This expression is directly comparable to (4–58), written for a point outside the ellipsoid for which

$$\int_{\beta}^{1} = \int_{1}^{1} = 0 \quad ,$$

so that E, T, and Q vanish, and $D = D(1) = 1$ as above:

$$V = \frac{GM}{r}\left[1 - \frac{2}{5}\frac{R^2}{r^2} S_1 P_2(\cos\theta) + \frac{12}{35}\frac{R^4}{r^4} P_1 P_4(\cos\theta)\right] \quad . \tag{6–14}$$

Here we have introduced GM/r in a similar way as in (6–1) and expressed r in metric units, which explains the occurrence of R to make dimensions right. Comparing the coefficients of $P_4(\cos\theta)$ in (6–13) and (6–14) thus yields

$$-\frac{a^4}{r^4} J_4 = \frac{12}{35}\frac{R^4}{r^4} P_1$$

or

$$P_1 = -\frac{35}{12}\left(\frac{a}{R}\right)^4 J_4 \quad .$$

Since $J_4 = O(f^2)$, we may put $(a/R)^4 = 1$ without losing accuracy, obtaining

$$P_1 = -\frac{35}{12} J_4 \quad . \tag{6–15}$$

By means of (6–12) and (6–15) we are thus able to express J_4 as follows:

$$J_4 = -\frac{4}{5} f^2 + \frac{4}{7} fm - \frac{32}{35}\kappa_1 \quad . \tag{6–16}$$

This fundamental equation links the spherical-harmonic coefficient J_4 with the geometric parameters f and κ_1 and with m. A useful check is provided by the fact that for the ellipsoid with $\kappa_1 = 0$, (6–16) reduces to eq. (2–119) of (Heiskanen and Moritz, 1967).

In fact, for the level ellipsoid, $\kappa_1 = 0$ by (6–6), but we have retained $\kappa_1 \neq 0$ because we shall consider, besides the level ellipsoid, also the case of the real earth.

6.2 Level Ellipsoid and Equilibrium Figures

As we have remarked, the basic spheroidal equations such as (6–1), (6–3), or (6–16) hold for the level ellipsoid (and the real earth) as well as for equilibrium figures, since they have been derived without presupposing hydrostatic equilibrium.

Now there comes a surprise: it is possible to find two corresponding mass distributions, one for equilibrium and the other for the level ellipsoid, for which the values of the ellipticity (4–48),

$$e = f - \frac{5}{42} f^2 + \frac{4}{7}\kappa \tag{6–17}$$

are equal for each β. This is possible because the coefficients $A_0(\beta)$ and $A_2(\beta)$ in (6–1), as given by (4–63) and (4–64), do not depend on the deviation κ, provided they are expressed in terms of e instead of f; compare also (4–56) and (4–59)! This is an essential advantage of describing the elliptical shape of the internal surfaces of constant density by the "artificial" parameter (6–17) instead of the flattening f itself.

In other terms, we may take

$$A_0^E(\beta) \equiv A_0^H(\beta) \quad , \tag{6–18}$$

$$A_2^E(\beta) \equiv A_2^H(\beta) \quad , \tag{6–19}$$

the superscript E denoting the ellipsoid. Of course, A_4^E will be different from $A_4^H = 0$, and (6–19) vanishes because of the equilibrium conditions (4–69).

Thus using now subscripts to denote the ellipsoidal and the hydrostatic cases,

$$e_E(\beta) = e_H(\beta) \quad , \tag{6–20}$$

but since

$$\kappa_E \neq \kappa_H \quad , \tag{6–21}$$

also

$$f_E \neq f_H \quad , \tag{6–22}$$

except in second-order terms.

Thus we have (Moritz, 1973, p. 31) the

Theorem

> To each mass distribution in hydrostatic equilibrium there corresponds a mass distribution for the equipotential ellipsoid in such a way that the density ρ is the same function of β and that the values of the ellipticity e are the same for any two surfaces corresponding to the same value of β.

In the sequel we shall always assume that $e = e_E$ is selected in this way. We then obtain an ellipsoidal mass configuration which deviates very little from an equilibrium configuration.

We shall also omit the index E for ellipsoidal quantities.

On this assumption, (6–19) vanishes as we have seen, and (6–1) reduces to

$$W(\beta, \theta) = W_0(\beta) + W_4(\beta) P_4(\cos\theta) \quad , \tag{6–23}$$

where, by (4–63) and (4–68) with $A_2 = 0$:

$$W_0(\beta) = \frac{GM}{R}\beta^2 \left[D\left(1 + \frac{1}{3}\mu + \frac{4}{45}e^2 + \frac{4}{45}e\mu\right) + \frac{3}{2}E - \frac{4}{25}eS + \frac{8}{75}eT \right] \quad , \tag{6–24}$$

$$W_4(\beta) = \frac{GM}{R}\beta^2 \cdot \frac{8}{35}\left[\left(\frac{3}{2}e^2 - 4\kappa\right) D - 3eS + \frac{3}{2}P + \frac{4}{3}Q \right] \quad . \tag{6–25}$$

A further simplification of W_4 is obtained by subtracting the hydrostatic value

$$W_4^H(\beta) = \frac{GM}{R}\beta^2 \cdot \frac{8}{35}\left[\left(\frac{3}{2}e^2 - 4\kappa_H\right)D - 3eS + \frac{3}{2}P_H + \frac{4}{3}Q_H\right] \equiv 0 \quad , \tag{6–26}$$

noting that D and S are equal in both cases. Thus we get

$$W_4(\beta) = \frac{GM}{R}\beta^2 \cdot \frac{32}{105}\left[-3(\kappa - \kappa_H)D + \frac{9}{8}(P - P_H) + (Q - Q_H)\right] \quad , \tag{6–27}$$

where, by (4–56),

$$\frac{9}{8}(P - P_H) \;=\; \beta^{-7}\int_0^\beta \delta \frac{d}{d\beta}\left[(\kappa - \kappa_H)\beta^7\right] d\beta \quad , \tag{6–28}$$

$$Q - Q_H \;=\; \beta^2 \int_\beta^1 \delta \frac{d}{d\beta}\left[(\kappa - \kappa_H)\beta^{-2}\right] d\beta \quad . \tag{6–29}$$

6.3 Equipotential Surfaces and Surfaces of Constant Density

Denote a surface of constant density, $\rho = \rho_1$, by S_1 and a corresponding surface of constant potential, $W = W_1$, by S_2. Let the surface S_1 be characterized by a value β_1 such that

$$\rho(\beta_1) = \rho_1 \quad ; \tag{6–30}$$

then the constant W_1 will be determined by

$$W_0(\beta_1) = W_1 \quad , \tag{6–31}$$

the function $W_0(\beta)$ being expressed by (6–24). Thus a surface S_2 is made to correspond to each surface S_1 (Fig. 6.1).

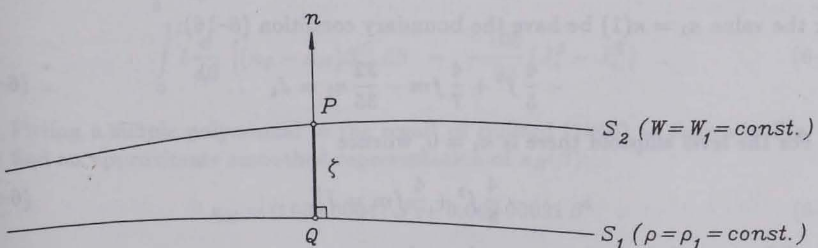

FIGURE 6.1: A surface of constant density, S_1, and the corresponding surface of constant potential, S_2