Chapter 7

Density Inhomogeneities

This chapter reviews the gravitational inverse problem with a view to applications to global geodesy and geophysics. It has a physical and heuristic character. A profound discussion would require the deep mathematical tools of modern potential theory (cf. Schulze, 1977; Schulze and Wildenhain, 1977; Anger, 1990). This is not attempted here: our treatment will be mathematically as elementary as possible.

After a rather detailed discussion of general aspects of the gravitational inverse problem, we shall in sec. 7.6 then consider the problem of finding continuous density distributions inside a sphere which produce a given external potential. We shall find an explicit, very simple and practically applicable, representation of the set of all density distributions that are compatible with the given potential, valid to any prescribed degree of accuracy.

Finally, the important but little known solution of Lauricella by means of Green's function will be described.

7.1 The Gravitational Inverse Problem

Assume a body bounded by a smooth surface S with a distribution of density ρ which is piecewise continuous. By "smooth" we mean "differentiable as far as required" (differentiable once or several times, depending on the circumstances), and by "piecewise continuous" we mean that the regions (within a body) in which the density is continuous, are separated by smooth surfaces. As an example we may take the earth: inner core, outer core, mantle and crust are separated by "discontinuities": the core-mantle boundary, the Mohorovičić discontinuity, etc.

To the mathematician, these assumptions are neither very sharp nor the weakest possible, but they are intuitive and physically meaningful and sufficient for the present discussion.

Then the gravitational potential V of this body *(volume potential)* is given by the standard Newtonian integral (1–1), written in the form

$$V(P) = G \iiint\limits_{v} \frac{\rho(Q)}{l_{PQ}} \, dv_Q \quad , \tag{7–1}$$

where P denotes the point at which V is considered, Q is the point to which the volume element dv, and hence also the density ρ, refer, l_{PQ} is the distance between P and Q, v is the volume enclosed by our surface, and G denotes the gravitational constant as usual. To the physicist it is clear that we are working in Euclidean three-space R^3.

The essential point is that (7–1) is *linear* in ρ. We may thus write symbolically

$$V = N\rho \quad , \tag{7–2}$$

where N denotes the linear *"Newtonian operator"* defined by eq. (7–1): N acts on the function ρ to give the function V. Both functions are defined all over R^3: V is continuous and differentiable everywhere, and ρ piecewise continuous, being zero outside S.

The *gravitational* or *gravimetric inverse problem* then may be formulated (and formally solved) by inverting (7–2):

$$\rho = N^{-1}V \quad . \tag{7–3}$$

The operator N^{-1} would be one-to-one if V were given all over R^3, because by Poisson's equation

$$\Delta V = -4\pi G\rho \quad , \tag{7–4}$$

so that in this case

$$N^{-1} = -\frac{1}{4\pi G}\Delta \quad , \tag{7–5}$$

Δ denoting the Laplace operator, or *Laplacian*:

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

in Cartesian coordinates xyz.

In reality, of course, V is given only *outside* S, and this is what makes the gravitational inverse problem a real problem: the operator N^{-1} then becomes one to infinitely many. In fact, it is well known that there are infinitely many density distributions that are compatible with a given external potential V; the solution (7–3) is not unique.

Since $\rho = 0$ outside S, eq. (7–4) gives

$$\Delta V = 0 \quad \text{outside } S \quad , \tag{7–6}$$

V is a *harmonic function* there. *Thus it is sufficient to know V on S*: the solution of the exterior Dirichlet problem gives V outside S. It is also sufficient, e.g., *to know the gravity vector*

$$\mathrm{g} = \mathrm{grad}V = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}\right)$$

on S: the solution of Molodensky's problem then gives V outside and on S. (We are disregarding the centrifugal force to keep the argument as transparent as possible.)

The situation is quite similar to an underdetermined system of linear equations $y = Nx$ with formal solution $x = N^{-1}y$. (Note that ordinary letters are employed for "algebraic" vectors x and y, as will be done, e.g., also in sec. 7.6.2; cf. a corresponding remark in sec. 1.2. Briefly, boldface is only used when there is a danger of confusion: of the vector $\mathbf{g}$ with gravity g, or of the position vector $\mathbf{x}$ with the coordinate x.) Then N^{-1} is a *generalized inverse matrix* which is well known to be non-unique. In fact we may try to expand the functions ρ and V into a complete set of three-dimensional orthonormal base functions: this would transform the Newtonian operator N into an infinite matrix, and N^{-1} would be a generalized inverse of this matrix. This is the correspondence between linear operators and infinite matrices well-known since the foundations of quantum mechanics before 1930 (Schroedinger's formulation in terms of linear operators and Heisenberg's matrix mechanics). For a geodetic reference cf. (Moritz, 1980, sec. 4).

In his pioneering work, Dufour (1977) has treated the gravitational inverse problem for the sphere by such an orthonormal expansion. In sec. 7.6, we shall first present a similar approach which at the same time is more general and more elementary, using a polynomial representation for the radial dependence of the density. The problem will be reduced to a finite system of linear equations for which the generalized matrix inverse is extremely simple. Finally, the transition to the elegant approach of Dufour will be made.

Relation to improperly posed problems. A problem is called *properly posed* if the solution satisfies the following three requirements:

1. existence,
2. uniqueness,
3. stability.

This means that a solution must exist for arbitrary (within a certain range) data, there must be only one solution, and the solution must depend continuously on the data. If one or more of these requirements are violated, then we have an *improperly posed*, or *ill-posed*, problem.

For a long time it was thought that only properly posed problems are physically meaningful. In fact, deterministic processes, as considered in classical mechanics, depend uniquely and continuously on the initial data – this is the essence of causality – and thus correspond to properly posed problems (or at least it was thought so).

Only relatively recently it was recognized that many important problems are not properly posed. Not only most inverse problems, from geophysics to medicine and to scientific inference in general, are improperly posed – this thesis is convincingly proposed in the introductory chapter ("On the interpretation of nature") of the book (Anger, 1990) – but even deterministic processes of classical mechanics need not be stable – this is the nowadays extremely fashionable idea of "deterministic chaos"; cf. (Schuster, 1988).

The gravitational inverse problem was recognized as one of the first improperly posed problems (Lavrentiev, 1967). Of other geophysical inverse problems we mention seismic inversion, from the determination of global earth models such as PREM (Preliminary Reference Earth Model, cf. sec. 1.5) to seismic tomography. There is a huge literature on this subject; we can only mention a recent textbook (Tarantola, 1987) but cannot help quoting the fundamental paper (Backus, 1970).

7.2 Zero-Potential Densities

Since N^{-1} is non-unique, it is fundamental to investigate the *kernel* (or *nullspace*) of the operator N: the set of all density distributions ρ_0 within S that produce zero external potential:

$$N\rho_0 = 0 \quad \text{outside } S \quad . \tag{7–7}$$

Such density distributions ρ_0 will be called zero-potential densities. We repeat: *the set of all possible zero-potential densities forms the kernel of the Newtonian operator* N, symbolized by $\ker(N) = N^{-1}(0)$.

Clearly, ρ_0 must be alternatively positive and negative, so that the total mass is zero; otherwise (7–7) would be impossible. Contrary to the usage of much of standard potential theory, we do not require ρ to be positive now. In fact, in practical applications, V will represent *potential anomalies* rather than potentials, and the corresponding ρ will be *density anomalies* which may be positive or negative.

It is extremely easy to find a rather general method of determining $\ker(N)$. Take any function V_0 that is zero outside S and continued in a continuous and differentiable manner to the inside of S in such a way that it is also twice piecewise differentiable within S. This is illustrated in Fig. 7.1 for one instead of three dimensions; then the

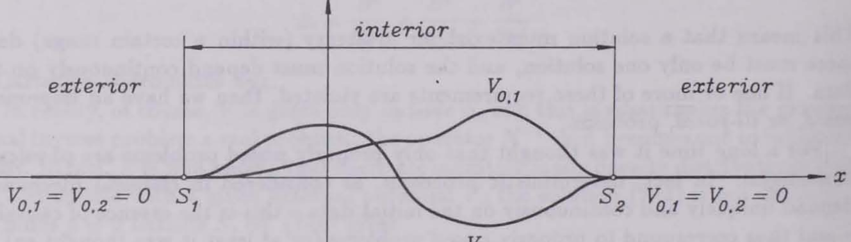

FIGURE 7.1: Two possible functions V_0 in one dimension

boundary S consists of two points S_1 and S_2.

Return to R^3. Since after continuation to the inside of S, V_0 is now defined throughout R^3, the corresponding density ρ_0 is given by (7–4):

$$\rho_0 = -\frac{1}{4\pi G}\,\Delta V_0 \quad . \tag{7–8}$$

Outside S this gives $\rho_0 = 0$ as it should, and inside, the zero potential density ρ_0 is piecewise continuous according to our differentiability assumptions concerning V_0.