Chapter 6

Equipotential Ellipsoid: Second-Order Approximation

As we have seen in Chapter 5, it is possible to find density distributions for the equipotential ellipsoid in closed form, using an ellipsoidal coordinate system. This coordinate system, however, exhibits a "focal disc singularity" which presents a serious obstacle to constructing realistic and flexible density distributions (sec. 5.10).

For practical purposes it is, therefore, better to abandon this closed theory and change over to series expansions based on a spherical coordinate system. Expansions are needed at least to second order in the flattening f, but the first part of the second-order spheroidal theory of Chapter 4 is not restricted to equilibrium figures and can also be applied to the ellipsoid.

6.1 Basic Formulas

We recall the basic formulas of sec. 4.2.1, in particular eqs. (4–62) through (4–68):

$$W(\beta,\theta) = \frac{GM}{R}\,\beta^2\left[A_0(\beta) + A_2(\beta)P_2(\cos\theta) + A_4(\beta)P_4(\cos\theta)\right] \quad , \qquad (6\text{–}1)$$

which is easily seen to be identical to (4–62), where θ is now again the *spherical* coordinate (polar distance),

$$\beta = \frac{q}{R} \qquad (6\text{–}2)$$

denotes the mean radius of the spheroid $\rho =$ const. (with $\beta = 1$ for the surface of the ellipsoid), and $A_0(\beta)$, $A_2(\beta)$, and $A_4(\beta)$ are given by (4–63), (4–64), and (4–65). Of particular interest is (4–68):

$$A_4(\beta) + \frac{24}{35}\,eA_2(\beta) = \frac{8}{35}\left[\left(\frac{3}{2}\,e^2 - 4\kappa\right)D - 3eS + \frac{3}{2}\,P + \frac{4}{3}\,Q\right] \quad . \qquad (6\text{–}3)$$

The notations are the same as before. The radius vector of a surface of constant density is expressed by (4–3):

$$r = a(\beta)\left[1 - f\cos^2\theta - \left(\frac{3}{8}\,f^2 + \kappa\right)\sin^2 2\theta\right] \quad , \qquad (6\text{–}4)$$

where $a(\beta)$ denotes the semimajor axis of this interior spheroid; at the surface

$$a(1) = a \tag{6–5}$$

is, of course, nothing else than the semimajor axis of our reference ellipsoid. Also flattening f and deviation κ (Fig. 4.1) for the interior spheroids are functions of β; at the surface of the equipotential ellipsoid there is

$$\kappa(1) = \kappa_E = 0 \tag{6–6}$$

by the very definition of κ. The ellipticity e in (6–3) is related to the flattening f by (4–48); since (6–3) contains only second-order terms, we may put $e = f$ there in view of (4–49). The quantities D, S, P, and Q are defined by (4–56).

For an equilibrium figure we have had (4–69) for all values of β, which we write now

$$A_2^H(\beta) = 0 \; , \qquad A_4^H(\beta) = 0 \quad , \tag{6–7}$$

the superscript H denoting hydrostatic equilibrium.

For the equipotential ellipsoid, which is not an equilibrium figure, eqs. (6–7) will not hold. However, since the surface $\beta = 1$ is equipotential, the gravity potential (6–1) must be constant there, so that, at least,

$$A_2(1) = 0 \; , \qquad A_4(1) = 0 \quad , \tag{6–8}$$

must hold, which implies also the vanishing of (6–3) for $\beta = 1$. By (4–57), $D = 1$ for $\beta = 1$; and $Q(1) = 0$ by putting $\beta = 1$ in (4–56). Thus the vanishing of (6–3) for $\beta = 1$ implies

$$\frac{3}{2} f^2 - 4\kappa_1 - 3fS_1 + \frac{3}{2} P_1 = 0 \quad , \tag{6–9}$$

with $S_1 = S(1)$, $P_1 = P(1)$ and $\kappa_1 = \kappa(1)$; only $f(1)$ is simply denoted by f.

Now (4–94) (which is independent of equilibrium!) gives, to first order which is sufficient since S_1 is multiplied by f in (6–9):

$$f - \frac{3}{5} S_1 = \frac{1}{2} m \quad , \tag{6–10}$$

with m defined by (4–67). Hence

$$S_1 = \frac{5}{3}\left(f - \frac{1}{2} m\right) \quad , \tag{6–11}$$

and (6–9) provides

$$P_1 = \frac{7}{3} f^2 - \frac{5}{3} fm + \frac{8}{3} \kappa_1 \; . \tag{6–12}$$

Now we use for the gravitational potential V the expression (1–39), truncated after J_4:

$$V = \frac{GM}{r}\left[1 - \frac{a^2}{r^2} J_2 P_2(\cos\theta) - \frac{a^4}{r^4} J_4 P_4(\cos\theta)\right] \quad . \tag{6–13}$$

This expression is directly comparable to (4–58), written for a point outside the ellipsoid for which

$$\int_\beta^1 = \int_1^1 = 0 \quad ,$$

so that E, T, and Q vanish, and $D = D(1) = 1$ as above:

$$V = \frac{GM}{r}\left[1 - \frac{2}{5}\frac{R^2}{r^2}\,S_1 P_2(\cos\theta) + \frac{12}{35}\frac{R^4}{r^4}\,P_1 P_4(\cos\theta)\right] \quad . \tag{6–14}$$

Here we have introduced GM/r in a similar way as in (6–1) and expressed r in metric units, which explains the occurrence of R to make dimensions right. Comparing the coefficients of $P_4(\cos\theta)$ in (6–13) and (6–14) thus yields

$$-\frac{a^4}{r^4}\,J_4 = \frac{12}{35}\frac{R^4}{r^4}\,P_1$$

or

$$P_1 = -\frac{35}{12}\left(\frac{a}{R}\right)^4 J_4 \quad .$$

Since $J_4 = O(f^2)$, we may put $(a/R)^4 = 1$ without losing accuracy, obtaining

$$P_1 = -\frac{35}{12}\,J_4 \quad . \tag{6–15}$$

By means of (6–12) and (6–15) we are thus able to express J_4 as follows:

$$J_4 = -\frac{4}{5}\,f^2 + \frac{4}{7}\,fm - \frac{32}{35}\,\kappa_1 \quad . \tag{6–16}$$

This fundamental equation links the spherical-harmonic coefficient J_4 with the geometric parameters f and κ_1 and with m. A useful check is provided by the fact that for the ellipsoid with $\kappa_1 = 0$, (6–16) reduces to eq. (2–119) of (Heiskanen and Moritz, 1967).

In fact, for the level ellipsoid, $\kappa_1 = 0$ by (6–6), but we have retained $\kappa_1 \neq 0$ because we shall consider, besides the level ellipsoid, also the case of the real earth.

6.2 Level Ellipsoid and Equilibrium Figures

As we have remarked, the basic spheroidal equations such as (6–1), (6–3), or (6–16) hold for the level ellipsoid (and the real earth) as well as for equilibrium figures, since they have been derived without presupposing hydrostatic equilibrium.

Now there comes a surprise: it is possible to find two corresponding mass distributions, one for equilibrium and the other for the level ellipsoid, for which the values of the ellipticity (4–48),

$$e = f - \frac{5}{42}\,f^2 + \frac{4}{7}\,\kappa \tag{6–17}$$