$$\begin{aligned} A &= \frac{1}{2e'^3}\left[(1+e'^2)\arctan e' - e'\right] \quad , \\ B &= \frac{1+e'^2}{e'^3}\left(e' - \arctan e'\right) \quad , \\ C &= \frac{a^2}{e'}\arctan e' \quad . \end{aligned} \tag{5-152}$$

More about homogeneous ellipsoidal equilibrium figures can be found in the monograph (Chandrasekhar, 1969). For us the Maclaurin ellipsoid will play only an auxiliary role.

5.5 Reduction to a Maclaurin Ellipsoid

For an equipotential ellipsoid it is possible to find a density distribution which is almost homogeneous: the density deviates only very little from the constant density of a Maclaurin ellipsoid. This fact will be an essential step towards finding more realistic heterogeneous density models.

We again consider a density model of the form (5–121):

$$\rho(u,\theta) = g(u) + \frac{u^2+E^2}{u^2+E^2\cos^2\theta}h(u) \quad , \tag{5-153}$$

in which the functions $g(u)$ and $h(u)$ are to a large extent arbitrary. For the present purpose we take a $g(u)$ of the form

$$g(u) = \rho_0 - h(u) \qquad (\rho_0 = \text{const.}) \quad , \tag{5-154}$$

so that (5–153) becomes

$$\rho(u,\theta) = \rho_0 + \Delta\rho \quad , \tag{5-155}$$

where

$$\Delta\rho = \left(-1 + \frac{u^2+E^2}{u^2+E^2\cos^2\theta}\right)h(u) \quad . \tag{5-156}$$

This may be interpreted as follows. The constant density ρ_0 corresponds to a Maclaurin ellipsoid. To get our original density distribution, we must, according to (5–155), superpose to the Maclaurin density ρ_0 a density difference $\Delta\rho$. The latter can obviously be made very small since the second term within parentheses in (5–156) is, in general, very nearly 1. By (5–112) and (5–113) we have

$$\begin{aligned} G(u) &= \left(u^2+\frac{1}{3}E^2\right)g(u) = \left(u^2+\frac{1}{3}E^2\right)\rho_0 - \left(u^2+\frac{1}{3}E^2\right)h(u) \quad , \\ H(u) &= (u^2+E^2)h(u) \quad . \end{aligned} \tag{5-157}$$

Inserting this into (5–107) we obtain

$$A_0 = A_0^{ML} + \Delta A_0 \quad , \qquad A_2 = A_2^{ML} + \Delta A_2 \quad , \tag{5-158}$$

where

$$A_0^{ML} = -A_2^{ML} = \frac{4\pi}{3} i \frac{G}{E} \rho_0 a^2 b \qquad (5\text{–}159)$$

corresponds to the "Maclaurin part" and

$$\Delta A_0 = \frac{8\pi}{3} iGE \int_0^b h(u) du \ , \qquad (5\text{–}160)$$

$$\Delta A_2 = 4\pi i \frac{G}{E} \int_0^b \left(u^2 + \frac{1}{3} E^2 \right) h(u) du \qquad (5\text{–}161)$$

correspond to the "deviatoric part".

The only condition imposed on the function $h(u)$, apart from conditions of regularity, is, by (5–110),

$$\int_0^b (u^2 + E^2) h(u) du = \frac{M}{4\pi} \left(-\frac{3}{2} + \frac{15}{2} \frac{J_2}{e^2} \right) \ . \qquad (5\text{–}162)$$

After a function satisfying this condition has been found, we compute ΔA_0 by (5–160) and obtain ρ_0 from

$$\frac{4\pi}{3} i \frac{G}{E} \rho_0 a^2 b = A_0 - \Delta A_0 \ , \qquad (5\text{–}163)$$

which is a consequence of (5–158) and (5–159). It is also easy to find a more direct expression. Substituting A_0 from (5–95) and ΔA_0 from (5–160) we obtain

$$\rho_0 = \rho_m - 2 \frac{e^2}{b} \int_0^b h(u) du \ , \qquad (5\text{–}164)$$

where ρ_m is the earth's mean density (cf. sec. 5.3.1) and $e = E/a$ is the first excentricity as usual.

Polynomial representation of $h(u)$. We shall assume a $h(u)$ of the form

$$h(u) = \frac{C}{b^6} u^4 (b^2 - u^2) \qquad (5\text{–}165)$$

with a certain constant C. The factor u^4 ensures smooth behavior of $\Delta\rho$, as given by (5–156), around the "focal disk" $u = 0$: we have

$$\lim_{u \to 0} \Delta\rho = 0 \ , \qquad \lim_{u \to 0} \frac{\partial \Delta\rho}{\partial u} = 0 \ , \qquad \lim_{u \to 0} \frac{\partial \Delta\rho}{\partial \theta} = 0 \ , \qquad (5\text{–}166)$$

regardless of the value of θ. (This would not have been possible with a $h(u)$ of the simpler form const. $\times\, u^2(b^2 - u^2)$.)

The factor $b^2 - u^2$ effects that

$$\lim_{u \to b} \Delta\rho = 0 \quad , \tag{5-167}$$

regardless of θ, so that we have constant density at the surface of the ellipsoid.

The insertion of (5–165) into (5–162) gives the condition

$$\left(\frac{2}{63} + \frac{2}{35} e'^2\right) b^3 C = \frac{M}{4\pi} \left(-\frac{3}{2} + \frac{15}{2} \frac{J_2}{e^2}\right) \tag{5-168}$$

($e' = E/b$), from which C is to be determined. Then ρ_0 is obtained from (5–164), which becomes

$$\rho_0 = \rho_m - \frac{4}{35} e^2 C \quad . \tag{5-169}$$

Numerical values. Using the constants of sec. 5.3 we find from (5–168) and (5–169)

$$C = -16.534 \,\mathrm{g/cm^3} \tag{5-170}$$

and the "Maclaurin density"

$$\rho_0 = 5.527 \,\mathrm{g/cm^3} \quad . \tag{5-171}$$

The Maclaurin part of the ellipsoidal-harmonic coefficients is computed from (5–159)

$$A_0^{ML} = -A_2^{ML} = i \times 0.7656 \times 10^9 \mathrm{m^2 s^{-2}} \quad . \tag{5-172}$$

The deviations from the actual values (5–119) are thus

$$\begin{aligned} \Delta A_0 &= A_0 - A_0^{ML} = -i \times 0.0018 \times 10^9 \mathrm{m^2 s^{-2}} \quad , \\ \Delta A_2 &= A_2 - A_2^{ML} = -i \times 0.2175 \times 10^9 \mathrm{m^2 s^{-2}} \quad . \end{aligned} \tag{5-173}$$

A check is provided by computing ΔA_0 and ΔA_2 from (5–160) and (5–161), evaluated for (5–165):

$$\begin{aligned} \Delta A_0 &= i \frac{16\pi}{105} G E b C \quad , \\ \Delta A_2 &= 4\pi i \frac{G}{E} b^3 \left(\frac{2}{63} + \frac{2}{105} e'^2\right) C \quad . \end{aligned} \tag{5-174}$$

Finally, we shall determine the maximum value of the "deviatoric density" (5–156). We note that along the rotation axis, for $\theta = 0$, $\Delta\rho$ vanishes identically. The largest values of $\Delta\rho$ are found in the equatorial plane, for $\theta = \pi/2$; there we have, by (5–156) and (5–165),

$$\Delta\rho = \frac{E^2}{u^2} h(u) = \frac{E^2 C}{b^6} u^2 (b^2 - u^2) \quad . \tag{5-175}$$

This function attains its maximum (with regard to absolute value) for $u = b/\sqrt{2}$, which is

$$\Delta\rho_{max} = -0.028 \,\mathrm{g/cm^3} \quad . \tag{5-176}$$

On comparing this with (5–171), we see that an "almost homogeneous" density distribution for the equipotential ellipsoid exists, for which the deviation from homogeneity nowhere exceeds 0.5%.