7.6.6 An Essential Simplification

The solution (7–38) may be written in the matrix form

$$x = Ca(a^T Ca)^{-1} b \quad , \tag{7–54}$$

where C is a square matrix and $a^T Ca$ is a number which must be $\neq 0$; b is also a number.

Every x satisfying (7–37) can be represented in the form (7–54), but, so to speak, it is "overrepresented": to each x there correspond infinitely many matrices C.

In fact (Rao and Mitra, 1971, p. 20), the matrix

$$CA^T(ACA^T)^{-1} \tag{7–55}$$

expresses the general form of a right inverse of A, according to the theory of generalized matrix inverses; the rank of ACA^T must be equal to the rank of A.

In our case, $A = a^T$ is a vector, supposed non-zero, that is, of rank 1. In this case *it is sufficient if C has rank* 1, that is, if it is of form

$$C = v\, v^T \quad , \tag{7–56}$$

where v is an arbitrary $(n+1)$ column vector which only satisfies

$$a^T v \neq 0 \quad . \tag{7–57}$$

Thus we obtain the solution

$$x = \frac{v\, v^T a}{a^T v\, v^T a} b$$

or finally

$$x = \frac{b}{a^T v} v \quad . \tag{7–58}$$

This solution admits an immediately obvious geometrical interpretation (Fig. 7.8): it is the solution x that has the direction of the given vector v (since both b and $a^T v$ are numbers).

This extremely simple solution is due to G. Zielke (Zielke and Moritz, 1989). It goes without saying that this is definitely preferable to (7–54) for practical applications, unless we exceptionally have some a-priori statistical or other information which we would like to incorporate into the matrix C. With (7–58) we get along with $N+1$ components of the vector v, instead of working with the elements (of order $N^2/2$) of a full-rank symmetric matrix C.

Obviously, $v = a$ gives the solution of minimum Euclidean norm (shortest length of x), i.e.,

$$x_{min} = \frac{b}{a^T a} a \quad , \tag{7–59}$$

which mathematically is the simplest solution but which does not seem to have a physical interpretation.

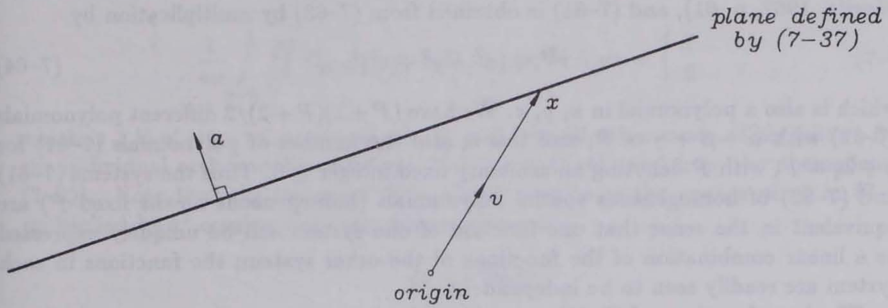

FIGURE 7.8: Representation of the solution x by an arbitrary vector v

Our set of density distributions comprises densities that are partly negative. As we have seen, this is not unphysical if V is regarded as a disturbing potential and ρ as a density anomaly, with respect to an underlying reference density model such as PREM. In fact, as mentioned before, this interpretation is of practical relevance if for V we take one of the global spherical harmonic expansions as discussed, e.g., in (Rapp, 1986).

The set of possible solutions can then be suitably restricted: by the obvious condition that the total density (reference density plus density anomaly) must be positive, and less trivially, by important information from seismology and other observational sources, as well as by theoretical considerations such as theories of mantle convection.

7.6.7 Application of Orthonormal Expansions

A very interesting special case of (7–51) has been treated by Dufour (1977). He considers representations of the density ρ of form

$$\rho(r, \theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{q=0}^{Q} \beta_{nmq} r^{n+2q} Y_{nm}(\theta, \lambda) \quad . \tag{7–60}$$

A first glance shows that (7–60) is less general than (7–51) because the powers r^k for $k < n$ are missing, as well as the powers r^{n+1}, r^{n+3}, r^{n+5}, ... However, the base functions

$$r^{n+2q} Y_{nm}(\theta, \lambda) = r^n Y_{nm}(\theta, \lambda) \cdot r^{2q} \tag{7–61}$$

are easily seen to be polynomials in the Cartesian coordinates x, y, z of form

$$x^\alpha y^\beta z^\gamma \quad , \tag{7–62}$$

α, β, γ being integers ≥ 0.

In fact, the solid harmonics (1–35a), or

$$r^n Y_n(\theta, \lambda) \quad , \tag{7–63}$$