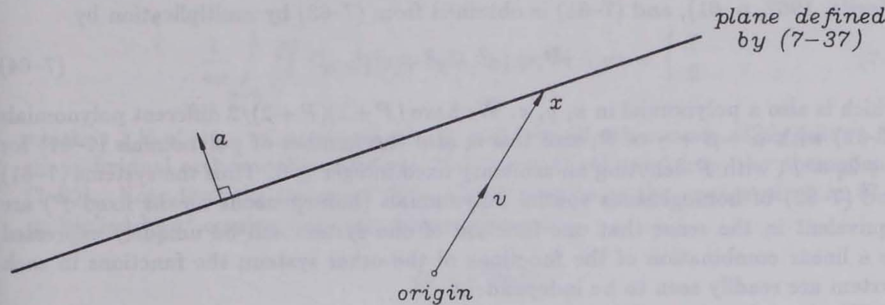

FIGURE 7.8: Representation of the solution x by an arbitrary vector v

Our set of density distributions comprises densities that are partly negative. As we have seen, this is not unphysical if V is regarded as a disturbing potential and ρ as a density anomaly, with respect to an underlying reference density model such as PREM. In fact, as mentioned before, this interpretation is of practical relevance if for V we take one of the global spherical harmonic expansions as discussed, e.g., in (Rapp, 1986).

The set of possible solutions can then be suitably restricted: by the obvious condition that the total density (reference density plus density anomaly) must be positive, and less trivially, by important information from seismology and other observational sources, as well as by theoretical considerations such as theories of mantle convection.

7.6.7 Application of Orthonormal Expansions

A very interesting special case of (7–51) has been treated by Dufour (1977). He considers representations of the density ρ of form

$$\rho(r, \theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{q=0}^{Q} \beta_{nmq} r^{n+2q} Y_{nm}(\theta, \lambda) \quad . \tag{7–60}$$

A first glance shows that (7–60) is less general than (7–51) because the powers r^k for $k < n$ are missing, as well as the powers r^{n+1}, r^{n+3}, r^{n+5}, ... However, the base functions

$$r^{n+2q} Y_{nm}(\theta, \lambda) = r^n Y_{nm}(\theta, \lambda) \cdot r^{2q} \tag{7–61}$$

are easily seen to be polynomials in the Cartesian coordinates x, y, z of form

$$x^\alpha y^\beta z^\gamma \quad , \tag{7–62}$$

α, β, γ being integers ≥ 0.

In fact, the solid harmonics (1–35a), or

$$r^n Y_n(\theta, \lambda) \quad , \tag{7–63}$$

are well known to be harmonic polynomials (cf. Kellogg, 1929, p. 141; Heiskanen and Moritz, 1967, p. 61), and (7–61) is obtained from (7–63) by multiplication by

$$r^{2q} = (x^2 + y^2 + z^2)^q \quad , \tag{7–64}$$

which is also a polynomial in x, y, z. We have $(P+1)(P+2)/2$ different polynomials (7–62) with $\alpha + \beta + \gamma = P$, and this is also the number of polynomials (7–61) for $n + 2q = P$, with P denoting an arbitrary fixed integer ≥ 0. Thus the systems (7–61) and (7–62) of homogeneous spatial polynomials (homogeneous means fixed P) are equivalent in the sense that one function of one system can be uniquely expressed as a linear combination of the functions of the other system; the functions in each system are readily seen to be independent.

The base functions of (7–60), as regards the dependence on r, are the functions

$$r^n[1, r^2, r^4, \ldots, r^{2Q}] \quad . \tag{7–65}$$

We may orthonormalize these functions, e.g., by the well-known Gram–Schmidt orthogonalization process (cf. Courant and Hilbert, 1953, pp. 4 and 50), obtaining the orthonormal system of polynomials

$$r^n P_{n,q}(r^2) \qquad (q = 0, 1, \ldots, Q) \tag{7–66}$$

equivalent to (7–65) in the sense just mentioned. They will satisfy the orthonormality relations

$$\int_0^1 r^{2n+2} P_{n,q}(r^2) P_{n,q'}(r^2) dr = \begin{cases} 1 & \text{if} \quad q' = q \, , \\ 0 & \text{if} \quad q' \neq q \, ; \end{cases} \tag{7–67}$$

we have put $R = 1$ without loss of generality. Substituting $r^2 = u$ we get

$$\frac{1}{2} \int_0^1 u^{n+\frac{1}{2}} P_{n,q}(u) P_{n,q'}(u) du = \begin{cases} 1 & \text{if} \quad q' = q \, , \\ 0 & \text{if} \quad q' \neq q \, . \end{cases} \tag{7–68}$$

Such polynomials (the factor $1/2$ is inessential) are special cases of Jacobi polynomials (cf. Courant and Hilbert, 1953, p. 88; Abramowitz and Stegun, 1965, sec. 22). We need not give them explicitly because we are mainly interested in the fundamental conceptual features of Dufour's theory.

The "conventional" spherical harmonics Y_{nm} ($= R_{nm}$ or S_{nm}) are orthogonal but not normalized; cf. (1–41) and (1–42). By multiplying them by an obvious factor related to κ_{nm} in (1–42) it is possible to normalize them in the sense that

$$\frac{1}{4\pi} \iint_\sigma [\bar{Y}_{nm}(\theta, \lambda)]^2 d\sigma = 1 \quad , \tag{7–69}$$

$\bar{Y}_{nm}(\theta, \lambda)$ being "fully normalized" spherical harmonics (cf. Heiskanen and Moritz, 1967, sec. 1–14). We thus see that Dufour's spatial functions

$$r^n \bar{Y}_{nm}(\theta, \lambda) P_{n,q}(r^2) \equiv D_{nmq}(r, \theta, \lambda) \tag{7–70}$$

are orthonormal in the solid sphere $r \leq 1$:

$$\frac{1}{4\pi} \int_{r=0}^{1} \iint_{\sigma} D_{nmq}(r, \theta, \lambda) D_{n'm'q'}(r, \theta, \lambda) dv = \left\{ \begin{array}{l} 1 \\ 0 \end{array} \right. , \qquad (7\text{–}71)$$

which is 1 if $n' = n$, $m' = m$ and $q' = q$, and 0 in all other cases. This follows from the individual orthogonality relations (7–67) and (1–41) and from the normalization (7–69). Note that the factor r^{2n+2} in (7–67) results as the square of r^n in (7–70) multiplied by r^2 coming from the volume element

$$dv = r^2 dr d\sigma$$

in (7–71). Note also that the Jacobi polynomial $P_{n,q}$ in (7–70) has nothing to do with the Legendre function P_{nm} defined by (1–30)!

Let now finally the density ρ be expressed in terms of the orthonormal base functions (7–70):

$$\rho(r, \theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{q=0}^{Q} \alpha_{nmq} D_{nmq}(r, \theta, \lambda) \quad . \qquad (7\text{–}72)$$

By what has been said, this expression is completely equivalent to (7–60), of course with different coefficients α_{nmq}.

Any function ρ that can be approximated by a linear combination of polynomials (7–62), can also be approximated by (7–72), and we may even let $Q \to \infty$ under certain circumstances. In fact, Weierstrass' approximation theorem has been extended to three– (and higher–) dimensional space, so that any continuous function in space can be uniformly approximated by a linear combination of polynomials (7–62) (cf. Davis, 1963, sec. 6.6).

Now it is clear that

$$P_{n,0}(r^2) \equiv \text{const.} \quad ; \qquad (7\text{–}73)$$

this follows directly from the orthogonalization process leading from (7–65) to (7–66). Thus (7–67) gives immediately

$$\int_{0}^{1} r^{2n+2} P_{n,q}(r^2) dr = 0 \quad \text{if} \quad q \neq 0 \quad . \qquad (7\text{–}74)$$

Substituting

$$f_{nm}(r') = \sum_{q=0}^{Q} \alpha_{nmq} r'^n P_{n,q}(r'^2)$$

into (7–31), noting that $n' = n$ because of orthogonality, and considering (7–74), we see that *only the terms with $q = 0$ in (7–72) give a non–zero contribution to V_{nm}*.

Now $q = 0$ by (7–61) means the solid harmonic (7–63), so that the terms with $q = 0$ in (7–72), all other coefficients being taken to be zero, represent the *harmonic density* (7–47). The terms with $q \neq 0$ then give a zero–potential density, *the arbitrariness of* *the corresponding coefficients expressing the non-uniqueness of the density producing zero external potential.*

Thus the main elegance of Dufour's method consists in a neat separation of the space D of possible density functions (as represented by polynomials) into two mutually orthogonal subspaces:

1. the set of harmonic densities D_1;
2. the set of zero–potential densities D_2.

The first subspace is represented by the terms with $q = 0$, the second subspace by the terms with $q \neq 0$. Taking a general expression (7–72) and putting all terms with $q \neq 0$ equal to zero thus corresponds to a projection of the function onto the subspace D_1, and putting the terms with $q = 0$ equal to zero amounts to a projection on the subspace D_2.

This beautiful result represents the principal theoretical value of Dufour's approach. From a practical point of view, the more general polynomial approximation considered in the previous subsections seems to be preferable because, for a given potential coefficient V_{nm}, one would certainly not like to miss the terms with $k < n$ in the polynomial (7–27). From a theoretical point of view, however, Dufour's approach represents the functional–analytic, "geometrical", aspect of the inverse Newtonian operator with unsurpassable clarity.

A simple example. Let us finally illustrate the situation by an extremely simple analogue. Let the space D be simplified to Euclidean three–dimensional space. Let the harmonic subspace D_1 correspond to the z–axis, and the orthogonal subspace D_2 to the xy–plane. A general 3–vector

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_0 \end{bmatrix}$$

will stand for the coefficients α_{nmq} in (7–72). The harmonic densities are then represented by the projection of this vector onto the z–axis,

$$\begin{bmatrix} 0 \\ 0 \\ \alpha_0 \end{bmatrix} ,$$

which also corresponds to the Newtonian operator N of sec. 7.1 (apart from a constant factor given by a_{nmn} in (7–35)). On the other hand, the zero–potential densities are represented by the projection of the vector onto the xy–plane:

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{bmatrix} ;$$

such vectors form the kernel, or nullspace, of the Newtonian operator N (sec. 7.2).

A given external potential essentially amounts to prescribing α_0 only. The non–uniqueness of the inverse of the Newtonian operator then means nothing else than that the components α_1 and α_2 in the first vector can be arbitrarily chosen.

To be sure, this example is so simplified as to be almost trivial, but it illustrates the geometrical situation very clearly.

7.7 Lauricella's Use of Green's Function

Finally we shall treat a very general explicit solution of the gravimetric inverse problem due to Lauricella (1911, 1912), which forms part of important work done by Italian mathematicians such as T. Boggio, U. Crudeli, E. Laura, R. Marcolongo, C. Mineo, P. Pizzetti, and C. Somigliana between 1900 and 1930. This work is not so well known as it deserves; an excellent review is (Marussi, 1980), where also references to the original papers are found.

We shall here follow the book (Frank and Mises, 1961, pp. 845–862), translating that treatment from the two-dimensional to the three-dimensional case.

7.7.1 Application of Green's Identity

Green's second identity may be written:

$$\iiint_v (U\Delta F - F\Delta U)dv = \iint_S \left(U\frac{\partial F}{\partial n} - F\frac{\partial U}{\partial n} \right) dS \quad ; \qquad (7\text{–}75)$$

this is eq. (1–28) of (Heiskanen and Moritz, 1967, p. 11) with F instead of V. It is valid for arbitrary functions U and F (which are, of course, "smooth", that is, sufficiently often differentiable, but this will be taken for granted in the sequel without mentioning). Here v denotes the volume enclosed by the surface S, with volume element dv and surface element dS as usual, Δ is Laplace's operator and $\partial/\partial n$ denotes the derivative along the normal pointing away from v. The formula (7–75) is standard in physical geodesy; derivations may be found in (Sigl, 1985, pp. 30–32) or (Kellogg, 1929, pp. 211–215).

We now put

$$F = \Delta V \quad , \qquad (7\text{–}76)$$

the Laplacian of the gravitational potential V, obtaining

$$\iiint_v (U\Delta^2 V - \Delta V\Delta U)dv = \iint_S \left(U\frac{\partial \Delta V}{\partial n} - \Delta V\frac{\partial U}{\partial n} \right) dS \quad . \qquad (7\text{–}77)$$

In this equation we interchange U and V and subtract the new equation from (7–77). The result is

$$\iiint_v (U\Delta^2 V - V\Delta^2 U)dv = \iint_S \left(-V\frac{\partial \Delta U}{\partial n} + \Delta U\frac{\partial V}{\partial n} - \Delta V\frac{\partial U}{\partial n} + U\frac{\partial \Delta V}{\partial n} \right) dS \quad . \qquad (7\text{–}78)$$

Let us now daydream. Suppose we can select U such that

$$\Delta^2 U = 0 \qquad (7\text{–}79)$$