5.6 Heterogeneous Mass Distributions for the Maclaurin Ellipsoid

We have just found that by splitting off a density function $\Delta\rho$ which attains a maximum of $0.028\,\mathrm{g/cm^3}$, we are able to obtain a Maclaurin ellipsoid. Now we wish to find a density distribution that is capable of representing, to a certain extent, the actual density distribution within the earth, which is heterogeneous. We therefore look for a heterogeneous mass distribution for the Maclaurin ellipsoid, to which we then finally superpose the deviatoric density $\Delta\rho$. If the heterogeneous Maclaurin distribution represents the desired features of the actual terrestrial mass distribution, then the combined ellipsoidal distribution, obtained by adding $\Delta\rho$, will continue to do so since $\Delta\rho$ is very small.

Now the term "heterogeneous mass distribution for a Maclaurin ellipsoid" may be criticized, since the Maclaurin ellipsoid is by definition homogeneous. However, as far as the external gravity field is concerned, it is characterized by the only condition (cf. eq. (5–136)):

$$A_2 = -A_0 \quad , \tag{5–177}$$

and all mass distributions that maintain this condition (together with the numerical value of A_0) generate the same external gravity field. Therefore, by a slight extension of meaning, we shall call Maclaurin ellipsoid any ellipsoidal mass configuration for which (5–177) holds.

The simplest way of introducing heterogeneity is to use a density that depends on u only:

$$\rho(u,\theta) = g(u) \quad . \tag{5–178}$$

Comparing this with (5–121) we see that now

$$h(u) = 0 = H(u) \quad . \tag{5–179}$$

Equation (5–107) shows that then indeed (5–177) holds, so that (5–178) represents a possible mass distribution for the Maclaurin ellipsoid. The surfaces of constant density are the confocal ellipsoids $u = \text{const.}$ This expresses the well-known fact that redistributions of mass that preserve confocal stratification do not change the external gravity field. The function $g(u)$ in (5–178) is quite arbitrary; it must only satisfy the condition (5–109).

As an example, we may let $g(u)$ be a polynomial such as (1–109). Replacing r/R by u/b we may thus consider a density distribution

$$\rho = 12.19 - 16.71\left(\frac{u}{b}\right)^2 + 7.82\left(\frac{u}{b}\right)^4 \quad , \tag{5–180}$$

or generally,

$$\rho = g(u) = A + B\left(\frac{u}{b}\right)^2 + C\left(\frac{u}{b}\right)^4 \quad . \tag{5–181}$$

One condition to be satisfied by the coefficients of the polynomial (5–181) is obtained by substituting this polynomial into (5–109) after multiplication by $(u^2 + E^2/3)$ according to (5–112), and performing the integration:

$$\left(\frac{1}{3}+\frac{1}{3}e'^2\right)A+\left(\frac{1}{5}+\frac{1}{9}e'^2\right)B+\left(\frac{1}{7}+\frac{1}{15}e'^2\right)C=\left(\frac{1}{3}+\frac{1}{3}e'^2\right)\rho_0 \quad , \tag{5–182}$$

where ρ_0 is the Maclaurin density (5–169). It is readily verified that the coefficients of (5–180) satisfy this condition to two-place accuracy.

The disadvantage of a density law such as (5–178) is that the surfaces of constant density are confocal ellipsoids, whose flattening becomes infinite as $u \to 0$. To be sure, the practical effect of this fact can be made small by selecting a suitable function $g(u)$. If we select $g(u) = \text{const.}$ for $0 \le u \le u_0$ and decreasing for $u > u_0$, we shall not even have any singularity at all as $u \to 0$. Still the flattening of the surfaces of constant density increases with depth, which is not desirable.

More "natural" distributions will obviously have to be somewhat more complicated than (5–178). To keep the matter relatively simple and transparent, it will be convenient to consider any heterogeneous mass distribution of the Maclaurin ellipsoid as the superposition of

1. a homogeneous distribution of the usual Maclaurin density ρ_0, which generates the required external potential, and

2. a heterogeneous distribution $\bar{\rho}(u, \theta)$ whose external potential is zero.

The purpose of such a "zero-potential distribution" of density $\bar{\rho}(u, \theta)$ is thus to provide the desired heterogeneity without changing the external potential or the coefficients A_0^{ML} and A_2^{ML} defined by the Maclaurin density ρ_0. In other words, a heterogeneous distribution for the Maclaurin ellipsoid will be given by

$$\rho_{ML}(u, \theta) = \rho_0 + \bar{\rho}(u, \theta) \tag{5–183}$$

as the sum of the (homogeneous) Maclaurin density ρ_0 and a zero-potential density $\bar{\rho}(u, \theta)$.

The constant ρ_0 being uniquely defined by (5–164), the following section will study zero-potential density distributions.

5.7 Zero-Potential Densities

We shall thus determine density distributions inside the given ellipsoid that generate a potential which is everywhere zero outside the ellipsoid. To obtain spheroidal (nearly ellipsoidal) surfaces of equal density, we consider functions of the form

$$\bar{\rho}(u, \theta) = \rho_1 - (x^2 + y^2)A(u) - z^2 B(u) \quad , \tag{5–184}$$

where ρ_1 is a constant and A and B are functions of u to be determined.