8.1.3 Regional Compensation According to Vening Meinesz

Both systems just discussed are highly idealized in that they assume the compensation to be strictly *local*; that is, they assume that compensation takes place along vertical columns. This presupposes free vertical mobility of the masses to a degree that is obviously unrealistic in this strict form.

For this reason, Vening Meinesz (1931, 1940, 1941) modified the Airy floating theory, introducing regional instead of local compensation. The principal difference between these two kinds of compensation is illustrated in Fig. 8.3. In Vening Meinesz'

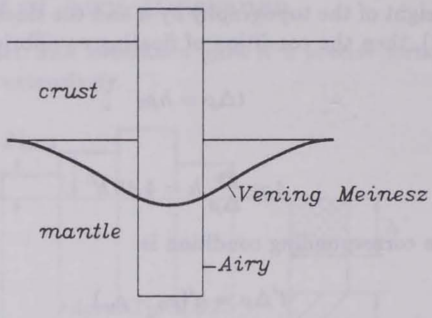

FIGURE 8.3: Local and regional compensation

theory, the topography is regarded as a load on an unbroken but yielding elastic crust.

To understand the situation, consider a point load P on an infinite plane elastic plate (representing the crust) which floats on a viscous underlayer of higher density (representing the mantle, see above); see Fig. 8.4. Since the topography is counted

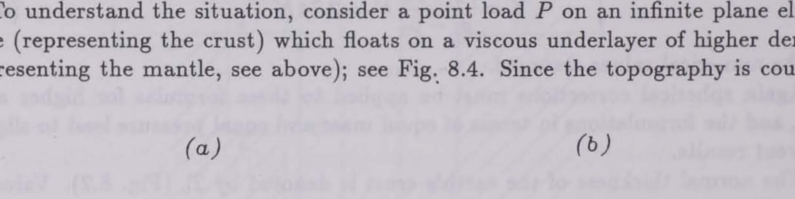

FIGURE 8.4: Bending (direct effect, (a)) and thickening (indirect effect, (b)) of an elastic plate

above sea level, we must fill the upper hollow in Fig. 8.4, (a), by crustal material of density ρ_0 which causes, as an additional load, a further bending (indirect effect)

(Fig. 8.4, (b)). Since the upper boundary is to remain horizontal, the total effect is a thickening of the plate. If m_P denotes the mass of the point load, then its weight, or the force it exerts on the plate, obviously is $m_P g$, g being gravity as usual.

Fig. 8.5 shows the lower boundary of this plate. This boundary surface is obtained

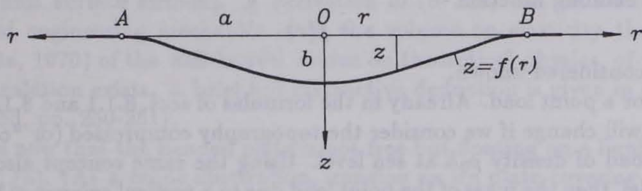

FIGURE 8.5: The bending curve

by rotating the bending curve around the z-axis; we obviously presuppose isotropy. We further assume the curve to be nonzero only in the region $r < a$, $a = AO = OB$, and to be tangent to the coordinate axes at the end points A and B. (In modern terminology, $f(r)$ is a "function of compact support", cf. sec. 7.5.)

The equilibrium condition obviously is

$$(\rho_1 - \rho_0) \iint\limits_S f(r) dS = 1 \quad , \tag{8–18}$$

if the mass m_P of the point load P is considered 1 (1 kg or 1 ton, say), S being the circle of radius a around O. This equation expresses the fact that the point load of mass 1 (right–hand side) is balanced by the hydrostatic uplift caused by the density difference $\rho_1 - \rho_0$ (left–hand side).

The bending curve is given by Hertz' theory of the bending of an elastic plate, as we shall see below. What we need now are only the principal functional values (Table 8.1). The constants l (Vening Meinesz' "degree of regionality") and b must be

TABLE 8.1: The bending curve after Hertz and Vening Meinesz

r	$f(r)$
0	b
l	$0.646\,b$
$2\,l$	$0.258\,b$
$3\,l$	$0.066\,b$
$3.887\,l$	0.000

selected appropriately; obviously

$$a = 3.887\,l \quad . \tag{8–19}$$

To be sure, $f(r)$ is not exactly zero for $r > a$, but periodic, representing small circular waves with constantly decreasing amplitudes.

Vening Meinesz, however, put $f(r) = 0$ outside $3.887\,l$ (more precisely, already outside $2.905\,l$ in order to enforce (8–18) for a *finite* circle around O) and approximated $f(r)$ piecewise by polynomials (nowadays we would use a spline approximation). At any rate, the bending function

$$z = f(r) \tag{8–20}$$

is now to be considered known.

So much for a point load. Already in the formulas of secs. 8.1.1 and 8.1.2 it is clear that nothing will change if we consider the topography compressed (or "condensed") as a surface load of density $\rho_0 h$ at sea level. Using the same concept also in Vening Meinesz' model, then the mass of the point load due to a vertical column of topography of cross section dS becomes

$$dm = \rho_0 h dS \quad .$$

Since $z = f(r)$ corresponds to a unit mass load, the bending due to the column under consideration is

$$z\,dm = \rho_0 h dS f(r) \quad ,$$

and the total bending Z due to the entire topography will be

$$Z(x, y) = \iint z\,dm = \rho_0 \iint h(x', y') f(r) dx'dy' \quad , \tag{8–21}$$

the integral being formally extended over the whole plane. Note that z has dimension: length per unit mass. Since

$$r = \sqrt{(x - x')^2 + (y - y')^2} \quad ,$$

the above formula represents Z as a *linear convolution* of the functions h and f.

Finally we note that

$$T = T_0 + Z \tag{8–22}$$

will be the depth of the Moho below sea level, T_0 being the "normal thickness of the earth's crust" of Airy–Heiskanen, as given, for instance, by (8–15).

Physical background. For those readers who have some knowledge of elastostatics or are otherwise interested in the physical basis of Vening Meinesz' theory, we shall outline the background, which is of considerable mathematical interest, also in view of the fact that, in Chapter 7, we have used the bipotential equation in a quite different context; cf. eq. (7–109).

It is well known that a plane elastic plate satisfies the "plate equation"

$$D\Delta^2 z = p \quad . \tag{8–23a}$$

Here

$$\Delta^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 y^2} + \frac{\partial^4}{\partial y^4}$$

represents the *biharmonic operator* in two dimensions (the upper boundary of the unbended plate is the xy-plane); cf. eq. (7–11) for three dimensions. The quantity z expresses the vertical displacement of the plate by bending; for a unit mass load, it is identical to (8–20) above. The "plate stiffness" D is a constant depending on the elastic properties of the plate and of its thickness, and p represents the load force on a unit surface element. A derivation of (8–23a) can be found in any text on advanced engineering mechanics or in the volume on elasticity theory (Landau and Lifschitz, 1970) of the well-known course on theoretical physics, of which also an English translation exists. A brief but instructive deduction is given in (Courant and Hilbert, 1953, pp. 250-251).

Suppose now that the bended plate is not free but floating on a liquid underlayer, cf. Fig. 8.4, (a). (As a crude illustration, imagine an ice plate covering a lake, which is bent by the weight of a man standing on it.) Then the hydrostatic uplift causes a force

$$g\rho_1 z$$

on a unit surface element, which acts opposite to the load p and must be subtracted from it. Thus (8–23a) is to be replaced by

$$D\Delta^2 z = p - g\rho_1 z \quad . \tag{8–23b}$$

This case was first considered by Hertz (1884) and is given a lengthy elementary treatment by Föppl (1922, pp. 103–119), to whom Vening Meinesz refers. Eq. (8–23b) is also used, without derivation, in (Jeffreys, 1976, p. 270).

Eq. (8–23b) represents to the "direct effect", cf. Fig. 8.4, (a). To get a horizontal upper surface, we must fill up the upper hollow. This produces a force

$$g\rho_0 z$$

per unit area, which acts in the same direction as p and thus must be added to the right-hand side of (8–23b), with the result

$$D\Delta^2 z = p - g(\rho_1 - \rho_0)z \quad . \tag{8–23c}$$

Thus the "indirect effect" is taken into account by simply replacing ρ_1 in (8–23b) by the density contrast (8–10)! This case was not considered by Hertz and may have first been treated by Vening Meinesz. For a somewhat different physical model leading to the same result (cf. Turcotte and Schubert, 1982, pp. 121-122).

Consider now a point load of mass 1 concentrated at the origin (in modern terminology, we would call it a "delta function load"). Outside the origin, p is zero, so that (8–23c) becomes

$$D\Delta^2 z = -g(\rho_1 - \rho_0)z$$

except for $x = y = 0$, or

$$\Delta^2 z + l^{-4} z = 0 \quad , \tag{8–24a}$$

where

$$l = \sqrt[4]{\frac{D}{g(\rho_1 - \rho_0)}}$$

has the dimension of a length and is nothing else than Vening Meinesz' "degree of regionality" mentioned above; he considers values of l from 10 to 60 km.

Solution of Hertz' equation. Because of rotational symmetry, it is best to transform (8–24a) to polar coordinates. Since

$$z = f(r)$$

is a function of

$$r = \sqrt{x^2 + y^2}$$

only, we get

$$\frac{\partial z}{\partial x} = \frac{dz}{dr}\frac{\partial r}{\partial x} = \frac{dz}{dr}\frac{x}{r} \,, \quad \text{etc.,}$$

so that we can express the Laplace operator

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$$

for functions of r only. Thus, with $l^{-1} = k$, eq. (8–24a) becomes

$$\left[\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) + k^4\right] z = 0 \tag{8–24b}$$

or, since with $i^2 = -1$,

$$a^4 + b^4 = (a^2 + i\,b^2)(a^2 - i\,b^2) \quad ,$$

further

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} + i\,k^2\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - i\,k^2\right) z = 0 \quad . \tag{8–24c}$$

Now

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} + u = 0 \tag{8–25a}$$

is the well-known *Bessel equation* (of zero order), whose solutions are, e.g., Bessel's function

$$u = J_0(x)$$

and Hankel's functions

$$u = H_0^{(1)}(x) \quad \text{and} \quad u = H_0^{(2)}(x) \quad ;$$

cf., e.g., (Courant and Hilbert, 1953, pp. 467–471). Solutions of the equation

$$\frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} \pm i\,k^2 u = 0 \tag{8–25b}$$

will consequently be the functions

$$J_0(kx\sqrt{\pm i}) \ , \quad H_0^{(1)}(kx\sqrt{\pm i}) \quad \text{and} \quad H_0^{(2)}(kx\sqrt{\pm i}) \quad , \tag{8–26a}$$

and these functions will obviously also solve (8–24c).

The functions, or linear combinations of them, are known as *Kelvin functions*; splitting into a real and an imaginary part we have, e.g.,

$$\begin{aligned} \operatorname{ber} x + i \operatorname{bei} x &= J_0(x\sqrt{-i}) \quad , \\ \ker x + i \operatorname{kei} x &= \frac{1}{2}\pi i \, H_0^{(1)}(x\sqrt{-i}) \quad . \end{aligned} \tag{8–26b}$$

This all sounds very complicated, but we simply need a solution which is finite, with horizontal tangent, at the origin and vanishes at infinity. Looking at standard tables (Janke and Emde, 1945) and (Abramowitz and Stegun, 1965), we find without difficulty the required functions: (Janke and Emde, 1945) shows in the graph on p. 250 and the table on p. 252 that the real part of $H_0^{(1)}(x\sqrt{i})$ does the job, and so likewise do (Abramowitz and Stegun, 1965) in the graph on p. 382 and the table on p. 431: here $\operatorname{kei}(x)$ is the required solution. Both functions are identical, apart from a constant factor. If we norm them to have $f(0) = 1$, we get from both tables the values shown in Table 8.2 (multiply the values in Janke–Emde by 2, and the values in

TABLE 8.2: Enlarged and corrected version of Table 8.1, with $l = b = 1$

x	$f(x)$
0	1.0000
0.5	0.8551
1.0	0.6302
1.5	0.4219
2.0	0.2577
2.5	0.1409
3.0	0.0651
3.5	0.0204
3.915	0.0000

Abramowitz–Stegun by $-4/\pi$). No further knowledge of Bessel functions is required: just use the table as if it were a table of sines or cosines! (Cf. also Turcotte, 1979, p. 66.)

The difference between the values of Tables 8.1 and 8.2 is not surprising if we note that Hertz (1884), for functions which are not easy to calculate after all, had only limited computational facilities, and that Vening Meinesz simply took Hertz' values.

To return to our physical model, we finally remark that Hertz (1884, p. 452) gives, in our notations, m_P denoting the mass of the point load:

$$b = f(0) = \frac{m_P}{8\rho_1 l^2} \quad .$$

If we consider a unit point mass load ($m_P = 1$) and replace ρ_1 by the density contrast $\rho_1 - \rho_0$ as we have seen above, we get

$$b = \frac{1}{8(\rho_1 - \rho_0)l^2} \quad .$$

This represents a relation between l, the density contrast, and the maximum depth of bending under a unit point load; it is identical to Vening Meinesz' (1940) eq. (1B). This value obviously must be in agreement with (8–18).

A simplified case. As we have seen, the two-dimensional equation (8–24a), in the case of rotational symmetry, can only be solved by somewhat unusual functions. Suppressing the y-coordinate, however, we get an extremely simple solution which gives an excellent qualitative (though not quantitative) picture of the problem and thus will facilitate our understanding (Turcotte and Schubert, 1982, pp. 125–126).

Disregarding the dependence on y, we have $\Delta^4 z = d^4z/dx^4$, so that (8–24a) reduces to

$$\frac{d^4z}{dx^4} + l^{-4}z = 0 \quad .$$

This is a linear ordinary differential equation with constant coefficients, for which the general solution is readily found by standard methods. It is

$$\begin{aligned} z &= e^{x/\alpha}\left(c_1 \cos\frac{x}{\alpha} + c_2 \sin\frac{x}{\alpha}\right) + \\ &\quad + e^{-x/\alpha}\left(c_3 \cos\frac{x}{\alpha} + c_4 \sin\frac{x}{\alpha}\right) \quad ; \end{aligned}$$

the constants c_i are to be determined by the boundary conditions and $\alpha = l\sqrt{2}$.

The requirement that the deformation z vanishes at infinity ($x \to \infty$) immediately eliminates, for positive x, the terms multiplied by $e^{x/\alpha}$, so that $c_1 = c_2 = 0$. Furthermore, the condition of a horizontal tangent at the origin $x = 0$ gives $c_3 = c_4$, so that our final solution simply is

$$z = be^{-x/\alpha}\left(\cos\frac{x}{\alpha} + \sin\frac{x}{\alpha}\right) \qquad (x \geq 0) \tag{8–27}$$

as the equation of our "one-dimensional bending curve"; we have put $c_3 = c_4 = b$ in agreement with our former notations.

In fact, for small x we may expand this function into a Taylor series:

$$z = b\left(1 - \frac{x^2}{\alpha^2} + \frac{2x^3}{3\alpha^3} \cdots\right) \quad ,$$

which is immediately seen to give $dz/dx = 0$ for $x = 0$; the term linear in x is missing only if $c_3 = c_4$! To have symmetry with respect to $x = 0$ (corresponding to the origin $r = 0$ in Fig. 8.5), we must replace x by $|x|$, which produces a step discontinuity in d^3z/dx^3 and hence the required delta-like singularity in d^4z/dx^4 at $x = 0$, corresponding to a point load; cf. sec. 3.3.2.

To repeat, this extremely simple solution is not the equation of the actual bending curve (8–20) but gives an excellent qualitative picture. This can be seen by drawing the graph of (8–27), with x replaced by $-x$ for negative values of x: a central depression surrounded by very small waves of decreasing amplitude.

8.1.4 Attraction of the Compensating Masses

As a preparatory step for computing isostatic reductions, to be discussed in sec. 8.1.5, we need the attraction of the compensating masses. For simplicity we consider the problem in the usual local plane approximation, replacing the geoid by its tangential plane. The spherical approximation will be used later (sec. 8.2).

We shall assume a basic definition concerning our three-dimensional local Cartesian coordinate system (Fig. 8.6): The xy-plane represents sea level, the z-axis points

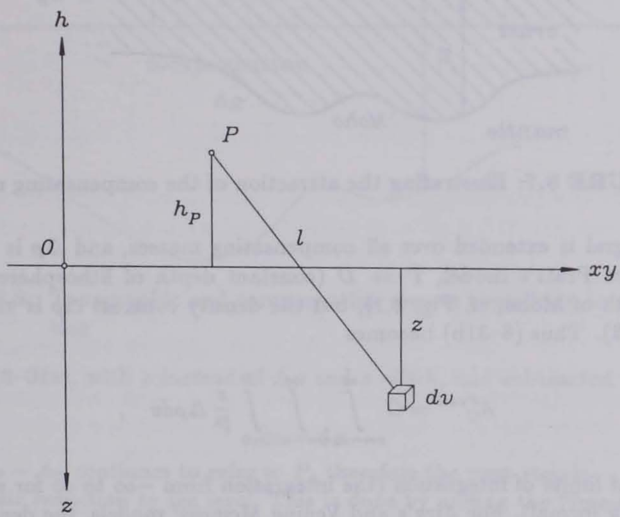

FIGURE 8.6: The basic coordinate systems xyz and xyh

vertically *downwards*, whereas the h-axis points vertically *upwards*, so that, for an arbitrary point,

$$z = -h \quad . \tag{8–28}$$

Keeping this definition in mind, the distance l between the computation point P and the volume element dv becomes

$$l^2 = (z + h_P)^2 + (x - x_P)^2 + (y - y_P)^2 \quad . \tag{8–29}$$