For equilibrium figures, the surfaces S_1 and S_2 are identical. In the case of ellipsoidal mass distributions, they will be slightly different, and we shall now determine their deviation ζ. The idea is the same as that used in determining the height N of the geoid above the reference ellipsoid (cf. Heiskanen and Moritz, 1967, p. 84).

At P we have $W_P = W_1$, so that at Q

$$W_Q = W_1 - \frac{\partial W}{\partial n}\zeta = W_1 + g\zeta \quad . \tag{6-32}$$

Here $\partial/\partial n$ denotes the derivative along the normal n to the equidensity surface S_1 (Fig. 6.1), which can practically be identified with the plumb line; hence $-\partial W/\partial n = g$ is gravity inside the earth, for which the spherical approximation (2–62) is sufficient. On the other hand, since Q lies on the surface $\rho = \rho_1$, we can apply (6–23) to get

$$\begin{aligned} W_Q &= W_0(\beta_1) + W_4(\beta_1)P_4(\cos\theta) \\ &= W_1 + W_4(\beta_1)P_4(\cos\theta) \end{aligned} \tag{6-33}$$

in view of (6–31). By comparing the right-hand sides of (6–32) and (6–33) we see that

$$\zeta = \frac{1}{g}\,W_4(\beta)P_4(\cos\theta) \tag{6-34}$$

(since β_1 may be replaced by a general β) is the desired result for the height of S_2 above S_1. The reader will recognize the analogy of this result with the standard Bruns formula (1–25).

6.4 The Deviation κ

The deviation $\kappa = \kappa(\beta)$ for any second-order spheroid must satisfy the integral condition (6–15), where P_1 is given by (4–56) with $\beta = 1$:

$$\int_0^1 \delta\frac{d}{d\beta}(f^2\beta^7)\,d\beta + \frac{8}{9}\int_0^1 \delta\frac{d}{d\beta}(\kappa\beta^7)d\beta = -\frac{35}{12}J_4 \quad . \tag{6-35}$$

For the value $\kappa_1 = \kappa(1)$ be have the boundary condition (6–16):

$$-\frac{4}{5}f^2 + \frac{4}{7}fm - \frac{32}{35}\kappa_1 = J_4 \quad . \tag{6-36}$$

For the level ellipsoid there is $\kappa_1 = 0$, whence

$$-\frac{4}{5}f^2 + \frac{4}{7}fm = J_4^E \quad . \tag{6-37}$$

The difference of the last two equations gives

$$J_4 = J_4^E - \frac{32}{35}\kappa_1 \quad . \tag{6-38}$$

Now, for the Geodetic Reference System 1980 (cf. Moritz, 1984) we have

$$J_4^E = -0.000\,00237 \quad . \tag{6–39}$$

For hydrostatic equilibrium, we take Bullard's value

$$\kappa_1^H = 0.000\,00068 \quad , \tag{6–40}$$

whence (6–38) gives

$$J_4^H = -0.000\,00299 \quad . \tag{6–41}$$

For the actual earth we have from satellite observations

$$J_4^S = -0.000\,00162 \tag{6–42}$$

(IAG, 1980, p. 379). Hence, from (6–38),

$$\kappa_1^S = -\frac{35}{32}(J_4^S - J_4^E) \quad , \tag{6–43}$$

since for the ellipsoid

$$\kappa_1^E = 0 \tag{6–44}$$

and f^2 and fm are the same in all three cases: ellipsoid, equilibrium spheroid, and real earth regarded as a spheroid. From (6-39), (6–42), and (6–43) we compute

$$\kappa_1^S = -0.000\,00082 \tag{6–45}$$

for the real earth spheroid (defined by (6–13) with the observed values J_2 and J_4^S).

Let us now turn to the functions $\kappa_E(\beta)$, $\kappa_H(\beta)$, and $\kappa_S(\beta)$, which represent the deviation κ inside the body in our three cases. From (6–35) we immediately get the conditions

$$\int_0^1 \delta \frac{d}{d\beta}\left[(\kappa_E - \kappa_H)\beta^7\right] d\beta = -\frac{105}{32}(J_4^E - J_4^H) \quad , \tag{6–46}$$

$$\int_0^1 \delta \frac{d}{d\beta}\left[(\kappa_S - \kappa_H)\beta^7\right] d\beta = -\frac{105}{32}(J_4^S - J_4^H) \quad . \tag{6–47}$$

Fitting a simple polynomial to the result of Bullard (1948) as shown in Fig. 4.5, we find an approximate smoothed representation of $\kappa_H(\beta)$:

$$\kappa_H = 0.000\,00047\,\beta^2 + 0.000\,00021\,\beta^4 \quad . \tag{6–48}$$

Let us try a polynomial approximation also for κ_E and κ_S:

$$\kappa - \kappa_H = h_2\beta^2 + h_4\beta^4 \quad ; \tag{6–49}$$

h_0 must be zero, otherwise Q as defined by (4–56) would not converge as $\beta \to 0$. Letting $\beta = 1$ in (6–49) immediately gives the boundary condition

$$h_2 + h_4 = \kappa_1 - \kappa_1^H \quad . \tag{6–50}$$

The integrals (6–46) and (6–47) may easily be evaluated if we use a polynomial representation also for the density ρ and hence for $\delta = \rho/\rho_m$. Dividing Bullard's polynomial (1–109) by ρ_m we thus find the expression

$$\delta = 2.21 - 3.03\,\beta^2 + 1.42\,\beta^4 \quad , \tag{6–51}$$

which contains only dimensionless quantities.

Using all these polynomials, the integrals (6–46) and (6–47) can be evaluated, κ denoting κ_E or κ_S:

$$\begin{aligned} (\kappa - \kappa_H)\beta^7 &= h_2\beta^9 + h_4\beta^{11} \quad , \\ \delta\frac{d}{d\beta}\left[(\kappa - \kappa_H)\beta^7\right] &= h_2(19.89\,\beta^8 - 27.27\,\beta^{10} + 12.78\,\beta^{12}) + \\ &+ h_4(24.31\,\beta^{10} - 33.33\,\beta^{12} + 15.62\,\beta^{14}) \quad , \\ \int_0^1 \delta\frac{d}{d\beta}\left[(\kappa - \kappa_H)\beta^7\right] d\beta &= 0.7140\,h_2 + 0.6875\,h_4 \quad . \end{aligned} \tag{6–52}$$

With (6–46) and (6–50) this gives for κ_E:

$$\begin{aligned} 0.7140\,h_2^E + 0.6875\,h_4^E &= -0.000\,00203 \quad , \\ h_2^E + h_4^E &= -0.000\,00068 \quad , \end{aligned}$$

and similarly for κ_S:

$$\begin{aligned} 0.7140\,h_2^S + 0.6875\,h_4^S &= -0.000\,00450 \quad , \\ h_2^S + h_4^S &= -0.000\,00150 \quad , \end{aligned}$$

with the solutions

$$\begin{aligned} \kappa_E - \kappa_H &= -0.000\,0590\beta^2 + 0.000\,0583\,\beta^4 \quad , \\ \kappa_S - \kappa_H &= -0.000\,1309\beta^2 + 0.000\,1294\,\beta^4 \quad . \end{aligned} \tag{6–53}$$

We see that $\kappa_H \ll \kappa_E$, so that to an acceptable accuracy we may put

$$\kappa_E - \kappa_H \doteq \kappa_E \quad , \tag{6–54}$$

and similarly for κ_S.

When is κ monotonic? There is a striking contrast between the behavior of the function $\kappa(\beta)$ in the hydrostatic case (κ_H) and in the case of the real earth spheroid (κ_S), based on a satellite-determined J_4, although the surface values (6–40) and (6–45) have a similar magnitude (the sign is different!). In the first case, κ_H decreases monotonically from the surface to the center; in the second case it first increases considerably in absolute value, reaching a maximum, before it decreases to zero at the center. For the ellipsoid , κ also behaves in a way similar to the second case.

Since a monotonic behavior may appear somewhat more "natural", the question arises as to when the function κ can be monotonic.

Any of the two equations (6–46) and (6–47) yields

$$\int_0^1 \delta \frac{d}{d\beta}(\kappa\beta^7)d\beta = \int_0^1 \delta \frac{d}{d\beta}(\kappa_H\beta^7)d\beta + \frac{105}{32}J_4^H - \frac{105}{32}J_4 \quad , \tag{6–55}$$

and on substituting (6–38)

$$\int_0^1 \delta \frac{d}{d\beta}(\kappa\beta^7)d\beta - 3\kappa_1 = \int_0^1 \delta \frac{d}{d\beta}(\kappa_H\beta^7)d\beta + \frac{105}{32}(J_4^H - J_4^E) \quad . \tag{6–56}$$

The right-hand side is given by (6–39), (6–41), (6–48), and (6–51) and can easily be evaluated, also considering (6–52). The result is

$$\int_0^1 \delta \frac{d}{d\beta}(\kappa\beta^7)d\beta - 3\kappa_1 = -0.000\,00155 \quad . \tag{6–57}$$

Using again a polynomial representation

$$\kappa = k_2\beta^2 + k_4\beta^4 \quad , \tag{6–58}$$

we thus get in view of (6–52)

$$-2.2860\,k_2 - 2.3125\,k_4 = -0.000\,00155$$

or

$$k_2 + 1.0116\,k_4 = 0.000\,00068 \quad .$$

Since by (6–58)

$$k_2 + k_4 = \kappa_1 \quad ,$$

we get

$$k_4 = 86(0.000\,00068 - \kappa_1) \quad . \tag{6–59}$$

Thus we see that k_4 will be very large in absolute value as compared to κ_1, *except in the case that κ_1 is very close to the hydrostatic value* (6–40).

Now what does this mean? If $k_4 \gg \kappa_1$, then, by (6–58)

$$k_2 = \kappa_1 - k_4 \doteq -k_4 \quad . \tag{6–60}$$

A polynomial

$$k_2\beta^2 + k_4\beta^4$$

has an extremum at

$$\beta = \left(-\frac{k_2}{2k_4}\right)^{1/2} \quad , \tag{6–61}$$

so that in the case of (6–60), κ will have an extremum around $1/\sqrt{2} \doteq 0.7$, that is, between 0 and 1, so that it cannot be monotonic. Even if κ_1 deviates from κ_1^H only by 10^{-8},

$$\kappa_1 = 0.000\,00067 \quad ,$$

the function $\kappa(\beta)$ is readily seen to be no longer monotonic.

In this way we see that a monotonic behavior of κ is possible only for mass configurations which are extremely close to equilibrium configurations. As (6–53) shows, this is not the case for the equipotential ellipsoid, and for the real earth the situation is even "worse" by a factor of more than two! This serves as another confirmation of the validity of Ledersteger's theorem (sec. 4.2.4) for the case of the earth.

6.5 Numerical Results and Conclusions

Using the polynomial representations of sec. 6.4 we can evaluate the ellipsoidal potential anomaly $W_4(\beta)$ by (6–27) and gravity $g(\beta)$ inside the ellipsoid by (2–62). Then Bruns' theorem (6–34) gives the separation $\zeta = W_4 P_4(\cos\theta)/g$ between corresponding surfaces of equal potential and of equal density. The result, by (Moritz, 1973, pp. 44–45), with our present numerical values, is

$$W_4 = \beta^4(627 - 1072\,\beta^2 + 585\,\beta^4 - 140\,\beta^6) \times 10\,\mathrm{m^2s^{-2}} \quad , \tag{6–62}$$

$$g = \beta(21.7 - 17.9\,\beta^2 + 6.0\,\beta^4)\,\mathrm{ms^{-2}} \quad . \tag{6–63}$$

The values of Table 6.1 have been computed from these expressions.

We see that the maximum separation between surfaces of constant potential and corresponding surfaces of constant density is almost 60 m, occurring on a depth of about 1400 km. This is on the order of the geoidal heights, which is not unplausible. It is not to be expected that a more realistic earth model and an expression for κ that is more sophisticated than (6–49) will give significantly different values. The values of ζ for the real earth are even larger by a factor of more than 2, as (6–53) shows!

By methods described in (Jeffreys, 1976, Chapter VI) or (Moritz, 1973, pp. 35–40) we may also compute corresponding stress differences. They are on the order of $2 \cdot 10^7\,\mathrm{dyn/cm^2}$, which is considerably less than the stress differences that may occur in the actual earth (Jeffreys, 1976, p. 270; we are using the old cgs unit here in order to facilitate the comparison).

Summarizing we may say (Marussi et al., 1974): To find an earth model consistent with an equipotential ellipsoid such as represented by the Geodetic Reference System 1980, the following procedure may be used. From the given value of the