and (5–199) with (5–200), and with the density constant at the surface of the ellipsoid, generates a zero external potential.

These conditions may be used in many different ways. At any case, three parameters can be determined from them. Since $A(u)$ represents the given density law, the coefficients b_0 and b_2 are prescribed.

We may, for instance, specialize the polynomial (5–200) as

$$F(u) = a_0 + a_2 u^2 \tag{5-220}$$

and determine the coefficients a_0 and a_2 and the density constant ρ_1.

Or we may wish to prescribe the excentricity e_0' of the surfaces of constant density at the center of the ellipsoids (considered known from hydrostatic theory, see below). Then a_0, being determined by (5–216), is to be considered as given, and we may take

$$F(u) = a_0 + a_2 u^2 + a_4 u^4 \quad , \tag{5-221}$$

so that the constants a_2, a_4, and ρ_1 are to be determined from (5–217). This possibility seems to be the best.

5.8.1 A Fourth-Degree Polynomial

We shall thus investigate polynomials of the form (5–221), so that

$$B(u) = (a_0 + a_2 u^2 + a_4 u^4) A(u) \quad . \tag{5-222}$$

Then the system (5–217) may be written

$$\begin{aligned} a_2 + a_4 &= 1 + e'^2 - a_0 \quad , \\ b_{25} a_2 + b_{45} a_4 + c_1 \rho_1 &= h_1 - b_{05} a_0 \quad , \\ b_{27} a_2 + b_{47} a_4 + c_2 \rho_1 &= h_2 - b_{07} a_0 \quad . \end{aligned} \tag{5-223}$$

These are three equations for the three unkowns a_2, a_4, and ρ_1. The coefficient a_0, which is related to the flattening at the center of the ellipsoid by (5–216), is assumed to be known. It will, however, be desirable to vary it, corresponding to different assumptions as to the central flattening, so that we shall substitute

$$a_0 = 1 + e_0'^2 \tag{5-224}$$

into the above system, whence

$$\begin{aligned} a_2 + a_4 &= e'^2 - e_0'^2 \quad , \\ b_{25} a_2 + b_{45} a_4 + c_1 \rho_1 &= h_1 - b_{05} - b_{05} e_0'^2 \quad , \\ b_{27} a_2 + b_{47} a_4 + c_2 \rho_1 &= h_2 - b_{07} - b_{07} e_0'^2 \quad . \end{aligned} \tag{5-225}$$

The elimination of a_4 by

$$a_4 = -a_2 + e'^2 - e_0'^2 \tag{5-226}$$

reduces this system to

$$\begin{aligned}(b_{25}-b_{45})a_2+c_1\rho_1 &= h_1-b_{05}-e'^2b_{45}+(b_{45}-b_{05})e_0'^2 \quad ,\\ (b_{27}-b_{47})a_2+c_2\rho_1 &= h_2-b_{07}-e'^2b_{47}+(b_{47}-b_{07})e_0'^2 \quad .\end{aligned} \tag{5–227}$$

Further investigations require numerical studies. We shall use Bullard's density law (1–109) (with R as unit):

$$\rho = 12.19 - 16.71\, r^2 + 7.82\, r^4 \quad . \tag{5–228}$$

To identify coefficients, we note that with $B(u) \doteq A(u)$ eq. (5–184) becomes approximately

$$\bar{\rho} \doteq \rho_1 - r^2 A(u) \qquad (r^2 = x^2 + y^2 + z^2) \quad , \tag{5–229}$$

so that, with (5–203) and $u \doteq r$,

$$\bar{\rho} \doteq \rho_1 - b_0 r^2 - b_2 r^4 \tag{5–230}$$

and, by (5–183),

$$\rho \doteq \rho_0 + \rho_1 - b_0 r^2 - b_2 r^4 \quad . \tag{5–231}$$

This expression is directly comparable to (5–228). We shall thus throughout use the values

$$\begin{aligned} b_0 &= 16.71 \quad ,\\ b_2 &= -7.82 \quad ,\end{aligned} \tag{5–232}$$

assumed as exact.

All ellipsoidal constants will be taken from sec. 1.5 (Geodetic Reference System 1980).

We find

$$\begin{aligned} b_{05} &= 2.2411 \,, & b_{07} &= 1.5290 \,,\\ b_{25} &= 1.5273 \,, & b_{27} &= 1.1531 \,,\\ b_{45} &= 1.1519 \,, & b_{47} &= 0.9231 \end{aligned} \tag{5–233}$$

and

$$\begin{aligned} c_1 &= -1.0067 \,, & h_1 &= -4.5148 \,,\\ c_2 &= +0.0010 \,, & h_2 &= +1.5506 \,. \end{aligned} \tag{5–234}$$

The system (5–227) may now be solved for a_2 and ρ_1. Then (5–226) gives a_4, and (5–224) expresses a_0. The result is

$$\begin{aligned} a_0 &= 1 + e_0'^2 \quad ,\\ a_2 &= 0.0387 - 2.63\, e_0'^2 \quad ,\\ a_4 &= -0.0320 + 1.63\, e_0'^2 \quad ,\\ \rho_1 &= 6.7328 + 0.10\, e_0'^2 \quad . \end{aligned} \tag{5–235}$$

Thus the result depends on the central excentricity. E.g., assume an $e_0'^2$ that corresponds to Bullen's (1975, p. 58, correcting an obvious printing error) central flattening

$$f_0 = 0.00242 \quad (\doteq 1/413) \quad , \tag{5–236}$$

which is in agreement with (Denis and Ibrahim, 1981, p. 189). Then

$$e_0'^2 = 0.00486 \quad . \tag{5–237}$$

For this we find

$$\begin{aligned} \rho_1 &= 6.7332 \quad , \\ a_0 &= 1.0049 \quad , \\ a_2 &= 0.0259 \quad , \\ a_4 &= -0.0241 \quad . \end{aligned} \tag{5–238}$$

Other values of f_0 such as 1/469 (Bullard, 1954, p. 96) will slightly change these values.

At any rate, the values (5–238) show that $F(u)$ as given by (5–221) is indeed close to unity.

5.9 Combined Density Models

According to the discussions of secs. 5.5 and 5.6, the density $\rho(u, \theta)$ of a mass distribution for the equipotential ellipsoid has been represented as follows

$$\rho(u, \theta) = \rho_0 + \bar{\rho}(u, \theta) + \Delta\rho(u, \theta) \quad . \tag{5–239}$$

The constant ρ_0 is the constant density of the homogeneous Maclaurin ellipsoid that corresponds to the given equipotential ellipsoid, the function $\bar{\rho}(u, \theta)$ is the "zero-potential density" that introduces the desired heterogeneity without changing the external gravity field of the Maclaurin ellipsoid, and $\Delta\rho(u, \theta)$ is the "deviatoric density" that changes the external field of the Maclaurin ellipsoid to the prescribed field of the original equipotential ellipsoid without changing appreciably (that is, by more than about 0.028 g/cm^3) the density distribution.

To present an example of a density distribution that arises in this way, we use a function $\Delta\rho(u, \theta)$ according to (5–156) and (5–165), and a function $\bar{\rho}(u,\theta)$ according to (5–184), the functions $A(u)$ and $B(u)$ being given by (5–203) and (5–222). We thus have

$$\begin{aligned} \rho(u, \theta) &= \rho_0 + \rho_1 - \left[b_0 + b_2\left(\frac{u}{b}\right)^2\right](x^2+y^2) - \\ &- \left[b_0 + b_2\left(\frac{u}{b}\right)^2\right]\left[a_0 + a_2\left(\frac{u}{b}\right)^2 + a_4\left(\frac{u}{b}\right)^4\right] z^2 + \\ &+ C\left(\frac{u}{b}\right)^4\left[1 - \left(\frac{u}{b}\right)^2\right]\left(-1 + \frac{u^2+E^2}{u^2+E^2\cos^2\theta}\right) \quad . \end{aligned} \tag{5–240}$$

The replacement of u by u/b in the polynomials representing $A(u)$ and $B(u)$ expresses the fact that we are no longer using b as a unit, but have returned to metric units.