One condition to be satisfied by the coefficients of the polynomial (5–181) is obtained by substituting this polynomial into (5–109) after multiplication by $(u^2 + E^2/3)$ according to (5–112), and performing the integration:

$$\left(\frac{1}{3}+\frac{1}{3}e'^2\right)A+\left(\frac{1}{5}+\frac{1}{9}e'^2\right)B+\left(\frac{1}{7}+\frac{1}{15}e'^2\right)C=\left(\frac{1}{3}+\frac{1}{3}e'^2\right)\rho_0 \quad , \tag{5–182}$$

where ρ_0 is the Maclaurin density (5–169). It is readily verified that the coefficients of (5–180) satisfy this condition to two-place accuracy.

The disadvantage of a density law such as (5–178) is that the surfaces of constant density are confocal ellipsoids, whose flattening becomes infinite as $u \to 0$. To be sure, the practical effect of this fact can be made small by selecting a suitable function $g(u)$. If we select $g(u) =$ const. for $0 \le u \le u_0$ and decreasing for $u > u_0$, we shall not even have any singularity at all as $u \to 0$. Still the flattening of the surfaces of constant density increases with depth, which is not desirable.

More "natural" distributions will obviously have to be somewhat more complicated than (5–178). To keep the matter relatively simple and transparent, it will be convenient to consider any heterogeneous mass distribution of the Maclaurin ellipsoid as the superposition of

1. a homogeneous distribution of the usual Maclaurin density ρ_0, which generates the required external potential, and

2. a heterogeneous distribution $\bar{\rho}(u, \theta)$ whose external potential is zero.

The purpose of such a "zero-potential distribution" of density $\bar{\rho}(u, \theta)$ is thus to provide the desired heterogeneity without changing the external potential or the coefficients A_0^{ML} and A_2^{ML} defined by the Maclaurin density ρ_0. In other words, a heterogeneous distribution for the Maclaurin ellipsoid will be given by

$$\rho_{ML}(u, \theta) = \rho_0 + \bar{\rho}(u, \theta) \tag{5–183}$$

as the sum of the (homogeneous) Maclaurin density ρ_0 and a zero-potential density $\bar{\rho}(u, \theta)$.

The constant ρ_0 being uniquely defined by (5–164), the following section will study zero-potential density distributions.

5.7 Zero-Potential Densities

We shall thus determine density distributions inside the given ellipsoid that generate a potential which is everywhere zero outside the ellipsoid. To obtain spheroidal (nearly ellipsoidal) surfaces of equal density, we consider functions of the form

$$\bar{\rho}(u, \theta) = \rho_1 - (x^2 + y^2)A(u) - z^2 B(u) \quad , \tag{5–184}$$

where ρ_1 is a constant and A and B are functions of u to be determined.

With

$$\begin{aligned} x^2+y^2 &= (u^2+E^2)\sin^2\theta \quad , \\ z^2 &= u^2\cos^2\theta \end{aligned}$$

by (5–1) and

$$\begin{aligned} \cos^2\theta &= \frac{2}{3}P_2(\cos\theta)+\frac{1}{3} \quad , \\ \sin^2\theta &= -\frac{2}{3}P_2(\cos\theta)+\frac{2}{3} \end{aligned}$$

this becomes

$$\bar{\rho}(u,\,\theta)=\rho_1-\left[\frac{2}{3}(u^2+E^2)A+\frac{1}{3}u^2B\right]+\left[\frac{2}{3}(u^2+E^2)A-\frac{2}{3}u^2B\right]P_2(\cos\theta) \quad . \tag{5–185}$$

Multiplying by

$$u^2+E^2\cos^2\theta=\left(u^2+\frac{1}{3}E^2\right)+\frac{2}{3}E^2P_2(\cos\theta)$$

gives

$$\bar{\rho}(u,\,\theta)\,(u^2+E^2\cos^2\theta)=S_0(u)+S_2(u)P_2(\cos\theta)+S_4(u)\,[P_2(\cos\theta)]^2 \quad ,$$

where

$$\begin{aligned} S_0(u) &= \left(u^2+\frac{1}{3}E^2\right)\rho_1-\left(u^2+\frac{1}{3}E^2\right)\left[\frac{2}{3}(u^2+E^2)A+\frac{1}{3}u^2B\right] \quad , \\ S_2(u) &= \frac{2}{3}E^2\rho_1-\frac{4}{9}E^2(u^2+E^2)A-\frac{2}{9}E^2u^2B+\frac{2}{3}\left(u^2+\frac{1}{3}E^2\right)(u^2+E^2)A- \\ &- \frac{2}{3}u^2\left(u^2+\frac{1}{3}E^2\right)B \quad , \\ S_4(u) &= \frac{4}{9}E^2(u^2+E^2)A-\frac{4}{9}E^2u^2B \quad . \end{aligned}$$

Finally we use the formula (4–37),

$$[P_2(\cos\theta)]^2=\frac{1}{5}+\frac{2}{7}P_2(\cos\theta)+\frac{18}{35}P_4(\cos\theta) \quad ,$$

to obtain the expression

$$\bar{\rho}(u,\,\theta)\,(u^2+E^2\cos^2\theta)=R_0(u)+R_2(u)P_2(\cos\theta)+R_4(u)P_4(\cos\theta) \quad , \tag{5–186}$$

where

$$\begin{aligned} R_0(u) &= S_0(u)+\frac{1}{5}S_4(u) \quad , \\ R_2(u) &= S_2(u)+\frac{2}{7}S_4(u) \quad , \\ R_4(u) &= \frac{18}{35}S_4(u) \quad , \end{aligned}$$

that is,

$$
\begin{aligned}
R_0(u) &= \left(u^2 + \frac{1}{3}E^2\right)\rho_1 + \left(-\frac{2}{3}u^4 - \frac{4}{5}E^2u^2 - \frac{2}{15}E^4\right)A(u) + \\
&+ \left(-\frac{1}{3}u^4 - \frac{1}{5}E^2u^2\right)B(u) \quad , \qquad (5\text{–}187)\\
R_2(u) &= \frac{2}{3}E^2\rho_1 + \left(\frac{2}{3}u^4 + \frac{4}{7}E^2u^2 - \frac{2}{21}E^4\right)A(u) + \\
&+ \left(-\frac{2}{3}u^4 - \frac{4}{7}E^2u^2\right)B(u) \quad , \qquad (5\text{–}188)\\
R_4(u) &= \frac{8}{35}\left[(E^2u^2 + E^4)A(u) - E^2u^2B(u)\right] \quad . \qquad (5\text{–}189)
\end{aligned}
$$

Comparing (5–186) with (5–86) we see that for the present case

$$
\begin{aligned}
\bar{\alpha}_n(u) &= 4\pi R_n(u) \qquad \text{if } n = 0, 2, 4 \quad ;\\
\bar{\alpha}_n(u) &= 0 \qquad \text{if } n > 4 \quad .
\end{aligned}
$$

Since for zero external potential all coefficients of the ellipsoidal harmonics must vanish, equation (5–87) gives the conditions

$$\int_0^b R_n(u)P_n\left(i\frac{u}{E}\right)du = 0 \qquad \text{if } n = 0, 2, 4 \quad . \qquad (5\text{–}190)$$

The three conditions (5–190) are, however, not independent. This is seen as follows. With

$$
\begin{aligned}
P_0\left(i\frac{u}{E}\right) &= 1 \quad ,\\
P_2\left(i\frac{u}{E}\right) &= -\frac{3}{2E^2}\left(u^2 + \frac{1}{3}E^2\right) \quad , \qquad (5\text{–}191)\\
P_4\left(i\frac{u}{E}\right) &= \frac{35}{8E^4}\left(u^4 + \frac{6}{7}E^2u^2 + \frac{3}{35}E^4\right)
\end{aligned}
$$

we compute

$$
\begin{aligned}
R_0P_0 &= \left(u^2 + \frac{1}{3}E^2\right)\rho_1 + \left(-\frac{2}{3}u^4 - \frac{4}{5}E^2u^2 - \frac{2}{15}E^4\right)A + \\
&+ \left(-\frac{1}{3}u^4 - \frac{1}{5}E^2u^2\right)B \quad , \qquad (5\text{–}192)\\
R_2P_2 &= -\left(u^2 + \frac{1}{3}E^2\right)\rho_1 + \left(-\frac{u^6}{E^2} - \frac{25}{21}u^4 - \frac{1}{7}E^2u^2 + \frac{1}{21}E^4\right)A + \\
&+ \left(\frac{u^6}{E^2} + \frac{25}{21}u^4 + \frac{2}{7}E^2u^2\right)B \quad , \qquad (5\text{–}193)\\
R_4P_4 &= \left(\frac{u^6}{E^2} + \frac{13}{7}u^4 + \frac{33}{35}E^2u^2 + \frac{3}{35}E^4\right)A + \\
&+ \left(-\frac{u^6}{E^2} - \frac{6}{7}u^4 - \frac{3}{35}E^2u^2\right)B \quad . \qquad (5\text{–}194)
\end{aligned}
$$

On adding these three equations we see that

$$R_0(u)P_0\left(i\frac{u}{E}\right) + R_2(u)P_2\left(i\frac{u}{E}\right) + R_4(u)P_4\left(i\frac{u}{E}\right) = 0 \quad . \tag{5–195}$$

Because of this linear relation, the three conditions (5–192) to (5–194) are in fact not independent. Therefore, one of these three conditions is superfluous and can be omitted. We omit the condition corresponding to $n = 2$ and retain those corresponding to $n = 0$ and $n = 4$. Substituting

$$\begin{aligned} G_1(u) &= R_0(u)P_0\left(i\frac{u}{E}\right) = R_0(u) \quad , \\ G_2(u) &= E^2 R_4(u)P_4\left(i\frac{u}{E}\right) \quad , \end{aligned}$$

that is

$$\begin{aligned} G_1(u) &= \left(u^2 + \frac{1}{3}E^2\right)\rho_1 + \left(-\frac{2}{3}u^4 - \frac{4}{5}E^2u^2 - \frac{2}{15}E^4\right)A(u) + \\ &+ \left(-\frac{1}{3}u^4 - \frac{1}{5}E^2u^2\right)B(u) \quad , \end{aligned} \tag{5–196}$$

$$\begin{aligned} G_2(u) &= \left(u^6 + \frac{13}{7}E^2u^4 + \frac{33}{35}E^4u^2 + \frac{3}{35}E^6\right)A(u) + \\ &+ \left(-u^6 - \frac{6}{7}E^2u^4 - \frac{3}{35}E^4u^2\right)B(u) \quad , \end{aligned} \tag{5–197}$$

we are thus finally left with the two conditions

$$\int_0^b G_1(u)du = 0 \; , \qquad \int_0^b G_2(u)du = 0 \; . \tag{5–198}$$

The functions $A(u)$ and $B(u)$ and the constant ρ_1 must satisfy these two equations; otherwise they are arbitrary.

5.8 Representation by Polynomials

First we set

$$B(u) = F(u)A(u) \tag{5–199}$$

and specify the function $F(u)$ to be a polynomial

$$F(u) = \sum_{i=0}^{N} a_i u^i \equiv \sum a_i u^i \tag{5–200}$$

(briefly). Then the functions G_1 and G_2 of (5–196) and (5–197) become