and that, in some miraculous way, the third and the fourth term on the right-hand side of (7–78) could be made to vanish, whereas in some no less miraculous way V_P (V at some interior point P) would show up as an additive term. Then the result would obviously be

$$V_P = L_1 V_S + L_2 \left(\frac{\partial V}{\partial n}\right)_S + L_3 \Delta\rho \quad , \tag{7–80}$$

expressing V_P as a combination of linear functionals applied to the boundary values V and $\partial V/\partial n$ on S and to $\Delta\rho$ (which, by (7–4), is proportional to $\Delta^2 V$ entering on the left-hand side of (7–78)). Since the boundary values V_S and $(\partial V/\partial n)_S$ are given, a very general solution would be obtained since the Laplacian of the density, $\Delta\rho$, may be arbitrarily assigned.

This daydream can be made true through the use of a so-called *Green's function.* Thus it is hoped that the reader is sufficiently motivated to follow the mildly intricate mathematical development to be presented now.

7.7.2 Transformation of Green's Identity

Let us first put

$$U = l \quad , \tag{7–81}$$

where l denotes the distance from the point $P(x_P, y_P, z_P)$ under consideration to a variable point (x, y, z) (Fig. 7.9):

$$l^2 = (x - x_P)^2 + (y - y_P)^2 + (z - z_P)^2 \quad . \tag{7–82}$$

Then, with

$$\Delta l = \frac{\partial^2 l}{\partial x^2} + \frac{\partial^2 l}{\partial y^2} + \frac{\partial^2 l}{\partial z^2} \tag{7–83}$$

as usual, we immediately calculate

$$\Delta l = \frac{2}{l} \quad , \tag{7–84}$$

$$\Delta^2 l = 2\Delta\left(\frac{1}{l}\right) = 0 \quad , \tag{7–85}$$

so that (7–79) is satisfied. The only problem is the singularity of $1/l$ at P (that is, for $l = 0$). Therefore, we cannot apply (7–78) directly but must use a simple trick (which, by the way, is also responsible for the difference between Green's second and third identities; cf. (Heiskanen and Moritz, 1967, pp. 11–12) and, for more detail, (Sigl, 1985, pp. 92–94)).

We apply (7–78) not to v, but to the region v' obtained from v by cutting out a small sphere S_h of radius h around P. This region v' is bounded by S and by S_h, where the normal n_h to S_h points away from v', that is towards P (Fig. 7.9). Thus (7–78) is replaced by

$$\iiint_{v'} l\Delta^2 V\, dv = \iint_{S,S_h} \left(-2V\frac{\partial}{\partial n}\left(\frac{1}{l}\right) + \frac{2}{l}\frac{\partial V}{\partial n} - \Delta V\frac{\partial l}{\partial n} + l\frac{\partial \Delta V}{\partial n}\right) dS \quad , \tag{7–86}$$

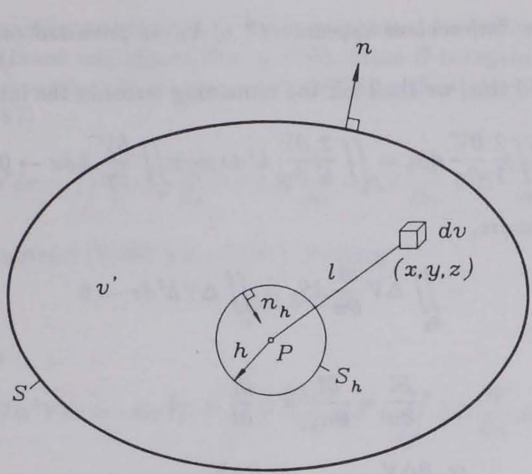

FIGURE 7.9: Illustrating the method of Green's function

where we have already taken into account (7–81), (7–84), and (7–85) and where we have used the abbreviation

$$\iint\limits_{S,S_h} dS = \iint\limits_{S} dS + \iint\limits_{S_h} dS_h \quad . \tag{7–87}$$

Now

$$\iint\limits_{S_h} \left(-2V\frac{\partial}{\partial n_h}\left(\frac{1}{l}\right)\right) dS_h \doteq -2V_P \iint\limits_{S_h} \frac{\partial}{\partial n_h}\left(\frac{1}{l}\right) dS_h \tag{7–88}$$

since, because of the continuity of V, $V \doteq V_P$ inside and on S_h, the approximation is becoming better and better as $h \to 0$. Fig. 7.9 shows that

$$\frac{\partial}{\partial n_h} = -\frac{\partial}{\partial l} \quad , \tag{7–89}$$

so that

$$\frac{\partial}{\partial n_h}\left(\frac{1}{l}\right) = -\frac{d}{dl}\left(\frac{1}{l}\right) = \frac{1}{l^2} = \frac{1}{h^2}$$

since $l = h$ on S_h. Furthermore

$$dS_h = h^2 d\sigma \quad , \tag{7–90}$$

with $d\sigma$ denoting the element of the unit sphere as usual. Thus the integral (7–88) becomes

$$-2V_P \iint\limits_{\sigma} \frac{1}{h^2} h^2 d\sigma = -2V_P \iint\limits_{\sigma} d\sigma = -8\pi V_P \quad , \tag{7–91}$$

which provides the "miraculous appearance" of V_P as promised towards the end of sec. 7.7.1!

Having achieved this, we shall kill the remaining terms in the integral over S_h. In fact,

$$\iint_{S_h} \frac{2}{l} \frac{\partial V}{\partial n} dS_h = \iint_{\sigma} \frac{2}{h} \frac{\partial V}{\partial n} h^2 d\sigma = 2 \iint_{\sigma} \frac{\partial V}{\partial n} h d\sigma \rightarrow 0 \tag{7-92}$$

as $h \rightarrow 0$. Furthermore,

$$-\iint_{S_h} \Delta V \frac{\partial l}{\partial n} dS_h = \iint_{\sigma} \Delta V h^2 d\sigma \rightarrow 0 \tag{7-93}$$

since

$$\frac{\partial l}{\partial n} = \frac{\partial l}{\partial n_h} = -\frac{\partial l}{\partial l} = -1 \quad ,$$

and

$$\iint_{S_h} l \frac{\partial \Delta V}{\partial n} dS_h = \iint_{\sigma} \frac{\partial \Delta V}{\partial n} h^3 d\sigma \rightarrow 0 \quad . \tag{7-94}$$

Hence in the limit $h \rightarrow 0$, eq. (7–86) reduces to

$$\iiint_v l \Delta^2 V dv = -8\pi V_P + + \iint_S \left(-2V \frac{\partial}{\partial n}\left(\frac{1}{l}\right) + \frac{2}{l} \frac{\partial V}{\partial n} - \Delta V \frac{\partial l}{\partial n} + l \frac{\partial \Delta V}{\partial n} \right) dS \quad . \tag{7-95}$$

This equation has exactly the same relation to (7–78) as Green's third identity has to Green's second identity (cf. Heiskanen and Moritz, 1967, pp. 11–12).

7.7.3 Lauricella's Theorems

What we still have to achieve is to eliminate the third and fourth terms of the integral on the right-hand side of (7–95). For this purpose we introduce an auxiliary function H which is *biharmonic and regular* (twice continuously differentiable) throughout v and assumes, together with its normal derivative, on the boundary surface S the same boundary values as the function (7–81):

$$H_S = l_S \; , \qquad \left(\frac{\partial H}{\partial n}\right)_S = \left(\frac{\partial l}{\partial n}\right)_S \quad . \tag{7-96}$$

The difference between the functions $U = l$ and H thus is that H is regular throughout v, whereas U has a singularity in its Laplacian at the point P; cf. (7–84). The point P is considered fixed in this context.

The existence and uniqueness of a solution H of the biharmonic equation

$$\Delta^2 H = 0 \tag{7-97}$$