5.4 The Maclaurin Ellipsoid

Assume a *homogeneous* equipotential ellipsoid of constant density ρ. Then in (5-114) we have

$$\begin{aligned} g(u) &= \rho = \text{const.} \quad , & (5\text{-}133) \\ h(u) &= 0 \quad , & (5\text{-}134) \end{aligned}$$

and all other $\alpha_{2\nu} = 0$. By (5-112) and (5-113) this implies

$$\begin{aligned} G(u) &= \left(u^2 + \tfrac{1}{3}E^2\right)\rho \quad , \\ H(u) &= 0 \quad , \end{aligned} \qquad (5\text{-}135)$$

so that (5–107) and (5-95) give

$$A_0 = -A_2 = i\frac{GM}{E} = i \cdot \frac{4\pi}{3} G\rho \frac{a^2 b}{E} \quad , \qquad (5\text{-}136)$$

since

$$M = \rho v = \rho \cdot \frac{4\pi}{3} a^2 b \quad , \qquad (5\text{-}137)$$

v denoting the volume of the ellipsoid as usual.

Now we take (5–89) and (5–92) into consideration:

$$A_2 = -i\frac{\omega^2 a^2}{3q_0} \qquad (5\text{-}138)$$

by (5–92). Combining (5–136) and (5–138) we find

$$\frac{4\pi}{3} G\rho \frac{a^2 b}{E} = \frac{\omega^2 a^2}{3q_0} \qquad (5\text{-}139)$$

or

$$\frac{\omega^2}{2\pi G\rho} = 2\frac{b}{E} q_0 = 2\frac{q_0}{e'} \quad . \qquad (5\text{-}140)$$

Finally q_0 is expressed by (5–48) with $e' = E/b$ and we get

$$\frac{\omega^2}{2\pi G\rho} = \frac{1}{e'^3}\left[(3 + e'^2)\arctan e' - 3e'\right] \quad , \qquad (5\text{-}141)$$

the well-known *Maclaurin condition.*

For our earth, with the actual values of ω from (1–77) and

$$e' = 0.082\ 094\ 439 \qquad (5\text{-}142)$$

from (5–115), this would give

$$\rho = 7.10\,\text{g/cm}^3 \quad , \qquad (5\text{-}143)$$

which clearly is much larger than the actual mean density (5–118) (the numerical equality with (5–126) is no coincidence; why?). This, of course, shows that the earth cannot be a homogeneous equilibrium figure.

In fact, the Maclaurin ellipsoid is a homogeneous figure *of equilibrium*. Its surface is a surface of constant density and, by the very definition of the level ellipsoid, also of constant gravity potential. Thus the fundamental condition of hydrostatic equilibrium (sec. 2.5) is satisfied for the surface. It is also satisfied in the interior: whatever be the shape of the internal level surfaces, they are also surfaces of constant density since $\rho =$ const. throughout.

The internal level surfaces must be ellipsoids that are geometrically similar to the outer surface (sec. 3.2.4). The gravitational potential V in the interior of the ellipsoid must have the form

$$V = 2\pi G\rho\left[C - A(x^2+y^2) - Bz^2\right] \tag{5–144}$$

with certain constants A, B, C (which have nothing to do with moments of inertia!), so that the gravity potential W becomes

$$U = V + \frac{1}{2}\omega^2(x^2+y^2) = 2\pi G\rho\left[C - \left(A - \frac{\omega^2}{4\pi G\rho}\right)(x^2+y^2) - Bz^2\right] \quad . \tag{5–145}$$

In fact,

$$\Delta V = V_{xx} + V_{yy} + V_{zz} = -4\pi G\rho(2A+B) \tag{5–146}$$

is then constant and the equipotential surfaces (including the boundary)

$$\left(A - \frac{\omega^2}{4\pi G\rho}\right)(x^2+y^2) + Bz^2 = C - \frac{U}{2\pi G\rho} = \text{const.} \tag{5–147}$$

are similar ellipsoids; cf. sec. 3.2.4 (A and B have different meanings there).

Comparing (5–146) with (1–12) we get the condition

$$2A + B = 1 \quad . \tag{5–148}$$

The condition that the external surface as given by (5–147) with $U = U_0$ must be identical to the ellipsoid

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1 \tag{5–149}$$

provides two further equations

$$a^2\left(A - \frac{\omega^2}{4\pi G\rho}\right) = b^2 B \quad , \tag{5–150}$$

$$b^2 B = C - \frac{U_0}{2\pi G\rho} \quad . \tag{5–151}$$

Eqs. (5–148) and (5–150) can be solved for A and B, and then (5–151) gives C. The result, also using (5–60) and (5–141), is

$$\begin{aligned} A &= \frac{1}{2e'^3}\left[(1+e'^2)\arctan e' - e'\right] \quad , \\ B &= \frac{1+e'^2}{e'^3}\left(e' - \arctan e'\right) \quad , \\ C &= \frac{a^2}{e'} \arctan e' \quad . \end{aligned} \tag{5–152}$$

More about homogeneous ellipsoidal equilibrium figures can be found in the monograph (Chandrasekhar, 1969). For us the Maclaurin ellipsoid will play only an auxiliary role.

5.5 Reduction to a Maclaurin Ellipsoid

For an equipotential ellipsoid it is possible to find a density distribution which is almost homogeneous: the density deviates only very little from the constant density of a Maclaurin ellipsoid. This fact will be an essential step towards finding more realistic heterogeneous density models.

We again consider a density model of the form (5–121):

$$\rho(u, \theta) = g(u) + \frac{u^2 + E^2}{u^2 + E^2 \cos^2 \theta} h(u) \quad , \tag{5–153}$$

in which the functions $g(u)$ and $h(u)$ are to a large extent arbitrary. For the present purpose we take a $g(u)$ of the form

$$g(u) = \rho_0 - h(u) \qquad (\rho_0 = \text{const.}) \quad , \tag{5–154}$$

so that (5–153) becomes

$$\rho(u, \theta) = \rho_0 + \Delta\rho \quad , \tag{5–155}$$

where

$$\Delta\rho = \left(-1 + \frac{u^2 + E^2}{u^2 + E^2 \cos^2 \theta}\right) h(u) \quad . \tag{5–156}$$

This may be interpreted as follows. The constant density ρ_0 corresponds to a Maclaurin ellipsoid. To get our original density distribution, we must, according to (5–155), superpose to the Maclaurin density ρ_0 a density difference $\Delta\rho$. The latter can obviously be made very small since the second term within parentheses in (5–156) is, in general, very nearly 1. By (5–112) and (5–113) we have

$$\begin{aligned} G(u) &= \left(u^2 + \frac{1}{3} E^2\right) g(u) = \left(u^2 + \frac{1}{3} E^2\right) \rho_0 - \left(u^2 + \frac{1}{3} E^2\right) h(u) \quad , \\ H(u) &= (u^2 + E^2) h(u) \quad . \end{aligned} \tag{5–157}$$

Inserting this into (5–107) we obtain

$$A_0 = A_0^{ML} + \Delta A_0 \quad , \qquad A_2 = A_2^{ML} + \Delta A_2 \quad , \tag{5–158}$$