To be sure, this example is so simplified as to be almost trivial, but it illustrates the geometrical situation very clearly.

7.7 Lauricella's Use of Green's Function

Finally we shall treat a very general explicit solution of the gravimetric inverse problem due to Lauricella (1911, 1912), which forms part of important work done by Italian mathematicians such as T. Boggio, U. Crudeli, E. Laura, R. Marcolongo, C. Mineo, P. Pizzetti, and C. Somigliana between 1900 and 1930. This work is not so well known as it deserves; an excellent review is (Marussi, 1980), where also references to the original papers are found.

We shall here follow the book (Frank and Mises, 1961, pp. 845–862), translating that treatment from the two-dimensional to the three-dimensional case.

7.7.1 Application of Green's Identity

Green's second identity may be written:

$$\iiint_v (U\Delta F - F\Delta U)dv = \iint_S \left(U\frac{\partial F}{\partial n} - F\frac{\partial U}{\partial n}\right) dS \quad ; \qquad (7\text{–}75)$$

this is eq. (1–28) of (Heiskanen and Moritz, 1967, p. 11) with F instead of V. It is valid for arbitrary functions U and F (which are, of course, "smooth", that is, sufficiently often differentiable, but this will be taken for granted in the sequel without mentioning). Here v denotes the volume enclosed by the surface S, with volume element dv and surface element dS as usual, Δ is Laplace's operator and $\partial/\partial n$ denotes the derivative along the normal pointing away from v. The formula (7–75) is standard in physical geodesy; derivations may be found in (Sigl, 1985, pp. 30–32) or (Kellogg, 1929, pp. 211–215).

We now put

$$F = \Delta V \quad , \qquad (7\text{–}76)$$

the Laplacian of the gravitational potential V, obtaining

$$\iiint_v (U\Delta^2 V - \Delta V\Delta U)dv = \iint_S \left(U\frac{\partial \Delta V}{\partial n} - \Delta V\frac{\partial U}{\partial n}\right) dS \quad . \qquad (7\text{–}77)$$

In this equation we interchange U and V and subtract the new equation from (7–77). The result is

$$\iiint_v (U\Delta^2 V - V\Delta^2 U)dv = \iint_S \left(-V\frac{\partial \Delta U}{\partial n} + \Delta U\frac{\partial V}{\partial n} - \Delta V\frac{\partial U}{\partial n} + U\frac{\partial \Delta V}{\partial n}\right) dS \quad . \qquad (7\text{–}78)$$

Let us now daydream. Suppose we can select U such that

$$\Delta^2 U = 0 \qquad (7\text{–}79)$$

and that, in some miraculous way, the third and the fourth term on the right-hand side of (7–78) could be made to vanish, whereas in some no less miraculous way V_P (V at some interior point P) would show up as an additive term. Then the result would obviously be

$$V_P = L_1 V_S + L_2 \left(\frac{\partial V}{\partial n}\right)_S + L_3 \Delta\rho \quad , \tag{7–80}$$

expressing V_P as a combination of linear functionals applied to the boundary values V and $\partial V/\partial n$ on S and to $\Delta\rho$ (which, by (7–4), is proportional to $\Delta^2 V$ entering on the left-hand side of (7–78)). Since the boundary values V_S and $(\partial V/\partial n)_S$ are given, a very general solution would be obtained since the Laplacian of the density, $\Delta\rho$, may be arbitrarily assigned.

This daydream can be made true through the use of a so-called *Green's function.* Thus it is hoped that the reader is sufficiently motivated to follow the mildly intricate mathematical development to be presented now.

7.7.2 Transformation of Green's Identity

Let us first put

$$U = l \quad , \tag{7–81}$$

where l denotes the distance from the point $P(x_P, y_P, z_P)$ under consideration to a variable point (x, y, z) (Fig. 7.9):

$$l^2 = (x - x_P)^2 + (y - y_P)^2 + (z - z_P)^2 \quad . \tag{7–82}$$

Then, with

$$\Delta l = \frac{\partial^2 l}{\partial x^2} + \frac{\partial^2 l}{\partial y^2} + \frac{\partial^2 l}{\partial z^2} \tag{7–83}$$

as usual, we immediately calculate

$$\Delta l = \frac{2}{l} \quad , \tag{7–84}$$

$$\Delta^2 l = 2\Delta\left(\frac{1}{l}\right) = 0 \quad , \tag{7–85}$$

so that (7–79) is satisfied. The only problem is the singularity of $1/l$ at P (that is, for $l = 0$). Therefore, we cannot apply (7–78) directly but must use a simple trick (which, by the way, is also responsible for the difference between Green's second and third identities; cf. (Heiskanen and Moritz, 1967, pp. 11–12) and, for more detail, (Sigl, 1985, pp. 92–94)).

We apply (7–78) not to v, but to the region v' obtained from v by cutting out a small sphere S_h of radius h around P. This region v' is bounded by S and by S_h, where the normal n_h to S_h points away from v', that is towards P (Fig. 7.9). Thus (7–78) is replaced by

$$\iiint_{v'} l\Delta^2 V\,dv = \iint_{S,S_h} \left(-2V\frac{\partial}{\partial n}\left(\frac{1}{l}\right) + \frac{2}{l}\frac{\partial V}{\partial n} - \Delta V\frac{\partial l}{\partial n} + l\frac{\partial \Delta V}{\partial n}\right) dS \quad , \tag{7–86}$$