components of these $N+1$ base vectors in our original Cartesian coordinate system. Then A is not singular, and the condition that the given vectors be orthonormal with respect to P can be expressed as follows:

$$A^T P A = I \tag{7-43}$$

(I denotes the unit matrix), whence

$$P = (AA^T)^{-1} \tag{7-44}$$

is determined. Clearly, P and hence C are symmetric and positive definite matrices.

If one takes care of convergence, one may even let $N \to \infty$, but this is not really necessary because of Weierstrass' theorem mentioned above.

A minor point is that the degree $n = 1$ is usually missing: it can be made zero by a suitable choice of origin (Heiskanen and Moritz, 1967, p. 62). Also in order to have a well-defined density at the origin, it is necessary, except for $n = 0$, to start the summation in equations such as (7–27) or (7–37) with $k = 1$ rather than $k = 0$ (which reduces the dimension of our base space from $N + 1$ to N).

7.6.3 Harmonic Densities

A possible solution of (7–36) is, of course, obtained by putting $a_{nmk} = 0$ except for $k = n$, which gives

$$x_{nm} = \frac{V_{nm}}{a_{nmn}} = \frac{(2n+1)(2n+3)}{4\pi G R^{2n+3}} V_{nm} \quad , \tag{7-45}$$

by (7–35); this solution is unique. Thus

$$f_{nm} = x_{nm} r^n = \text{const.} r^n \quad , \tag{7-46}$$

and (7–26) gives

$$\rho(r, \theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} x_{nm} r^n Y_{nm}(\theta, \lambda) \quad , \tag{7-47}$$

which is a series of (internal) spherical harmonics; cf. sec. 1.3. Thus (7–47) represents the *harmonic density* for the spherical case. It is uniquely defined as we have announced in sec. 7.3 (theorem of Lauricella).

Considering the behavior of the powers r^n (Fig. 7.6; we have put $R = 1$), we see that the higher the degree n, the more concentrated towards the earth's surface will be the corresponding contribution of the density. This about corresponds to the physical feeling that higher-frequency density anomalies should be situated in the earth's upper crust and mantle, but otherwise the harmonic densities do not have any meaningful physical interpretation. Their main usefulness is mathematical, as a uniquely defined continuous solution of the inverse problem; cf. sec. 7.3.

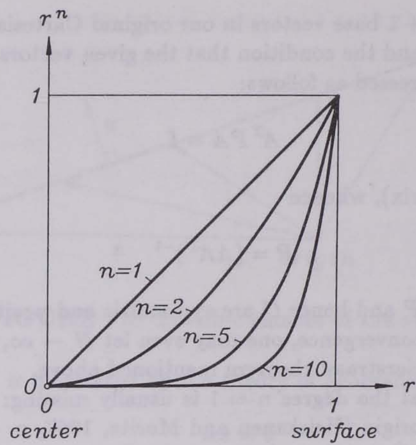

FIGURE 7.6: The powers r^n $(0 \leq r \leq 1)$

7.6.4 Zero-Potential Densities

The solution (7–38) gives $x_k = 0$ if the right-hand side of (7–37), $b = V_{nm}$, is zero. This is the case of the homogeneous equation corresponding to (7–36),

$$\sum_{k=0}^{N} a_{nmk} x_{nmk} = 0 \quad , \tag{7–48}$$

or briefly, corresponding to (7–37),

$$\sum_{k=0}^{N} a_k x_k = 0 \quad , \tag{7–49}$$

which represents the case of a mass distribution that produces zero external potential.

These are the "zero-potential densities" (sec. 7.2), forming the kernel of the Newtonian operator, for the present case. It is very easy to find non-zero solutions of (7–48) or (7-49): eq. (7–49) means simply that the vector x is normal to the given vector a (in the usual Euclidean metric)! Thus any vector x in the plane normal to a is admissible.

Finally we mention that the set of solutions of (7–49), forming *the vector* $x^{(2)}$ in (7–52) *is "orthogonal" to the vector* (7–38), denoted in (7–52) by $x^{(1)}$, if we again take P as metric tensor. This is geometrically evident and is also immediately verified by direct computation: using (7–38) in matrix notation, we have

$$x^{(1)T} P x^{(2)} = \frac{b}{a^T C a}\, a^T C P x^{(2)} = 0 \tag{7-50}$$

since $CP = I$ (unit matrix) and $a^T x^{(2)} = 0$ by (7–49).