7.3 Unique Solutions

Harmonic densities. Take a general exterior potential V (i.e., $\Delta V = 0$ outside S), then there is a unique continuous density distribution ρ_H producing V outside S and being harmonic inside S:

$$\Delta \rho_H = 0 \quad \text{inside } S \quad . \tag{7–10}$$

(Obviously, ρ_H must be smooth in the sense mentioned in the beginning: twice differentiable in the present case.) Substituting (7–4) we get

$$\Delta^2 V \equiv \Delta(\Delta V) = 0 \quad , \tag{7–11}$$

i.e., V must satisfy the *biharmonic equation*, which also occurs in elasticity theory; cf. sec. 8.1.3. Under certain natural mathematical conditions, the solution of the biharmonic equation exists and is unique, so that the same holds for ρ_H. This is a *theorem of Lauricella*; see sec. 7.7.3. The harmonic density thus provides a uniquely defined inverse of the Newtonian operator, symbolically

$$\rho_H = N_H^{-1} V \quad . \tag{7–12}$$

The function $\rho = \rho_H$ minimizes the integral

$$\iiint_v \rho^2 dv$$

(sec. 7.7.5). This integral obviously defines a "norm in the space of square-integrable functions" (L_2-norm), being the continuous analogue to the vector norm $(\sum x_i^2)^{1/2}$. Other norms are also possible: this is the "problem of choice of norm" (Sansò et al., 1986; Hein et al., 1988). We shall not pursue this question here, because we shall concentrate on the set of possible solutions (secs. 7.6 and 7.7) rather than on singling out special solutions. We should, however, mention some rather sophisticated approaches and results: (Matyska, 1987; Remmer, 1986a, b; Rubincam, 1982; Sansò, 1980; Sansò and Tscherning, 1989; Škorvanek, 1981; Tscherning, 1974; Tscherning and Strykowski, 1987; and Tscherning and Sünkel, 1981).

Surface densities. Assume the masses concentrated entirely on the surface S, as a layer of surface density μ, the inside and outside of S being entirely empty. Then, instead of (7–1), we have

$$V(P) = G \iint_S \frac{\mu(Q)}{l_{PQ}} dS_Q \quad ; \tag{7–13}$$

the notations are obvious; Q now is a point on the surface S. Then V is harmonic inside and outside S; it is continuous on S but on S has discontinuities of the derivative $\partial V/\partial n$ along the surface normal, so that

$$\left(\frac{\partial V}{\partial n}\right)_e - \left(\frac{\partial V}{\partial n}\right)_i = -4\pi G\mu \quad , \tag{7–14}$$

(cf. Kellogg, 1929, p. 164; Heiskanen and Moritz, 1967, p. 6).

Knowing V outside and on S, we can find that harmonic function inside S which coincides with the given V on S, by solving Dirichlet's problem for the interior of S. Then $\partial V/\partial n$ are known both inside and outside S, as well as their limits $(\partial V/\partial n)_i$ if we approach S from the inside and correspondingly $(\partial V/\partial n)_e$ as we approach S from the outside. Now (7–14) gives the surface density μ, which is thus uniquely defined.

In this way we have found another unique solution of our gravitational inverse problem, symbolically

$$\mu = N_S^{-1} V \quad . \tag{7–15}$$

The winner is unique, but the competition is not quite fair since a surface density μ is not a continuous volume density ρ (although it may be considered the limit of the volume density of a shell of finite thickness h and finite density ρ, with $h \to 0$ and $h\rho \to \mu$).

At any rate, surface densities play a very important role in potential theory. We may say that the surface layer potential V_S represents that unique solution N^{-1} which satisfies

$$\Delta V_S = 0 \quad \text{inside}\, S \tag{7–16}$$

(compare to (7–10)!) and which coincides with the given V on S. By *Dirichlet's principle* (Kellogg, 1929, p. 279), V_S minimizes the potential energy (which is another norm, theoretically fundamental but of little relevance for a realistic terrestrial mass distribution; cf. also sec. 5.12.1).

Example of the sphere. For a homogeneous sphere, the external potential is given by the same expression as that of a mass point:

$$V = \frac{GM}{r} \quad , \tag{7–17}$$

where $r = \sqrt{x^2 + y^2 + z^2}$. If S is the unit sphere $r = 1$, then the harmonic density is

$$\rho_H = \frac{M}{4\pi/3} = \frac{\text{mass}}{\text{volume}} = \text{const. inside}\, S \quad , \tag{7–18}$$

representing a homogeneous sphere of radius 1 whose internal potential, by (2–43), is

$$V_H = 2\pi G \rho_H \left(1 - \frac{1}{3} r^2\right) \quad . \tag{7–19}$$

It is trivial that $\Delta \rho_H = 0$ by (7–18). (For the level ellipsoid, the problem of finding the harmonic density is not at all trivial!) On the other hand, the surface density is

$$\mu = \frac{M}{4\pi} = \frac{\text{mass}}{\text{surface}} = \text{const. on}\, S \quad , \tag{7–20}$$

so that, by (2–34),

$$V_S = GM = \text{const. inside}\, S \quad . \tag{7–21}$$

Fig. 7.2 shows V_H and V_S, as well as V_P, the potential of a mass point. Note that V_H

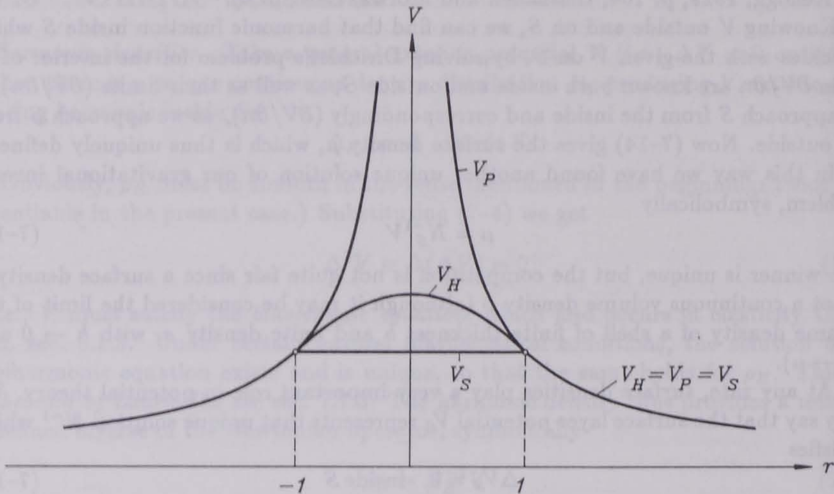

FIGURE 7.2: The potentials V_P, V_H and V_P; negative arguments are for the symmetry of the figure only (negative r are without geometric meaning!)

is continuous and differentiable everywhere, but it is not an analytic function in R^3 because it is represented by two different analytic functions: by (7–19) for $r \leq 1$ and by (7–17) for $r \geq 1$; both functions are welded smoothly together at $r = 1$, so that their combination forms the nice bell-shaped curve for V_H in Fig. 7.2. On the other hand, V_S has a discontinuous derivative at S ($r = 1$), which shows that it cannot be the potential of a volume distribution. At any rate, V_H and V_S "bridge", in different ways, the singularity of V_P at the origin $r = 0$.

7.4 A "General" Solution

It is well known that the general solution of an inhomogeneous linear equation is obtained *as the sum of one particular solution of the inhomogeneous equation and the general solution of the corresponding homogeneous equation.* In our case, the particular solution is provided by the harmonic density described in the preceding section. The general solution of the homogeneous equation (7–7) (homogeneous means zero right-hand side) is the set of zero-potential densities forming the kernel of the Newtonian operator N.

Thus we find the general solution of the gravitational inverse problem by determining the uniquely defined harmonic density that corresponds to the given external potential, and adding any zero-potential density determined by the continuation method described in sec. 7.2; cf. also Fig. 7.1.

We may also proceed directly in the following way. We take the given harmo-