On adding these three equations we see that

$$R_0(u)P_0\left(i\frac{u}{E}\right) + R_2(u)P_2\left(i\frac{u}{E}\right) + R_4(u)P_4\left(i\frac{u}{E}\right) = 0 \quad . \tag{5–195}$$

Because of this linear relation, the three conditions (5–192) to (5–194) are in fact not independent. Therefore, one of these three conditions is superfluous and can be omitted. We omit the condition corresponding to $n = 2$ and retain those corresponding to $n = 0$ and $n = 4$. Substituting

$$\begin{aligned} G_1(u) &= R_0(u)P_0\left(i\frac{u}{E}\right) = R_0(u) \quad , \\ G_2(u) &= E^2 R_4(u)P_4\left(i\frac{u}{E}\right) \quad , \end{aligned}$$

that is

$$\begin{aligned} G_1(u) &= \left(u^2 + \frac{1}{3}E^2\right)\rho_1 + \left(-\frac{2}{3}u^4 - \frac{4}{5}E^2u^2 - \frac{2}{15}E^4\right)A(u) + \\ &+ \left(-\frac{1}{3}u^4 - \frac{1}{5}E^2u^2\right)B(u) \quad , \end{aligned} \tag{5–196}$$

$$\begin{aligned} G_2(u) &= \left(u^6 + \frac{13}{7}E^2u^4 + \frac{33}{35}E^4u^2 + \frac{3}{35}E^6\right)A(u) + \\ &+ \left(-u^6 - \frac{6}{7}E^2u^4 - \frac{3}{35}E^4u^2\right)B(u) \quad , \end{aligned} \tag{5–197}$$

we are thus finally left with the two conditions

$$\int_0^b G_1(u)du = 0 \; , \qquad \int_0^b G_2(u)du = 0 \; . \tag{5–198}$$

The functions $A(u)$ and $B(u)$ and the constant ρ_1 must satisfy these two equations; otherwise they are arbitrary.

5.8 Representation by Polynomials

First we set

$$B(u) = F(u)A(u) \tag{5–199}$$

and specify the function $F(u)$ to be a polynomial

$$F(u) = \sum_{i=0}^{N} a_i u^i \equiv \sum a_i u^i \tag{5–200}$$

(briefly). Then the functions G_1 and G_2 of (5–196) and (5–197) become

$$\begin{aligned} G_1(u) &= \left(u^2 + \frac{1}{3}E^2\right)\rho_1 + A(u)\left[-\frac{2}{3}u^4 - \frac{4}{5}E^2u^2 - \frac{2}{15}E^4 - \right. \\ &\left. - \sum a_i\left(\frac{1}{3}u^{i+4} + \frac{1}{5}E^2u^{i+2}\right)\right] \quad , \qquad (5\text{–}201) \\ G_2(u) &= A(u)\left[u^6 + \frac{13}{7}E^2u^4 + \frac{33}{35}E^4u^2 + \frac{3}{35}E^6 - \right. \\ &\left. - \sum a_i\left(u^{i+6} + \frac{6}{7}E^2u^{i+4} + \frac{3}{35}E^4u^{i+2}\right)\right] \quad . \qquad (5\text{–}202) \end{aligned}$$

Secondly, we represent also the function $A(u)$ by a polynomial:

$$A(u) = b_0 + b_2u^2 \quad . \qquad (5\text{–}203)$$

To simplify our computations, we set

$$b = 1 \quad ; \qquad (5\text{–}204)$$

thus everything is expressed in terms of the semiminor axis as the unit of length (we did the same in sec. 3.2.1!); of course, b has nothing to do with b_0 or b_2.

On substituting (5–203), the equations (5–201) and (5–202) must be integrated according to (5–198). This involves the definite integrals

$$\begin{aligned} \int_0^b A(u)u^i du &= \int_0^1 \left(b_0u^i + b_2u^{i+2}\right) du \\ &= \frac{b_0}{i+1} + \frac{b_2}{i+3} \quad . \qquad (5\text{–}205) \end{aligned}$$

It is convenient to denote the value of this definite integral by b_{i+1}, that is, we define

$$b_{i+1} = \frac{b_0}{i+1} + \frac{b_2}{i+3} \qquad (5\text{–}206)$$

(for even integers i).

Now the integration of (5–201) and (5–202) is straightforward and gives the result

$$\begin{aligned} \sum a_i\left(\frac{1}{3}b_{i+5} + \frac{1}{5}e'^2b_{i+3}\right) &= \frac{1}{3}(1+e'^2)\rho_1 - \frac{2}{3}b_5 - \frac{4}{5}e'^2b_3 - \frac{2}{15}e'^4b_1 \ , \\ \sum a_i\left(b_{i+7} + \frac{6}{7}e'^2b_{i+5} + \frac{3}{35}e'^4b_{i+3}\right) &= b_7 + \frac{13}{7}e'^2b_5 + \frac{33}{35}e'^4b_3 + \frac{3}{35}e'^6b_1 \ . \end{aligned} \qquad (5\text{–}207)$$

To simplify this system, we modify the second equation by subtracting from it the first equation multiplied by $3e'^2/7$; the first equation itself is modified by multiplying it by 3. We thus obtain

$$\begin{aligned} \sum \left(b_{i+5} + \frac{3}{5} e'^2 b_{i+3} \right) a_i &= (1 + e'^2)\rho_1 + h_1 \quad , \\ \sum \left(b_{i+7} + \frac{5}{7} e'^2 b_{i+5} \right) a_i &= -\frac{1}{7} e'^2 (1 + e'^2)\rho_1 + h_2 \quad , \end{aligned} \tag{5–208}$$

where

$$\begin{aligned} h_1 &= -2b_5 - \frac{12}{5} e'^2 b_3 - \frac{2}{5} e'^4 b_1 \quad , \\ h_2 &= b_7 + \frac{15}{7} e'^2 b_5 + \frac{9}{7} e'^4 b_3 + \frac{1}{7} e'^6 b_1 \quad . \end{aligned} \tag{5–209}$$

This is the final form of the conditions (5–198) for our present case. An explanatory remark will now be in order. We have put $b = 1$, which means that all lengths are to be measured with b as unit or, in other words, all lengths must be divided by b. Thus E is to be replaced by the second excentricity $e' = E/b$, which explains the occurrence of e' in the above equations. Similarly, in polynomials such as (5–200) and (5–203), u must be replaced by u/b if lengths are measured in metric units.

What is the meaning of the polynomials themselves? The function $A(u)$ represents the change of density with depth; it is taken as a prescribed function representing a given density law. We cannot, however, likewise prescribe the function $B(u)$ without violating the conditions (5–198). At any case, $B(u)$ should be almost equal to $A(u)$ to ensure spheroidal (that is, nearly spherical) stratification of density; therefore the function $F(u)$ in (5–199) must be close to unity. In order to fulfil the conditions (5–198), we have tried to represent it as a polynomial, whose coefficients satisfy the conditions (5–208) equivalent to (5–198).

If we wish the density to be constant at the surface of the ellipsoid, we must add a third condition. The density $\bar{\rho}(u, \theta)$ will be constant at the ellipsoid if and only if the coefficient of $P_2(\cos\theta)$ in (5–185) vanishes for $u = b$. This means

$$\frac{2}{3}(b^2 + E^2)A(b) - \frac{2}{3} b^2 B(b) = 0$$

or

$$B(b) = (1 + e'^2)A(b) \quad . \tag{5–210}$$

On substituting B from (5–199) and dividing by $A(b)$ we thus have

$$F(b) = 1 + e'^2 \quad . \tag{5–211}$$

Putting $u = b = 1$ in (5–200) we get

$$\sum a_i = 1 + e'^2 \quad . \tag{5–212}$$

This is the condition which the coefficients a_i must satisfy if the density is to be constant on the surface of the ellipsoid.

We shall now show that the coefficient a_0 is related to the limit of the flattening or excentricity of the surfaces of constant density as we approach the center of the ellipsoid. We assume that the polynomial (5–200) contains only even powers of u, that is,

$$F(u) = a_0 + a_2 u^2 + a_4 u^4 + \cdots + a_{2n} u^{2n} \quad . \tag{5–213}$$

By (5–203) and (5–199) we shall then have in the neighborhood of the center

$$\begin{aligned} A(u) &= b_0 + O(u^2) = b_0 + O(r^2) \quad , \\ B(u) &= a_0 b_0 + O(u^2) = a_0 b_0 + O(r^2) \quad , \end{aligned} \tag{5–214}$$

where $O(r^2)$ denotes terms of the order of $r^2 = x^2 + y^2 + z^2$ as usual. Then (5–184) becomes

$$\bar{\rho}(u, \theta) = \rho_1 - b_0(x^2 + y^2) - a_0 b_0 z^2 + O(r^4) \quad . \tag{5–215}$$

The surface of constant density $\bar{\rho} = c$ thus is expressed by

$$x^2 + y^2 + a_0 z^2 = \frac{\rho_1 - c}{b_0} + O(r^4) = \text{const.} + O(r^4) \quad .$$

The equation of an ellipsoid of second excentricity e_0' and semimajor axis A is given by

$$x^2 + y^2 + (1 + e_0'^2) z^2 = A^2 \quad .$$

The comparison of these two expressions as $r \to 0$ shows that

$$a_0 = 1 + e_0'^2 \tag{5–216}$$

is the desired relation between the coefficient a_0 and the excentricity of the surfaces of constant density at the center of the ellipsoid.

We shall finally put together the three conditions (5–208) and (5–212). They may be written as

$$\begin{aligned} \sum a_i &= 1 + e'^2 \quad , \\ \sum b_{i5} a_i + c_1 \rho_1 &= h_1 \quad , \\ \sum b_{i7} a_i + c_2 \rho_1 &= h_2 \quad , \end{aligned} \tag{5–217}$$

where we have used the abbreviations

$$\begin{aligned} b_{i5} &= b_{i+5} + \frac{3}{5} e'^2 b_{i+3} \quad , \\ b_{i7} &= b_{i+7} + \frac{5}{7} e'^2 b_{i+5} \quad , \end{aligned} \tag{5–218}$$

and

$$c_1 = -(1 + e'^2) \, , \qquad c_2 = \frac{1}{7} e'^2 (1 + e'^2) \, , \tag{5–219}$$

h_1 and h_2 being given by (5–209) and b_{i+3}, etc., being defined by (5–206).

The three conditions (5–217) are necessary and sufficient in order that a mass distribution of the form (5–184), with A and B being given by the polynomials (5–203) and (5–199) with (5–200), and with the density constant at the surface of the ellipsoid, generates a zero external potential.

These conditions may be used in many different ways. At any case, three parameters can be determined from them. Since $A(u)$ represents the given density law, the coefficients b_0 and b_2 are prescribed.

We may, for instance, specialize the polynomial (5–200) as

$$F(u) = a_0 + a_2 u^2 \tag{5–220}$$

and determine the coefficients a_0 and a_2 and the density constant ρ_1.

Or we may wish to prescribe the excentricity e'_0 of the surfaces of constant density at the center of the ellipsoids (considered known from hydrostatic theory, see below). Then a_0, being determined by (5–216), is to be considered as given, and we may take

$$F(u) = a_0 + a_2 u^2 + a_4 u^4 \quad , \tag{5–221}$$

so that the constants a_2, a_4, and ρ_1 are to be determined from (5–217). This possibility seems to be the best.

5.8.1 A Fourth-Degree Polynomial

We shall thus investigate polynomials of the form (5–221), so that

$$B(u) = (a_0 + a_2 u^2 + a_4 u^4) A(u) \quad . \tag{5–222}$$

Then the system (5–217) may be written

$$\begin{aligned} a_2 + a_4 &= 1 + e'^2 - a_0 \quad , \\ b_{25} a_2 + b_{45} a_4 + c_1 \rho_1 &= h_1 - b_{05} a_0 \quad , \\ b_{27} a_2 + b_{47} a_4 + c_2 \rho_1 &= h_2 - b_{07} a_0 \quad . \end{aligned} \tag{5–223}$$

These are three equations for the three unkowns a_2, a_4, and ρ_1. The coefficient a_0, which is related to the flattening at the center of the ellipsoid by (5–216), is assumed to be known. It will, however, be desirable to vary it, corresponding to different assumptions as to the central flattening, so that we shall substitute

$$a_0 = 1 + e_0'^2 \tag{5–224}$$

into the above system, whence

$$\begin{aligned} a_2 + a_4 &= e'^2 - e_0'^2 \quad , \\ b_{25} a_2 + b_{45} a_4 + c_1 \rho_1 &= h_1 - b_{05} - b_{05} e_0'^2 \quad , \\ b_{27} a_2 + b_{47} a_4 + c_2 \rho_1 &= h_2 - b_{07} - b_{07} e_0'^2 \quad . \end{aligned} \tag{5–225}$$

The elimination of a_4 by

$$a_4 = -a_2 + e'^2 - e_0'^2 \tag{5–226}$$

reduces this system to

$$\begin{aligned} (b_{25}-b_{45})a_2+c_1\rho_1 &= h_1-b_{05}-e'^2b_{45}+(b_{45}-b_{05})e_0'^2 \quad , \\ (b_{27}-b_{47})a_2+c_2\rho_1 &= h_2-b_{07}-e'^2b_{47}+(b_{47}-b_{07})e_0'^2 \quad . \end{aligned} \tag{5–227}$$

Further investigations require numerical studies. We shall use Bullard's density law (1–109) (with R as unit):

$$\rho = 12.19 - 16.71\, r^2 + 7.82\, r^4 \quad . \tag{5–228}$$

To identify coefficients, we note that with $B(u) \doteq A(u)$ eq. (5–184) becomes approximately

$$\bar{\rho} \doteq \rho_1 - r^2 A(u) \qquad (r^2 = x^2 + y^2 + z^2) \quad , \tag{5–229}$$

so that, with (5–203) and $u \doteq r$,

$$\bar{\rho} \doteq \rho_1 - b_0 r^2 - b_2 r^4 \tag{5–230}$$

and, by (5–183),

$$\rho \doteq \rho_0 + \rho_1 - b_0 r^2 - b_2 r^4 \quad . \tag{5–231}$$

This expression is directly comparable to (5–228). We shall thus throughout use the values

$$\begin{aligned} b_0 &= 16.71 \quad , \\ b_2 &= -7.82 \quad , \end{aligned} \tag{5–232}$$

assumed as exact.

All ellipsoidal constants will be taken from sec. 1.5 (Geodetic Reference System 1980).

We find

$$\begin{aligned} b_{05} &= 2.2411 \,, & b_{07} &= 1.5290 \,, \\ b_{25} &= 1.5273 \,, & b_{27} &= 1.1531 \,, \\ b_{45} &= 1.1519 \,, & b_{47} &= 0.9231 \end{aligned} \tag{5–233}$$

and

$$\begin{aligned} c_1 &= -1.0067 \,, & h_1 &= -4.5148 \,, \\ c_2 &= +0.0010 \,, & h_2 &= +1.5506 \,. \end{aligned} \tag{5–234}$$

The system (5–227) may now be solved for a_2 and ρ_1. Then (5–226) gives a_4, and (5–224) expresses a_0. The result is

$$\begin{aligned} a_0 &= 1 + e_0'^2 \quad , \\ a_2 &= 0.0387 - 2.63\, e_0'^2 \quad , \\ a_4 &= -0.0320 + 1.63\, e_0'^2 \quad , \\ \rho_1 &= 6.7328 + 0.10\, e_0'^2 \quad . \end{aligned} \tag{5–235}$$

Thus the result depends on the central excentricity. E.g., assume an $e_0'^2$ that corresponds to Bullen's (1975, p. 58, correcting an obvious printing error) central flattening

$$f_0 = 0.00242 \quad (\doteq 1/413) \quad , \tag{5–236}$$

which is in agreement with (Denis and Ibrahim, 1981, p. 189). Then

$$e_0'^2 = 0.00486 \quad . \tag{5–237}$$

For this we find

$$\begin{aligned} \rho_1 &= 6.7332 \quad , \\ a_0 &= 1.0049 \quad , \\ a_2 &= 0.0259 \quad , \\ a_4 &= -0.0241 \quad . \end{aligned} \tag{5–238}$$

Other values of f_0 such as 1/469 (Bullard, 1954, p. 96) will slightly change these values.

At any rate, the values (5–238) show that $F(u)$ as given by (5–221) is indeed close to unity.

5.9 Combined Density Models

According to the discussions of secs. 5.5 and 5.6, the density $\rho(u, \theta)$ of a mass distribution for the equipotential ellipsoid has been represented as follows

$$\rho(u, \theta) = \rho_0 + \bar{\rho}(u, \theta) + \Delta\rho(u, \theta) \quad . \tag{5–239}$$

The constant ρ_0 is the constant density of the homogeneous Maclaurin ellipsoid that corresponds to the given equipotential ellipsoid, the function $\bar{\rho}(u, \theta)$ is the "zero-potential density" that introduces the desired heterogeneity without changing the external gravity field of the Maclaurin ellipsoid, and $\Delta\rho(u, \theta)$ is the "deviatoric density" that changes the external field of the Maclaurin ellipsoid to the prescribed field of the original equipotential ellipsoid without changing appreciably (that is, by more than about $0.028\,\mathrm{g/cm^3}$) the density distribution.

To present an example of a density distribution that arises in this way, we use a function $\Delta\rho(u, \theta)$ according to (5–156) and (5–165), and a function $\bar{\rho}(u, \theta)$ according to (5–184), the functions $A(u)$ and $B(u)$ being given by (5–203) and (5–222). We thus have

$$\begin{aligned} \rho(u, \theta) &= \rho_0 + \rho_1 - \left[b_0 + b_2\left(\frac{u}{b}\right)^2\right](x^2 + y^2) - \\ &- \left[b_0 + b_2\left(\frac{u}{b}\right)^2\right]\left[a_0 + a_2\left(\frac{u}{b}\right)^2 + a_4\left(\frac{u}{b}\right)^4\right] z^2 + \\ &+ C\left(\frac{u}{b}\right)^4\left[1 - \left(\frac{u}{b}\right)^2\right]\left(-1 + \frac{u^2 + E^2}{u^2 + E^2\cos^2\theta}\right) \quad . \end{aligned} \tag{5–240}$$

The replacement of u by u/b in the polynomials representing $A(u)$ and $B(u)$ expresses the fact that we are no longer using b as a unit, but have returned to metric units.