{\eta=0}^{h_P} \frac{\eta - h_P}{l^3} \, d\sigma d\eta \quad , \tag{8-69}$$

$$B'' = -G\rho R^2 \iint_{\sigma} \int_{\eta=h_P}^{h} \frac{\eta - h_P}{l^3} \, d\sigma d\eta \quad . \tag{8-70}$$

Here B' represents the effect of the "spherical Bouguer plate". The attraction of this plate is expressed by

$$A' = -\frac{\partial V'}{\partial r_P} = \frac{GM}{r_P^2} \quad ,$$

in agreement with (8–50). With (8–51), considering $R/r_P \doteq 1$, we find

$$A' = 4\pi G\rho h_P \quad , \tag{8-71}$$

which represents the *attraction of the spherical Bouguer plate*, which is well known to be twice the attraction of the plane Bouguer plate of the same thickness h_P. We now obtain B' from (8–65) as

$$B' = A' - \frac{1}{2r_P} V' \quad . \tag{8-72}$$

Using (8–71) and (8–52) we obtain with $r_P \doteq R$

$$B' = 2\pi G\rho h_P \quad . \tag{8-73}$$

Thus the contribution of the spherical Bouguer plate to B is numerically equal to the attraction of the corresponding plane Bouguer plate. This simple fact will be of basic significance for a deeper understanding of the Bouguer reduction; see sec. 8.2.5.

Let us now consider B'', given by (8–70). As the integrand is easily seen to decrease very rapidly to zero with increasing distance l, it is sufficient to consider a neighborhood of, say, 50 km around the computation point P. Thus it is admissible to replace the sphere by its tangential plane at P, which is taken as the xy-plane; see Fig. 8.12. Then

$$R^2 d\sigma = dx dy \quad ,$$
$$l = \sqrt{x^2 + y^2 + (\eta - h_P)^2} \quad ,$$

and (8–70) becomes

$$B'' = -G\rho \iint_{-\infty}^{\infty} \int_{h_P}^{h} \frac{\eta - h_P}{[x^2 + y^2 + (\eta - h_P)^2]^{3/2}} \, dx dy d\eta \quad . \qquad (8\text{–}74)$$

Since the integral is extended over the region that is crosshatched in Fig. 8.12,

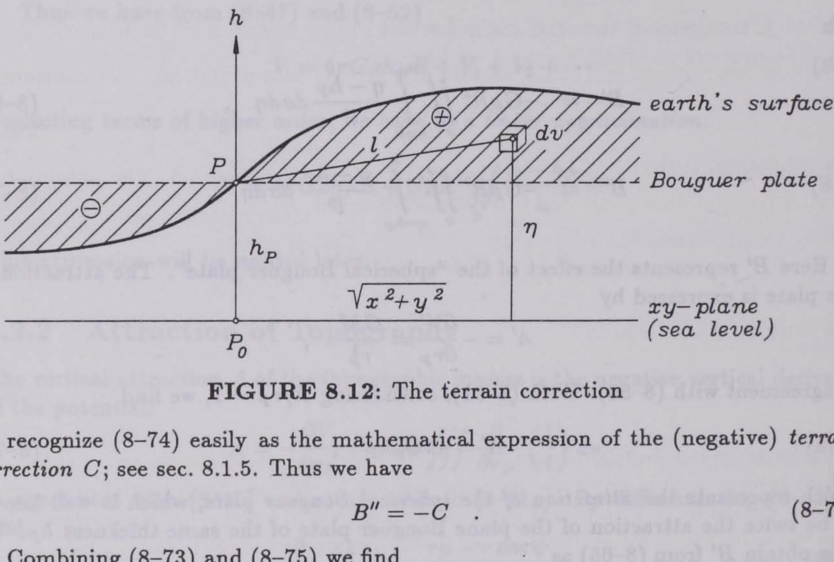

FIGURE 8.12: The terrain correction

we recognize (8–74) easily as the mathematical expression of the (negative) *terrain correction* C; see sec. 8.1.5. Thus we have

$$B'' = -C \quad . \qquad (8\text{–}75)$$

Combining (8–73) and (8–75) we find

$$B = 2\pi G \rho h_P - C \quad . \qquad (8\text{–}76)$$

The conventional Bouguer reduction is based on (8–38), which is formally identical with the right-hand side of (8–76); this again indicates the fact that the auxiliary quantity B has some connection with Bouguer reduction; see sec. 8.2.5.

The planar approximation of (8–70) is obtained by replacing l by $l_0 = 2R \sin \frac{\psi}{2}$. Now we can readily integrate with respect to η to get B'' or C, by (8–75). The result is

$$C = \frac{1}{2} G \rho R^2 \iint_\sigma \frac{(h - h_P)^2}{l_0^3} \, d\sigma \quad . \qquad (8\text{–}77)$$

Nor is it difficult to integrate (8–70) with respect to η if l is expressed by (8–54). The result is

$$B = 2\pi G\rho h_P + G\rho R^2 \iint_\sigma \left(\frac{1}{l_1} - \frac{1}{l_0}\right) d\sigma \quad , \tag{8–78}$$

where l_0 and l_1 are given by (8–55) and (8–54) with $\eta = h$. This was already found by Pellinen (1962).

Now it is easy to obtain the attraction A. Combining (8–65), with $r_P \doteq R$, and (8–76) we have

$$A = 2\pi G\rho h_P - C + \frac{1}{2R} V \quad . \tag{8–79}$$

We finally note that B has to A the same relation as the gravity anomaly Δg to the gravity disturbance δg: compare (8–65) with eq. (2–151e) of (Heiskanen and Moritz, 1967).

8.2.3 Condensation on Sea Level

The linear approximation (8–61) admits of a simple interpretation. We consider a layer of surface density

$$\kappa = \rho h \tag{8–80}$$

on the mean terrestrial sphere $r = R$ which represents the sea level. The potential of this surface layer at a point P_0 of the surface is given by

$$V_S = G \iint_\sigma \frac{\kappa}{l_0} R^2 d\sigma = G\rho R^2 \iint_\sigma \frac{h}{l_0} d\sigma \quad . \tag{8–81}$$

This can be transformed as

$$V_S = G\rho R^2 h_P \iint_\sigma \frac{d\sigma}{l_0} + G\rho R^2 \iint_\sigma \frac{h - h_P}{l_0} d\sigma \quad . \tag{8–82}$$

The first term on the right-hand side is the potential of a homogeneous spherical surface layer, which is given by the same formula (8–50) as the potential of a homogeneous sphere or of a spherical shell. Since even (8–51) holds for our surface layer (now $r_P = R$ exactly), the first term of (8–82) is given by (8–52), and we have

$$V_S = 4\pi G\rho h_P R + G\rho R^2 \iint_\sigma \frac{h - h_P}{l_0} d\sigma \quad . \tag{8–83}$$

This formula, which is rigorously valid for a spherical surface layer of density (8–80), is seen to agree with the linear approximation (8–61) to the potential of the topographic masses.

This immediately suggests a relation to the well-known condensation reduction of Helmert (Heiskanen and Moritz, 1967, p. 145), in which the topographic masses are compressed into a surface layer of density (8–80) on the geoid. We thus see that the change of potential because of the condensation, $V - V_S$, is a small quantity of