where the last integral is solvable by recursion:

$$\int \frac{x^m}{x^2+1}\,dx = \frac{x^{m-1}}{m-1} - \int \frac{x^{m-2}}{x^2+1}\,dx \quad . \tag{5–296}$$

5.12 Potential Energy

The condition that the potential energy of gravity, E_U, is made stationary, has been applied to the theory of equilibrium figures in sec. 3.3.

As we have seen repeatedly, an equipotential ellipsoid, other than the homogeneous Maclaurin ellipsoid, cannot be a figure of hydrostatic equilibrium. The condition of minimum (or maximum, depending on the sign) potential energy,

$$E_U = \text{minimum} \quad , \tag{5–297}$$

which characterizes equilibrium figures, might, however, still be applied to the equipotential ellipsoid. The corresponding mass distribution, if it exists, will be characterized by least potential energy and will, so to speak, come as close to hydrostatic equilibrium as possible. If a solution exists under certain conditions, it will also be unique.

As we have seen, the advantage of applying ellipsoidal coordinates to the theory of the level ellipsoid consists in the fact that the limits of integration are constant and that advantage may be taken of orthogonality relations, so that the integrals can be evaluated in closed form. This applies also to the potential energy.

By eq. (3–99), the potential energy of gravity is

$$E_U = E_V + E_\Phi = \iiint\limits_E \left(\frac{1}{2}V + \Phi\right)\rho dv \quad . \tag{5–298}$$

For the gravitational potential V we have (5–281) with (5–284) through (5–287), and the centrifugal potential Φ is expressed by (5–39). If this is substituted into (5–298) we obtain

$$\begin{aligned} E_U &= \frac{1}{2}\iiint\limits_E \sum_{n=0}^{\infty} \rho(u,\theta)\left[A_n(u)Q_n\left(i\frac{u}{E}\right) + B_n(u)P_n\left(i\frac{u}{E}\right)\right]P_n(\cos\theta)dv + \\ &+ \frac{1}{2}\omega^2 \iiint\limits_E \rho(u,\theta)(u^2+E^2)\sin^2\theta dv \quad , \end{aligned} \tag{5–299}$$

where

$$dv = (u^2 + E^2\cos^2\theta)\sin\theta du d\theta d\lambda \quad .$$

(It is somewhat unfortunate that the letter E is used to denote energy, ellipsoid, and excentricity in this formula, but the reader, unlike a computer, will certainly not be confused.)

If it is permissible to represent the density by the series (5–86) and to interchange integration and summation in (5–299), we can considerably simplify this expression.

On substituting (5–86) into (5–299), the integration with respect to λ and θ is straightforward, the orthogonality of the Legendre polynomials $P_2(\cos\theta)$ being taken into account. The result is

$$\begin{aligned} E_U &= \sum_{n=0}^{\infty} \frac{1}{2(2n+1)} \int_0^b \alpha_n(u) \left[A_n(u) Q_n\left(i\frac{u}{E}\right) + B_n(u) P_n\left(i\frac{u}{E}\right)\right] du + \\ &+ \frac{1}{3}\omega^2 \int_0^b \left[\alpha_0(u) - \frac{1}{5}\alpha_2(u)\right](u^2+E^2) du \quad . \end{aligned} \tag{5–300}$$

The functions $A_n(u)$ and $B_n(u)$ are related to $\alpha_n(u)$ by (5–288).

The "variational problem" (5–297) is now to determine those functions $\alpha_n(u)$ that minimize (5–300) and satisfy the boundary conditions (5–87) with (5–88) through (5–90). This leads to a system of infinitely many equations (so-called Eulerian equations for the variational problem) for the functions $\alpha_n(u)$.

Since the functions $\alpha_n(u)$ are to be varied independently of each other, since (5–300) is quadratic in $\alpha_n(u)$ by (5–288) for $n > 2$, and since the boundary condition (5–111) holds, it seems to follow that (unless, e.g., we have reasons to impose a nonzero α_4)

$$\alpha_n \equiv 0 \quad \text{for} \quad n = 3, 4, 5, \ldots \tag{5–301}$$

as well as for $n = 1$, and there remain only $\alpha_0(u)$ and $\alpha_2(u)$ to be determined.

Calling an *optimal mass configuration* one that is uniquely determined by the condition (5–297) of stationary potential energy, we may pose the question: *Is the search of an optimal mass configuration, under the only condition that the ellipsoidal boundary is fixed and the usual boundary conditions are satisfied, meaningful?*

The answer is very probably no, as we shall see at the end of sec. 5.12.1. But let us first try to understand the situation by means of the example of spheroidal equilibrium figures. In fact, the basic spherical stratification is quite arbitrary (sec. 3.2.3) and must be given initially (see the end of sec. 3.2.6). This is also borne out by the fact that in the minimization of potential energy, the basic spherical stratification must be prescribed as a side condition (sec. 3.3.4).

In the case of the equipotential ellipsoid the situation is somewhat different since there is no initial or underlying spherical configuration. Still, it is very unlikely that the condition (5–297) alone would determine uniquely a "meaningful" distribution of density inside the ellipsoid: we probably also need a side condition. We shall return to this question at the end of sec. 5.12.1.

5.12.1 The Spherical Case

To get a concrete idea, it is worthwhile to examine the spherical case a little more closely. Assume a nonrotating spherically symmetric earth; its radius is again taken as 1 for convenience. This is the case of the equipotential ellipsoid for the limit $E \to 0$, $\omega \to 0$. Then putting

$$4\pi G \rho(r) r^2 = f(r) \quad , \tag{5–302}$$

eq. (2–55) gives with $u = r'$ for the gravitational potential

$$V(r) = \frac{1}{r} \int_0^r f(u)du + \int_r^1 \frac{f(u)}{u} du \quad . \tag{5–303}$$

The gravitational energy (3–97)

$$E = \frac{1}{2} \iiint_{sphere} V \rho \, dv \tag{5–304}$$

is easily found to be, omitting an irrelevant constant factor,

$$E = \int_0^1 f(r)V(r)dr = \int_0^1 f(r) \left[\frac{1}{r} \int_0^r f(u)du + \int_r^1 \frac{f(u)}{u} du \right] dr \quad . \tag{5–305}$$

The side condition is the conservation of total mass M, which is readily seen to give

$$\int_0^1 f(r)dr = 1 \quad , \tag{5–306}$$

making also

$$GM = 1$$

by an appropriate scaling.

Homogeneous sphere. Then ρ = const., (5–302) gives $f(r) = Cr^2$, and (5–306) shows that $C = 3$. Thus

$$f(r) = 3r^2 \quad . \tag{5–307}$$

Then (5–303) yields

$$V(r) = \frac{3}{2} \left(1 - \frac{1}{3} r^2 \right) \quad , \tag{5–308}$$

in agreement with (2–43), so that, by (5–305),

$$E = E_{\text{hom}} = \frac{6}{5} \quad . \tag{5–309}$$

Spherical shell. Consider a homogeneous spherical shell bounded by concentric spheres $r = 1$ and $p < 1$ (Fig. 5.5). With the condition of total constant mass, (5–306), we find

$$E = \frac{3}{5} \frac{2 - 5\,p^3 + 3\,p^5}{(1 - p^3)^2} \quad ; \tag{5–310}$$

the computation is left as an exercise to the reader. For the limit $p \to 1$ we get

$$E_{\text{min}} = \lim_{p \to 1} E = 1 \tag{5–311}$$

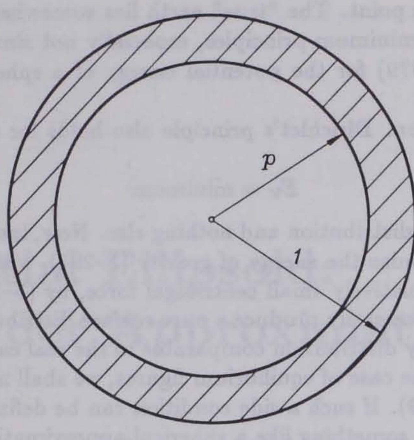

FIGURE 5.5: A spherical shell

for the potential energy of a *surface layer* on the sphere. For $p < 1$ we always have

$$E > E_{\text{min}} \quad . \tag{5–312}$$

This is not surprising after all: *Dirichlet's principle* (cf. Kellogg, 1929, p. 279) explicitly states that E *is minimized if the masses are concentrated on the boundary* and the interior is empty!

For the homogeneous sphere we have by (5–309)

$$\frac{E_{\text{hom}}}{E_{\text{min}}} = \frac{6}{5} = 1.2 \quad , \tag{5–313}$$

which certainly is > 1. For the actual earth we get approximately (we may use a Roche-type polynomial)

$$\frac{E_{\text{earth}}}{E_{\text{min}}} \doteq 1.3 \quad . \tag{5–314}$$

Further, if we let the core radius go to zero, always keeping the total mass constant and the mantle density zero, we get

$$E \to \infty \; ! \tag{5–315}$$

This is clear because, if the mass is concentrated at a point, we have

$$V = \frac{GM}{r} \tag{5–316}$$

and (5–305) becomes infinite (verify)!

This minimum and maximum potential energy (if we consider $E = \infty$ as some kind of maximum) correspond to physically (for the earth) meaningless cases: a surface distribution and a mass point. The "true" earth lies somewhere in between. Nature does not always follow minimum principles, especially not simplistic ones! (We also mention (Rubincam, 1979) for the potential energy of a spherical but not radially symmetric earth.)

Provisional conclusion. Dirichlet's principle also holds for the ellipsoid: the condition

$$E_V = \text{minimum} \tag{5–317}$$

produces a pure surface distribution and nothing else. Now, instead of the gravitational energy E_V, we minimize the energy of gravity (5–297), which differs from E_V by E_Φ, the energy of the relatively small centrifugal force, by (5–298). Thus the condition (5–297) may not necessarily produce a pure surface distribution, but I very much doubt it will be a density distribution comparable to the real earth. It appears highly probable that, just in the case of equilibrium figures, we shall need a meaningful side condition such as (3–109). If such a side condition can be defined, and if the density law is known at least to something like a spherical approximation, then (5–297) may lead to a reasonable solution for $\alpha_0(u)$ and $\alpha_2(u)$, possibly also with other nonzero $\alpha_n(u)$. This conjecture is left as an open problem to the reader.