To get the second one, we differentiate (7–113):

$$\frac{\partial H}{\partial n} = \frac{\partial H}{\partial R} = \frac{l}{l_1}\frac{\partial l}{\partial R} - \frac{1}{2}\frac{l^2}{l_1^2}\frac{\partial l_1}{\partial R} + \frac{1}{2}\frac{\partial l_1}{\partial R} \quad . \tag{7–124}$$

(The differentiation is considered to be carried out in such a way that, for the moment only, R varies since $\partial/\partial n = \partial/\partial R$ for the sphere, whereas the points P and P', and hence r and r', are unchanged and kept constant.) After differentiation, we set again $l_1 = l$ on S by (7–122) to get from (7–124):

$$\frac{\partial H}{\partial n} = \frac{\partial H}{\partial R} = \frac{\partial l}{\partial R} = \frac{\partial l}{\partial n} \quad \text{on} \quad S \quad , \tag{7–125}$$

so that our second boundary condition is satisfied as well. This proves that (7–114) in fact represents Green's function for the sphere.

7.7.5 Stokes' Constants and the Harmonic Density

Let F be an arbitrary function which is twice continuously differentiable inside a surface S and continuous and differentiable on S. Let further U be an arbitrary regular harmonic function inside S, that is

$$\Delta U = 0 \quad \text{inside } S \quad , \tag{7–126}$$

and continuous and differentiable on S. Then Green's identity (7–75) immediately gives

$$\iiint_v U \Delta F dv = \iint_S \left(U \frac{\partial F}{\partial n} - F \frac{\partial U}{\partial n} \right) dS \quad . \tag{7–127}$$

Thus the integral (7–127) *does not explicitly depend on the values of U inside v* but only on the boundary values U and $\partial U/\partial n$ on S, as the right-hand side shows. Such an integral is called a *Stokes' constant* (cf. Wavre, 1932, p. 43).

Examples of Stokes' constants are the quantity GM and the other spherical-harmonic coefficients A_{nm} and B_{nm} in (1–36); in this case, the functions U are the inner zonal harmonics (1–35a), as the expressions (2–38) of (Heiskanen and Moritz, 1967, p. 59) show; F is proportional to the inner potential V since $-4\pi G\rho = \Delta V$.

Let now F be the potential V_0 of a zero-potential density, that is, $V_0 \neq 0$ inside S but $V_0 \equiv 0$ on and outside S, so that also $\partial V_0/\partial n = 0$ on S. Then (7–127) reduces to

$$\iiint_v U \Delta V_0 dv = 0 \tag{7–128}$$

or

$$\iiint_v U \rho_0 dv = 0 \tag{7–129}$$

for any zero-potential density ρ_0 and any regular harmonic function U. This is a nice *characterization of zero-potential densities:* all their Stokes' constants are zero, in particular all their spherical-harmonic coefficients must vanish (Pizzetti, 1909, 1910).

As we have seen in sec. 7.4, any density ρ may be written

$$\rho = \rho_H + \rho_0 \tag{7–130}$$

as the sum of a harmonic density ρ_H and a zero-potential density ρ_0. Consider now

$$\iiint\limits_v \rho^2 dv = \iiint\limits_v (\rho_H + \rho_0)^2 dv \quad , \tag{7–131}$$

which equals

$$\iiint\limits_v \rho_H^2 dv + 2 \iiint\limits_v \rho_H \rho_0 dv + \iiint\limits_v \rho_0^2 dv \quad . \tag{7–132}$$

Regarding (7–131) as the definition of a norm $\|\ \|$ for the function ρ:

$$\| \rho \|^2 = \iiint\limits_v \rho^2 dv \quad , \tag{7–133}$$

we may write (7–132) in the form

$$\| \rho \|^2 = \| \rho_H \|^2 + 2(\rho_H, \rho_0) + \| \rho_0^2 \| \quad , \tag{7–134}$$

with an obvious definition and notation for the *inner product* of the functions ρ_H and ρ_0. Now (7–129), with $U = \rho_H$ (which is harmonic!), immediately shows that

$$(\rho_H, \rho_0) = 0 \quad , \tag{7–135}$$

that is, *the densities ρ_H and ρ_0 are mutually "orthogonal".*

Thus (7–134) reduces to

$$\| \rho \|^2 = \| \rho_H \|^2 + \| \rho_0 \|^2 \geq \| \rho_H \|^2 \quad , \tag{7–136}$$

proving the *minimum norm property of the harmonic density* mentioned in sec. 7.3 (Marussi, 1980; Sansò, 1980).