3. *Effect of isostatic compensation.* This effect A_C as expressed by (8–31b) is to be added to (8–36) to give the *isostatic anomaly*

$$\Delta g_I = \Delta g_B + A_C = g_P - A_T + A_C + F - \gamma \quad . \tag{8–37}$$

Bouguer plate and topographic correction. The attraction A_T is conventionally computed as

$$A_T = A_B - C \tag{8–38}$$

as the difference of the attraction of a *"Bouguer plate"* (Fig. 8.9):

$$A_B = 2\pi G \rho_0 h_P \tag{8–39}$$

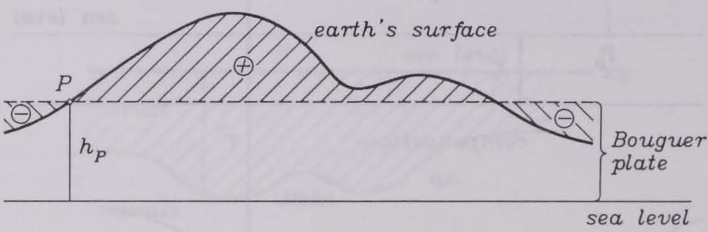

FIGURE 8.9: Bouguer plate and terrain correction; note that the effect of both the "positive" and the "negative" masses on C is always positive

and a *"topographic correction"*, or *"terrain correction"*, C which is usually quite small but *always positive.* For more details cf. (Heiskanen and Moritz, 1967, pp. 130–133); see also sec. 8.2.2 below. Isostatic and other reduced gravity anomalies may also be defined so as to refer to the topographic earth surface rather than to sea level. This is the modern conception related to Molodensky's theory, which is outside the scope of the present book (cf. Heiskanen and Moritz, 1967, secs. 8–2 and 8–11; Moritz, 1980, Part D).

8.2 Isostasy as a Dipole Field

In the case of local compensation, the isostatically compensating mass inside a vertical column is exactly equal to the topographic mass contained in the same column. This holds for both the Pratt and the Airy concept, by the very principle of local compensation. Fig. 8.10 illustrates the situation for the Airy–Heiskanen model. Approximately, the topography may be "condensed" as a surface layer on sea level S_0, whereas the compensation, with appropriate opposite sign, is thought to be concentrated as a surface layer on the surface S_T parallel to S_0 at constant depth T (T is our former T_0). Both surface elements dm for topography and $-dm$ for compensation thus form a dipole. This fact is also expressed by the difference $A_C - A_T$ in (8–37).

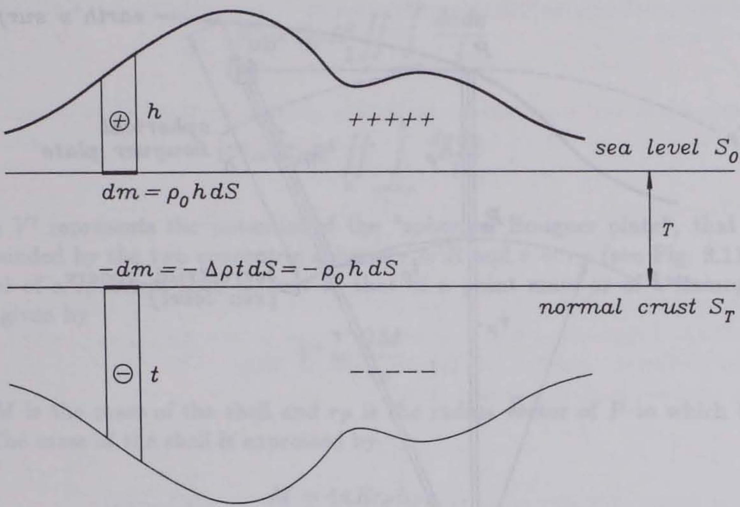

FIGURE 8.10: Topographic and isostatic masses form a dipole

This simplified concept of isostasy as a dipole field goes indirectly back to Helmert (1903) and was directly used by Jung (1956) and others. It is very useful for a deeper qualitative understanding of isostatic anomalies (cf. Turcotte and Schubert, 1982, p. 223). We shall follow (Moritz, 1968c).

8.2.1 Potential of the Topographic Masses

As a preparatory step, we first restrict ourselves to the topographic masses only, disregarding isostatic compensation until sec. 8.2.4. We shall restrict ourselves throughout to the usual *spherical approximation*, that is, we replace formally the geoid by a mean terrestrial sphere of radius R; see Fig. 8.11. The potential of the topographic masses (the masses outside the geoid) is

$$V = G\rho \iiint \frac{dv}{l} \quad . \tag{8-40}$$

The integral is extended over the exterior of the geoid ($R < r < R + h$); dv is the element of volume, and l is the distance between dv and the point P to which V refers. The density ρ is assumed to be constant (we shall now write ρ instead of ρ_0).

We have in (8–40)

$$dv = r^2 d\sigma dr \quad , \tag{8-41}$$

where $d\sigma$, as before is the element of solid angle, and

$$l = \sqrt{r_P^2 + r^2 - 2r_P r \cos\psi} \quad , \tag{8-42}$$

in agreement with Fig. 8.11.

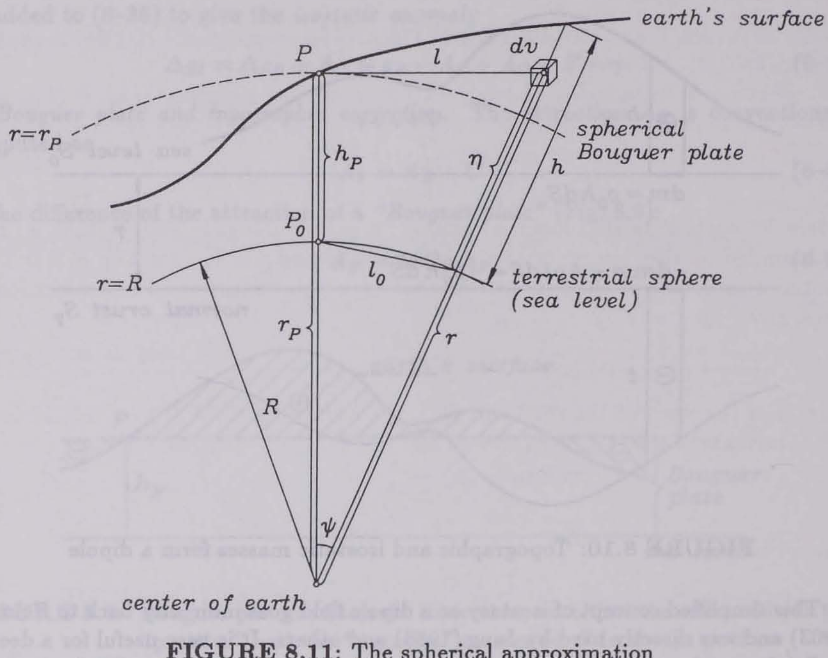

FIGURE 8.11: The spherical approximation

We shall now introduce, in addition, the so-called *planar approximation*, that is, we neglect a relative error of

$$\frac{h}{R} < 0.14\,\% \tag{8–43}$$

(cf. Moritz, 1980, p. 359). Then we may simplify (8–41) as

$$dv = R^2 d\sigma d\eta \ , \tag{8–44}$$

so that (8–40) becomes

$$V = G\rho R^2 \iint\limits_{\sigma} \int\limits_{\eta=0}^{h} \frac{d\sigma d\eta}{l} \quad . \tag{8–45}$$

Here the integral with respect to σ denotes integration over the full solid angle, and

$$\eta = r - R \tag{8–46}$$

is the elevation of the volume element dv above sea level (represented by the sphere $r = R$).

We may now split up (8–45) as

$$V = V' + V'' \tag{8–47}$$

with

$$V' = G\rho R^2 \iint_\sigma \int_{\eta=0}^{h_P} \frac{d\sigma d\eta}{l} \tag{8–48}$$

and

$$V'' = G\rho R^2 \iint_\sigma \int_{\eta=h_P}^{h} \frac{d\sigma d\eta}{l} \quad . \tag{8–49}$$

Here V' represents the potential of the "spherical Bouguer plate", that is, the shell bounded by the two concentric spheres $r = R$ and $r = r_P$ (see Fig. 8.11). The potential of a spherical shell is, just as that of a point mass or of a homogeneous sphere, given by

$$V' = \frac{GM}{r_P} \quad , \tag{8–50}$$

where M is the mass of the shell and r_P is the radius vector of P to which V' is to refer. The mass of the shell is expressed by

$$M = 4\pi R r_P h_P \rho \quad . \tag{8–51}$$

Thus we simply have

$$V' = 4\pi G\rho h_P R \quad . \tag{8–52}$$

Now we shall consider V'' as given by (8–49). Substituting

$$u = \eta - h_P$$

we find

$$V'' = G\rho R^2 \iint_\sigma \int_{u=0}^{h-h_P} \frac{d\sigma du}{l} \quad . \tag{8–53}$$

As a planar approximation (Moritz, 1980, p. 359) we may put

$$l^2 = l_0^2 + (\eta - h_P)^2 = l_0^2 + u^2 \quad , \tag{8–54}$$

with l_0 given by

$$l_0 = 2R \sin \frac{\psi}{2} \tag{8–55}$$

(Fig. 8.11). We write

$$\frac{1}{l} = \frac{1}{l_0}\left(1 + \frac{u^2}{l_0^2}\right)^{-1/2} \tag{8–56}$$

and expand the expressions between parentheses as a binomial series, obtaining

$$\frac{1}{l} = \frac{1}{l_0} - \frac{u^2}{2l_0^3} + \frac{3}{8}\frac{u^4}{l_0^5} - + \cdots \tag{8–57}$$

This is permissible since u/l_0 in V'' is never greater than the terrain inclination, which is considered small. By substituting the series (8–57) into (8–53) and integrating with respect to u we find

$$V'' = V_1 + V_2 + V_3 + \cdots \tag{8–58}$$

with

$$\begin{aligned} V_1 &= G\rho R^2 \iint_\sigma \frac{h - h_P}{l_0}\, d\sigma \quad , \\ V_2 &= -\frac{1}{6}\, G\rho R^2 \iint_\sigma \frac{(h - h_P)^3}{l_0^3}\, d\sigma \quad , \\ & \ldots\ldots\ldots \end{aligned} \tag{8–59}$$

This method of expanding into a series of powers of $(h - h_P)/l_0$ was used by Molodensky in a different context (cf. Moritz, 1980, p. 360).

Thus we have from (8–47) and (8–52)

$$V = 4\pi G\rho h_P R + V_1 + V_2 + \cdots \tag{8–60}$$

Neglecting terms of higher order, we have as a *linear approximation*:

$$V = 4\pi G\rho h_P R + G\rho R^2 \iint_\sigma \frac{h - h_P}{l_0}\, d\sigma \quad . \tag{8–61}$$

This expression will be needed later.

8.2.2 Attraction of Topography

The vertical attraction A of the topographic masses is the negative vertical derivative of the potential:

$$A = -\frac{\partial V}{\partial r_P} = -G\rho \iiint \frac{\partial}{\partial r_P}\left(\frac{1}{l}\right) dv \quad , \tag{8–62}$$

in agreement with (8–40) and comparable to (8–31a). By differentiating (8–42) we find

$$\frac{\partial}{\partial r_P}\left(\frac{1}{l}\right) = -\frac{r_P - r\cos\psi}{l^3} \quad . \tag{8–63}$$

This can be written as

$$\frac{\partial}{\partial r_P}\left(\frac{1}{l}\right) = \frac{r^2 - r_P^2}{2r_P l^3} - \frac{1}{2r_P l} \quad . \tag{8–64}$$

This transformation, simple as it is, will be fundamental for what follows.

By substituting (8–64) into (8–62) we find

$$A = B + \frac{1}{2r_P} V \quad , \tag{8–65}$$

where V is the potential considered in the preceding section, and

$$B = -G\rho \iiint \frac{r^2 - r_P^2}{2r_P l^3} \, dv \quad . \tag{8–66}$$

The quantity B can be essentially simplified by the use of the planar approximation. With $r_P \doteq R$, $r + r_P \doteq 2R$ and with (8–44) and (8–46) we obtain

$$B = -G\rho R^2 \iint_\sigma \int_{\eta=0}^{h} \frac{\eta - h_P}{l^3} \, d\sigma d\eta \quad . \tag{8–67}$$

This expression is comparable to (8–45) and will be split up in an analogous way:

$$B = B' + B'' \tag{8–68}$$

with

$$B' = -G\rho R^2 \iint_\sigma \int_{\eta=0}^{h_P} \frac{\eta - h_P}{l^3} \, d\sigma d\eta \quad , \tag{8–69}$$

$$B'' = -G\rho R^2 \iint_\sigma \int_{\eta=h_P}^{h} \frac{\eta - h_P}{l^3} \, d\sigma d\eta \quad . \tag{8–70}$$

Here B' represents the effect of the "spherical Bouguer plate". The attraction of this plate is expressed by

$$A' = -\frac{\partial V'}{\partial r_P} = \frac{GM}{r_P^2} \quad ,$$

in agreement with (8–50). With (8–51), considering $R/r_P \doteq 1$, we find

$$A' = 4\pi G\rho h_P \quad , \tag{8–71}$$

which represents the *attraction of the spherical Bouguer plate*, which is well known to be twice the attraction of the plane Bouguer plate of the same thickness h_P. We now obtain B' from (8–65) as

$$B' = A' - \frac{1}{2r_P} V' \quad . \tag{8–72}$$

Using (8–71) and (8–52) we obtain with $r_P \doteq R$

$$B' = 2\pi G\rho h_P \quad . \tag{8–73}$$

Thus the contribution of the spherical Bouguer plate to B is numerically equal to the attraction of the corresponding plane Bouguer plate. This simple fact will be of basic significance for a deeper understanding of the Bouguer reduction; see sec. 8.2.5.

Let us now consider B'', given by (8–70). As the integrand is easily seen to decrease very rapidly to zero with increasing distance l, it is sufficient to consider a neighborhood of, say, 50 km around the computation point P. Thus it is admissible to replace the sphere by its tangential plane at P, which is taken as the xy-plane; see Fig. 8.12. Then

$$R^2 d\sigma = dx dy \quad ,$$
$$l = \sqrt{x^2 + y^2 + (\eta - h_P)^2} \quad ,$$

and (8–70) becomes

$$B'' = -G\rho \iint_{-\infty}^{\infty} \int_{h_P}^{h} \frac{\eta - h_P}{[x^2 + y^2 + (\eta - h_P)^2]^{3/2}} dx dy d\eta \quad . \tag{8–74}$$

Since the integral is extended over the region that is crosshatched in Fig. 8.12,

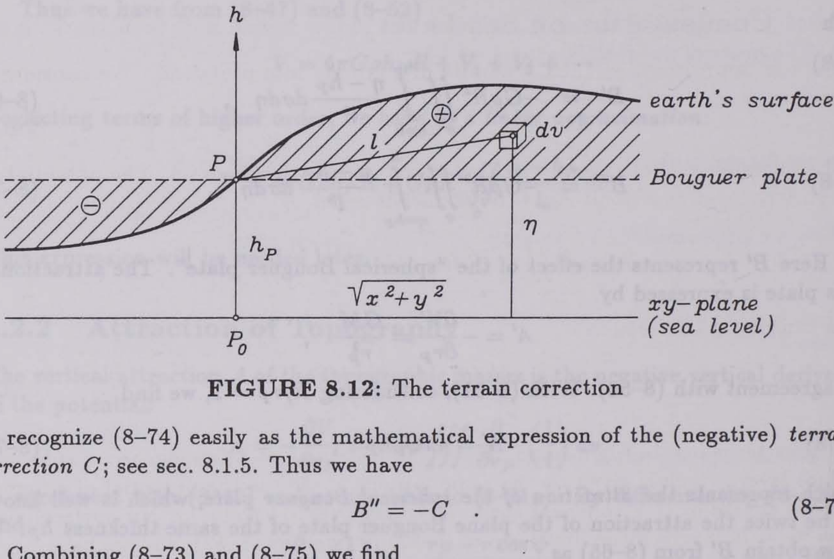

FIGURE 8.12: The terrain correction

we recognize (8–74) easily as the mathematical expression of the (negative) *terrain correction* C; see sec. 8.1.5. Thus we have

$$B'' = -C \quad . \tag{8–75}$$

Combining (8–73) and (8–75) we find

$$B = 2\pi G\rho h_P - C \quad . \tag{8–76}$$

The conventional Bouguer reduction is based on (8–38), which is formally identical with the right-hand side of (8–76); this again indicates the fact that the auxiliary quantity B has some connection with Bouguer reduction; see sec. 8.2.5.

The planar approximation of (8–70) is obtained by replacing l by $l_0 = 2R \sin \frac{\psi}{2}$. Now we can readily integrate with respect to η to get B'' or C, by (8–75). The result is

$$C = \frac{1}{2} G\rho R^2 \iint_{\sigma} \frac{(h - h_P)^2}{l_0^3} d\sigma \quad . \tag{8–77}$$

Nor is it difficult to integrate (8–70) with respect to η if l is expressed by (8–54). The result is

$$B = 2\pi G\rho h_P + G\rho R^2 \iint_\sigma \left(\frac{1}{l_1} - \frac{1}{l_0}\right) d\sigma \quad , \tag{8–78}$$

where l_0 and l_1 are given by (8–55) and (8–54) with $\eta = h$. This was already found by Pellinen (1962).

Now it is easy to obtain the attraction A. Combining (8–65), with $r_P \doteq R$, and (8–76) we have

$$A = 2\pi G\rho h_P - C + \frac{1}{2R} V \quad . \tag{8–79}$$

We finally note that B has to A the same relation as the gravity anomaly Δg to the gravity disturbance δg: compare (8–65) with eq. (2–151e) of (Heiskanen and Moritz, 1967).

8.2.3 Condensation on Sea Level

The linear approximation (8–61) admits of a simple interpretation. We consider a layer of surface density

$$\kappa = \rho h \tag{8–80}$$

on the mean terrestrial sphere $r = R$ which represents the sea level. The potential of this surface layer at a point P_0 of the surface is given by

$$V_S = G \iint_\sigma \frac{\kappa}{l_0} R^2 d\sigma = G\rho R^2 \iint_\sigma \frac{h}{l_0} d\sigma \quad . \tag{8–81}$$

This can be transformed as

$$V_S = G\rho R^2 h_P \iint_\sigma \frac{d\sigma}{l_0} + G\rho R^2 \iint_\sigma \frac{h - h_P}{l_0} d\sigma \quad . \tag{8–82}$$

The first term on the right-hand side is the potential of a homogeneous spherical surface layer, which is given by the same formula (8–50) as the potential of a homogeneous sphere or of a spherical shell. Since even (8–51) holds for our surface layer (now $r_P = R$ exactly), the first term of (8–82) is given by (8–52), and we have

$$V_S = 4\pi G\rho h_P R + G\rho R^2 \iint_\sigma \frac{h - h_P}{l_0} d\sigma \quad . \tag{8–83}$$

This formula, which is rigorously valid for a spherical surface layer of density (8–80), is seen to agree with the linear approximation (8–61) to the potential of the topographic masses.

This immediately suggests a relation to the well-known condensation reduction of Helmert (Heiskanen and Moritz, 1967, p. 145), in which the topographic masses are compressed into a surface layer of density (8–80) on the geoid. We thus see that the change of potential because of the condensation, $V - V_S$, is a small quantity of second order, because as a linear approximation V agrees with V_S. Here we have assumed that the point P, originally situated on the earth's surface, goes over into the corresponding point P_0 at sea level after condensation.

Thus, if we limit ourselves to the linear approximation which is often sufficient, we may regard the potential V as being generated by a spherical surface layer, the points P or P_0 being assumed to lie in both cases on the boundary of the attracting masses.

We shall now further investigate this surface layer. Let us first consider the attraction A and the auxiliary quantity B introduced in sec. 8.2.2. The point P is situated on the spherical surface, but at the outer boundary of the attracting masses. Thus A_S, the attraction of the surface layer at P, is given by the negative *external* derivative of V_S, e.g., expressed by equation (1–17a) of (Heiskanen and Moritz, 1967, p. 6). Thus we have

$$A_S = 2\pi G\kappa - G \iint_\sigma \kappa \frac{\partial}{\partial r_P} \left(\frac{1}{l}\right) R^2 d\sigma \quad . \tag{8–84}$$

To get the integrand, we must put $r = R = r_P$ in (8–64). We then obtain

$$A_S = 2\pi G\kappa + \frac{1}{2} GR \iint_\sigma \frac{\kappa}{l_0} d\sigma$$

and, with (8–80) and (8–81),

$$A_S = 2\pi G\rho h_P + \frac{1}{2R} V_S \quad . \tag{8–85}$$

We now consider the auxiliary quantity B_S defined in analogy to (8–65) as

$$B_S = A_S - \frac{1}{2R} V_S \quad . \tag{8–86}$$

We see that simply

$$B_S = 2\pi G\rho h_P \quad , \tag{8–87}$$

which is formally identical with the attraction of a "plane Bouguer plate". Equation (8–84) indicates, however, that the quantity B is in reality related to the discontinuity $2\pi G\kappa$ of the normal derivative of the surface potential on an *arbitrary* surface rather than to the attraction of a *plane* plate.

Let us now compare the quantities B for the actual topography and B_S for the surface layer. From (8–76) and (8–87) we obtain immediately

$$B = B_S - C \quad . \tag{8–88}$$

This means that these two quantities differ by the terrain correction C.

This has a consequence which will be of basic significance. As a linear approximation, also the attractions A and A_S differ by C,

$$A = A_S - C \quad . \tag{8–89}$$

This follows at once from the fact that A and B differ only by $V/2R$ and that as a linear approximation $V = V_S$. Thus as a linear approximation, *the potentials of the original and of the condensed topography are equal, but the attractions differ by the terrain correction.*

8.2.4 Effect of Compensation

We shall now consider a crustal density model by which the linear correlation of the free-air gravity anomalies with elevation can be explained and which at the same time is simple. Obviously, isostatic compensation must in some way be taken into account.

If we look at the Airy–Heiskanen isostatic model, we see that the compensation is given by the mountain roots which are some 30 km below sea level. The effect of this type of compensation on the earth's surface is thus quite similar as that of a surface layer of density $(-\rho h)$ on the sphere of radius $R - T$, where T may be identified with the normal thickness of the earth's crust of about 30 km, formerly denoted by T_0; see Fig. 8.13 and Fig. 8.10 above. The idea of regarding, for mathematical simplicity,

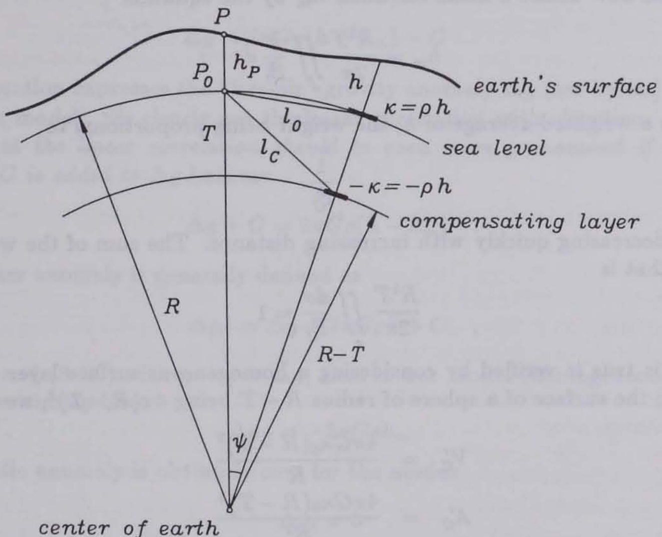

FIGURE 8.13: Spherical equivalent of Fig. 8.10; note again the dipole character

the isostatic compensation as a surface layer on a sphere concentric to the terrestrial sphere, was also used by Jung (1956, p. 590); we are following (Moritz, 1968c).

Let us now consider potential V_C and attraction A_C of this compensation layer. Since $h << T$, these quantities are almost the same whether referred to P or to P_0

(Fig. 8.13). We thus refer to P_0 and have

$$V_C = G\rho R^2 \iint_\sigma \frac{h}{l_C}\, d\sigma \quad , \tag{8-90}$$

$$B_C = G\rho R^2 T \iint_\sigma \frac{h}{l_C^3}\, d\sigma \quad . \tag{8-91}$$

The quantity B_C is defined in analogy to (8–65) as

$$B_C = A_C - \frac{1}{2R} V_C \tag{8-92}$$

and is expressed by an appropriate modification of (8–67): the mass element $\rho d\sigma d\eta$ in (8–67) is replaced by the mass element $\kappa d\sigma = \rho h d\sigma$ for a surface potential, and

$$\eta = -T \; , \quad h_P = 0 \quad .$$

With these changes, and on replacing the triple (volume) integral by a double (surface) integral, (8–67) indeed reduces to (8–91).

We shall now define a mean elevation h_m by the equation

$$h_m = \frac{R^2 T}{2\pi} \iint_\sigma \frac{h}{l_C^3}\, d\sigma \quad ; \tag{8-93}$$

h_m is thus a weighted average of h, the weight being proportional to

$$\frac{1}{l_C^3}$$

and thus decreasing quickly with increasing distance. The sum of the weights must be unity, that is

$$\frac{R^2 T}{2\pi} \iint_\sigma \frac{d\sigma}{l_C^3} = 1 \quad . \tag{8-94}$$

That this is true is verified by considering a homogeneous surface layer of constant density κ_0; the surface of a sphere of radius $R - T$ being $4\pi(R-T)^2$, we then have

$$V'_C = \frac{4\pi G \kappa_0 (R-T)^2}{R} \quad ,$$

$$A'_C = \frac{4\pi G \kappa_0 (R-T)^2}{R^2}$$

and thus, by (8–92),

$$B'_C = 2\pi G \kappa_0 \frac{(R-T)^2}{R^2} \doteq 2\pi G \kappa_0 \tag{8-95}$$

with a relative error of about 1%. On the other hand, from (8–91),

$$B'_C = G\kappa_0 R^2 T \iint_\sigma \frac{d\sigma}{l_C^3} \quad . \tag{8-96}$$

The comparison of (8–95) and (8–96) gives (8–94).

Substituting (8–93) into (8–91) we find

$$B_C = 2\pi G\rho h_m \quad , \tag{8–97}$$

so that by (8–92),

$$A_C = 2\pi G\rho h_m + \frac{1}{2R} V_C \quad . \tag{8–98}$$

According to our model, assuming crust and mantle to be homogeneous, the gravity anomaly Δg is caused only by the combined effect of topography and compensation:

$$\Delta g = A - A_C \quad , \tag{8–99}$$

where A is the attraction of topography. Substituting (8–79) and (8–98) we thus have

$$\Delta g = 2\pi G\rho(h_P - h_m) - C + \frac{1}{2R}(V - V_C) \quad . \tag{8–100}$$

The last term, which is very small (of order 1 mgal) because V and V_C are almost equal, will be neglected, and there remains (on omitting the subscript P)

$$\Delta g = 2\pi G\rho(h - h_m) - C \quad . \tag{8–101}$$

This equation expresses the "free-air" gravity anomaly Δg (see below) corresponding to our model. We clearly see the linear correlation with elevation, and we see at once that *the linear correlation should be even more pronounced if the terrain correction C is added to Δg* because

$$\Delta g + C = 2\pi G\rho(h - h_m) \quad . \tag{8–102}$$

The Bouguer anomaly is generally defined as

$$\Delta g_B = \Delta g - 2\pi G\rho h + C \quad , \tag{8–103}$$

by (8–36) and (8–38) with $g - \gamma = \Delta g$; thus in our model (homogeneous crust and mantle!) we simply have

$$\Delta g_B = -2\pi G\rho h_m \quad . \tag{8–104}$$

The isostatic anomaly is obviously zero for the model:

$$\Delta g_I = 0 \quad . \tag{8–105}$$

8.2.5 Conclusions Regarding Gravity Anomalies

Thus our model gives a reasonably realistic interpretation of the following empirical facts (Heiskanen and Moritz, 1967, pp. 281–285):

1. The free-air anomalies (see below) fluctuate around zero but are linearly correlated with elevation.

2. The Bouguer anomalies in mountain areas are systematically negative and increase in magnitude by

$$2\pi G\rho \doteq 100\,\text{mgals} \tag{8–106}$$

per km of mean elevation h_m.

These facts, which are well known from observation to hold quite generally and of which one is a consequence of the other, can be explained by isostatic compensation as we shall discuss now in more detail.

Correlation with elevation. The *free-air anomaly* is defined by

$$\Delta g = g_P + F - \gamma \quad ; \tag{8–107}$$

cf. sec. 8.1.5 (only the free-air reduction F is applied) and (Heiskanen and Moritz, 1967, pp. 146 and 293). Empirically, free-air anomalies are *linearly correlated with elevation*, that is, approximately they satisfy a linear relation

$$\Delta g = a + bh \quad , \tag{8–108}$$

where a and b are more or less constants.

On disregarding the terrain correction C, eq. (8–101) becomes

$$\Delta g = 2\pi G\rho(h - h_m) \quad . \tag{8–109}$$

The comparison with (8–108) shows that

$$b = 2\pi G\rho \tag{8–110}$$

and that

$$a = -2\pi G\rho h_m \tag{8–111}$$

essentially is nothing else than the Bouguer anomaly (8–104).

Linear correlation means that a linear functional relation is satisfied, not exactly but on the average. Fluctuations occur for three main reasons:

1. Density anomalies in the crust and the mantle and, possibly, in the core have been disregarded.

2. Isostatic equilibrium is not exact: local deviations from equilibrium occur. These are the main reasons.

3. The terrain correction C has been disregarded. This indicates that the "modified free-air anomaly" $\Delta g + C$ should exhibit this correlation even better than Δg itself, according to (8–102).

It is also clear that the parameter b in (8–108) is, for constant density ρ, really a constant; cf. (8–110). The parameter a, however, is essentially the Bouguer anomaly, by (8–104) and (8–111), and is therefore at best a "regional constant", that is, it varies, but much more slowly than Δg.

Thus an expression such as (8–111) explains the facts we have mentioned at the beginning of this section: the Bouguer anomalies in mountain areas are essentially negative and approximately proportional to a mean elevation h_m in such a way that a change in h_m of 1000 meters corresponds to a change in the Bouguer anomaly of about 100 mgals; for an application see (Heiskanen and Moritz, 1967, p. 328).

On the other hand, a look on (8–109) explains why the free-air anomaly exhibits no systematic tendency to either positive or negative (such a tendency is removed by h_m being subtracted from h) although it is approximately a linear function of h.

Our model corresponds to complete isostatic compensation but the manner of compensation is quite unrealistic: we have assumed the compensating masses forming a surface layer situated at a constant depth T below sea level. The purpose of this model, however, was only to furnish the simplest mathematical description of the surface gravity field, and as such it is quite adequate. If a more realistic model, for instance of Airy, Pratt, or even Vening Meinesz type, is considered, then the definition (8–93) of h_m will be replaced by a more complicated one, but this is rather the only change. The relevant formulas, such as (8–101), will still be valid, with h_m being still some sort of a mean elevation, but with different weighting. The only essential prerequisite is *that the compensating masses produce approximately the same potential and the same attraction at the corresponding points P and P_0* (Fig. 8.13). If the major part of the compensating masses is sufficiently deep, this will certainly be true. The validity of our results is thus far wider than the rather special model would indicate.

The reason may be summarized as: equation (8–101) is valid in any isostatic model if h_m is suitably defined; and the succeeding argument is based only on (8–101) and on the prerequisite just mentioned.

The dipole character of isostasy is particularly evident from equations such as (8–109).

A remark on the Bouguer reduction. As we have seen (eq. (8–71)), the attraction of a spherical Bouguer plate is $4\pi G\rho h$ and not $2\pi G\rho h$. Thus, strictly speaking, it is wrong to consider the term (8–39) as the attraction of an "infinite Bouguer plate". In fact, eq. (8–84) indicates that $2\pi G\rho h$ is in reality related to the discontinuity $2\pi G\kappa$ of the attraction of an *arbitrary* surface layer rather than to the attraction of a *plane* plate.

Thus, so to speak, the term $2\pi G\rho h$ represents the "local" effect of the Bouguer plate, and this is exactly what we want. Standing at a point of elevation h_P, it would be grossly unrealistic to assume that the actual earth's surface can be approximated by a "spherical Bouguer plate" extending with constant elevation h_P all around the earth! The major part of the earth is covered by the oceans for which $h = 0$, so that we can operate with a Bouguer plate only locally, and this local effect is $2\pi G\rho h_P$ even for the sphere. This justifies the conventional way of computing Bouguer anomalies. A further justification is provided by the fact that Bouguer anomalies usually are not an end in themselves, but that they are, e.g., a means for computing isostatic anomalies, for which

$$A - A_C \doteq B - B_C \tag{8–112}$$

by (8–65) and (8–92), since $V \doteq V_C$ and hence $(V - V_C)/2R$ is very nearly zero; and B is associated with the factor 2π and not 4π, as (8–76) shows.

8.3 Inverse Problems in Isostasy

Consider Pratt's model (sec. 8.1.1). The compensation takes place along vertical columns; this is *local compensation.* There is a *variable* density contrast $\Delta\rho$ given in terms of elevation h by (8–3). The corresponding isostatic gravity anomaly Δg_I (8–37) will in general not be zero, partly because of imperfections in the model. The inverse problem consists in trying to make

$$\Delta g_I \equiv 0 \tag{8–113}$$

by *determining a suitable distribution* $\Delta\rho(z)$ *of the density anomaly* in each vertical column.

On the other hand, consider isostatic models of Airy and Vening Meinesz type. Here the density contrast $\Delta\rho$ is *constant*, but the Moho depth T is variable, depending on the topography locally (Airy) or regionally (Vening Meinesz) in a prescribed way (now T and T_0 are again used in the sense of sec. 8.1!). Here the inverse problem would consist in making Δg_I zero by *determining a suitable variable Moho depth* T for a prescribed constant density contrast $\Delta\rho$, which need not be 0.6 g/cm^3 but can be any given value between 0 and 0.7 g/cm^3 (say).

Rather than making Δg_I zero, we may also prescribe the Bouguer anomaly field. This amounts to the same since by (8–37), $\Delta g_I = 0$ implies

$$A_C = -\Delta g_B \quad . \tag{8–114}$$

So the problem is in fact: given A_C, to determine the compensating masses that produce it. In the inverse Pratt problem this is done by seeking an appropriate density contrast $\Delta\rho$, in the inverse Vening Meinesz problem this is achieved by suitably selecting the Moho depth T. Thus we have genuine inverse problems (with given constraints) in the sense of Chapter 7 (cf. also Barzaghi and Sansò, 1986).

8.3.1 The Inverse Pratt Problem

The basic paper is (Dorman and Lewis, 1970). Consider a column defined by fixing the spherical coordinates (θ, λ); the column extends from the earth's surface radially to the earth's center (theoretically: this corresponds to $D = R$ in sec. 8.1.1). In each column $\Delta\rho$ is a function of the radius vector r (or of depth), which accounts for the functional dependence

$$\Delta\rho = \Delta\rho(r, \theta, \lambda) \quad . \tag{8–115}$$

One assumes $\Delta\rho$ to be linearly related to the topography (height h) by a *"convolution"*

$$\Delta\rho(r', \theta', \lambda') = \iint_\sigma h(\theta'', \lambda'')K(r', \psi')d\sigma \quad , \tag{8–116}$$