7.7.4 Green's Function for the Sphere

It is easy to give Green's function G (7–99) if the boundary surface S is a sphere.

Submit the point P (to which V_P refers) to a *Kelvin transformation*, or *inversion in a sphere*. Cf. (Kellogg, 1929, pp. 231–223); for a different application see (Heiskanen and Moritz, 1967, pp. 143–144).

Fig. 7.10 shows the geometric situation. The inversion in the sphere transforms P into a point P' on the same radius as P, such that

$$rr' = R^2 \quad . \tag{7–111}$$

Define a function l_1 by

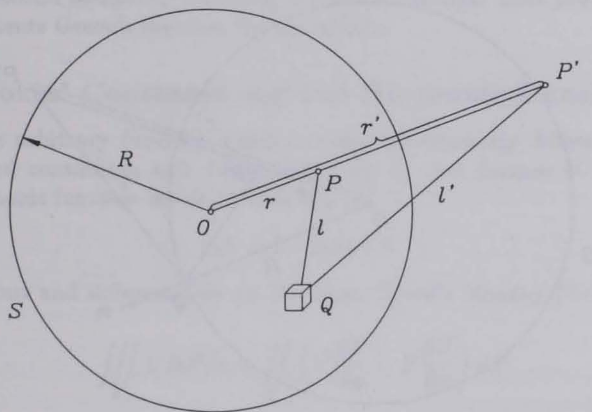

FIGURE 7.10: Kelvin transformation as an inversion in the sphere

$$l_1 = \frac{r}{R} l' \quad . \tag{7–112}$$

Then the auxiliary function H in (7–99) simply is

$$H = \frac{1}{2}\frac{l^2}{l_1} + \frac{1}{2} l_1 \quad , \tag{7–113}$$

so that Green's function (7–99) becomes

$$G = l - \frac{1}{2}\frac{l^2}{l_1} - \frac{1}{2} l_1 \tag{7–114}$$

(Marcolongo, 1901).

With coordinates for $P(x_P, y_P, z_P)$, $P'(x'_P, y'_P, z'_P)$ and $Q(x, y, z)$ we thus have

$$x'_P = \frac{R^2}{r^2} x_P \ , \quad y'_P = \frac{R^2}{r^2} y_P \ , \quad z'_P = \frac{R^2}{r^2} z_P \ , \tag{7–115}$$

$$r^2 = x_P^2 + y_P^2 + z_P^2 \quad , \tag{7–116}$$
$$l^2 = (x - x_P)^2 + (y - y_P)^2 + (z - z_P)^2 \quad , \tag{7–117}$$
$$l'^2 = (x - x'_P)^2 + (y - y'_P)^2 + (z - z'_P)^2 \quad . \tag{7–118}$$

It is straightforward though somewhat cumbersome to compute

$$\Delta^2 H = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{\partial^2 H}{\partial z^2}\right) \tag{7–119}$$

and to find that it is zero and regular even at P, so that H is indeed a regular solution of the biharmonic equation $\Delta^2 H = 0$.

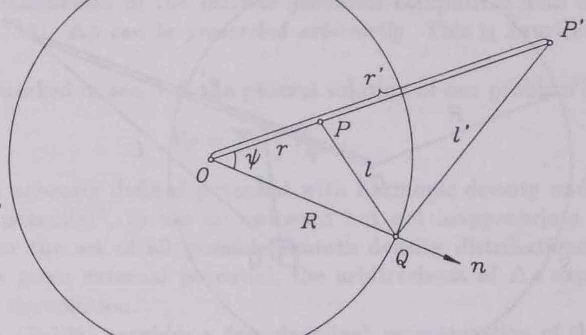

FIGURE 7.11: The point Q lies on the sphere S

There remains to verify the boundary conditions (7–96) on the sphere S. If Q lies on S, then (Fig. 7.11)

$$l^2 = r^2 + R^2 - 2rR\cos\psi \quad , \tag{7–120}$$

$$\begin{aligned} l'^2 &= r'^2 + R^2 - 2r'R\cos\psi = \frac{R^4}{r^2} + R^2 - 2\frac{R^3}{r}\cos\psi \\ &= \frac{R^2}{r^2} l^2 \quad , \end{aligned} \tag{7–121}$$

so that by (7–112),

$$l_1 = \frac{r}{R} l' = \frac{r}{R}\frac{R}{r} l = l \quad \text{on} \quad S \quad . \tag{7–122}$$

Hence (7–113) gives

$$H = l \quad \text{on} \quad S \tag{7–123}$$

which is our first boundary condition.

To get the second one, we differentiate (7–113):

$$\frac{\partial H}{\partial n} = \frac{\partial H}{\partial R} = \frac{l}{l_1}\frac{\partial l}{\partial R} - \frac{1}{2}\frac{l^2}{l_1^2}\frac{\partial l_1}{\partial R} + \frac{1}{2}\frac{\partial l_1}{\partial R} \quad . \tag{7–124}$$

(The differentiation is considered to be carried out in such a way that, for the moment only, R varies since $\partial/\partial n = \partial/\partial R$ for the sphere, whereas the points P and P', and hence r and r', are unchanged and kept constant.) After differentiation, we set again $l_1 = l$ on S by (7–122) to get from (7–124):

$$\frac{\partial H}{\partial n} = \frac{\partial H}{\partial R} = \frac{\partial l}{\partial R} = \frac{\partial l}{\partial n} \quad \text{on} \quad S \quad , \tag{7–125}$$

so that our second boundary condition is satisfied as well. This proves that (7–114) in fact represents Green's function for the sphere.

7.7.5 Stokes' Constants and the Harmonic Density

Let F be an arbitrary function which is twice continuously differentiable inside a surface S and continuous and differentiable on S. Let further U be an arbitrary regular harmonic function inside S, that is

$$\Delta U = 0 \quad \text{inside } S \quad , \tag{7–126}$$

and continuous and differentiable on S. Then Green's identity (7–75) immediately gives

$$\iiint_v U\Delta F dv = \iint_S \left(U\frac{\partial F}{\partial n} - F\frac{\partial U}{\partial n} \right) dS \quad . \tag{7–127}$$

Thus the integral (7–127) *does not explicitly depend on the values of U inside v* but only on the boundary values U and $\partial U/\partial n$ on S, as the right-hand side shows. Such an integral is called a *Stokes' constant* (cf. Wavre, 1932, p. 43).

Examples of Stokes' constants are the quantity GM and the other spherical-harmonic coefficients A_{nm} and B_{nm} in (1–36); in this case, the functions U are the inner zonal harmonics (1–35a), as the expressions (2–38) of (Heiskanen and Moritz, 1967, p. 59) show; F is proportional to the inner potential V since $-4\pi G\rho = \Delta V$.

Let now F be the potential V_0 of a zero-potential density, that is, $V_0 \neq 0$ inside S but $V_0 \equiv 0$ on and outside S, so that also $\partial V_0/\partial n = 0$ on S. Then (7–127) reduces to

$$\iiint_v U\Delta V_0 dv = 0 \tag{7–128}$$

or

$$\iiint_v U\rho_0 dv = 0 \tag{7–129}$$

for any zero-potential density ρ_0 and any regular harmonic function U. This is a nice *characterization of zero-potential densities:* all their Stokes' constants are zero, in particular all their spherical-harmonic coefficients must vanish (Pizzetti, 1909, 1910).