To be sure, this example is so simplified as to be almost trivial, but it illustrates the geometrical situation very clearly.

7.7 Lauricella's Use of Green's Function

Finally we shall treat a very general explicit solution of the gravimetric inverse problem due to Lauricella (1911, 1912), which forms part of important work done by Italian mathematicians such as T. Boggio, U. Crudeli, E. Laura, R. Marcolongo, C. Mineo, P. Pizzetti, and C. Somigliana between 1900 and 1930. This work is not so well known as it deserves; an excellent review is (Marussi, 1980), where also references to the original papers are found.

We shall here follow the book (Frank and Mises, 1961, pp. 845–862), translating that treatment from the two-dimensional to the three-dimensional case.

7.7.1 Application of Green's Identity

Green's second identity may be written:

$$\iiint_v (U\Delta F - F\Delta U)dv = \iint_S \left(U\frac{\partial F}{\partial n} - F\frac{\partial U}{\partial n}\right) dS \quad ; \qquad (7\text{–}75)$$

this is eq. (1–28) of (Heiskanen and Moritz, 1967, p. 11) with F instead of V. It is valid for arbitrary functions U and F (which are, of course, "smooth", that is, sufficiently often differentiable, but this will be taken for granted in the sequel without mentioning). Here v denotes the volume enclosed by the surface S, with volume element dv and surface element dS as usual, Δ is Laplace's operator and $\partial/\partial n$ denotes the derivative along the normal pointing away from v. The formula (7–75) is standard in physical geodesy; derivations may be found in (Sigl, 1985, pp. 30–32) or (Kellogg, 1929, pp. 211–215).

We now put

$$F = \Delta V \quad , \qquad (7\text{–}76)$$

the Laplacian of the gravitational potential V, obtaining

$$\iiint_v (U\Delta^2 V - \Delta V \Delta U)dv = \iint_S \left(U\frac{\partial \Delta V}{\partial n} - \Delta V\frac{\partial U}{\partial n}\right) dS \quad . \qquad (7\text{–}77)$$

In this equation we interchange U and V and subtract the new equation from (7–77). The result is

$$\iiint_v (U\Delta^2 V - V\Delta^2 U)dv = \iint_S \left(-V\frac{\partial \Delta U}{\partial n} + \Delta U\frac{\partial V}{\partial n} - \Delta V\frac{\partial U}{\partial n} + U\frac{\partial \Delta V}{\partial n}\right) dS \quad . \qquad (7\text{–}78)$$

Let us now daydream. Suppose we can select U such that

$$\Delta^2 U = 0 \qquad (7\text{–}79)$$

and that, in some miraculous way, the third and the fourth term on the right-hand side of (7–78) could be made to vanish, whereas in some no less miraculous way V_P (V at some interior point P) would show up as an additive term. Then the result would obviously be

$$V_P = L_1 V_S + L_2 \left(\frac{\partial V}{\partial n}\right)_S + L_3 \Delta\rho \quad , \tag{7–80}$$

expressing V_P as a combination of linear functionals applied to the boundary values V and $\partial V/\partial n$ on S and to $\Delta\rho$ (which, by (7–4), is proportional to $\Delta^2 V$ entering on the left-hand side of (7–78)). Since the boundary values V_S and $(\partial V/\partial n)_S$ are given, a very general solution would be obtained since the Laplacian of the density, $\Delta\rho$, may be arbitrarily assigned.

This daydream can be made true through the use of a so-called *Green's function.* Thus it is hoped that the reader is sufficiently motivated to follow the mildly intricate mathematical development to be presented now.

7.7.2 Transformation of Green's Identity

Let us first put

$$U = l \quad , \tag{7–81}$$

where l denotes the distance from the point $P(x_P, y_P, z_P)$ under consideration to a variable point (x, y, z) (Fig. 7.9):

$$l^2 = (x - x_P)^2 + (y - y_P)^2 + (z - z_P)^2 \quad . \tag{7–82}$$

Then, with

$$\Delta l = \frac{\partial^2 l}{\partial x^2} + \frac{\partial^2 l}{\partial y^2} + \frac{\partial^2 l}{\partial z^2} \tag{7–83}$$

as usual, we immediately calculate

$$\Delta l = \frac{2}{l} \quad , \tag{7–84}$$

$$\Delta^2 l = 2\Delta\left(\frac{1}{l}\right) = 0 \quad , \tag{7–85}$$

so that (7–79) is satisfied. The only problem is the singularity of $1/l$ at P (that is, for $l = 0$). Therefore, we cannot apply (7–78) directly but must use a simple trick (which, by the way, is also responsible for the difference between Green's second and third identities; cf. (Heiskanen and Moritz, 1967, pp. 11–12) and, for more detail, (Sigl, 1985, pp. 92–94)).

We apply (7–78) not to v, but to the region v' obtained from v by cutting out a small sphere S_h of radius h around P. This region v' is bounded by S and by S_h, where the normal n_h to S_h points away from v', that is towards P (Fig. 7.9). Thus (7–78) is replaced by

$$\iiint_{v'} l\Delta^2 V\, dv = \iint_{S,S_h} \left(-2V\frac{\partial}{\partial n}\left(\frac{1}{l}\right) + \frac{2}{l}\frac{\partial V}{\partial n} - \Delta V\frac{\partial l}{\partial n} + l\frac{\partial \Delta V}{\partial n}\right) dS \quad , \tag{7–86}$$

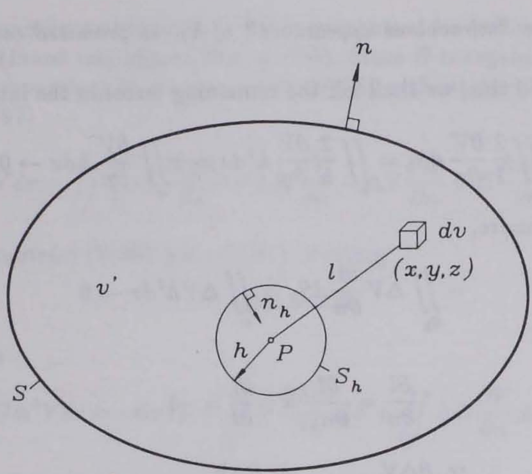

FIGURE 7.9: Illustrating the method of Green's function

where we have already taken into account (7–81), (7–84), and (7–85) and where we have used the abbreviation

$$\iint\limits_{S,S_h} dS = \iint\limits_{S} dS + \iint\limits_{S_h} dS_h \quad . \tag{7–87}$$

Now

$$\iint\limits_{S_h} \left(-2V \frac{\partial}{\partial n_h}\left(\frac{1}{l}\right)\right) dS_h \doteq -2V_P \iint\limits_{S_h} \frac{\partial}{\partial n_h}\left(\frac{1}{l}\right) dS_h \tag{7–88}$$

since, because of the continuity of V, $V \doteq V_P$ inside and on S_h, the approximation is becoming better and better as $h \to 0$. Fig. 7.9 shows that

$$\frac{\partial}{\partial n_h} = -\frac{\partial}{\partial l} \; , \tag{7–89}$$

so that

$$\frac{\partial}{\partial n_h}\left(\frac{1}{l}\right) = -\frac{d}{dl}\left(\frac{1}{l}\right) = \frac{1}{l^2} = \frac{1}{h^2}$$

since $l = h$ on S_h. Furthermore

$$dS_h = h^2 d\sigma \quad , \tag{7–90}$$

with $d\sigma$ denoting the element of the unit sphere as usual. Thus the integral (7–88) becomes

$$-2V_P \iint\limits_{\sigma} \frac{1}{h^2} h^2 d\sigma = -2V_P \iint\limits_{\sigma} d\sigma = -8\pi V_P \quad , \tag{7–91}$$

which provides the "miraculous appearance" of V_P as promised towards the end of sec. 7.7.1!

Having achieved this, we shall kill the remaining terms in the integral over S_h. In fact,

$$\iint\limits_{S_h} \frac{2}{l}\frac{\partial V}{\partial n}\, dS_h = \iint\limits_{\sigma} \frac{2}{h}\frac{\partial V}{\partial n} h^2 d\sigma = 2\iint\limits_{\sigma} \frac{\partial V}{\partial n} h d\sigma \to 0 \tag{7-92}$$

as $h \to 0$. Furthermore,

$$-\iint\limits_{S_h} \Delta V \frac{\partial l}{\partial n}\, dS_h = \iint\limits_{\sigma} \Delta V h^2 d\sigma \to 0 \tag{7-93}$$

since

$$\frac{\partial l}{\partial n} = \frac{\partial l}{\partial n_h} = -\frac{\partial l}{\partial l} = -1 \quad ,$$

and

$$\iint\limits_{S_h} l \frac{\partial \Delta V}{\partial n}\, dS_h = \iint\limits_{\sigma} \frac{\partial \Delta V}{\partial n} h^3 d\sigma \to 0 \quad . \tag{7-94}$$

Hence in the limit $h \to 0$, eq. (7-86) reduces to

$$\iiint\limits_{v} l\Delta^2 V\, dv = -8\pi V_P +$$
$$+ \iint\limits_{S} \left(-2V \frac{\partial}{\partial n}\left(\frac{1}{l}\right) + \frac{2}{l}\frac{\partial V}{\partial n} - \Delta V \frac{\partial l}{\partial n} + l \frac{\partial \Delta V}{\partial n} \right) dS \quad . \tag{7-95}$$

This equation has exactly the same relation to (7-78) as Green's third identity has to Green's second identity (cf. Heiskanen and Moritz, 1967, pp. 11-12).

7.7.3 Lauricella's Theorems

What we still have to achieve is to eliminate the third and fourth terms of the integral on the right-hand side of (7-95). For this purpose we introduce an auxiliary function H which is *biharmonic and regular* (twice continuously differentiable) throughout v and assumes, together with its normal derivative, on the boundary surface S the same boundary values as the function (7-81):

$$H_S = l_S \ , \qquad \left(\frac{\partial H}{\partial n}\right)_S = \left(\frac{\partial l}{\partial n}\right)_S \quad . \tag{7-96}$$

The difference between the functions $U = l$ and H thus is that H is regular throughout v, whereas U has a singularity in its Laplacian at the point P; cf. (7-84). The point P is considered fixed in this context.

The existence and uniqueness of a solution H of the biharmonic equation

$$\Delta^2 H = 0 \tag{7-97}$$

satisfying the boundary conditions (7–96) is guaranteed for sufficiently smooth boundary surfaces S (Frank and Mises, 1961, p. 858). Since H is regular in the whole region v including its boundary S, we may apply (7–78) without problems, obtaining with (7–96) and (7–97)

$$\iiint\limits_v H\Delta^2 V dv = \iint\limits_S \left(-V\frac{\partial \Delta H}{\partial n} + \Delta H\frac{\partial V}{\partial n} - \Delta V\frac{\partial H}{\partial n} + H\frac{\partial \Delta V}{\partial n}\right) dS \quad . \tag{7–98}$$

Now we may subtract (7–98) from (7–95). Putting

$$G = l - H \tag{7–99}$$

we thus obtain

$$\iiint\limits_v G\Delta^2 V dv = -8\pi V_P - \iint\limits_S V\frac{\partial \Delta G}{\partial n} dS + \iint\limits_S \Delta G\frac{\partial V}{\partial n} dS \quad ; \tag{7–100}$$

the remainder cancels in virtue of the very conditions (7–96) (which hold only on S, exactly where we need them!). The function (7–99) now is *Green's function* for our present problem.

We thus get

$$V_P = -\frac{1}{8\pi}\iint\limits_S V\frac{\partial \Delta G}{\partial n} dS + \frac{1}{8\pi}\iint\limits_S \Delta G\frac{\partial V}{\partial n} dS - \frac{1}{8\pi}\iiint\limits_v G\Delta^2 V dv \quad , \tag{7–101}$$

which furnishes the promised representation of $V = V_P$ since

$$\Delta V = -4\pi G\rho \ , \qquad \Delta^2 V = -4\pi G\Delta\rho \tag{7–102}$$

by Poisson's equation (7–4).

In order to avoid a conflict of notation, we shall now restrict the use of the symbol G to the gravitational constant as in (7–102), using

$$G_2 = \frac{G}{2} \times \text{Green's function } G \quad . \tag{7–103}$$

Then (7–101) becomes

$$V_P = -\frac{1}{4\pi G}\iint\limits_S \frac{\partial \Delta G_2}{\partial n} V dS + \frac{1}{4\pi G}\iint\limits_S \Delta G_2\frac{\partial V}{\partial n} dS + \iiint\limits_v G_2 \Delta\rho dv \quad ; \tag{7–104}$$

this is Lauricella's formula. Note that P is a point in the interior v of S.

We shall distinguish two cases:

1. $\Delta\rho = 0$ (*harmonic density*). Then V may be prescribed on S, and the solution of the exterior Dirichlet problem gives the harmonic function outside V with prescribed boundary values V_S. This also provides the gravity vector

$$\mathbf{g} = \text{grad } V \tag{7–105}$$

outside and, by continuity, also on S; $(\partial V/\partial n)_S$ is the normal component of g on S and is therefore uniquely defined by V_S. Thus (7–104) gives

$$V_H = -\frac{1}{4\pi G}\iint_S \frac{\partial \Delta G_2}{\partial n} V\,dS + \frac{1}{4\pi G}\iint_S \Delta G_2 \frac{\partial V}{\partial n}\,dS \quad , \tag{7–106}$$

uniquely furnishing V in the interior of S and hence also the harmonic density ρ_H by (7–4). This is *Lauricella's First Theorem.*

2. $V = 0 = \partial V/\partial n$ on and outside S (the case of a *zero potential density*, cf. sec. 7.2). Then (7–104) reduces to

$$V_0 = \iiint_v G_2 \Delta\rho\, dv \quad , \tag{7–107}$$

as an explicit determination of the *interior potential* compatible with zero outside potential, cf. eq. (7–9). *$\Delta\rho$ can be prescribed arbitrarily.* This is *Lauricella's Second Theorem.*

As we have remarked in sec. 7.4, the general solution of our problem is

$$V_P = V_H + V_0 \quad , \tag{7–108}$$

as the sum of the uniquely defined potential with harmonic density and the "zero-potential density potential", to use an awkward but not inappropriate expression. Thus (7–108) gives the set of all possible smooth density distributions which are compatible with a given external potential, the arbitrariness of $\Delta\rho$ expressing the non-uniqueness of the solution.

In other terms, (7–108) provides a "constructive" representation of the set of all solutions of the gravimetric inner problem!

As a matter of fact, this sweeping statement must be taken with a grain of salt. What has been achieved is a solution of the inhomogeneous *"bipotential equation"*

$$\Delta^2 V = f \quad , \tag{7–109}$$

where

$$f = -4\pi G \Delta\rho \tag{7–110}$$

inside S. A solution of (7–109), however, is only possible if f and hence $\Delta\rho$ satisfy certain regularity conditions, for instance, if they are continuous with continuous derivatives everywhere within S. This is a much stronger condition than the mere continuity of ρ presupposed in sec. 7.6.

This immediately excludes discontinuous density jumps within the earth. However, this limitation is practically less serious than it looks since the density jumps can always be smoothed out to an arbitrarily high accuracy (also the polynomials used in sec. 7.6 are infinitely differentiable!).

Hence it is reasonable to say that Lauricella's solution (7–108) can be used to provide arbitrarily good approximations to the density anomalies inside the earth, and this may be just what we practically need.

7.7.4 Green's Function for the Sphere

It is easy to give Green's function G (7–99) if the boundary surface S is a sphere.

Submit the point P (to which V_P refers) to a *Kelvin transformation*, or *inversion in a sphere*. Cf. (Kellogg, 1929, pp. 231–223); for a different application see (Heiskanen and Moritz, 1967, pp. 143–144).

Fig. 7.10 shows the geometric situation. The inversion in the sphere transforms P into a point P' on the same radius as P, such that

$$rr' = R^2 \quad . \tag{7–111}$$

Define a function l_1 by

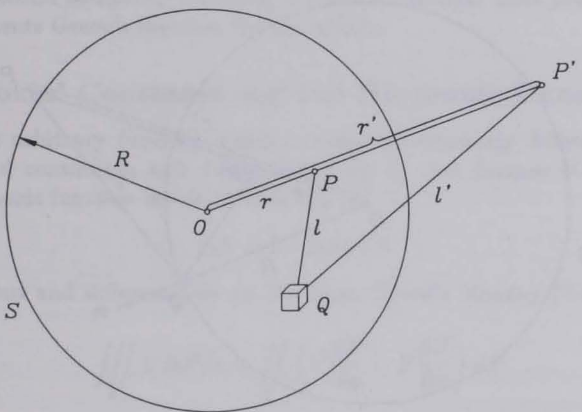

FIGURE 7.10: Kelvin transformation as an inversion in the sphere

$$l_1 = \frac{r}{R} l' \quad . \tag{7–112}$$

Then the auxiliary function H in (7–99) simply is

$$H = \frac{1}{2}\frac{l^2}{l_1} + \frac{1}{2} l_1 \quad , \tag{7–113}$$

so that Green's function (7–99) becomes

$$G = l - \frac{1}{2}\frac{l^2}{l_1} - \frac{1}{2} l_1 \tag{7–114}$$

(Marcolongo, 1901).

With coordinates for $P(x_P, y_P, z_P)$, $P'(x'_P, y'_P, z'_P)$ and $Q(x, y, z)$ we thus have

$$x'_P = \frac{R^2}{r^2} x_P \; , \quad y'_P = \frac{R^2}{r^2} y_P \; , \quad z'_P = \frac{R^2}{r^2} z_P \; , \tag{7–115}$$

$$r^2 = x_P^2 + y_P^2 + z_P^2 \quad , \tag{7–116}$$
$$l^2 = (x - x_P)^2 + (y - y_P)^2 + (z - z_P)^2 \quad , \tag{7–117}$$
$$l'^2 = (x - x'_P)^2 + (y - y'_P)^2 + (z - z'_P)^2 \quad . \tag{7–118}$$

It is straightforward though somewhat cumbersome to compute

$$\Delta^2 H = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{\partial^2 H}{\partial z^2}\right) \tag{7–119}$$

and to find that it is zero and regular even at P, so that H is indeed a regular solution of the biharmonic equation $\Delta^2 H = 0$.

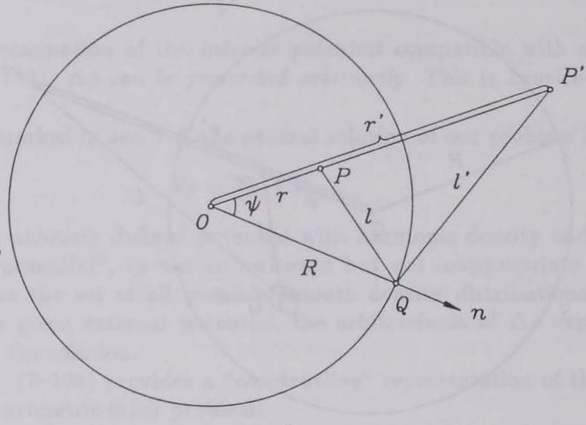

FIGURE 7.11: The point Q lies on the sphere S

There remains to verify the boundary conditions (7–96) on the sphere S. If Q lies on S, then (Fig. 7.11)

$$l^2 = r^2 + R^2 - 2rR\cos\psi \quad , \tag{7–120}$$
$$l'^2 = r'^2 + R^2 - 2r'R\cos\psi = \frac{R^4}{r^2} + R^2 - 2\frac{R^3}{r}\cos\psi$$
$$= \frac{R^2}{r^2} l^2 \quad , \tag{7–121}$$

so that by (7–112),

$$l_1 = \frac{r}{R} l' = \frac{r}{R}\frac{R}{r} l = l \quad \text{on} \quad S \quad . \tag{7–122}$$

Hence (7–113) gives

$$H = l \quad \text{on} \quad S \tag{7–123}$$

which is our first boundary condition.

To get the second one, we differentiate (7–113):

$$\frac{\partial H}{\partial n} = \frac{\partial H}{\partial R} = \frac{l}{l_1}\frac{\partial l}{\partial R} - \frac{1}{2}\frac{l^2}{l_1^2}\frac{\partial l_1}{\partial R} + \frac{1}{2}\frac{\partial l_1}{\partial R} \quad . \tag{7–124}$$

(The differentiation is considered to be carried out in such a way that, for the moment only, R varies since $\partial/\partial n = \partial/\partial R$ for the sphere, whereas the points P and P', and hence r and r', are unchanged and kept constant.) After differentiation, we set again $l_1 = l$ on S by (7–122) to get from (7–124):

$$\frac{\partial H}{\partial n} = \frac{\partial H}{\partial R} = \frac{\partial l}{\partial R} = \frac{\partial l}{\partial n} \quad \text{on} \quad S \quad , \tag{7–125}$$

so that our second boundary condition is satisfied as well. This proves that (7–114) in fact represents Green's function for the sphere.

7.7.5 Stokes' Constants and the Harmonic Density

Let F be an arbitrary function which is twice continuously differentiable inside a surface S and continuous and differentiable on S. Let further U be an arbitrary regular harmonic function inside S, that is

$$\Delta U = 0 \quad \text{inside } S \quad , \tag{7–126}$$

and continuous and differentiable on S. Then Green's identity (7–75) immediately gives

$$\iiint_v U \Delta F dv = \iint_S \left(U \frac{\partial F}{\partial n} - F \frac{\partial U}{\partial n} \right) dS \quad . \tag{7–127}$$

Thus the integral (7–127) *does not explicitly depend on the values of U inside v* but only on the boundary values U and $\partial U/\partial n$ on S, as the right-hand side shows. Such an integral is called a *Stokes' constant* (cf. Wavre, 1932, p. 43).

Examples of Stokes' constants are the quantity GM and the other spherical-harmonic coefficients A_{nm} and B_{nm} in (1–36); in this case, the functions U are the inner zonal harmonics (1–35a), as the expressions (2–38) of (Heiskanen and Moritz, 1967, p. 59) show; F is proportional to the inner potential V since $-4\pi G\rho = \Delta V$.

Let now F be the potential V_0 of a zero-potential density, that is, $V_0 \neq 0$ inside S but $V_0 \equiv 0$ on and outside S, so that also $\partial V_0/\partial n = 0$ on S. Then (7–127) reduces to

$$\iiint_v U \Delta V_0 dv = 0 \tag{7–128}$$

or

$$\iiint_v U \rho_0 dv = 0 \tag{7–129}$$

for any zero-potential density ρ_0 and any regular harmonic function U. This is a nice *characterization of zero-potential densities:* all their Stokes' constants are zero, in particular all their spherical-harmonic coefficients must vanish (Pizzetti, 1909, 1910).

As we have seen in sec. 7.4, any density ρ may be written

$$\rho = \rho_H + \rho_0 \tag{7–130}$$

as the sum of a harmonic density ρ_H and a zero-potential density ρ_0. Consider now

$$\iiint_v \rho^2 dv = \iiint_v (\rho_H + \rho_0)^2 dv \quad , \tag{7–131}$$

which equals

$$\iiint_v \rho_H^2 dv + 2\iiint_v \rho_H \rho_0 dv + \iiint_v \rho_0^2 dv \quad . \tag{7–132}$$

Regarding (7–131) as the definition of a norm $\|\ \|$ for the function ρ:

$$\|\rho\|^2 = \iiint_v \rho^2 dv \quad , \tag{7–133}$$

we may write (7–132) in the form

$$\|\rho\|^2 = \|\rho_H\|^2 + 2(\rho_H, \rho_0) + \|\rho_0^2\| \quad , \tag{7–134}$$

with an obvious definition and notation for the *inner product* of the functions ρ_H and ρ_0. Now (7–129), with $U = \rho_H$ (which is harmonic!), immediately shows that

$$(\rho_H, \rho_0) = 0 \quad , \tag{7–135}$$

that is, *the densities ρ_H and ρ_0 are mutually "orthogonal".*

Thus (7–134) reduces to

$$\|\rho\|^2 = \|\rho_H\|^2 + \|\rho_0\|^2 \geq \|\rho_H\|^2 \quad , \tag{7–136}$$

proving the *minimum norm property of the harmonic density* mentioned in sec. 7.3 (Marussi, 1980; Sansò, 1980).