as we have seen in the preceding chapters. The invaluable advantage of ellipsoidal coordinates is that they permit closed formulas. Therefore it is worthwhile to still use them to investigate problems in which closed formulas are important. This will be done in the last two sections of the present chapter.

5.11 Potential and Gravity Inside the Ellipsoid

Eq. (5–77) holds for the potential inside as well as outside the ellipsoid E, but the series for $1/l$, eq. (5–32), requires $u > u'$. If $u < u'$, then in this series we must interchange u and u'. This is completely analogous to the corresponding series for spherical harmonics, cf. (4–8) and (4–27). If the computation point $P(u, \theta, \lambda)$ lies inside the ellipsoid, we have to pass the coordinate ellipsoid S_P through it and use (5–32) directly for its interior I_P and, with u and u' interchanged, for the "shell" E_P between S_P and E; cf. Fig. 4.2 with the ellipsoid E instead of the spheroid S as boundary.

In agreement with eq. (4–6) we thus split up V as

$$V(u, \theta) = V_i(u, \theta) + V_e(u, \theta) \tag{5–281}$$

with

$$V_i(u, \theta) = G \int_{\lambda'=0}^{2\pi} \int_{\theta'=0}^{\pi} \int_{u'=0}^{u} \frac{1}{l} \rho(u', \theta') dv \quad , \tag{5–282}$$

$$V_e(u, \theta) = G \int_{\lambda'=0}^{2\pi} \int_{\theta'=0}^{\pi} \int_{u'=u}^{b} \frac{1}{l} \rho(u', \theta') dv \quad . \tag{5–283}$$

Now we proceed exactly as we did in sec. 5.3. For V_i we get the same expressions as (5–84), but with the upper limit of integration b replaced by u. Nonzonal terms are removed by orthogonality and there remains

$$V_i(u, \theta) = \sum_{n=0}^{\infty} A_n(u) Q_n\left(i\frac{u}{E}\right) P_n(\cos\theta) \tag{5–284}$$

with

$$A_n(u) = i\frac{G}{E}(2n+1) \int_{\lambda'=0}^{2\pi} \int_{\theta'=0}^{\pi} \int_{u'=0}^{u} \rho(u', \theta') P_n\left(i\frac{u'}{E}\right) P_n(\cos\theta') dv \quad , \tag{5–285}$$

in complete analogy to (5–74) and (5–85); of course, A_n is now a function of u.

Looking at (5–32), we immediately recognize that the interchange of u and u' is equivalent to the interchange of P_{nm} and Q_{nm} for u, with perfect symmetry. Applying these considerations to (5–284) and (5–285), we directly find

$$V_e(u, \theta) = \sum_{n=0}^{\infty} B_n(u) P_n\left(i\frac{u}{E}\right) P_n(\cos\theta) \tag{5–286}$$

with

$$B_n(u) = i\frac{G}{E}(2n+1)\int_{\lambda'=0}^{2\pi}\int_{\theta'=0}^{\pi}\int_{u'=u}^{b}\rho(u',\theta')Q_n\left(i\frac{u'}{E}\right)P_n(\cos\theta')dv \quad . \tag{5-287}$$

It is instructive to compare these expressions for the internal potential to (5–74) and (5–85) for the external potential. For the internal potential, A_n and B_n are functions of u, whereas for the external potential, the A_n are constants and the B_n are zero. This is immediately confirmed by putting $u = b$ in (5–285) and (5–287).

We continue to proceed in exactly the same manner as we did in sec. 5.3. Again we represent the density by (5–86), which we substitute in (5–285) and (5–287), after replacing u and θ by u' and θ'. The integration with respect to θ' is straightforward because of the orthogonality of the Legendre polynomials $P_n(\cos\theta')$. The result is

$$\begin{aligned} A_n(u) &= i\frac{G}{E}\int_0^u \alpha_n(u')P_n\left(i\frac{u'}{E}\right)du' \quad , \\ B_n(u) &= i\frac{G}{E}\int_u^b \alpha_n(u')Q_n\left(i\frac{u'}{E}\right)du' \quad . \end{aligned} \tag{5-288}$$

These equations, together with (5–281), (5–284), and (5–286), determine the internal potential for any given density function of the form (5–86).

We have found it convenient to replace the functions $\alpha_0(u)$ and $\alpha_2(u)$ by functions $G(u)$ and $H(u)$, or $g(u)$ and $h(u)$, which are equivalent according to (5–112) and (5–113). Hence, we consider the density law (5–114):

$$\begin{aligned} \rho(u,\theta) &= g(u) + \frac{u^2+E^2}{u^2+E^2\cos^2\theta}h(u) + \\ &+ \frac{1}{u^2+E^2\cos^2\theta}\cdot\frac{1}{4\pi}\sum_{\nu=2}^{\infty}\alpha_{2\nu}(u)P_{2\nu}(\cos\theta) \quad . \end{aligned} \tag{5-289}$$

Thus we have substituted (5–104) and (5–105) into (5–97) with (5–98). Since

$$\begin{aligned} P_0\left(i\frac{u'}{E}\right) &= 1 \, , \quad P_2\left(i\frac{u'}{E}\right) = -\frac{3}{2E^2}\left(u'^2+\frac{1}{3}E^2\right) \quad , \\ Q_0\left(i\frac{u'}{E}\right) &= -i\arctan\frac{E}{u'} \quad , \\ Q_2\left(i\frac{u'}{E}\right) &= i\left[\frac{3}{2E^2}\left(u'^2+\frac{1}{3}E^2\right)\arctan\frac{E}{u'}-\frac{3u'}{2E}\right] \end{aligned}$$

by (5–20), (5–21), and (5–191), we obtain from (5–288)

$$
\begin{aligned}
A_0(u) &= 4\pi i \frac{G}{E} \int_0^u [G(u') + H(u')]\, du' \quad , \\
A_2(u) &= -4\pi i \frac{G}{E} \int_0^u G(u') du' \quad , \\
B_0(u) &= 4\pi \frac{G}{E} \int_u^b [G(u') + H(u')] \arctan \frac{E}{u'}\, du' \quad , \\
B_2(u) &= -4\pi \frac{G}{E} \int_u^b G(u') \arctan \frac{E}{u'}\, du' + 4\pi G \int_u^b u' g(u') du' \quad .
\end{aligned}
\tag{5–290}
$$

The functions $G(u)$, $H(u)$ and $\alpha_n(u)$, $n \geq 4$, must, of course, satisfy the boundary conditions (5–109) through (5–111), which ensure that the given ellipsoid is an equipotential surface. Evidently these boundary conditions could also be obtained by specializing (5–288) and (5–290) for $u = b$. Thus, in principle, the theory of sec. 5.3 is nothing but a special case of the results of the present section.

After the gravitational potential V has been found, the gravity potential U is obtained by adding the potential of centrifugal force:

$$U = V + \frac{1}{2}\omega^2(u^2 + E^2)\sin^2\theta \quad ; \tag{5–291}$$

cf. (5–40).

Finally the components of the gravity vector are found to be given by

$$
\begin{aligned}
\gamma_u &= \sqrt{\frac{u^2 + E^2}{u^2 + E^2\cos^2\theta}}\, \frac{\partial U}{\partial u} \quad , \\
\gamma_\theta &= \frac{1}{\sqrt{u^2 + E^2\cos^2\theta}}\, \frac{\partial U}{\partial \theta} \quad , \\
\gamma_\lambda &= 0 \quad ,
\end{aligned}
\tag{5–292}
$$

(cf. Heiskanen and Moritz, 1967, p. 68), and gravity inside the ellipsoid has the value

$$\gamma = \sqrt{\gamma_u^2 + \gamma_\theta^2} \quad . \tag{5–293}$$

Representation of the density by polynomials. In this chapter we have represented the functions $g(u)$, $h(u)$ and $\alpha_n(u)$ in various ways by polynomials. In all these cases the integration of (5–288) and (5–290) is possible in closed form in terms of elementary functions. Since

$$\arctan \frac{E}{u} = \frac{\pi}{2} - \arctan \frac{u}{E} \quad , \tag{5–294}$$

the occurring integrals are not more difficult than

$$\int u^k \arctan \frac{u}{E}\, du = \frac{u^{k+1}}{k+1} \arctan \frac{u}{E} - \frac{E}{k+1} \int \frac{u^{k+1}}{u^2 + E^2}\, du \quad , \tag{5–295}$$

where the last integral is solvable by recursion:

$$\int \frac{x^m}{x^2+1}\,dx = \frac{x^{m-1}}{m-1} - \int \frac{x^{m-2}}{x^2+1}\,dx \quad . \tag{5–296}$$

5.12 Potential Energy

The condition that the potential energy of gravity, E_U, is made stationary, has been applied to the theory of equilibrium figures in sec. 3.3.

As we have seen repeatedly, an equipotential ellipsoid, other than the homogeneous Maclaurin ellipsoid, cannot be a figure of hydrostatic equilibrium. The condition of minimum (or maximum, depending on the sign) potential energy,

$$E_U = \text{minimum} \quad , \tag{5–297}$$

which characterizes equilibrium figures, might, however, still be applied to the equipotential ellipsoid. The corresponding mass distribution, if it exists, will be characterized by least potential energy and will, so to speak, come as close to hydrostatic equilibrium as possible. If a solution exists under certain conditions, it will also be unique.

As we have seen, the advantage of applying ellipsoidal coordinates to the theory of the level ellipsoid consists in the fact that the limits of integration are constant and that advantage may be taken of orthogonality relations, so that the integrals can be evaluated in closed form. This applies also to the potential energy.

By eq. (3–99), the potential energy of gravity is

$$E_U = E_V + E_\Phi = \iiint\limits_E \left(\frac{1}{2}V + \Phi\right)\rho dv \quad . \tag{5–298}$$

For the gravitational potential V we have (5–281) with (5–284) through (5–287), and the centrifugal potential Φ is expressed by (5–39). If this is substituted into (5–298) we obtain

$$\begin{aligned} E_U &= \frac{1}{2}\iiint\limits_E \sum_{n=0}^{\infty} \rho(u,\theta)\left[A_n(u)Q_n\left(i\frac{u}{E}\right) + B_n(u)P_n\left(i\frac{u}{E}\right)\right]P_n(\cos\theta)dv + \\ &+ \frac{1}{2}\omega^2 \iiint\limits_E \rho(u,\theta)(u^2+E^2)\sin^2\theta dv \quad , \end{aligned} \tag{5–299}$$

where

$$dv = (u^2 + E^2\cos^2\theta)\sin\theta du d\theta d\lambda \quad .$$

(It is somewhat unfortunate that the letter E is used to denote energy, ellipsoid, and excentricity in this formula, but the reader, unlike a computer, will certainly not be confused.)

If it is permissible to represent the density by the series (5–86) and to interchange integration and summation in (5–299), we can considerably simplify this expression.