(longitude). With respect to θ and λ, the base functions are the surface spherical harmonics, which are known to form a complete basis for functions defined on the sphere. For describing the dependence on the radius vector r, we take a polynomial representation. The powers r^n form a complete though not orthogonal basis in the space of continuous functions $f(r)$, in view of the famous *theorem of Weierstrass*: the polynomials are dense in the space of continuous functions, or in less abstract terms, any continuous function can be uniformly approximated by polynomials to any degree of accuracy. The coefficients of these polynomials form a finite-dimensional space. Our method thus is entirely elementary, avoiding Hilbert spaces considered in (Ballani and Stromeyer, 1983). As we have already mentioned, there is a certain similarity to the elegant approach of Dufour (1977) who uses orthogonalized (Jacobi) polynomials, but the present method is somewhat more general, admitting all even and odd powers of the radius vector r. We shall follow (Moritz, 1989).

7.6.1 Use of Spherical Harmonics

So let us use spherical coordinates r, θ and λ. For an internal sphere $r = \text{const.} = r_0$ the density ρ will be a function of θ and λ:

$$\rho = f_0(\theta,\, \lambda) = \rho(\theta,\, \lambda;\, r_0) \quad , \tag{7–22}$$

where r_0 enters as a parameter labeling the set of concentric spheres. This function may be expanded into a series of spherical surface harmonics (1–45):

$$f_0(\theta,\, \lambda) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (a_{nm} \cos m\lambda + b_{nm} \sin m\lambda) P_{nm} \cos\theta) \quad , \tag{7–23}$$

where P_{nm} are the standard Legendre functions and the coefficients a_{nm} and b_{nm} depend on r_0. Using the notation

$$\begin{aligned} Y_{nm}(\theta, \lambda) &= P_{nm}(\cos\theta)\cos m\lambda\,, \quad m = 0, 1, \ldots, n\,, \\ Y_{n,-m}(\theta, \lambda) &= P_{nm}(\cos\theta)\sin m\lambda\,, \quad m = 1, \ldots, n\,, \end{aligned} \tag{7–24}$$

we may write (7–23) in a more compact way:

$$f_0(\theta,\, \lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_{nm}(r_0) Y_{nm}(\theta,\, \lambda) = \sum_{n,m} f_{nm} Y_{nm} \quad . \tag{7–25}$$

Since f_{nm} ($= a_{nm}$ or b_{nm}) depends on r_0, we may generally write

$$\rho(r,\, \theta,\, \lambda) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_{nm}(r) Y_{nm}(\theta,\, \lambda) \quad , \tag{7–26}$$

an equation which is valid and convergent under very weak conditions on the smoothness of the functions involved, which we shall always take for granted without explicitly mentioning it.

The functions $f_{nm}(r)$ are quite arbitrary; they only must decrease with increasing n in such a way that (7–26) converges and, above all, they must be compatible with the given external potential. Still, there remains an infinite set of possible $f_{nm}(r)$, which expresses the non-uniqueness of the problem.

Let us represent f_{nm} by a polynomial

$$f_{nm}(r) = \sum_{k=0}^{N} x_{nmk} r^k \tag{7–27}$$

or, more briefly,

$$f(r) = \sum_{k=0}^{N} x_k r^k \quad , \tag{7–28}$$

x_k denoting constant coefficients. According to Weierstrass' theorem, we get an arbitrarily good approximation by choosing N sufficiently large.

We write eq. (7–1) in the form

$$V(r,\,\theta,\,\lambda) = G \iiint \frac{\rho}{l}\, dv \quad , \tag{7–29}$$

where V denotes the gravitational potential, G the gravitational constant, the integral is extended over the volume of the body bounded by the sphere $r = R$ ($\doteq$ 6371 km for terrestrial applications), ρ is the density and dv the volume element as usual. The symbol l denotes the distance between the point $P(r,\,\theta,\,\lambda)$ to which V refers, and the point $Q(r',\,\theta',\,\lambda')$ to which ρ and dv refer. If P is outside the sphere, we have by (1–53)

$$\frac{1}{l} = \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n(\cos\psi) \quad , \tag{7–30}$$

where P_n denotes Legendre's polynomials and ψ is the angle between r and r'.

We substitute (7–26) and (7–30) into (7–29), to get

$$V(r,\,\theta,\,\lambda) = G \sum_{n'=0}^{\infty} \frac{1}{r^{n'+1}} \sum_{n,m} \iint_{\sigma} Y_{nm}(\theta',\,\lambda') P_{n'}(\cos\psi) d\sigma \cdot \int_{r'=0}^{R} f_{nm}(r') r'^{n'+2} dr' . \tag{7–31}$$

Here we have replaced the index n in (7–30) by n', interchanged sum and integration without mathematical scruples, and put for the volume element

$$dv = r'^2 \sin\theta' dr' d\theta' d\lambda' = r'^2 dr' d\sigma \quad , \tag{7–32}$$

$d\sigma$ denoting the element of solid angle or the surface element of the unit sphere σ.

Now the integral over σ is zero unless $n' = n$ because of orthogonality, and there remains

$$V(r,\,\theta,\,\lambda) = \sum_{n,m} V_{nm} \frac{Y_{nm}(\theta,\,\lambda)}{r^{n+1}} \quad , \tag{7–33}$$

using (1–49) with $f(\theta', \lambda') = Y_{nm}(\theta', \lambda')$, so that

$$V_{nm} = \frac{4\pi G}{2n+1} \int_0^R f_{nm}(r') r'^{n+2} dr' \quad . \tag{7–34}$$

The substitution of the polynomial (7–27) finally yields on integration

$$V_{nm} = \frac{4\pi G}{2n+1} \sum_{k=0}^{N} \frac{R^{n+k+3}}{n+k+3} x_{nmk} = \sum_{k=0}^{N} a_{nmk} x_{nmk} \quad . \tag{7–35}$$

Now the coefficients V_{nm} are nothing else than the spherical-harmonic coefficients of the external gravitational potential, which are well-known on a smoothed global scale; cf. (Rapp, 1986).

Assuming them given, we thus have the system of equations

$$\sum_{k=0}^{N} a_{nmk} x_{nmk} = V_{nm} \tag{7–36}$$

for the unknown coefficients x_{nmk}.

7.6.2 A Very General Solution

The system (7–36) is much less formidable than it looks. First of all, all degrees n and orders m are separated! *This means that we can treat each term* (m, n) *individually.* (This seems to be an essential advantage as compared to the approach of Dufour to be treated in sec. 7.6.7.) We thus omit the symbols n, m as we already did in (7–28) to get, instead of (7–36), a linear equation of form

$$\sum_{k=0}^{N} a_k x_k = b \quad , \tag{7–37}$$

where, of course, b represents V_{nm}.

Given b and the coefficients a_k (by (7–35)), we can satisfy (7–37) by infinitely many $(N+1)$-tuples x_k. Geometrically speaking, (7–37) is the equation of a (hyper)plane in $(N+1)$-dimensional space, and the only condition that the vector $x = [x_0, x_1, \ldots, x_N]$ must satisfy is that it must lead to a point in the plane (7–37), cf. Fig. 7.5.

A very general solution of (7–37) is

$$x_k = \frac{\sum_{j=0}^{N} c_{kj} a_j}{\sum_{i=0}^{N} \sum_{j=0}^{N} c_{ij} a_i a_j} b \quad , \tag{7–38}$$

as one immediately sees on substituting into (7–37). The matrix $[c_{ij}]$ can be chosen symmetric and positive definite and is otherwise arbitrary. The set of all these matrices (for all n) characterizes the set of possible solutions!