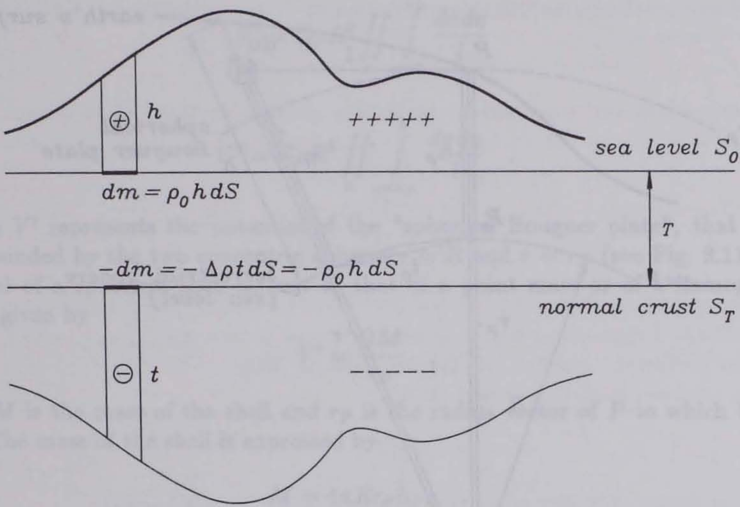

FIGURE 8.10: Topographic and isostatic masses form a dipole

This simplified concept of isostasy as a dipole field goes indirectly back to Helmert (1903) and was directly used by Jung (1956) and others. It is very useful for a deeper qualitative understanding of isostatic anomalies (cf. Turcotte and Schubert, 1982, p. 223). We shall follow (Moritz, 1968c).

8.2.1 Potential of the Topographic Masses

As a preparatory step, we first restrict ourselves to the topographic masses only, disregarding isostatic compensation until sec. 8.2.4. We shall restrict ourselves throughout to the usual *spherical approximation*, that is, we replace formally the geoid by a mean terrestrial sphere of radius R; see Fig. 8.11. The potential of the topographic masses (the masses outside the geoid) is

$$V = G\rho \iiint \frac{dv}{l} \quad . \tag{8–40}$$

The integral is extended over the exterior of the geoid ($R < r < R + h$); dv is the element of volume, and l is the distance between dv and the point P to which V refers. The density ρ is assumed to be constant (we shall now write ρ instead of ρ_0).

We have in (8–40)

$$dv = r^2 d\sigma dr \quad , \tag{8–41}$$

where $d\sigma$, as before is the element of solid angle, and

$$l = \sqrt{r_P^2 + r^2 - 2r_P r \cos\psi} \quad , \tag{8–42}$$

in agreement with Fig. 8.11.

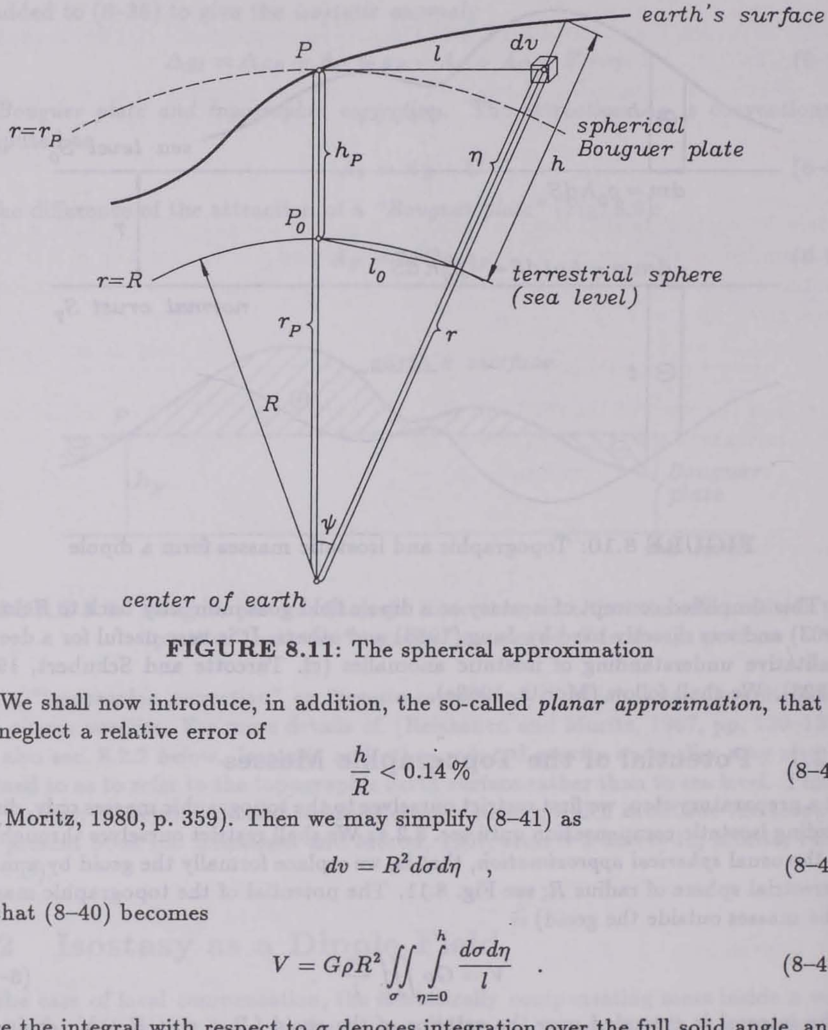

FIGURE 8.11: The spherical approximation

We shall now introduce, in addition, the so-called *planar approximation*, that is, we neglect a relative error of

$$\frac{h}{R} < 0.14\,\% \tag{8–43}$$

(cf. Moritz, 1980, p. 359). Then we may simplify (8–41) as

$$dv = R^2 d\sigma d\eta \ , \tag{8–44}$$

so that (8–40) becomes

$$V = G\rho R^2 \iint_\sigma \int_{\eta=0}^{h} \frac{d\sigma d\eta}{l} \quad . \tag{8–45}$$

Here the integral with respect to σ denotes integration over the full solid angle, and

$$\eta = r - R \tag{8–46}$$

is the elevation of the volume element dv above sea level (represented by the sphere $r = R$).

We may now split up (8–45) as

$$V = V' + V'' \tag{8–47}$$

with

$$V' = G\rho R^2 \iint_\sigma \int_{\eta=0}^{h_P} \frac{d\sigma d\eta}{l} \tag{8–48}$$

and

$$V'' = G\rho R^2 \iint_\sigma \int_{\eta=h_P}^{h} \frac{d\sigma d\eta}{l} \ . \tag{8–49}$$

Here V' represents the potential of the "spherical Bouguer plate", that is, the shell bounded by the two concentric spheres $r = R$ and $r = r_P$ (see Fig. 8.11). The potential of a spherical shell is, just as that of a point mass or of a homogeneous sphere, given by

$$V' = \frac{GM}{r_P} \ , \tag{8–50}$$

where M is the mass of the shell and r_P is the radius vector of P to which V' is to refer. The mass of the shell is expressed by

$$M = 4\pi R r_P h_P \rho \ . \tag{8–51}$$

Thus we simply have

$$V' = 4\pi G\rho h_P R \ . \tag{8–52}$$

Now we shall consider V'' as given by (8–49). Substituting

$$u = \eta - h_P$$

we find

$$V'' = G\rho R^2 \iint_\sigma \int_{u=0}^{h-h_P} \frac{d\sigma du}{l} \ . \tag{8–53}$$

As a planar approximation (Moritz, 1980, p. 359) we may put

$$l^2 = l_0^2 + (\eta - h_P)^2 = l_0^2 + u^2 \ , \tag{8–54}$$

with l_0 given by

$$l_0 = 2R \sin\frac{\psi}{2} \tag{8–55}$$

(Fig. 8.11). We write

$$\frac{1}{l} = \frac{1}{l_0}\left(1 + \frac{u^2}{l_0^2}\right)^{-1/2} \tag{8–56}$$

and expand the expressions between parentheses as a binomial series, obtaining

$$\frac{1}{l} = \frac{1}{l_0} - \frac{u^2}{2l_0^3} + \frac{3}{8}\frac{u^4}{l_0^5} - + \cdots \tag{8–57}$$

This is permissible since u/l_0 in V'' is never greater than the terrain inclination, which is considered small. By substituting the series (8–57) into (8–53) and integrating with respect to u we find

$$V'' = V_1 + V_2 + V_3 + \cdots \tag{8–58}$$

with

$$\begin{aligned} V_1 &= G\rho R^2 \iint_\sigma \frac{h - h_P}{l_0}\, d\sigma \quad , \\ V_2 &= -\frac{1}{6}\, G\rho R^2 \iint_\sigma \frac{(h - h_P)^3}{l_0^3}\, d\sigma \quad , \\ &\cdots\cdots\cdots \end{aligned} \tag{8–59}$$

This method of expanding into a series of powers of $(h - h_P)/l_0$ was used by Molodensky in a different context (cf. Moritz, 1980, p. 360).

Thus we have from (8–47) and (8–52)

$$V = 4\pi G\rho h_P R + V_1 + V_2 + \cdots \tag{8–60}$$

Neglecting terms of higher order, we have as a *linear approximation*:

$$V = 4\pi G\rho h_P R + G\rho R^2 \iint_\sigma \frac{h - h_P}{l_0}\, d\sigma \quad . \tag{8–61}$$

This expression will be needed later.

8.2.2 Attraction of Topography

The vertical attraction A of the topographic masses is the negative vertical derivative of the potential:

$$A = -\frac{\partial V}{\partial r_P} = -G\rho \iiint \frac{\partial}{\partial r_P}\left(\frac{1}{l}\right) dv \quad , \tag{8–62}$$

in agreement with (8–40) and comparable to (8–31a). By differentiating (8–42) we find

$$\frac{\partial}{\partial r_P}\left(\frac{1}{l}\right) = -\frac{r_P - r\cos\psi}{l^3} \quad . \tag{8–63}$$

This can be written as

$$\frac{\partial}{\partial r_P}\left(\frac{1}{l}\right) = \frac{r^2 - r_P^2}{2r_P l^3} - \frac{1}{2r_P l} \quad . \tag{8–64}$$

This transformation, simple as it is, will be fundamental for what follows.

By substituting (8–64) into (8–62) we find

$$A = B + \frac{1}{2r_P} V \quad , \tag{8–65}$$