using (1–49) with $f(\theta', \lambda') = Y_{nm}(\theta', \lambda')$, so that

$$V_{nm} = \frac{4\pi G}{2n+1} \int_0^R f_{nm}(r') r'^{n+2} dr' \quad . \tag{7-34}$$

The substitution of the polynomial (7–27) finally yields on integration

$$V_{nm} = \frac{4\pi G}{2n+1} \sum_{k=0}^{N} \frac{R^{n+k+3}}{n+k+3} x_{nmk} = \sum_{k=0}^{N} a_{nmk} x_{nmk} \quad . \tag{7-35}$$

Now the coefficients V_{nm} are nothing else than the spherical-harmonic coefficients of the external gravitational potential, which are well-known on a smoothed global scale; cf. (Rapp, 1986).

Assuming them given, we thus have the system of equations

$$\sum_{k=0}^{N} a_{nmk} x_{nmk} = V_{nm} \tag{7-36}$$

for the unknown coefficients x_{nmk}.

7.6.2 A Very General Solution

The system (7–36) is much less formidable than it looks. First of all, all degrees n and orders m are separated! *This means that we can treat each term (m, n) individually.* (This seems to be an essential advantage as compared to the approach of Dufour to be treated in sec. 7.6.7.) We thus omit the symbols n, m as we already did in (7–28) to get, instead of (7–36), a linear equation of form

$$\sum_{k=0}^{N} a_k x_k = b \quad , \tag{7-37}$$

where, of course, b represents V_{nm}.

Given b and the coefficients a_k (by (7–35)), we can satisfy (7–37) by infinitely many $(N+1)$-tuples x_k. Geometrically speaking, (7–37) is the equation of a (hyper)plane in $(N+1)$-dimensional space, and the only condition that the vector $x = [x_0, x_1, \ldots, x_N]$ must satisfy is that it must lead to a point in the plane (7–37), cf. Fig. 7.5.

A very general solution of (7–37) is

$$x_k = \frac{\sum_{j=0}^{N} c_{kj} a_j}{\sum_{i=0}^{N} \sum_{j=0}^{N} c_{ij} a_i a_j} b \quad , \tag{7-38}$$

as one immediately sees on substituting into (7–37). The matrix $[c_{ij}]$ can be chosen symmetric and positive definite and is otherwise arbitrary. The set of all these matrices (for all n) characterizes the set of possible solutions!

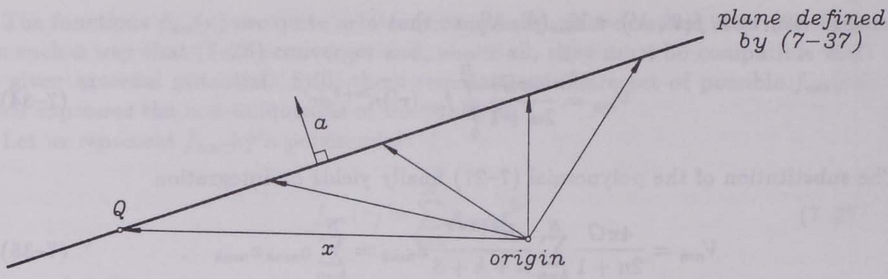

FIGURE 7.5: Possible choices of the vector x

The form (7–38) is motivated by the theory of generalized matrix inverses: if

$$Ax = b \tag{7–39}$$

is an underdetermined system of equations, the solution is formally given by

$$x = A^{-1}b \quad , \tag{7–40}$$

where the generalized inverse has the form (T denotes the transpose)

$$A^{-1} = CA^T(ACA^T)^{-1} \quad , \tag{7–41}$$

with any positive-definite symmetric square matrix C of appropriate dimension (cf. Bjerhammar, 1973, p. 110; Moritz, 1980, p. 164). Clearly (7–37) and (7–38) are special cases of (7–39) and (7–40) with (7–41).

The solution (7–38) satisfies the minimum condition

$$x^T P x = \text{minimum} \quad , \tag{7–42}$$

where $P = C^{-1}$. This means that x represents the "shortest" distance of the plane (7–37) from the origin, but of course in a non-orthogonal coordinate system whose metric tensor is P. That any point in the plane can be reached by a suitable choice of P can be seen in the following way (Krarup, 1972).

As we have mentioned, eq. (7–37) defines an N-dimensional hyperplane in our $(N+1)$-dimensional space (Fig. 7.5). Choose, for the first N base vectors, any set of N mutually orthogonal unit vectors (in the Euclidean sense) spanning the hyperplane. For the remaining $(N+1)$st base vector simply take the vector x leading from the origin to the desired point Q in the plane (Fig. 7.5). It is "orthogonal" to the hyperplane in the sense of the metric tensor P (though not in the Euclidean sense!) by the very condition (7–42), and its length is arbitrarily taken as unity.

Now we have found a set of $N+1$ linearly independent non-orthogonal vectors, and we must determine the metric tensor P for which they constitute an "orthonormal" set of base vectors. Let A now be the $(N+1)\times(N+1)$ matrix having as column vectors the components of these $N+1$ base vectors in our original Cartesian coordinate system. Then A is not singular, and the condition that the given vectors be orthonormal with respect to P can be expressed as follows:

$$A^T P A = I \tag{7-43}$$

(I denotes the unit matrix), whence

$$P = (AA^T)^{-1} \tag{7-44}$$

is determined. Clearly, P and hence C are symmetric and positive definite matrices.

If one takes care of convergence, one may even let $N \to \infty$, but this is not really necessary because of Weierstrass' theorem mentioned above.

A minor point is that the degree $n = 1$ is usually missing: it can be made zero by a suitable choice of origin (Heiskanen and Moritz, 1967, p. 62). Also in order to have a well-defined density at the origin, it is necessary, except for $n = 0$, to start the summation in equations such as (7–27) or (7–37) with $k = 1$ rather than $k = 0$ (which reduces the dimension of our base space from $N+1$ to N).

7.6.3 Harmonic Densities

A possible solution of (7–36) is, of course, obtained by putting $a_{nmk} = 0$ except for $k = n$, which gives

$$x_{nm} = \frac{V_{nm}}{a_{nmn}} = \frac{(2n+1)(2n+3)}{4\pi G R^{2n+3}} V_{nm} \quad , \tag{7-45}$$

by (7–35); this solution is unique. Thus

$$f_{nm} = x_{nm} r^n = \text{const.} r^n \quad , \tag{7-46}$$

and (7–26) gives

$$\rho(r, \theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} x_{nm} r^n Y_{nm}(\theta, \lambda) \quad , \tag{7-47}$$

which is a series of (internal) spherical harmonics; cf. sec. 1.3. Thus (7–47) represents the *harmonic density* for the spherical case. It is uniquely defined as we have announced in sec. 7.3 (theorem of Lauricella).

Considering the behavior of the powers r^n (Fig. 7.6; we have put $R = 1$), we see that the higher the degree n, the more concentrated towards the earth's surface will be the corresponding contribution of the density. This about corresponds to the physical feeling that higher-frequency density anomalies should be situated in the earth's upper crust and mantle, but otherwise the harmonic densities do not have any meaningful physical interpretation. Their main usefulness is mathematical, as a uniquely defined continuous solution of the inverse problem; cf. sec. 7.3.