and, with G determined by (1–2) to four significant digits only:

$$G = 6.673 \times 10^{-11}\mathrm{m^3 s^{-2} kg^{-1}} \quad , \tag{5–116}$$

also

$$M = \frac{GM}{G} = 5.973 \times 10^{24}\mathrm{kg} \quad , \tag{5–117}$$

$$\rho_m = 5.514\mathrm{g/cm^3} \quad , \tag{5–118}$$

for the earth's mass and mean density are meaningful to four digits only; cf. sec. 1.5.

Hence the spherical–harmonic coefficients (5–95) are

$$\begin{aligned} A_0 &= i \times 0.76382 \times 10^9\mathrm{m^2 s^{-2}} \quad , \\ A_2 &= -i \times 0.98310 \times 10^9\mathrm{m^2 s^{-2}} \quad , \end{aligned} \tag{5–119}$$

and the constants on the right-hand side of (5–109) and (5–110) are

$$\begin{aligned} \frac{M}{4\pi}\left(\frac{5}{2} - \frac{15}{2}\frac{J_2}{e^2}\right) &= 6.1181 \times 10^{23}\mathrm{kg} \quad , \\ \frac{M}{4\pi}\left(-\frac{3}{2} + \frac{15}{2}\frac{J_2}{e^2}\right) &= -1.3646 \times 10^{23}\mathrm{kg} \quad . \end{aligned} \tag{5–120}$$

5.3.1 A Simple Example

We shall now illustrate the general method by a simple example. Consider the representation (5–114), with $\alpha_n \equiv 0 \quad (n = 4, 6, 8, \ldots)$; this is obviously consistent with (5–111). Thus

$$\rho(u, \theta) = g(u) + \frac{u^2 + E^2}{u^2 + E^2 \cos^2\theta} h(u) \quad . \tag{5–121}$$

Assume

$$\begin{aligned} g(u) &= \rho_1 = \text{const.} \quad , \\ h(u) &= \begin{cases} 0 \quad , & 0 \leq u < b - \Delta b \quad , \\ -\rho_2 = \text{const.} \quad , & b - \Delta b \leq u \leq b \quad , \end{cases} \end{aligned} \tag{5–122}$$

so that

$$\rho(u, \theta) = \begin{cases} \rho_1 \quad , & 0 \leq u < b - \Delta b \quad , \\ \rho_1 - \dfrac{u^2 + E^2}{u^2 + E^2\cos^2\theta}\rho_2 \quad , & b - \Delta b \leq u \leq b \quad . \end{cases} \tag{5–123}$$

Since for Δb around 1000 km or smaller the expression $(u^2 + E^2)/(u^2 + E^2\cos^2\theta)$ is close to unity, this model represents a homogeneous core enclosed by an almost homogeneous mantle.

The regularity conditions are evidently satisfied here if $\rho > 0$, and the integral conditions determine the constant ρ_1 and give a relation between the other constants

ρ_2 and Δb. We substitute (5–121) with (5–122) into (5–109) and (5–110) and perform the elementary integrations. The result is

$$\begin{aligned} \frac{4\pi}{3}\, a^2 b \rho_1 &= \left(\frac{5}{2} - \frac{15}{2}\frac{J_2}{e^2}\right) M \quad , \\ -\frac{4\pi}{3}\, \rho_2 \Delta b (3a^2 - 3b\Delta b + \Delta b^2) &= \left(-\frac{3}{2} + \frac{15}{2}\frac{J_2}{e^2}\right) M \quad . \end{aligned} \tag{5–124}$$

As

$$\frac{4\pi}{3}\, a^2 b = v$$

is the volume of the ellipsoid and

$$\frac{M}{v} = \rho_m$$

is the mean density, we obtain from (5–124)

$$\begin{aligned} \rho_1 &= \left(\frac{5}{2} - \frac{15}{2}\frac{J_2}{e^2}\right) \rho_m \quad , \\ \rho_2 &= (\rho_1 - \rho_m) \frac{a^2 b}{\Delta b (3a^2 - 3b\Delta b + \Delta b^2)} \quad . \end{aligned} \tag{5–125}$$

The first formula determines ρ_1, which is seen to be independent of the mantle thickness Δb. With the value (5–118) for the earth's mean density we get

$$\rho_1 = 7.10\,\mathrm{g/cm^3} \quad . \tag{5–126}$$

The second formula then determines ρ_2 as a function of Δb. For instance, let

$$\Delta b = 1000\,\mathrm{km} \quad .$$

Then $\rho_2 = 3.94\,\mathrm{g/cm^3}$, so that the density at the earth's surface will be approximately $\rho_1 - \rho_2 = 3.16\,\mathrm{g/cm^3}$, which is about the value of the density at the base of the continental layers.

It is evident that such a primitive model does not represent an approximation to the mass configuration of the real earth. It was chosen merely as an illustration of the general method.

However, this model also has a certain theoretical interest because as $\Delta b \to 0$, we obtain as a limit the well-known singular mass distribution, by means of which Pizzetti (1894) has founded the theory of the equipotential ellipsoid. Pizzetti's model represents a homogeneous ellipsoid covered by a surface layer of negative density. It is, of course, quite unrealistic physically, but it has proved to be a highly successful mathematical device for deriving formulas (e.g., Lambert, 1961). As long as only the external potential is needed, any mathematical model for the mass distribution will work provided it produces an equipotential surface of the shape of an ellipsoid of revolution, and Pizzetti's model was constructed precisely so as to fulfil this requirement.

The presently preferred approach is the determination of the external potential without explicitly using any density model at all, as we did in sec. 5.2, but Pizzetti's model remains of historic interest.

Let us thus investigate the limiting case of (5–122) as $\Delta b \to 0$. As a limit, the shell enclosed between the confocal ellipsoids $u = b - \Delta b$ and $u = b$ will reduce to a surface layer on the ellipsoid $u = b$. The surface density will become, by (5–123), the negative of

$$\sigma = \lim_{\Delta b \to 0} \frac{u^2 + E^2}{u^2 + E^2 \cos^2\theta} \rho_2 \Delta n \quad , \tag{5–127}$$

where Δn is the thickness of the shell measured along the normal to the reference ellipsoid. We have

$$\Delta n \doteq \frac{dn}{du} \Delta u \quad ,$$

where by (5–65) we get

$$\frac{dn}{du} = \sqrt{\frac{u^2 + E^2 \cos^2\theta}{u^2 + E^2}} \quad ; \tag{5–128}$$

cf. also (Heiskanen and Moritz, 1967, p. 67). On the reference ellipsoid $u = b$ this reduces to

$$\sqrt{\frac{b^2 + E^2 \cos^2\theta}{b^2 + E^2}} = \frac{1}{a} \sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta} \quad .$$

On taking all this into account, the limit (5–127) becomes

$$\sigma = \frac{a}{\sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta}} \sigma_1 \quad , \tag{5–129}$$

where

$$\sigma_1 = \lim_{\Delta b \to 0} (\rho_2 \Delta b) \tag{5–130}$$

is a constant, which is determined from (5–125) as

$$\sigma_1 = \frac{1}{3} b(\rho_1 - \rho_m) \quad . \tag{5–131}$$

In this way we have recovered the singular Pizzetti distribution as a limiting case of the regular distribution (5–123), because as the limit we have a homogeneous volume distribution of density ρ_1 given by (5–125), combined with a surface layer of density $-\sigma$ given by (5–129) and (5–131).

Finally it should be mentioned that even the singular Pizzetti distribution can be expressed in the form (5–121). This is possible through the use of the well-known Dirac delta function $\delta(x)$, cf. sec. 3.3.2. This expression is

$$\rho(u, \theta) = \rho_0 - \frac{u^2 + E^2}{u^2 + E^2 \cos^2\theta} \sigma_1 \delta(u - b) \quad . \tag{5–132}$$

It shows that the use of the Dirac function makes it possible to treat formally the potential of a surface layer as the potential of a volume distribution; this fact is sometimes mathematically convenient.