and that, in some miraculous way, the third and the fourth term on the right-hand side of (7–78) could be made to vanish, whereas in some no less miraculous way V_P (V at some interior point P) would show up as an additive term. Then the result would obviously be

$$V_P = L_1 V_S + L_2 \left(\frac{\partial V}{\partial n}\right)_S + L_3 \Delta\rho \quad , \tag{7–80}$$

expressing V_P as a combination of linear functionals applied to the boundary values V and $\partial V/\partial n$ on S and to $\Delta\rho$ (which, by (7–4), is proportional to $\Delta^2 V$ entering on the left-hand side of (7–78)). Since the boundary values V_S and $(\partial V/\partial n)_S$ are given, a very general solution would be obtained since the Laplacian of the density, $\Delta\rho$, may be arbitrarily assigned.

This daydream can be made true through the use of a so-called *Green's function.* Thus it is hoped that the reader is sufficiently motivated to follow the mildly intricate mathematical development to be presented now.

7.7.2 Transformation of Green's Identity

Let us first put

$$U = l \quad , \tag{7–81}$$

where l denotes the distance from the point $P(x_P, y_P, z_P)$ under consideration to a variable point (x, y, z) (Fig. 7.9):

$$l^2 = (x - x_P)^2 + (y - y_P)^2 + (z - z_P)^2 \quad . \tag{7–82}$$

Then, with

$$\Delta l = \frac{\partial^2 l}{\partial x^2} + \frac{\partial^2 l}{\partial y^2} + \frac{\partial^2 l}{\partial z^2} \tag{7–83}$$

as usual, we immediately calculate

$$\Delta l = \frac{2}{l} \quad , \tag{7–84}$$

$$\Delta^2 l = 2\Delta\left(\frac{1}{l}\right) = 0 \quad , \tag{7–85}$$

so that (7–79) is satisfied. The only problem is the singularity of $1/l$ at P (that is, for $l = 0$). Therefore, we cannot apply (7–78) directly but must use a simple trick (which, by the way, is also responsible for the difference between Green's second and third identities; cf. (Heiskanen and Moritz, 1967, pp. 11–12) and, for more detail, (Sigl, 1985, pp. 92–94)).

We apply (7–78) not to v, but to the region v' obtained from v by cutting out a small sphere S_h of radius h around P. This region v' is bounded by S and by S_h, where the normal n_h to S_h points away from v', that is towards P (Fig. 7.9). Thus (7–78) is replaced by

$$\iiint\limits_{v'} l\Delta^2 V dv = \iint\limits_{S,S_h} \left(-2V\frac{\partial}{\partial n}\left(\frac{1}{l}\right) + \frac{2}{l}\frac{\partial V}{\partial n} - \Delta V\frac{\partial l}{\partial n} + l\frac{\partial \Delta V}{\partial n}\right) dS \quad , \tag{7–86}$$

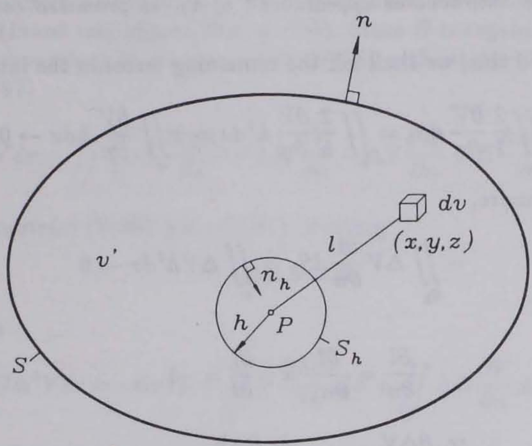

FIGURE 7.9: Illustrating the method of Green's function

where we have already taken into account (7–81), (7–84), and (7–85) and where we have used the abbreviation

$$\iint_{S,S_h} dS = \iint_S dS + \iint_{S_h} dS_h \quad . \tag{7–87}$$

Now

$$\iint_{S_h} \left(-2V\frac{\partial}{\partial n_h}\left(\frac{1}{l}\right)\right) dS_h \doteq -2V_P \iint_{S_h} \frac{\partial}{\partial n_h}\left(\frac{1}{l}\right) dS_h \tag{7–88}$$

since, because of the continuity of V, $V \doteq V_P$ inside and on S_h, the approximation is becoming better and better as $h \to 0$. Fig. 7.9 shows that

$$\frac{\partial}{\partial n_h} = -\frac{\partial}{\partial l} \ , \tag{7–89}$$

so that

$$\frac{\partial}{\partial n_h}\left(\frac{1}{l}\right) = -\frac{d}{dl}\left(\frac{1}{l}\right) = \frac{1}{l^2} = \frac{1}{h^2}$$

since $l = h$ on S_h. Furthermore

$$dS_h = h^2 d\sigma \quad , \tag{7–90}$$

with $d\sigma$ denoting the element of the unit sphere as usual. Thus the integral (7–88) becomes

$$-2V_P \iint_\sigma \frac{1}{h^2} h^2 d\sigma = -2V_P \iint_\sigma d\sigma = -8\pi V_P \quad , \tag{7–91}$$

which provides the "miraculous appearance" of V_P as promised towards the end of sec. 7.7.1!

Having achieved this, we shall kill the remaining terms in the integral over S_h. In fact,

$$\iint_{S_h} \frac{2}{l}\frac{\partial V}{\partial n}\,dS_h = \iint_{\sigma} \frac{2}{h}\frac{\partial V}{\partial n}\,h^2 d\sigma = 2\iint_{\sigma} \frac{\partial V}{\partial n}\,h d\sigma \to 0 \tag{7-92}$$

as $h \to 0$. Furthermore,

$$-\iint_{S_h} \Delta V \frac{\partial l}{\partial n}\,dS_h = \iint_{\sigma} \Delta V h^2 d\sigma \to 0 \tag{7-93}$$

since

$$\frac{\partial l}{\partial n} = \frac{\partial l}{\partial n_h} = -\frac{\partial l}{\partial l} = -1 \quad ,$$

and

$$\iint_{S_h} l\frac{\partial \Delta V}{\partial n}\,dS_h = \iint_{\sigma} \frac{\partial \Delta V}{\partial n}\,h^3 d\sigma \to 0 \quad . \tag{7-94}$$

Hence in the limit $h \to 0$, eq. (7-86) reduces to

$$\iiint_v l\Delta^2 V\,dv = -8\pi V_P + \iint_S \left(-2V\frac{\partial}{\partial n}\left(\frac{1}{l}\right) + \frac{2}{l}\frac{\partial V}{\partial n} - \Delta V\frac{\partial l}{\partial n} + l\frac{\partial \Delta V}{\partial n}\right) dS \quad . \tag{7-95}$$

This equation has exactly the same relation to (7-78) as Green's third identity has to Green's second identity (cf. Heiskanen and Moritz, 1967, pp. 11-12).

7.7.3 Lauricella's Theorems

What we still have to achieve is to eliminate the third and fourth terms of the integral on the right-hand side of (7-95). For this purpose we introduce an auxiliary function H which is *biharmonic and regular* (twice continuously differentiable) throughout v and assumes, together with its normal derivative, on the boundary surface S the same boundary values as the function (7-81):

$$H_S = l_S \ , \qquad \left(\frac{\partial H}{\partial n}\right)_S = \left(\frac{\partial l}{\partial n}\right)_S \quad . \tag{7-96}$$

The difference between the functions $U = l$ and H thus is that H is regular throughout v, whereas U has a singularity in its Laplacian at the point P; cf. (7-84). The point P is considered fixed in this context.

The existence and uniqueness of a solution H of the biharmonic equation

$$\Delta^2 H = 0 \tag{7-97}$$