5.3 Mass Distributions for the Level Ellipsoid

Consider any rotationally symmetric mass configuration such as a solid heterogeneous ellipsoid of revolution. Its external gravitational potential must have the representation

$$V(u, \theta) = \sum_{n=0}^{\infty} A_n Q_n \left(i\frac{u}{E}\right) P_n(\cos\theta) \quad . \tag{5-74}$$

This is a slightly different form of (5–37): we have put

$$A_n = \frac{a_n}{Q_n\left(i\frac{b}{E}\right)} \tag{5-75}$$

for the coefficients. Furthermore the overbar on θ will be dropped from now on since we are using ellipsoidal coordinates exclusively and no confusion with spherical coordinates is likely to arise. Henceforth,

$$\theta = 90° - \beta \tag{5-76}$$

will denote *the complement of the reduced latitude* and no longer the spherical distance. Thus, our ellipsoidal coordinates will be denoted by u, θ, λ.

To derive the coefficients A_n in terms of the density ρ, we start from the basic equation (1–1):

$$V(u, \theta) = G \iiint_E \frac{\rho(u', \theta')}{l}\, dv \quad , \tag{5-77}$$

where the integral is extended over the reference ellipsoid $u = b$ (which need not yet be a level surface), denoted by E, and both potential V and density ρ are functions only of u and θ, but not of λ because of rotational symmetry.

The volume element in ellipsoidal coordinates may be found by transforming

$$dv = dx\,dy\,dz = J\,du\,d\theta\,d\lambda \tag{5-78}$$

with Jacobian determinant

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \lambda} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \lambda} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \lambda} \end{vmatrix} \tag{5-79}$$

in analogy to (4–15) and (4–16). The result is

$$dv = (u'^2 + E^2\cos^2\theta')\sin\theta'\,du'\,d\theta'\,d\lambda' \tag{5-80}$$

or

$$dv = (u'^2 + E^2\cos^2\theta')\,du'\,d\sigma \quad , \tag{5-81}$$

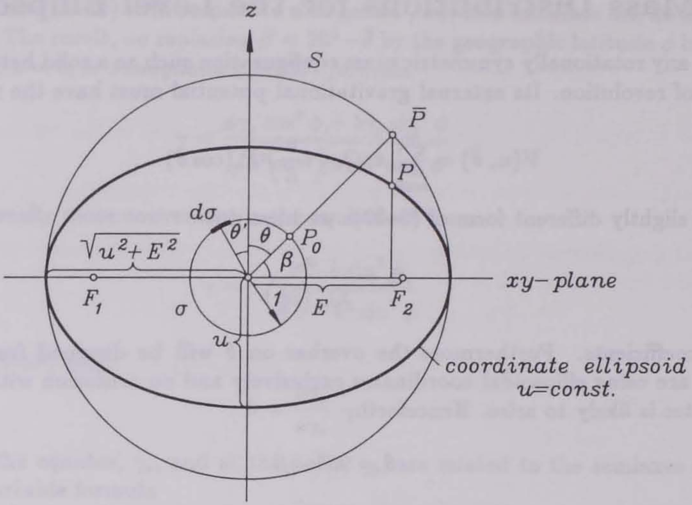

FIGURE 5.3: A coordinate ellipsoid u = const. and the auxiliary spheres S and σ

with

$$d\sigma = \sin\theta' d\theta' d\lambda' \tag{5–82}$$

denoting the element of solid angle as usual; more precisely, it is the surface element of the auxiliary unit sphere σ on which the point P_0 in Fig. 5.3 is situated. The primes indicate that dv refers to the integration point (u', θ', λ'). For $E \to 0$, eq. (5–80) reduces to the usual expression for the volume element in spherical coordinates.

At this point it is appropriate to use Fig. 5.3 to recall the geometric situation and make it completely clear. Take an arbitrary point P (u, β, λ) in space and pass the appropriate coordinate ellipsoid u = const. through it. Its semiaxes are u and $\sqrt{u^2+E^2}$. The auxiliary "affine" sphere S thus has the radius $\sqrt{u^2+E^2}$. For the reduced latitude β or its complement θ we have the familiar construction $P \to \bar{P}$; θ is the polar distance, not of P, but of the auxiliary point $\bar{P}$. As we have seen, we also need the concentric unit sphere σ; to P there corresponds the auxiliary point P_0 on σ.

Repeat the same construction for the point Q (u', θ', λ') which carries the volume element dv, but note that the coordinate ellipsoid u' = const. and the auxiliary sphere S will be different! The concentric unit sphere σ, however, remains of course the same. In this way, to Q there corresponds on σ the auxiliary point Q_0 which carries the surface element $d\sigma$. The coordinate ellipsoid u' = const. and the details of the construction $Q \to \bar{Q} \to Q_0$ are not shown in order not to overload the figure.

Orthogonality relations such as (1–41) will be used later; the corresponding integrals

$$\iint_\sigma d\sigma = \int_{\lambda'=0}^{2\pi} \int_{\theta'=0}^{\pi} \sin\theta' d\theta' d\lambda' \tag{5–83}$$

can either be regarded in a purely formal way or else interpreted by means of the construction shown in Fig. 5.3, as integrals over the auxiliary unit sphere σ.

Now we express $1/l$ by (5–32) which is permissible since for an external point there is $u > u'$. Interchanging integration and summation we get

$$\begin{aligned} V(u, \theta) = G \sum_{n=0}^{\infty} \sum_{m=0}^{n} \int_{\lambda'=0}^{2\pi} \int_{\theta'=0}^{\pi} \int_{u'=0}^{b} \rho(u', \theta') \cdot \\ \cdot C_{nm} Q_{nm}\left(i\frac{u}{E}\right) P_{nm}\left(i\frac{u'}{E}\right) P_{nm}(\cos\theta) P_{nm}(\cos\theta') \cdot \\ \cdot (\cos m\lambda \cos m\lambda' + \sin m\lambda \sin m\lambda')(u'^2 + E^2 \cos^2\theta') d\sigma du' \quad , \end{aligned} \tag{5–84}$$

as $\cos m(\lambda - \lambda') = \cos m\lambda \cos m\lambda' + \sin m\lambda \sin m\lambda'$. Since ρ does not depend on λ', orthogonality (as explained above) is immediately seen to remove all nonzonal terms ($m \neq 0$), and there remains (5–74) with

$$\begin{aligned} A_n &= i\frac{G}{E}(2n+1) \int_{\lambda=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{u=0}^{b} \rho(u, \theta) P_n\left(i\frac{u}{E}\right) P_n(\cos\theta) \cdot \\ &\cdot (u^2 + E^2\cos^2\theta) \sin\theta du d\theta d\lambda \quad ; \end{aligned} \tag{5–85}$$

here C_{n0} as given by (5-33) has been taken into account, and the primes have been omitted for simplicity, which obviously is possible since A_n are constants.

It is appropriate to expand ρ as a series in the following way:

$$(u^2 + E^2\cos^2\theta)\rho(u, \theta) = \frac{1}{4\pi} \sum_{\nu=0}^{\infty} \alpha_{2\nu}(u) P_{2\nu}(\cos\theta) \quad ; \tag{5–86}$$

cf. also (Heine, 1961, vol. II, p. 107). By taking only *even* harmonics (subscript 2ν) we restrict ourselves to density distributions that are symmetric with respect to the equator; for the ellipsoid of revolution this is as natural as rotational symmetry. The functions $\alpha_{2\nu}(u)$ are to subject to the condition that the mass distribution produces a given external potential.

On substituting (5–86), the expression (5–85) can readily be integrated. Because of orthogonality, only the term with $2\nu = n$ survives, and (1–42) applies. The result is

$$A_n = i\frac{G}{E} \int_0^b \alpha_n(u) P_n\left(i\frac{u}{E}\right) du \quad . \tag{5–87}$$

Given A_n, this is the only condition which the function $\alpha_n(u)$ for the density must satisfy. Obviously $A_n = 0$ for odd n.

For the *level ellipsoid* we get by (5–75) and (5–45)

$$A_0 = \frac{U_0 - \frac{1}{3}\omega^2 a^2}{Q_0\left(i\frac{b}{E}\right)} \quad , \tag{5–88}$$

$$A_2 = \frac{\omega^2 a^2}{3Q_2\left(i\frac{b}{E}\right)} \quad , \tag{5–89}$$

$$A_n = 0 \quad \text{otherwise} \quad . \tag{5–90}$$

By (5–20) and (5–57) we have

$$Q_0\left(i\frac{b}{E}\right) = -i \arctan e' \quad . \tag{5–91}$$

By (5–21) and (5–48) there is

$$Q_2\left(i\frac{b}{E}\right) = iq_0 \quad , \tag{5–92}$$

which can be expressed by (5–71) with (5–72):

$$\frac{1}{q_0} = \frac{GM}{\omega^2 a^2 E}\left(\frac{15}{2} - \frac{45}{2}\frac{J_2}{e^2}\right) \quad . \tag{5–93}$$

Finally,

$$U_0 - \frac{1}{3}\omega^2 a^2 = \frac{GM}{E} \arctan e' \tag{5–94}$$

by (5–60). Combining all these relations we find simply

$$A_0 = i\frac{GM}{E} \quad , \quad A_2 = i\frac{GM}{E}\left(-\frac{5}{2} + \frac{15}{2}\frac{J_2}{e^2}\right) \quad , \tag{5–95}$$

$$A_4 = A_6 = \ldots = 0 \quad ; \quad A_1 = A_3 = A_5 = \ldots = 0 \quad . \tag{5–96}$$

Thus the functions α_n with $n = 2\nu$ must satisfy the conditions (5–87) with the constants A_n given by (5–95) and (5–96).

Since only A_0 and A_2 are different from zero, it is convenient to split off the terms of degrees zero and two in the expansion (5–86), obtaining

$$(u^2 + E^2 \cos^2\theta)\rho(u, \theta) = R(u, \theta) + S(u, \theta) \quad , \tag{5–97}$$

where

$$R(u, \theta) = \frac{1}{4\pi}\left[\alpha_0(u) + \alpha_2(u) P_2(\cos\theta)\right] \quad , \tag{5–98}$$

$$S(u, \theta) = \frac{1}{4\pi}\sum_{\nu=2}^{\infty} \alpha_{2\nu}(u) P_{2\nu}(\cos\theta) \quad . \tag{5–99}$$

Let us first consider the function $R(u, \theta)$. By a simple transformation we get

$$\begin{aligned} R(u, \theta) &= \frac{1}{4\pi}\left[\alpha_0(u) - \frac{3}{2E^2}\left(u^2 + \frac{1}{3}E^2\right)\alpha_2(u) + \right. \\ &+ \left. \frac{3}{2E^2}\alpha_2(u)(u^2 + E^2\cos^2\theta)\right] \quad . \end{aligned} \tag{5–100}$$

It will be convenient to introduce new functions $G(u)$ and $H(u)$ by

$$G(u) = \frac{1}{4\pi}\frac{3}{2E^2}\left(u^2 + \frac{1}{3}E^2\right)\alpha_2(u) \quad , \tag{5–101}$$

$$H(u) = \frac{1}{4\pi}\left[\alpha_0(u) - \frac{3}{2E^2}\left(u^2 + \frac{1}{3}E^2\right)\alpha_2(u)\right] \quad , \tag{5–102}$$

in terms of which (5–100) becomes

$$R(u, \theta) = H(u) + \frac{u^2 + E^2\cos^2\theta}{u^2 + \frac{1}{3}E^2}G(u) \quad . \tag{5–103}$$

Since by (5–101) and (5–102) we have

$$\alpha_0(u) = 4\pi\left[G(u) + H(u)\right] \quad , \tag{5–104}$$

$$\alpha_2(u) = \frac{8\pi}{3}\frac{E^2}{u^2 + \frac{1}{3}E^2}G(u) \quad , \tag{5–105}$$

and since by (1–33)

$$P_2\left(i\frac{u}{E}\right) = -\frac{3}{2E^2}\left(u^2 + \frac{1}{3}E^2\right) \quad , \tag{5–106}$$

we obtain from (5–87) the simple expressions

$$\begin{aligned} A_0 &= 4\pi i\frac{G}{E}\int_0^b \left[G(u) + H(u)\right]du \quad , \\ A_2 &= -4\pi i\frac{G}{E}\int_0^b G(u)du \quad . \end{aligned} \tag{5–107}$$

Now we are in the position to formulate our solution for the problem of density distributions for the level ellipsoid. By (5-97), (5–99), and (5–103) we may express the density in the functional form

$$\begin{aligned} \rho(u, \theta) &= \frac{G(u)}{u^2 + \frac{1}{3}E^2} + \frac{H(u)}{u^2 + E^2\cos^2\theta} + \\ &+ \frac{1}{u^2 + E^2\cos^2\theta}\cdot\frac{1}{4\pi}\sum_{\nu=2}^{\infty}\alpha_{2\nu}(u)P_{2\nu}(\cos\theta) \quad . \end{aligned} \tag{5–108}$$

The conditions to be satisfied by the functions $G(u)$, $H(u)$, and $\alpha_n(u)$ are as follows. From (5–87), (5–95), (5–96), and (5–107) we have the integral conditions

$$\int_0^b G(u)du = \frac{M}{4\pi}\left(\frac{5}{2} - \frac{15}{2}\frac{J_2}{e^2}\right) , \qquad (5\text{–}109)$$

$$\int_0^b H(u)du = \frac{M}{4\pi}\left(-\frac{3}{2} + \frac{15}{2}\frac{J_2}{e^2}\right) , \qquad (5\text{–}110)$$

and

$$\int_0^b \alpha_n(u)P_n\left(i\frac{u}{E}\right) du = 0 \quad , \qquad n = 4, 6, 8, \ldots \qquad (5\text{–}111)$$

In addition, we have certain regularity conditions. The functions $G(u)$, $H(u)$, and $\alpha_{2\nu}(u)$ must be chosen in such a way that by (5–108) the density ρ is regular, i.e., everywhere finite, piecewise continuous, and positive. It is easy to see that this set of conditions is necessary and sufficient; that is, any function of the form (5–108) for which (5–109), (5–110), (5–111), and the regularity conditions are satisfied, represents a possible mass configuration for the level ellipsoid.

For later application, it will be often convenient to substitute

$$G(u) = \left(u^2 + \frac{1}{3}E^2\right) g(u) , \qquad (5\text{–}112)$$

$$H(u) = (u^2 + E^2)h(u) , \qquad (5\text{–}113)$$

so that the density model (5–108) assumes the form

$$\rho(u, \theta) = g(u) + \frac{u^2 + E^2}{u^2 + E^2\cos^2\theta} h(u) + + \frac{1}{u^2 + E^2\cos^2\theta} \cdot \frac{1}{4\pi}\sum_{\nu=2}^{\infty} \alpha_{2\nu}(u)P_{2\nu}(\cos\theta) . \qquad (5\text{–}114)$$

Thus we have obtained a rather general solution of our problem. It would be trivial to generalize our argument so as to obtain solutions that are not symmetric with respect to the axis of rotation and to the equatorial plane but, as we have mentioned, such solutions appear to be of no physical significance.

Numerical values. With the values of sec. 1.5 for the Geodetic Reference System 1980 (cf. also Moritz, 1984) we have

$$\begin{aligned} a &= 6378\,137 \text{ m} , \\ b &= 6356\,752 \text{ m} , \\ E &= 521\,854 \text{ m} , \\ e^2 &= 0.006\,694\,380 , \\ e'^2 &= 0.006\,739\,497 , \\ GM &= 3986\,005 \times 10^8 \text{m}^3\text{s}^{-2} , \\ J_2 &= 0.001\,082\,63 , \end{aligned} \qquad (5\text{–}115)$$

and, with G determined by (1–2) to four significant digits only:

$$G = 6.673 \times 10^{-11} \mathrm{m}^3 \mathrm{s}^{-2} \mathrm{kg}^{-1} \quad , \tag{5–116}$$

also

$$M = \frac{GM}{G} = 5.973 \times 10^{24} \mathrm{kg} \quad , \tag{5–117}$$

$$\rho_m = 5.514 \mathrm{g/cm}^3 \quad , \tag{5–118}$$

for the earth's mass and mean density are meaningful to four digits only; cf. sec. 1.5.

Hence the spherical–harmonic coefficients (5–95) are

$$\begin{aligned} A_0 &= i \times 0.76382 \times 10^9 \mathrm{m}^2 \mathrm{s}^{-2} \quad , \\ A_2 &= -i \times 0.98310 \times 10^9 \mathrm{m}^2 \mathrm{s}^{-2} \quad , \end{aligned} \tag{5–119}$$

and the constants on the right-hand side of (5–109) and (5–110) are

$$\begin{aligned} \frac{M}{4\pi}\left(\frac{5}{2} - \frac{15}{2}\frac{J_2}{e^2}\right) &= 6.1181 \times 10^{23} \mathrm{kg} \quad , \\ \frac{M}{4\pi}\left(-\frac{3}{2} + \frac{15}{2}\frac{J_2}{e^2}\right) &= -1.3646 \times 10^{23} \mathrm{kg} \quad . \end{aligned} \tag{5–120}$$

5.3.1 A Simple Example

We shall now illustrate the general method by a simple example. Consider the representation (5–114), with $\alpha_n \equiv 0 \quad (n = 4, 6, 8, \ldots)$; this is obviously consistent with (5–111). Thus

$$\rho(u, \theta) = g(u) + \frac{u^2 + E^2}{u^2 + E^2 \cos^2 \theta} h(u) \quad . \tag{5–121}$$

Assume

$$\begin{aligned} g(u) &= \rho_1 = \text{const.} \quad , \\ h(u) &= \begin{cases} 0 \quad , & 0 \le u < b - \Delta b \quad , \\ -\rho_2 = \text{const.} \quad , & b - \Delta b \le u \le b \quad , \end{cases} \end{aligned} \tag{5–122}$$

so that

$$\rho(u, \theta) = \begin{cases} \rho_1 \quad , & 0 \le u < b - \Delta b \quad , \\ \rho_1 - \dfrac{u^2 + E^2}{u^2 + E^2 \cos^2 \theta} \rho_2 \quad , & b - \Delta b \le u \le b \quad . \end{cases} \tag{5–123}$$

Since for Δb around 1000 km or smaller the expression $(u^2 + E^2)/(u^2 + E^2 \cos^2 \theta)$ is close to unity, this model represents a homogeneous core enclosed by an almost homogeneous mantle.

The regularity conditions are evidently satisfied here if $\rho > 0$, and the integral conditions determine the constant ρ_1 and give a relation between the other constants

ρ_2 and Δb. We substitute (5–121) with (5–122) into (5–109) and (5–110) and perform the elementary integrations. The result is

$$\begin{aligned} \frac{4\pi}{3} a^2 b \rho_1 &= \left(\frac{5}{2} - \frac{15}{2}\frac{J_2}{e^2}\right) M \quad , \\ -\frac{4\pi}{3} \rho_2 \Delta b (3a^2 - 3b\Delta b + \Delta b^2) &= \left(-\frac{3}{2} + \frac{15}{2}\frac{J_2}{e^2}\right) M \quad . \end{aligned} \tag{5–124}$$

As

$$\frac{4\pi}{3} a^2 b = v$$

is the volume of the ellipsoid and

$$\frac{M}{v} = \rho_m$$

is the mean density, we obtain from (5–124)

$$\begin{aligned} \rho_1 &= \left(\frac{5}{2} - \frac{15}{2}\frac{J_2}{e^2}\right) \rho_m \quad , \\ \rho_2 &= (\rho_1 - \rho_m) \frac{a^2 b}{\Delta b (3a^2 - 3b\Delta b + \Delta b^2)} \quad . \end{aligned} \tag{5–125}$$

The first formula determines ρ_1, which is seen to be independent of the mantle thickness Δb. With the value (5–118) for the earth's mean density we get

$$\rho_1 = 7.10\,\mathrm{g/cm^3} \quad . \tag{5–126}$$

The second formula then determines ρ_2 as a function of Δb. For instance, let

$$\Delta b = 1000\,\mathrm{km} \quad .$$

Then $\rho_2 = 3.94\,\mathrm{g/cm^3}$, so that the density at the earth's surface will be approximately $\rho_1 - \rho_2 = 3.16\,\mathrm{g/cm^3}$, which is about the value of the density at the base of the continental layers.

It is evident that such a primitive model does not represent an approximation to the mass configuration of the real earth. It was chosen merely as an illustration of the general method.

However, this model also has a certain theoretical interest because as $\Delta b \to 0$, we obtain as a limit the well-known singular mass distribution, by means of which Pizzetti (1894) has founded the theory of the equipotential ellipsoid. Pizzetti's model represents a homogeneous ellipsoid covered by a surface layer of negative density. It is, of course, quite unrealistic physically, but it has proved to be a highly successful mathematical device for deriving formulas (e.g., Lambert, 1961). As long as only the external potential is needed, any mathematical model for the mass distribution will work provided it produces an equipotential surface of the shape of an ellipsoid of revolution, and Pizzetti's model was constructed precisely so as to fulfil this requirement.

The presently preferred approach is the determination of the external potential without explicitly using any density model at all, as we did in sec. 5.2, but Pizzetti's model remains of historic interest.

Let us thus investigate the limiting case of (5–122) as $\Delta b \to 0$. As a limit, the shell enclosed between the confocal ellipsoids $u = b - \Delta b$ and $u = b$ will reduce to a surface layer on the ellipsoid $u = b$. The surface density will become, by (5–123), the negative of

$$\sigma = \lim_{\Delta b \to 0} \frac{u^2 + E^2}{u^2 + E^2 \cos^2 \theta} \rho_2 \Delta n \quad , \tag{5–127}$$

where Δn is the thickness of the shell measured along the normal to the reference ellipsoid. We have

$$\Delta n \doteq \frac{dn}{du} \Delta u \quad ,$$

where by (5–65) we get

$$\frac{dn}{du} = \sqrt{\frac{u^2 + E^2 \cos^2 \theta}{u^2 + E^2}} \quad ; \tag{5–128}$$

cf. also (Heiskanen and Moritz, 1967, p. 67). On the reference ellipsoid $u = b$ this reduces to

$$\sqrt{\frac{b^2 + E^2 \cos^2 \theta}{b^2 + E^2}} = \frac{1}{a} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \quad .$$

On taking all this into account, the limit (5–127) becomes

$$\sigma = \frac{a}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} \sigma_1 \quad , \tag{5–129}$$

where

$$\sigma_1 = \lim_{\Delta b \to 0} (\rho_2 \Delta b) \tag{5–130}$$

is a constant, which is determined from (5–125) as

$$\sigma_1 = \frac{1}{3} b(\rho_1 - \rho_m) \quad . \tag{5–131}$$

In this way we have recovered the singular Pizzetti distribution as a limiting case of the regular distribution (5–123), because as the limit we have a homogeneous volume distribution of density ρ_1 given by (5–125), combined with a surface layer of density $-\sigma$ given by (5–129) and (5–131).

Finally it should be mentioned that even the singular Pizzetti distribution can be expressed in the form (5–121). This is possible through the use of the well-known Dirac delta function $\delta(x)$, cf. sec. 3.3.2. This expression is

$$\rho(u, \theta) = \rho_0 - \frac{u^2 + E^2}{u^2 + E^2 \cos^2 \theta} \sigma_1 \delta(u - b) \quad . \tag{5–132}$$

It shows that the use of the Dirac function makes it possible to treat formally the potential of a surface layer as the potential of a volume distribution; this fact is sometimes mathematically convenient.