A further simplification of W_4 is obtained by subtracting the hydrostatic value

$$W_4^H(\beta) = \frac{GM}{R}\,\beta^2 \cdot \frac{8}{35}\left[\left(\frac{3}{2}\,e^2 - 4\kappa_H\right) D - 3eS + \frac{3}{2}\,P_H + \frac{4}{3}\,Q_H\right] \equiv 0 \quad , \tag{6-26}$$

noting that D and S are equal in both cases. Thus we get

$$W_4(\beta) = \frac{GM}{R}\,\beta^2 \cdot \frac{32}{105}\left[-3(\kappa - \kappa_H)D + \frac{9}{8}(P - P_H) + (Q - Q_H)\right] \quad , \tag{6-27}$$

where, by (4–56),

$$\frac{9}{8}(P - P_H) \;=\; \beta^{-7}\int_0^\beta \delta\frac{d}{d\beta}\left[(\kappa - \kappa_H)\beta^7\right] d\beta \quad , \tag{6-28}$$

$$Q - Q_H \;=\; \beta^2\int_\beta^1 \delta\frac{d}{d\beta}\left[(\kappa - \kappa_H)\beta^{-2}\right] d\beta \quad . \tag{6-29}$$

6.3 Equipotential Surfaces and Surfaces of Constant Density

Denote a surface of constant density, $\rho = \rho_1$, by S_1 and a corresponding surface of constant potential, $W = W_1$, by S_2. Let the surface S_1 be characterized by a value β_1 such that

$$\rho(\beta_1) = \rho_1 \quad ; \tag{6-30}$$

then the constant W_1 will be determined by

$$W_0(\beta_1) = W_1 \quad , \tag{6-31}$$

the function $W_0(\beta)$ being expressed by (6–24). Thus a surface S_2 is made to correspond to each surface S_1 (Fig. 6.1).

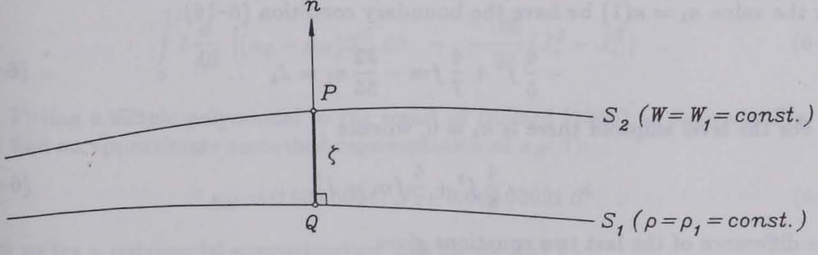

FIGURE 6.1: A surface of constant density, S_1, and the corresponding surface of constant potential, S_2

For equilibrium figures, the surfaces S_1 and S_2 are identical. In the case of ellipsoidal mass distributions, they will be slightly different, and we shall now determine their deviation ζ. The idea is the same as that used in determining the height N of the geoid above the reference ellipsoid (cf. Heiskanen and Moritz, 1967, p. 84).

At P we have $W_P = W_1$, so that at Q

$$W_Q = W_1 - \frac{\partial W}{\partial n}\zeta = W_1 + g\zeta \quad . \tag{6-32}$$

Here $\partial/\partial n$ denotes the derivative along the normal n to the equidensity surface S_1 (Fig. 6.1), which can practically be identified with the plumb line; hence $-\partial W/\partial n = g$ is gravity inside the earth, for which the spherical approximation (2–62) is sufficient. On the other hand, since Q lies on the surface $\rho = \rho_1$, we can apply (6–23) to get

$$\begin{aligned} W_Q &= W_0(\beta_1) + W_4(\beta_1)P_4(\cos\theta) \\ &= W_1 + W_4(\beta_1)P_4(\cos\theta) \end{aligned} \tag{6-33}$$

in view of (6–31). By comparing the right-hand sides of (6–32) and (6–33) we see that

$$\zeta = \frac{1}{g} W_4(\beta)P_4(\cos\theta) \tag{6-34}$$

(since β_1 may be replaced by a general β) is the desired result for the height of S_2 above S_1. The reader will recognize the analogy of this result with the standard Bruns formula (1–25).

6.4 The Deviation κ

The deviation $\kappa = \kappa(\beta)$ for any second-order spheroid must satisfy the integral condition (6–15), where P_1 is given by (4–56) with $\beta = 1$:

$$\int_0^1 \delta \frac{d}{d\beta}(f^2\beta^7)\,d\beta + \frac{8}{9}\int_0^1 \delta \frac{d}{d\beta}(\kappa\beta^7)d\beta = -\frac{35}{12}J_4 \quad . \tag{6-35}$$

For the value $\kappa_1 = \kappa(1)$ be have the boundary condition (6–16):

$$-\frac{4}{5}f^2 + \frac{4}{7}fm - \frac{32}{35}\kappa_1 = J_4 \quad . \tag{6-36}$$

For the level ellipsoid there is $\kappa_1 = 0$, whence

$$-\frac{4}{5}f^2 + \frac{4}{7}fm = J_4^E \quad . \tag{6-37}$$

The difference of the last two equations gives

$$J_4 = J_4^E - \frac{32}{35}\kappa_1 \quad . \tag{6-38}$$