7.5 Analytical Continuation

Finally we shall try to follow up on the singular solution V_P corresponding to the potential of a point mass given by (7–17) right to the singularity $r = 0$, which is harmonic in all R^3 excluding the point $r = 0$. If one calls the *support* of a function the region in which it is different from zero, then all mass distributions are *finite functions* in the sense that their support is finite (one also calls them "functions of compact support"). In fact, for a volume distribution such as ρ_H, the support is the compact volume bounded by the surface S (including S itself); for a surface distribution producing the potential V_S, the support is the surface itself. The point potential V_P has the property that all mass is concentrated at the origin $r = 0$: its support consists in that point only. Thus, V_P is the *potential of minimal support* corresponding to the function given by (7–17) outside S.

Since V_P is given by the same analytical expression (7–17) also inside S, it is called the *analytical continuation* of the external potential V to the interior of S. The analytical continuation of V is harmonic together with V but has a singularity at the origin.

"Being represented by the same analytical expression" is a rather primitive definition of analytical continuation. "Being harmonic and continuously differentiable together with V in a certain region" will be better since such harmonic functions are always *analytic*, i.e., expandable in a power series convergent in a subregion of that region. In fact, Laplace's equation $\Delta V = 0$ has the striking property of admitting only analytic solutions; cf. (Kellogg, 1929, p. 220).

The surface potential V_S is *not* an analytic function in the whole R^3: it satisfies Laplace's equation "almost everywhere" in R^3 (except the surface S itself!); it is even continuous throughout R^3; but it is not continuously differentiable on S as (7–14) shows.

Nor is V_P: it has a singularity at $r = 0$. In fact, the only function harmonic throughout R^3 is the constant function, which has the same value everywhere. *Any nonconstant harmonic function must have a singularity somewhere*, be it a point, a line, a surface or a more complicated point set; see below.

This is very similar to the behavior of *analytic functions of a complex variable*, which are quite analogous, in the plane, to harmonic functions in R^3; cf. (Kellogg, 1929, Chapter XII). Also the analytical continuation is a standard topic in complex function theory, as any textbook on the subject shows.

Thus the external potential together with its maximal analytical continuation to the interior of the body, of which V_P gives a simple example, can be supposed to furnish a potential of minimal support.

The problem of analytical continuation is mathematically extremely complex and has so far been solved for relatively simple cases only; this is also indicated by the fact that the booklet (Herglotz, 1914) which considers such simple cases, is considered a classic. The contemporary mathematical state of knowledge is found in (Schulze and Wildenhain, 1977, secs. III.5.4 and III.5.5); also the little we know is mathematically very deep and highly interesting. A discussion of geodetically relevant aspects is found in (Moritz, 1980, secs. 6 to 8). What follows is heuristic "small talk" about a fascinating and both theoretically and practically important subject.

The practical relevance is known from geophysical prospecting. If the analytical continuation of V (or of gravity anomalies Δg) into the interior of the earth is found to have a point-like (say) singularity at some interior point, then one may look for an anomalous mass there: the point-like singularity may be due to a spherical body which may represent an ore deposit (positive density anomaly) or a salt dome (negative density anomaly). The relation between the inverse problem and analytical continuation was also pointed out by Marussi (1982).

Are all singularities point-like? By no means: it is even impossible to classify all possible singularities, and they may have any degree of mathematical complexity (Schulze and Wildenhain, 1977, p. 121). Let us mention only a few of them (Fig. 7.4). Besides a simple pole (mass point) A we may have a dipole B (two points of oppositely

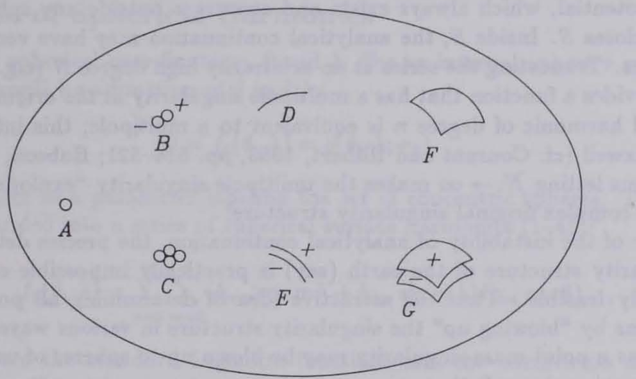

FIGURE 7.4: Possible types of singularities of the analytical continuation

equal mass "infinitely close" to each other), a quadrupole C or any higher multipole, a piece of line D or a "dipole line" E, a surface piece F, a piece of a "double layer surface" G which is some kind of "dipole surface", (cf. Heiskanen and Moritz, 1967, pp. 7 to 8), and so on to arbitrary complexity.

All these singularities have one property in common: they are not only of *zero measure* but also of *zero capacity*. Measure is a mathematical precision of the notion of volume, and capacity is not only a physical notion familiar from electrostatics but also a mathematical concept fundamental in modern potential theory. Also, if the masses are concentrated on sets of capacity zero, the *energy becomes infinite* (for V_S, the energy was minimum!) (cf. Schulze and Wildenhain, 1977, p. 122). On sets of capacity zero, V *also becomes infinite*.

A *closed* surface S is not a set of minimal support in the present sense: the potential does not become infinite on it and S, though being a set of zero measure, is not a set of zero capacity. An *open* surface, however, is a singularity in the present sense (cf. the set F in Fig. 7.4): it cannot be "compressed" to a smaller set (the sphere can be compressed into a point without changing the external potential!).

The maximal analytical extension gives a uniquely defined solution. Still, it is an *improperly posed problem* in the sense of sec. 7.1 since the solution is not stable (cf. Lavrentiev, 1967, Chapter II). An arbitrarily small change in the external potential V can provoke a large change in the analytical continuation and may even completely alter the singularities: shift them, make them vanish or create new ones. This is an implication for the present problem of the Runge theorem (cf. Schulze and Wildenhain, 1977, sec. III.2.9; Moritz, 1980, sec. 8). We may also say that the analytical continuation into the interior of S is an initial-value (Cauchy) problem for Laplace's equation, which has for a long time been known to be improperly posed (Courant and Hilbert, 1962, pp. 227–229). In fact, S may be regarded as a "Cauchy surface", from which the analytical continuation into the interior starts.

A striking and simple example is given by the spherical harmonic series of the external potential, which always exists and converges outside any sphere that completely encloses S. Inside S, the analytical continuation may have very complicated singularities. Truncating the series at an arbitrarily high degree N (e.g., equal to 10^6) always provides a function that has a multipole singularity *at the origin only.* In fact, a spherical harmonic of degree n is equivalent to a multipole; this interpretation is due to Maxwell (cf. Courant and Hilbert, 1953, pp. 514–521; Hobson, 1931, secs. 79 to 84). Thus letting $N \to \infty$ makes the multipole singularity "explode" to form the arbitrarily complex original singularity structure!

In view of the instability of analytical continuation, the precise determination of the singularity structure of the earth (say) is practically impossible even if it were theoretically feasible. Thus the attractive idea of determining all possible density distributions by "blowing up" the singularity structure in various ways (much in the same way as a point mass singularity may be blown up to spheres of various sizes) is likely to remain science fiction.

7.6 Continuous Density Distributions for the Sphere

We have seen that the general gravitational inverse problem is very difficult and has not been solved generally so far.

However, restricting ourselves to continuous density distributions for the sphere, a rather general solution can be found in a simple and elementary way. A spherical earth is a good approximation for many geophysical purposes, especially for determining density anomalies from given potential anomalies. Furthermore, even "discontinuities" such as the core-mantle boundary may be regarded as continuous, though rather abrupt, transitions.

The approach is based on trying to find an approximate finite matrix equivalent to the Newtonian operator N, as we have already announced in sec. 7.1. The approach employs the usual spherical coordinates r (radius vector), θ (polar distance), and λ